Advanced Mathematical Methods (ACM30020)

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Exercises #6

1. Consider the FIE
$$f(\theta) = h(\theta) + \int_0^{2\pi} K(\theta - \varphi) f(\varphi) \mathrm{d}\varphi, \tag{1}$$
where K is the kernel function
$$K(\varphi) = k_0 + k_1 \cos\varphi, \tag{2}$$
and where k_0 and k_1 are constant.
   (a) Using the trial solution
   $$f(\theta) = h(\theta) + p + q\cos\theta + r\sin\theta,$$
   solve the FIE (1).
   (b) Hence or otherwise, find the eigenvalues of the FIE
   $$y(\theta) = \lambda \int_0^{2\pi} K(\theta - \varphi) y(\varphi) \mathrm{d}\varphi,$$
   for the kernel function 2.

2. Consider the FIE
$$f(x) = g(x) + \lambda \int_{-\infty}^{\infty} K(x, y) f(y) \mathrm{d}y, \tag{3}$$
where $K(x, y) = 1$ inside the square $\{|x| < a, |y| < a\}$ and zero elsewhere. Using a trial solution
$$f(x) = g(x) + (\cdots),$$
find the solution of Equation (3).

3. The kernel of the FIE
$$\psi(x) = \lambda \int_a^b K(x, y) \psi(y) \mathrm{d}y$$
has the form
$$K(x, y) = \sum_{n=1}^{\infty} h_n(x) g_n(y),$$
where the $h_n(x)$ form a complete orthonormal set of functions over the interval $[a, b]$.

(a) Show that the eigenvalues λ_i are given by:

$$\left|M - \lambda^{-1}\mathbb{I}\right| = 0,$$

where M is the matrix with elements

$$M_{k\ell} = \int_a^b g_k(s)h_\ell(s)\mathrm{d}s.$$

If the corresponding solutions are $\psi^{(i)}(x) = \sum_{n=1}^{\infty} a_n^{(i)} h_n(x)$, find an expression for the $a_n^{(i)}$.

(b) Obtain the eigenvalues and eigenfunctions over the interval $[0, 2\pi]$ if

$$K(x, y) = \sum_{n=1}^{\infty} \frac{1}{n} \cos(nx)\cos(ny).$$

4. Use Fredholm theory to show that, for the kernel

$$K(x, z) = (x + z)\mathrm{e}^{x-z},$$

over the interval $[0, 1]$, the resolvent kernel is:

$$\Gamma(x, z; \lambda) = \frac{\mathrm{e}^{x-z}\left[(x + z) - \lambda\left(\frac{1}{2}x + \frac{1}{2}z - xz - \frac{1}{3}\right)\right]}{1 - \lambda - \frac{1}{12}\lambda^2}.$$

Hence, solve

$$y(x) = x^2 + 2\int_0^1 (x + z)\mathrm{e}^{x-z} y(z)\mathrm{d}z,$$

Leave your answer in terms of I_n, where $I_n = \int_0^1 u^n \mathrm{e}^{-u}\mathrm{d}u$.