Applied Analysis (ACM30020)

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Exercises #5

1. Given the following second order linear homogeneous ODE:

$$y'' + \omega^2 y = 0,$$

where ω is a real number. The initial conditions are: $y(0) = y_0$ and $y'(0) = 0$.

Transform the problem into a Volterra integral equation. Solve the integral equation using an iterative scheme.

From class notes, the general solution of the IVP is:

$$y(x) = y_0 + p_2(x_0)y_0' P(x) + \int_{x_0}^{x} [r(t) - p_0(t)y(t)][P(x) - P(t)]\mathrm{d}t,$$

where

$$P(x) = \int_{x_0}^{x} \frac{\mathrm{d}s}{p_2(s)}.$$

Reading off, we have $x_0 = 0$, $r(x) = 0$, $p_2(x) = 1$, and $p_0(x) = \omega^2$. Also, $y_0' = 0$. Hence, $P(x) = x$, and

$$y(x) = y_0 - \omega^2 \int_0^x y(t)(x - t)\mathrm{d}t. \tag{1}$$

This equation is in the form of a Volterra integral equation. We solve the equation iteratively using $y^{(0)}(x) = y_0$, and

$$\begin{aligned} y^{(1)}(x) &= y_0 - \omega^2 \int_0^x y^{(0)}(t)(x - t)\mathrm{d}t, \\ &= y_0 - \omega^2 y_0 \int_0^x (x - t)\mathrm{d}t, \\ &= y_0 \left(1 - \tfrac{1}{2}\omega^2 x^2\right). \end{aligned}$$

Next, we have:

$$\begin{aligned} y^{(2)}(x) &= y_0 - \omega^2 \int_0^x y^{(1)}(t)(x - t)\mathrm{d}t, \\ &= y_0 - \omega^2 y_0 \int_0^x \left(1 - \tfrac{1}{2}\omega^2 t^2\right)(x - t)\mathrm{d}t, \\ &= y_0 \left(1 - \tfrac{1}{2}\omega^2 x^2 + \tfrac{1}{4!}\omega^4 x^4\right). \end{aligned}$$

Guessing the pattern, and letting $n \to \infty$, we obtain:

$$y(x) = y_0 \cos(\omega x).$$

We can double-check this by considering the integral

$$I = y_0 - \omega^2 y_0 \int_0^x \cos(\omega t)(x-t)\mathrm{d}t,$$

which by direct evaluation gives:

$$\begin{aligned} \frac{I}{y_0} &= 1 - \omega^2 t \frac{\sin \omega t}{\omega}\bigg|_0^x + \omega^2 \left[\frac{1}{\omega^2}\cos(\omega t)\bigg|_0^x + \frac{t \sin \omega t}{\omega}\bigg|_0^x \right], \\ &= \cos(\omega x). \end{aligned}$$

Hence, $y(x) = y_0 \cos(\omega x)$ does indeed solve the integral equation (1).

2. Consider the inhomogeneous problem

$$y'' + \omega^2 y = f(x), \qquad x \in [0, \pi].$$

Here, the problem is a Boundary Value Problem (BVP), with boundary conditions $y(0) = y(\pi) = 0,$.

Use the Green's Function to solve the BVP in the case when:

(a) $f(x) = 1$;

(b) $f(x) = \sin(\omega x)$.

Linearly independent solutions of the homogeneous problem are:

$$u(x) = \sin(\omega x),$$

and

$$v(x) = \sin(\omega x) - \tan(\omega \pi)\cos(\omega x).$$

Hence, $u(x)$ satisfies the left-hand Boundary Condition (LHBC) at $x = 0$ and $v(x)$ satisfies the right-hand Boundary Condition (RHBC) at $x = \pi$. The Wronskian is:

$$W = \left| \begin{array}{cc} \sin \omega x & \sin \omega x - \tan \omega \pi \cos \omega x \\ \omega \cos \omega x & \omega \cos x + \omega \tan \omega \pi \sin \omega x \end{array} \right|.$$

By linear dependence, this simplifies:

$$W = \left| \begin{array}{cc} \sin \omega x & -\tan \omega \pi \cos \omega x \\ \omega \cos \omega x & \omega \tan \omega \pi \sin \omega x \end{array} \right| = \omega \tan(\omega \pi).$$

Thus, $W =$ Const., consistent with Abel's Theorem with $p_1/p_2 = 0/1$.

The Green's function for the BVP is now:

$$G(x,s) = \frac{[\sin(\omega x) - \tan(\omega\pi)\cos(\omega x)]\sin(\omega s)}{\omega\tan\omega\pi}, \qquad 0 \le s \le x \le \pi,$$

and

$$G(x,s) = \frac{\sin(\omega x)\,[\sin(\omega s) - \tan(\omega\pi)\cos(\omega s)]}{\omega\tan\omega\pi}, \qquad 0 \le x < s \le \pi.$$

By convolution, the solution to the BVP is:

$$y(x) = \int_0^\pi G(x,s)f(s)\mathrm{d}s.$$

Hence,

$$y(x) = \int_0^x \frac{[\sin(\omega x) - \tan(\omega\pi)\cos(\omega x)]\sin(\omega s)}{\omega\tan\omega\pi} f(s)\mathrm{d}s + \int_x^\pi \frac{\sin(\omega x)\,[\sin(\omega s) - \tan(\omega\pi)\cos(\omega s)]}{\omega\tan\omega\pi} f(s)\mathrm{d}s,$$

or

$$y(x) = \int_0^x \frac{[\sin(\omega x)\cos(\omega\pi) - \sin(\omega\pi)\cos(\omega x)]\sin(\omega s)}{\omega\sin\omega\pi} f(s)\mathrm{d}s + \int_x^\pi \frac{\sin(\omega x)\,[\sin(\omega s)\cos(\omega\pi) - \sin(\omega\pi)\cos(\omega s)]}{\omega\sin\omega\pi} f(s)\mathrm{d}s.$$

Or again,

$$y(x) = \int_0^x \frac{\sin(\omega(x-\pi))\sin(\omega s)}{\omega\sin\omega\pi} f(s)\mathrm{d}s + \int_x^\pi \frac{\sin(\omega x)\sin(\omega(s-\pi))}{\omega\sin\omega\pi} f(s)\mathrm{d}s.$$

Finally,

$$y(x) = \frac{\sin(\omega(x-\pi))}{\omega\sin\omega\pi}\int_0^x \sin(\omega s)f(s)\mathrm{d}s + \frac{\sin(\omega x)}{\omega\sin\omega\pi}\int_x^\pi \sin(\omega(s-\pi))f(s)\mathrm{d}s.$$

We fill in for $f(x) = 1$:

$$\begin{aligned}
y(x) &= \frac{\sin(\omega(x-\pi))}{\omega\sin\omega\pi}\int_0^x \sin(\omega s)\mathrm{d}s + \frac{\sin(\omega x)}{\omega\sin\omega\pi}\int_x^\pi \sin(\omega(s-\pi))\mathrm{d}s, \\
&= \frac{\sin(\omega(x-\pi))}{\omega^2\sin\omega\pi}\,[1-\cos(\omega x)] + \frac{\sin(\omega x)}{\omega^2\sin\omega\pi}\,[\cos(\omega(x-\pi)) - 1] \\
&= \frac{1}{\omega^2\sin(\omega\pi)}\,[\sin(\omega(x-\pi) - \sin(\omega x)] \\
&\qquad + \frac{1}{\omega^2\sin\omega\pi}\,[\sin(\omega x)\cos(\omega(x-\pi)) - \cos(\omega x)\sin(\omega(x-\pi))].
\end{aligned}$$

Hence,

$$y(x) = \frac{1}{\omega^2\sin(\omega\pi)}\,[\sin(\omega(x-\pi) - \sin(\omega x)] + \frac{1}{\omega^2}. \tag{2}$$

Clearly, $y''(x) + \omega^2 y = 1$, and $y(0) = 0$ and $y(\pi) = 0$, so our calculations are correct and Equation (2) does indeed solve the BVP.

Next, we fill in for $f(x) = \sin(\omega x)$:

$$y(x) = \frac{\sin(\omega(x-\pi))}{\omega \sin \omega\pi} \int_0^x \sin(\omega s)\sin(\omega s)\mathrm{d}s + \frac{\omega \sin(\omega x)}{\sin \omega\pi} \int_x^\pi \sin(\omega(s-\pi))\sin(\omega s)\mathrm{d}s.$$

This 'simplifies' to:

$$y(x) = \frac{\sin(\omega(x-\pi))}{\omega \sin \omega\pi}\left[\tfrac{1}{2}x - \frac{\sin(2\omega x)}{4\omega}\right] + \frac{\sin(\omega x)}{\omega \sin \omega\pi}\left[\tfrac{1}{2}(\pi - x)\cos(\omega\pi) - \frac{\sin(\omega(\pi - 2x))}{4\omega} - \frac{\sin(\omega\pi)}{4\omega}\right] \quad (3)$$

We have checked this result using a numerical 'shooting' method (Figure 1). Matlab listings are provided in the Appendix.

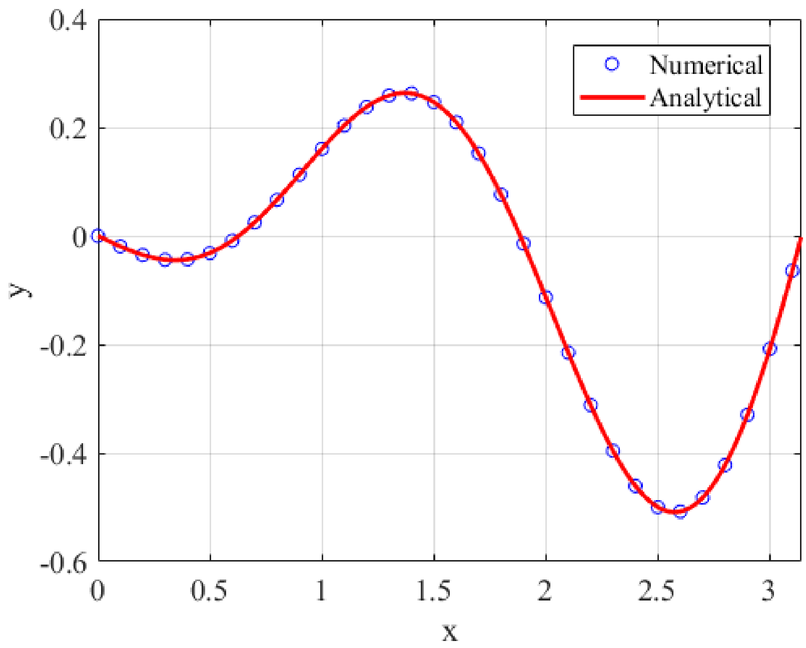

Figure 1: Comparison between numerical shooting method and Equation (3), for $\omega = 2.5$.

3. The stationary temperature distribution in a rod of unit length that has both ends kept at a constant zero temperature, with heat loss through its surface proportional to u, and that is subject to a given non-uniform heat source per unit length $f(x)$, is the solution of

$$-u'' + u = f, \qquad u(0) = u(1) = 0.$$

Show that the Green's function of this Boundary Value Problem is given by:

$$G(x,\xi) = \begin{cases} \frac{\sinh(x)\sinh(1-\xi)}{\sinh(1)}, & 0 \le x \le \xi \le 1, \\ \frac{\sinh(\xi)\sinh(1-x)}{\sinh(1)}, & 0 \le \xi \le x \le 1. \end{cases}$$

Linearly independent solutions of the homogeneous problem are:

$$u(x) = \sinh(x)$$

and

$$v(x) = \sinh(1)\cosh(x) - \cosh(1)\sinh(x).$$

Hence, $u(x)$ satisfies the left-hand Boundary Condition (LHBC) at $x = 0$ and $v(x)$ satisfies the right-hand Boundary Condition (RHBC) at $x = 1$. The Wronskian is:

$$\begin{aligned} W &= \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}, \\ &= \begin{vmatrix} \sinh(x) & \sinh(1)\cosh(x) \\ \cosh(x) & \sinh(1)\sinh(x) \end{vmatrix}, \\ &= -\sinh(1). \end{aligned}$$

Also, $p_2 = -1$, hence $p_2 W = \sinh(1)$.

Reading off from class notes, the Green's function is now:

$$G(x,\xi) = \frac{[\sinh(1)\cosh(x) - \cosh(1)\sinh(x)]\sinh(\xi)}{\sinh(1)}, \qquad 0 \le \xi \le x \le 1.$$

The term in the square bracket can be written as $\sinh(1-x)$, using a trig identity. Also,

$$G(x,\xi) = \frac{\sinh(x)[\sinh(1)\cosh(\xi) - \cosh(1)\sinh(\xi)]}{\sinh(1)}, \qquad 0 \le x \le \xi \le 1.$$

Again, a trig identity can be used on the term in the square bracket, this is $\sinh(1-\xi)$. Hence,

$$G(x,\xi) = \begin{cases} \frac{\sinh(\xi)\sinh(1-x)}{\sinh(1)}, & 0 \le \xi \le x \le 1, \\ \frac{\sinh(x)\sinh(1-\xi)}{\sinh(1)}, & 0 \le x \le \xi \le 1, \end{cases}$$

as required.

4. Consider the Fredholm integral equation

$$y(x) = f(x) + \lambda \int_{-1}^{1} (x+s) y(s) \mathrm{d}s.$$

(a) For which values of λ does the equation have a unique solution? Find the solution in this case.

(b) For each of those values of λ for which the equation does not have a unique solution, state a condition which $f(x)$ must satisfy in order for a solution to exist, and find the general solution when this is satisfied.

We have:

$$K(x,s) = u_1(x)v_1(s) + u_2(x)v_2(s),$$

where $u_1(x) = x$, $v_1(s) = 1$, $u_2(x) = 1$, and $v_2(s) = 1$. Thus, the FIE can be written as:

$$y(x) = f(x) + \lambda x c_1 + \lambda c_2 \tag{4}$$

where

$$c_1 = \int_{-1}^{1} y(s)\mathrm{d}s, \qquad c_2 = \int_{-1}^{1} s y(s)\mathrm{d}s.$$

Let

$$f_1 = \int_{-1}^{1} f(s)\mathrm{d}s, \qquad f_2 = \int_{-1}^{1} s f(s)\mathrm{d}s.$$

We multiply the FIE (4) by v_1 and integrate to obtain:

$$c_1 = f_1 + 2\lambda c_2. \tag{5a}$$

Next, we multiply the FIE (4) by v_2 and integrate to obtain:

$$c_2 = f_2 + 2\lambda c_1/3. \tag{5b}$$

In doing these integrals, it is helpful to remember the properties of integrals of odd/even functions over symmetric intervals, e.g.

$$\int_{-1}^{1} x\mathrm{d}x = 0,$$

etc. Equations (5) can be simplified and written in matrix form:

$$\underbrace{\begin{pmatrix} 1 & -2\lambda \\ -2\lambda/3 & 1 \end{pmatrix}}_{=M(\lambda)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

The characteristic polynomial is

$$\det\left[M(\lambda)\right] = 0,$$

hence

$$1 - \tfrac{4}{3}\lambda^2 = 0.$$

Thus, the roots of the characteristic polynomial are given by:

$$\lambda \in \mathcal{S} = \{\lambda_-, \lambda_+\} = \{-\tfrac{\sqrt{3}}{2}, \tfrac{\sqrt{3}}{2}\}.$$

Thus, the answer to **part (a)** is that there is a unique solution if $\lambda \neq \mathcal{S}$.

For **part (b)**, we look at the two separate cases.

- Case 1, $\lambda = -\sqrt{3}/2$. Here, we require $f_1 = \sqrt{3} f_2$, that is

$$\int_{-1}^{1} f(s)(1 - \sqrt{3}x)\mathrm{d}x = 0,$$

in which case the general solution is given for arbitrary f_2 by

$$\begin{aligned} y(x) &= f(x) + \lambda_- \left[\left(-\sqrt{3}c_2 + b_1\right)x + c_2\right], \\ &= f(x) + \lambda_- b_1 x + \lambda_- c_2 \underbrace{\left(1 - \sqrt{3}x\right)}_{=y_-}. \end{aligned}$$

Thus, the general solution is:

$$y(x) = f(x) + \lambda_- b_1 x + \alpha y_-(x),$$

where $y_-(x)$ satisfies the homogeneous FIE:

$$y_-(x) = \lambda_- \int_{-1}^{1} K(x,s) y_-(s)\mathrm{d}s.$$

The compatibility condition is:

$$\int_{-1}^{1} y_-(x) f(x)\mathrm{d}x = 0.$$

- Case 2, $\lambda = \sqrt{3}/2$. Here, we require $f_1 = -\sqrt{3} f_2$, that is

$$\int_{-1}^{1} f(s)(1 + \sqrt{3}x)\mathrm{d}x = 0,$$

in which case the general solution is given for arbitrary f_2 by

$$\begin{aligned} y(x) &= f(x) + \lambda_+ \left[\left(\sqrt{3}c_2 + b_1\right)x + c_2\right], \\ &= f(x) + \lambda_+ b_1 x + \lambda_+ c_2 \underbrace{\left(1 + \sqrt{3}x\right)}_{=y_-}. \end{aligned}$$

Thus, the general solution is:

$$y(x) = f(x) + \lambda_+ b_1 x + \alpha y_+(x),$$

where $y_+(x)$ satisfies the homogeneous FIE:

$$y_+(x) = \lambda_- \int_{-1}^{1} K(x,s) y_+(s) \mathrm{d}s.$$

The compatibility condition is:

$$\int_{-1}^{1} y_+(x) f(x) \mathrm{d}x = 0.$$

Remark: When we look at Hilbert–Schmidt theory, we will find, for $\lambda \notin \mathcal{S}$, that the solution of the FIE can be written as:

$$y(x) = f(x) + \frac{\lambda}{\lambda_+ - \lambda} f_+ y_+(x) + \frac{\lambda}{\lambda_- - \lambda} f_- y_+(x),$$

where

$$\lambda_\pm = \pm\sqrt{3}2.$$

Also,

$$y_\pm(x) = \frac{1 \pm \sqrt{3}x}{2}$$

are the *normalized* eigenfunctions of the FIE, with

$$\int_{-1}^{1} y_i(x) y_j(x) \mathrm{d}x = \delta_{ij}, \qquad i,j \in \{-,+\},$$

Furthermore,

$$f_\pm = \int_{-1}^{1} f(x) y_\pm(x) \mathrm{d}x.$$

Thus, we see transparently that in for part (b), the condition which f must satisfy in order for a solution to exist when $\lambda = \lambda_+$ (respectively $\lambda = \lambda_-$) is $f_+ = 0$ (respectively $f_- = 0$).

5. Solve for $\phi(x)$ in the integral equation

$$\phi(x) = f(x) + \lambda \int_0^1 \left[\left(\frac{x}{y}\right)^n + \left(\frac{y}{x}\right)^n \right] \phi(y)\mathrm{d}y,$$

where $f(x)$ is bounded for $0 < x < 1$ and $-1/2 < n < 1/2$, expressing your answer in terms of the quantities $F_m = \int_0^1 f(y) y^m \mathrm{d}y$.

(a) Give the explicit solution when $\lambda = 1$.

(b) For what values of λ are there no solutions unless $F_{\pm n}$ are in a particular ratio? What is this ratio?

Solution: We are looking at:

$$\phi(x) = f(x) + \lambda \int_0^1 K(x,y)\phi(y)\,\mathrm{d}y,$$

hence a Fredholm integral equation with kernel $K(x,y) = (x/y)^n + (y/x)^n$. This is a separable kernel,

$$K(x,s) = u_1(x)v_1(s) + u_2(x)v_2(s),$$

with $u_1(x) = x^n$, $v_1(s) = s^{-n}$, $u_2(x) = x^{-n}$, and $v_2(s) = s^n$. We have:

$$\begin{aligned} c_1 &= \int_0^1 v_1(s)\phi(s)\mathrm{d}s = \int_0^1 s^{-n}\phi(s)\mathrm{d}s, \\ c_2 &= \int_0^1 v_2(s)\phi(s)\mathrm{d}s = \int_0^1 s^{n}\phi(s)\mathrm{d}s. \end{aligned}$$

Also (using notation from class notes):

$$\begin{aligned} b_1 &= \int_0^1 v_1(s)f(s)\mathrm{d}s = \int_0^1 s^{-n} f(s)\mathrm{d}s = F_{-n}, \\ b_2 &= \int_0^1 v_2(s)f(s)\mathrm{d}s = \int_0^1 s^{n} f(s)\mathrm{d}s = F_n. \end{aligned}$$

We project the integral equation on to v_1 and v_2 to get:

$$c_i = b_i + \lambda \sum_{j=1}^{2} A_{ij} c_j,$$

where $A_{ij} = \int_0^1 v_i(s)u_j(s)\mathrm{d}s$. Hence, it remains to compute:

$$\begin{aligned} A_{11} &= \int_0^1 1\,\mathrm{d}s = 1, \\ A_{12} &= \int_0^1 s^{-2n}\,\mathrm{d}s = \frac{1}{1-2n}, \\ A_{21} &= \int_0^1 s^{2n}\,\mathrm{d}s = \frac{1}{2n+1}, \\ A_{22} &= \int_0^1 1\,\mathrm{d}s = 1. \end{aligned}$$

We don't encounter any divisions by zero here because $|n| < 1/2$. Thus, the linear problem to solve is:

$$\underbrace{\begin{pmatrix} 1-\lambda & -\frac{\lambda}{1-2n} \\ -\frac{\lambda}{1+2n} & 1-\lambda \end{pmatrix}}_{=M} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{6}$$

We focus on the characteristic equation:

$$\begin{vmatrix} 1-\lambda & -\frac{\lambda}{1-2n} \\ -\frac{\lambda}{1+2n} & 1-\lambda \end{vmatrix} = 0.$$

The solutions are:

$$\lambda_+ = \frac{\sqrt{1-4n^2}}{1+\sqrt{1-4n^2}}, \qquad \lambda_- = \frac{\sqrt{1-4n^2}}{-1+\sqrt{1-4n^2}}.$$

Thus, when $\lambda = 1$, it is the case that $\lambda \neq \lambda_\pm$, and hence, Equation (6) has a unique solution, characterized by:

$$M^{-1} = \begin{pmatrix} 0 & -(2n+1) \\ 2n-1 & 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned} (c_1, c_2)^T &= \begin{pmatrix} 0 & -(2n+1) \\ 2n-1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -(2n+1) \\ 2n-1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \end{aligned}$$

Hence,

$$c_1 = -(1+2n)b_2, \qquad c_2 = -(1-2n)b_1.$$

The solution in the case $\lambda = 1$ is therefore:

$$\begin{aligned} \phi_{\lambda=1}(x) &= f(x) + c_1 u_1(x) + c_2 u_2(x), \\ &= f(x) - (1+2n)b_2 x^n - (1-2n)b_1 x^{-n}. \end{aligned}$$

Or, in terms of the notation in the question,

$$\phi_{\lambda=1}(x) = f(x) - (1+2n)F_n x^n - (1-2n)F_{-n}x^{-n}.$$

For part (b), we look again at Equation (6) with $\lambda = \lambda_\pm$. In an obvious notation, we have:

$$M(\lambda_\pm)\boldsymbol{c} = \boldsymbol{b}, \tag{7}$$

and $\det M(\lambda_\pm) = 0$. Hence, for almost all values of n, Equation (6) with $\lambda = \lambda_\pm$ will have no solution.

However, there is one special case to look at. Hence, we re-write Equation (7) as:

$$\begin{aligned} M(\lambda_\pm)\boldsymbol{c} &= \boldsymbol{b}, \\ \lambda_\pm \begin{pmatrix} \frac{1}{\lambda_\pm} - 1 & -\frac{1}{1-2n} \\ -\frac{1}{1+2n} & \frac{1}{\lambda_\pm} - 1 \end{pmatrix} \boldsymbol{c} &= \boldsymbol{b}. \end{aligned}$$

Hence:

$$\lambda_\pm \begin{pmatrix} \pm\frac{1}{\sqrt{1-4n^2}} & -\frac{1}{1-2n} \\ -\frac{1}{1+2n} & \pm\frac{1}{\sqrt{1-4n^2}} \end{pmatrix} \boldsymbol{c} = \boldsymbol{b}.$$

In other words,

$$\begin{aligned} \pm\frac{1}{\sqrt{1-4n^2}}c_1 - \frac{1}{1-2n}c_2 &= b_1/\lambda_\pm, \\ (\mp)\frac{1+2n}{\sqrt{1-4n^2}} \times \left[-\frac{1}{1+2n}c_1 \pm \frac{1}{\sqrt{1-4n^2}}c_2 \right] &= (\mp)\frac{1+2n}{\sqrt{1-4n^2}} \times b_2/\lambda_\pm. \end{aligned}$$

The second row can now be re-written as:

$$\pm\frac{1}{\sqrt{1-4n^2}}c_1 - \frac{1}{1-2n}c_2 = \mp\sqrt{\frac{1+2n}{1-2n}}(b_2/\lambda_\pm).$$

Compare the two rows again:

$$\begin{aligned} \pm\frac{1}{\sqrt{1-4n^2}}c_1 - \frac{1}{1-2n}c_2 &= b_1/\lambda_\pm, \\ \pm\frac{1}{\sqrt{1-4n^2}}c_1 - \frac{1}{1-2n}c_2 &= \mp\sqrt{\frac{1+2n}{1-2n}}b_2/\lambda_\pm. \end{aligned}$$

If the two rows are the same, there are infinitely many solutions. Otherwise, there are no solutions. Hence, to find the case with solutions, we require:

$$\frac{b_1}{b_2} = \mp\sqrt{\frac{1+2n}{1-2n}}.$$

Lastly, go back to the notation in the question ($b_1 \to F_{-n}$, $b_2 \to F_n$):

$$\frac{F_n}{F_{-n}} = \mp\sqrt{\frac{1-2n}{1+2n}}.$$

6. Consider the FIE

$$y(x) = f(x) + \lambda \int_0^1 \cosh(x-s)y(s)\mathrm{d}s. \tag{8}$$

(a) Show that the eigenvalues of (8) are given by $2/(1 \pm \sinh 1)$.

(b) Using the Hilbert–Schmidt eigenfunction expansion, or otherwise, find the solution for $\lambda \notin \{2/(1 \pm \sinh 1)\}$.

(c) Find a necessary and sufficient condition on f for the equation

$$y(x) = f(x) + \frac{2}{1+\sinh 1} \int_0^1 \cosh(x-s)y(s)\mathrm{d}s$$

to have a solution and find all solutions when this condition is satisfied.

For part (a), note that the kernel is separable, since

$$K(x,s) = \cosh(x-s) = \tfrac{1}{2}\mathrm{e}^x\mathrm{e}^{-s} + \tfrac{1}{2}\mathrm{e}^{-x}\mathrm{e}^{s}.$$

To find the eigenvalues, let us write the homogeneous equation as

$$y(x) = \tfrac{1}{2}\mathrm{e}^x\lambda\, c_1 + \tfrac{1}{2}\mathrm{e}^{-x}\lambda\, c_2,$$

where

$$c_1 = \int_0^1 \mathrm{e}^{-s}y(s)\mathrm{d}s, \qquad c_2 = \int_0^1 \mathrm{e}^{s}y(s)\mathrm{d}s.$$

Then

$$\begin{aligned} c_1 &= \int_0^1 e^{-x}y(x)\,\mathrm{d}x = \tfrac{1}{2}\lambda c_1 \int_0^1 \mathrm{e}^{-x}\mathrm{e}^{x}\,\mathrm{d} + \tfrac{1}{2}\lambda\, c_2 \int_0^1 \mathrm{e}^{-x}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \tfrac{1}{2}\lambda\, c_1 + \tfrac{1}{2}\lambda\, c_2 \tfrac{1}{2}(1-\mathrm{e}^{-2}), \end{aligned}$$

and

$$\begin{aligned} c_2 &= \int_0^1 \mathrm{e}^{x}y(x)\,\mathrm{d}x = \tfrac{1}{2}\lambda c_1 \int_0^1 \mathrm{e}^{x}\mathrm{e}^{x}\,\mathrm{d}x + \tfrac{1}{2}\lambda\, c_2 \int_0^1 \mathrm{e}^{x}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \tfrac{1}{2}\lambda\, c_1 \tfrac{1}{2}(\mathrm{e}^2-1) + \tfrac{1}{2}\lambda\, c_2. \end{aligned}$$

These equations may be written in matrix form as

$$\underbrace{\begin{pmatrix} \lambda-2 & \lambda\frac{1}{2}(1-\mathrm{e}^{-2}) \\ \lambda\frac{1}{2}(\mathrm{e}^2-1) & \lambda-2 \end{pmatrix}}_{=M(\lambda)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9}$$

The charactrestic polynomial is:

$$\det\left[M(\lambda)\right] = 0.$$

This evaluates to:

$$(\lambda-2)^2 = \tfrac{1}{4}\lambda^2(\mathrm{e}^2-1)(1-\mathrm{e}^{-2}) = \tfrac{1}{4}\lambda^2(\mathrm{e}-\mathrm{e}^{-1})^2 = \lambda^2\sinh^2(1).$$

Hence, the eigenvalues of the FIE are:

$$\lambda \in \mathcal{S} = \{\lambda_1,\lambda_2\} = \left\{\frac{2}{1+\sinh 1}, \frac{2}{1-\sinh 1}\right\}.$$

For **part (b)**, the pertinent eigenfunctions are obtained by solving for $(c_1,c_2)^T$ in Equation (9). We look at the two cases:

- Case 1. We look at $\lambda=\lambda_1$. Notice that $\lambda_1\sinh 1 = -(\lambda_1-2)$ so, for $\lambda=\lambda_1$, Equation (9) becomes:

  $$\begin{pmatrix} \lambda_1-2 & -(\lambda_1-2)\mathrm{e}^{-1} \\ -(\lambda_1-2)\mathrm{e} & \lambda_1-2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  This imples that $c_2 = \mathrm{e}c_1$, so the corresponding eigenfunction is a multiple of

  $$\mathrm{e}^x + \mathrm{e}\mathrm{e}^{-x} = 2\mathrm{e}^{1/2}\frac{1}{2}(\mathrm{e}^{x-1/2}+\mathrm{e}^{-(x-1/2)}),$$

  or equivalently:

  $$y_1(x) \propto \cosh(x-\tfrac{1}{2}).$$

- For $\lambda=\lambda_2$, we have $c_2=-\mathrm{e}c_1$, and the corresponding eigenfunction is

  $$y_2(x) \propto \sinh(x-\tfrac{1}{2}).$$

We look at normalized eigenfunctions:

$$y_1(x) = \sqrt{\lambda_1}\cosh\left(x-\tfrac{1}{2}\right), \qquad y_2(x) = \sqrt{-\lambda_2}\sinh\left(x-\tfrac{1}{2}\right).$$

Notice that:

$$\int_0^1 y_i(x)y_j(x)\mathrm{d}x = \delta_{ij}.$$

In this case, Hilbert–Schmidt theory tells us that the solution fo the FIE can be expanded in terms of eigenfunctions:

$$y(x) = f(x) + \frac{\lambda}{\lambda_1-\lambda}f_1\,y_1(x) + \frac{\lambda}{\lambda_2-\lambda}f_2\,y_2(x), \qquad \lambda\notin\mathcal{S}.$$

Here,

$$f_i = \int_0^1 f(x)y_i(x)\mathrm{d}x.$$

For **part (c)**, Hilbert–Schmidt theory tells us that if $\lambda \in \mathcal{S}$ (in this case, $\lambda = \lambda_1$), then a necessary necessary and sufficient condition on f for the FIE to have a solution is:

$$\int_0^1 f(x)y_1(x)\mathrm{d}x = 0 \implies f_1 = 0.$$

In this case, the general solution is

$$y(x) = f(x) + Ay_1(x) + \frac{\lambda_1}{\lambda_2 - \lambda_1} f_2\, y_2(x).$$

A Code Listings

In this code, I solve the ODE $y''(x) = -\omega^2 y(x)+\sin(x)$ for the initial conditions $y(0) = 0$ and $y'(0) = p$. Here, p is a parameter that can be varied.

```
function [x,y]=odesolve(param)

omega=2.5;

[x,Y]=ode45(@myfun,[0 pi], [0 param]);
y=Y(:,1);

function dYdx=myfun(x,Y)
    dYdx=0*Y;
    dYdx(1)=Y(2);
    dYdx(2)=-omega*omega*Y(1)+sin(omega*x);
end

end
```

In this next code, I look at the cost function

$$J(p) = [y(x = \pi; p)]^2 ,$$

where $y(x = \pi, p)$ is output from `odesolve.m`, at paramter value p. I solve an optimization problem where I make $J(p)$ as small as possible. In this way, I solve the BVP $y''(x) = -\omega^2 y(x) + \sin(x)$ subject to $y(0) = 0$ and $y(\pi) = 0$. This is called the **shooting method**.

```
p_star=fminbnd(@mycost,-10 ,10000);
display(p_star);
J_min=mycost(p_star);
display(J_min);

% Obtain solution of BVP:
[x,y]=odesolve(p_star);

% Interpolate solution on to regular grid:
xi=0:0.1:pi;
yi=interp1(x,y,xi,'spline');

x=xi;
y=yi;
```

```
% Plot solution:
plot(x,y,'o','color','blue')

% Calculate analytical solution for omega=2.5:

omega=2.5;
xa=0:0.01:pi;
ya=sin(omega*(xa-pi)).*(0.5*xa-(sin(2*omega*xa)/(4*omega)))...
     +sin(omega*xa).*(0.5*(pi-xa)*cos(omega*pi)...
     -(sin(omega*(pi-2*xa))/(4*omega))
-(sin(omega*pi)/(4*omega)) );
ya=ya/(omega*sin(omega*pi));

% Plot analyical and numerical solution on same axes
% and compare:

hold on
plot(xa,ya,'color','red','linewidth',2)
grid on
xlabel('x')
ylabel('y')
set(gca,'fontsize',14,'fontname','times new roman')
hold off



function J=mycost(p)
     [~,y]=odesolve(p);

     J=y(end)*y(end);
end
```