Applied Analysis (ACM30020)

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Exercises #4

1. Consider the ODE

$$y'' + p(x)y' + q(x)y = 0.$$

If $y_1(x)$ is a solution,s how that a second solution can be written as:

$$y_2(x) = y_1(x)\int_a^x \frac{\mathrm{e}^{-\int_b^{x''} p(x')\mathrm{d}x'}}{[y_1(x'')]^2}\mathrm{d}x''. \tag{1}$$

Here, a and b are arbitrary.

Sub $y_2(x) = y_1(x)u(x)$ into the ODE to obtain:

$$u'' + \left[p(x) + 2\frac{y_2'}{y_2}\right]u' = 0. \tag{2}$$

Let $v = u'$ and reduce Equation (2) to a first-order ODE:

$$v' = -\left[p(x) + 2\frac{y_2'}{y_2}\right]v.$$

The solution is:

$$v(x) = v(b)\frac{\mathrm{e}^{\int_b^x p(x')\mathrm{d}x]}}{[y_2(x)]^2},$$

where b is arbitrary. But $v(x) = \mathrm{d}u/\mathrm{d}x$, hence

$$u(x) = C + v(b)\int_a^x \frac{\mathrm{e}^{-\int_b^{x''} p(x')\mathrm{d}x'}}{[y_1(x'')]^2}\mathrm{d}x''.$$

where C is a constant of integration.

Choose $C = 0$ and $v(b) = 1$ to get a second linearly-independent solution:

$$u(x) = \int_a^x \frac{\mathrm{e}^{-\int_b^{x''} p(x')\mathrm{d}x'}}{[y_1(x'')]^2}\mathrm{d}x''.$$

Hence,

$$y_2(x) = y_1(x)\int_a^x \frac{\mathrm{e}^{-\int_b^{x''} p(x')\mathrm{d}x'}}{[y_1(x'')]^2}\mathrm{d}x'',$$

as required.

2. Given that one solution of

$$R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R = 0$$

is $R = r^m$, show that Equation (1) provides a second solution, $R = r^{-m}$.

We read off from Question 1, with $p(x) = 1/x$. Hence, $\int p(x)\mathrm{d}x = \ln x$, and $\mathrm{e}^{-\int p(x)\mathrm{d}x} = -1/x$. Thus, the second solution (in an obvious notation, and letting $x \to r$) is:

$$\begin{aligned} R_2(r) &= r^m \int \frac{1}{r}\frac{1}{r^{2m}}\mathrm{d}r, \\ &= r^m \left(-\frac{1}{2m} r^{-2m} \right), \\ &\propto r^{-m}. \end{aligned}$$

As the solutions are only defined up to a constant, we choose the second linearly independent solution to be:

$$R_2(r) = r^{-m},$$

as required.

3. Consider Legendre's differential equation:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0. \tag{3}$$

(a) Solve the equation by by direct series substitution.

(b) Verify that the indicial equation is:

$$\alpha(\alpha - 1) = 0.$$

(c) Using $\alpha = 0$, obtain the following series of even powers of x ($a_1 = 0$):

$$y_{even} = a_0 \left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \cdots\right],$$

where

$$a_{j+2} = \frac{j(j+1) - n(n+1)}{(j+1)(j+2)}a_j.$$

(d) Using $\alpha = 1$, develop a series of odd powers of x ($a_1 = 0$).

$$y_{odd} = a_0\Big\{x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \cdots\Big\},$$

where

$$a_{j+2} = \frac{(j+1)(j+2) - n(n+1)}{(j+2)(j+3)}a_j.$$

(e) Show that both solutions, y_{even} and y_{odd}, diverge for $x = \pm 1$ if the series continue to infinity.

(f) Finally, show that by an appropriate choice of n, one series at a time may be converted into a polynomial, thereby avoiding the divergence catastrophe.

Because the coefficient functions $p(x) = -2x/(1-x^2)$ and $q(x) = n(n+1)/(1-x^2)$ are regular at $x = 0$, this is a regular point and hence, the ODE has a simple power-series solution:

$$y(x) = \sum_{p=0}^{\infty} a_p x^p$$

(we can't use n for the index because it's used already as the parameter in the ODE). We substitute this into Equation (3) to get:

$$\sum_{p=0}^{\infty} a_p p(p-1)x^{p-2} = \sum_{p=0}^{\infty} a_p p(p-1)x^p + 2\sum_{p=0}^{\infty} a_p p x^p - \sum_{p=0}^{\infty} n(n+1)a_p x^p.$$

Re-index. In the term on the LHS we use $q = p - 2$. In the terms on the RHS we use $q = p$. This gives:

$$\sum_{q=-2}^{\infty} a_{q+2}(q+2)(q+1)x^q = \sum_{q=0}^{\infty} a_q q(q-1)x^q + 2\sum_{q=0}^{\infty} a_q q x^q - \sum_{q=0}^{\infty} n(n+1)a_q x^q.$$

We equate powers of x^q. At $q = -2$ we have $a_0 \times 0 \times 1 = 0$, which implies that a_0 is arbitrary. At $q = -1$ we have $a_1 \times (-1) \times 0 = 0$, meaning that a_1 is arbitrary also. We also have the recurrence relation:

$$\begin{aligned} a_{q+2} &= a_q \frac{q(q-1) + 2q - n(n+1)}{(q+2)(q+1)}, \\ &= a_q \frac{q(q+1) - n(n+1)}{(q+2)(q+1)} \end{aligned}$$

These results mean that we can look at the possibilities $\{a_0 \neq 0, a_1 = 0\}$ and $\{a_0 = 0, a_1 \neq 0\}$ separately. Furthermore, because the recurrence relation steps up in steps of two, this means that we are looking at odd and even series solutions.

Consequently, the series solutions are $y(x) = a_0 + a_2x^2 + a_4x^4 + \cdots$ and $y(x) = a_1x + a_3x^3 + \cdots$, which can be encapsulated as $y(x) = x^\alpha(a_0 + a_1x + a_2x^2 + \cdots)$, with $\alpha = 0, 1$. This implies an indicial equation $\alpha(\alpha - 1) = 0$, which answers **Part (b)**.

Parts (a) and (c): We look at the possibilities $\{a_0 \neq 0, a_1 = 0\}$ and $\{a_0 = 0, a_1 \neq 0\}$ separately, and generate odd and even series solutions. We use the recurrence relation to generate the first few terms of the even solution:

$$\begin{aligned} a_2 &= a_0 \frac{-n(n+1)}{2!}, \\ a_4 &= a_2 \left[\frac{6 - n(n+1)}{4 \times 3}\right], \\ &= a_0 \frac{1}{4!}(-1)\left[6 - n(n+1)\right] n(n+1), \\ &= a_0 \frac{1}{4!} n(n+1)(n-2)(n+3). \end{aligned}$$

Hence,

$$y_{even} = a_0 \left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 + \cdots\right],$$

where

$$a_{q+2} = a_q \frac{q(q+1) - n(n+1)}{(q+2)(q+1)}.$$

Thus, the answer to Part (c) is complete.

Part (d) concerns the odd solution. The first few terms are:

$$\begin{aligned} a_3 &= a_1 \frac{2-n(n+1)}{3\times 2}, \\ &= a_1 \left[-\frac{(n-1)(n+2)}{3!} \right], \\ a_5 &= a_3 \left[\frac{3\times 4 - n(n+1)}{5\times 4} \right], \\ &= a_1 \left[-\frac{(n-1)(n+2)}{3!} \right] \left[-\frac{(n-3)(n+4)}{5\times 4} \right], \\ &= a_1 \left[\frac{(n-1)(n-3)(n+2)(n+4)}{5!} \right]. \end{aligned}$$

Hence, we obtain the odd solution:

$$y_{odd} = a_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 + \cdots \right].$$

Again, the ratio between successive terms is given by the recurrence relation:

$$a_{q+2} = a_q \frac{q(q+1) - n(n+1)}{(q+2)(q+1)}.$$

For **Part (e)**, we look again at the recurrence relation in the limit of large q. This gives

$$\frac{a_{q+2}}{a_q} \sim \frac{q}{q+2} \sim 1.$$

Thus, the tail of the series looks like a geometric progression with alternating sign. The geometric progression with alternating sign is:

$$G(x) = \sum_{q=0}^{\infty} (-1)^q x^q,$$

which diverges as $|x| \to 1$. Thus, and in general, the series y_{even} and y_{odd} will diverge as $|x| \to 1$. This answers Part (e).

For **Part (f)**, we notice that the series y_{even} and y_{odd} will terminate and reduce to polynomial expressions if n is zero or a positive integer, since then the expression

$$j(j+1) = n(n+1)$$

will be satisfied for $n = j$, and hence $a_{j+2} = 0$. These are the *Legendre Polynomials* (Figure 1).

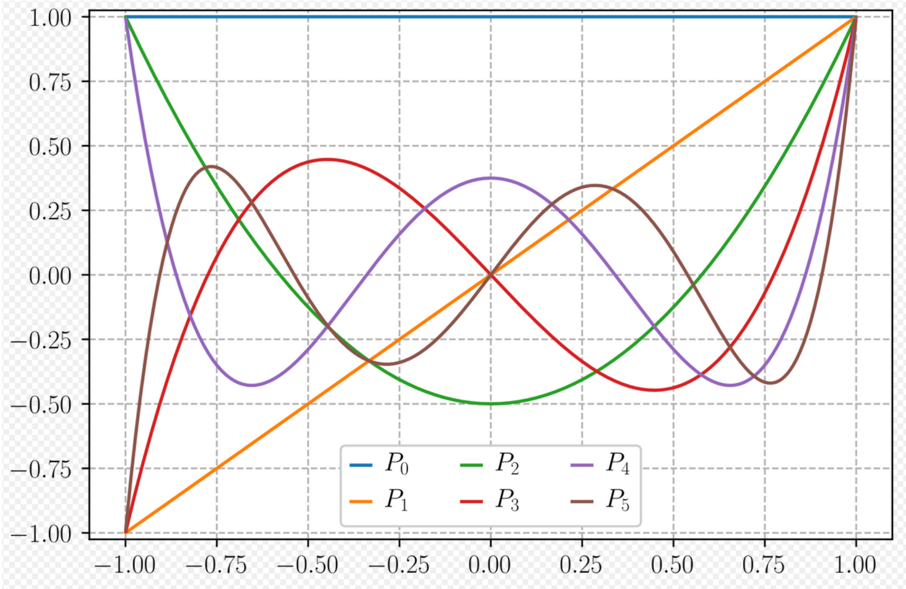

Figure 1: The first 6 Legendre Polynomials. From Wikipedia.

4. Obtain two series solutions of the confluent hypergeometric equation

$$xy'' + (c - x)y' - ay = 0.$$

Test your solutions for convergence.

We first look at $x[(c-x)/x] = c - x$ and $x^2[-a/x] = -ax$. The RHS of both these expressions have Taylor expansions around zero, so the singular point $x = 0$ is regular. Thus, a series solution

$$y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n,$$

is possible. We substitute this trial solution into the ODE and evaluate:

$$\sum_{n=0}^{\infty} a_n(n+\alpha)(n+\alpha-1)x^{n+\alpha-1} + c\sum_{n=0}^{\infty} a_n(n+\alpha)x^{n+\alpha-1} - \sum_{n=0}^{\infty} a_n(n+\alpha)x^{n+\alpha} - a\sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0.$$

We cancel out a power of x^α on both sides. Hence, we have:

$$\sum_{n=0}^{\infty} a_n(n+\alpha)(n+\alpha-1)x^{n-1} + c\sum_{n=0}^{\infty} a_n(n+\alpha)x^{n-1} - \sum_{n=0}^{\infty} a_n(n+\alpha)x^n - a\sum_{n=0}^{\infty} a_n x^n = 0.$$

We then re-index with $p = n - 1$, hence $n = p + 1$, and $p_{start} = -1$. Hence, we have:

$$\sum_{p=-1}^{\infty} a_{p+1}(p+1+\alpha)(p+\alpha)x^p + c\sum_{p=-1}^{\infty} a_{p+1}(p+1+\alpha)x^p - \sum_{p=0}^{\infty} a_p(p+\alpha)x^p - a\sum_{p=0}^{\infty} a_p x^p = 0.$$

For the $p = -1$ term we have:

$$a_0\left[\alpha(\alpha-1) + c\alpha\right] = 0.$$

Hence, the indicial equation is:

$$\alpha(\alpha-1) + c\alpha = 0,$$

with solutions

$$\alpha = 0, \qquad \alpha = 1 - c.$$

We look at the analytic solution with $\alpha = 0$. We look in particular at the recurrence relation:

$$a_{p+1}\left[p(p+1) + c(p+1)\right] = a_p(p+a).$$

or

$$a_n n(n-1+c) = a_{n-1}\left(n-1+a\right), \qquad n \geq 1.$$

Hence,

$$a_n = \frac{(n-1+a)}{n(n-1+c)}a_{n-1}, \tag{4}$$

Furthermore,

$$\begin{aligned} a_n &= \frac{(n-1+a)(n-2+a)}{n(n-1)(n-1+c)(n-2+c)}a_{n-2}, \\ &= \frac{(n-1+a)(n-2+a)\cdots(1+a)a}{n!(n-1+c)(n-2+c)c\cdots(1+c)c}a_0. \end{aligned}$$

Notice that the recurrence relation will terminate if a is a negative integer or zero (giving a polynomial solution), and also, the recurrence relation will fail if c is a negative integer or zero.

We identify the Pochammer symbol (rising factorial):

$$(a)_n = a(a+1)\cdots(a+n-1) = \prod_{k=0}^{n-1}(a+k), \qquad (a)_0 = 1,$$

and similarly for c. Hence, the first series solution can be written as:

$$y_1(x) = a_0\sum_{n=0}^{\infty}\frac{(a)_n}{n!(c)_n}x^n, \qquad c \notin -\mathbb{N}\cup\{0\}.$$

or

$$y_1(x) = a_0 \left[{}_1F_1(a; c; x) \right]. \tag{5}$$

where ${}_1F_1(a; c; x)$ is the confluent hypergeometric function.

Some remarks / clarifications are needed here:

- Notice that if $a \in -\mathbb{N} \cup \{0\}$, then $y_1(x)$ is a polynomial.
- We use the convention by Abramowtiz and Stegun[1] whereby $(a)_n$ represents the Pochammer symbol (rising factorial).

Furthermore, from Equation (4), we identify the ratio

$$\rho_n = \left| \frac{a_n}{a_{n-1}} \right| = \left| \frac{n-1+a}{n(n-1+c)} \right|.$$

Hence,

$$\rho = \lim_{n\to\infty} \rho_n = 0.$$

The radius of convergence of the series is therefore $R = 1/\rho = \infty$. Thus, except for $c = 0, -1, -2, \cdots$, the series in Equation (5) is an entire function, that is, an analytic on the entire complex plane.

For the second solution, we look at $\alpha = 1 - c$. The recurrence relation is:

$$\begin{aligned} a_n &= \frac{n+(a-c)}{n(n+1-c)} a_{n-1}, \\ &= \frac{n-1+(a-c+1)}{n[n-1+(2-c)]} a_{n-1}, \\ &= \frac{[n-1+(a-c+1)][n-2+(a-c+1)]}{n(n-1)[n-1+(2-c)][n-2+(2-c)]} a_{n-2}, \\ &= \cdots \\ &= a_0 \frac{(a-c+1)_n}{n!(2-c)_n} a_0. \end{aligned}$$

Hence, the second solution is:

$$y_2(x) = x^{1-c} \left[{}_1F_1(1+a-c; 2-c; x) \right], \qquad 2-c \notin -\mathbb{N} \cup \{0\}.$$

Again, notice that if $1 + a - c \in -\mathbb{N} \cup \{0\}$, then $y_2(x)$ is a polynomial.

Also, notice that if $c = 1$ then this method fails to generate a second independent solution. In this case, there are repeated roots of the indicial equation, and Frobenius's method is required to produce the second solution.

[1] Milton Abramowitz and Irene Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. National Bureau of Standards, USA, 1964

5. Bessel's equation can be written as

$$x^2y'' + xy' + \left(x^2 - \nu^2\right) y = 0.$$

Using power series, find the two linearly independent solutions of Bessel's equation with $\nu = 1/2$.

We use class notes to find the recurrence relations:

$$\begin{aligned} \alpha^2 - \nu^2 &= 0, \\ a_1 \left(\pm 2\nu + 1\right) &= 0. \end{aligned}$$

Also,

$$a_n n \left(n \pm 2\nu\right) = -a_{n-2}, \qquad n \geq 2. \tag{6}$$

The indicial equation is $\alpha^2 = \nu^2$, hence $\alpha = \pm 1/2$. There are two distinct roots of the inidical equation.

Next, we have $a_1(2\alpha + 1) = 0$. For $\alpha = 1/2$ this gives $2a_1 = 0$, so $a_1 = 0$ in this case. For $\alpha = -1/2$ this gives $a_1 0 = 0$, so a_1 is undetermined in this second case.

First Case: We have $\alpha = 1/2$, and the recurrence relation is well defined for all $n \geq 2$:

$$a_n = -\frac{a_{n-2}}{n(n+1)}, \qquad n \geq 2$$

This gives:

$$\begin{aligned} a_{2n} = -&\frac{1}{2n(2n+1)} a_{2(n-1)} \\ &= \frac{1}{2n(2n+1)} \frac{1}{(2n-2)(2n-1)} a_{2(n-2)} = \cdots (-1)^n \frac{1}{(2n+1)!} a_0, \end{aligned}$$

so we have first solution

$$J_{\frac{1}{2}}(z) = \left(\frac{z}{2}\right)^{1/2} \left(1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \cdots\right)$$

where, following convention, we have taken $a_0 = 2^{-1/2}$.

Furthermore, we take:

$$\begin{aligned} J_{\frac{1}{2}}(z) &= \left(\frac{z}{2}\right)^{1/2} \left(1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \cdots\right), \\ &= \left(\frac{z}{2}\right)^{1/2} \frac{1}{z} \left(z - \frac{1}{3!}x^2 + \frac{1}{5!}z^5 - \cdots\right), \\ &= \frac{1}{\sqrt{2}} \frac{\sin z}{z^{1/2}}. \end{aligned}$$

Second Case: For $\alpha = -\frac{1}{2}$, Equation (6) gives:

$$a_n n \left(n - 1\right) = -a_{n-2}, \qquad n \geq 2. \tag{7}$$

Here, the recurrence relation holds up for $n \geq 2$, so we can proceed to develop a regular series expansion for the second solution.

For the even powers of n, set $n \to 2n$ in Equation (7):

$$a_{2n} = -\frac{1}{2n(2n-1)} a_{2n-2}.$$

Hence,

$$a_{2n} = (-1)^n \frac{1}{(2n)!} a_0. \tag{8}$$

We know for the second case that a_1 is arbitrary. Furthermore, for the odd powers of n we get the following recurrence relation:

$$\begin{aligned} a_{2n+1} &\overset{\text{Eq. (7)}}{=} -\frac{1}{(2n+1)(2n)} a_{(2n+1)-2}, \\ &= -\frac{1}{(2n+1)(2n)} a_{2n-1}, \\ &= \cdots, \\ &= \frac{(-1)^n}{(2n+1)!} a_1. \end{aligned}$$

Thus, the odd coefficients just reproduce our previous solution, so we can set $a_1 = 0$ to produce a linearly independent second solution.

Thus, the linearly independent second solution is given entirely by the even powers (*cf.* Equation (8)):

$$J_{-\frac{1}{2}}(z) = \left(\frac{z}{2}\right)^{-1/2} \left(1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \cdots\right)$$

where, following convention, we have taken $a_0 = 2^{1/2}$. Hence,

$$J_{-\frac{1}{2}}(z) = \sqrt{2}\frac{\cos z}{z^{1/2}}.$$

Note that our two solutions are just multiples of $z^{-1/2} \sin z$ and $z^{-1/2} \cos z$ (behaving as $z^{1/2}$ and $z^{-1/2}$ respectively as $z \to 0$).