Advanced Mathematical Methods (ACM30020)

Dr Lennon Ó Náraigh

Exercises #5

1. Given the following second order linear homogeneous ODE:

$$y'' + \omega^2 y = 0,$$

where ω is a real number. The initial conditions are: $y(0) = y_0$ and $y'(0) = 0$.

Transform the problem into a Volterra integral equation. Solve the integral equation using an iterative scheme.

2. Consider the inhomogeneous problem

$$y'' + \omega^2 y = f(x), \qquad x \in [0, \pi].$$

Here, the problem is a Boundary Value Problem (BVP), with boundary conditions $y(0) = y(\pi) = 0$. Use the Green's Function to solve the BVP in the case when:

(a) $f(x) = 1$;

(b) $f(x) = \sin(\omega x)$.

3. The stationary temperature distribution in a rod of unit length that has both ends kept at a constant zero temperature, with heat loss through its surface proportional to u, and that is subject to a given non-uniform heat source per unit length $f(x)$, is the solution of

$$-u'' + u = f, \qquad u(0) = u(1) = 0.$$

Show that the Green's function of this Boundary Value Problem is given by:

$$G(x,\xi) = \begin{cases} \frac{\sinh(x)\sinh(1-\xi)}{\sinh(1)}, & 0 \le x \le \xi \le 1, \\ \frac{\sinh(\xi)\sinh(1-x)}{\sinh(1)}, & 0 \le \xi \le x \le 1. \end{cases}$$

4. Consider the Fredholm integral equation

$$y(x) = f(x) + \lambda \int_{-1}^{1} (x+s) y(s) \mathrm{d}s.$$

   (a) For which values of λ does the equation have a unique solution? Find the solution in this case.

   (b) For each of those values of λ for which the equation does not have a unique solution, state a condition which $f(x)$ must satisfy in order for a solution to exist, and find the general solution when this is satisfied.

5. Solve for $\phi(x)$ in the integral equation

$$\phi(x) = f(x) + \lambda \int_{0}^{1} \left[\left(\frac{x}{y}\right)^n + \left(\frac{y}{x}\right)^n \right] \phi(y) \mathrm{d}y,$$

   where $f(x)$ is bounded for $0 < x < 1$ and $-1/2 < n < 1/2$, expressing your answer in terms of the quantities $F_m = \int_0^1 f(y) y^m \mathrm{d}y$.

   (a) Give the explicit solution when $\lambda = 1$.

   (b) For what values of λ are there no solutions unless $F_{\pm n}$ are in a particular ratio? What is this ratio?

6. Consider the FIE

$$y(x) = f(x) + \lambda \int_{0}^{1} \cosh(x-s) y(s) \mathrm{d}s. \tag{1}$$

   (a) Show that the eigenvalues of (1) are given by $2/(1 \pm \sinh 1)$.

   (b) Using the Hilbert–Schmidt eigenfunction expansion, or otherwise, find the solution for $\lambda \notin \{2/(1 \pm \sinh 1)\}$.

   (c) Find a necessary and sufficient condition on f for the equation

$$y(x) = f(x) + \frac{2}{1 + \sinh 1} \int_{0}^{1} \cosh(x-s) y(s) \mathrm{d}$$

   to have a solution and find all solutions when this condition is satisfied.