Advanced Mathematical Methods (ACM30020)

Dr Lennon Ó Náraigh

Exercises #4

1. Consider the ODE
$$y'' + p(x)y' + q(x)y = 0.$$
If $y_1(x)$ is a solution,s how that a second solution can be written as:
$$y_2(x) = y_1(x) \int_a^x \frac{\mathrm{e}^{-\int_b^{x''} p(x')\mathrm{d}x'}}{[y_1(x'')]^2} \mathrm{d}x''. \tag{1}$$
Here, a and b are arbitrary.

2. Given that one solution of
$$R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R = 0$$
is $R = r^m$, show that Equation (1) provides a second solution, $R = r^{-m}$.

3. Consider Legendre's differential equation:
$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$
   (a) Solve the equation by by direct series substitution.
   (b) Verify that the indicial equation is:
   $$\alpha(\alpha - 1) = 0.$$
   (c) Using $\alpha = 0$, obtain the following series of even powers of x ($a_1 = 0$):
   $$y_{even} = a_0 \left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \cdots\right],$$
   where
   $$a_{j+2} = \frac{j(j+1) - n(n+1)}{(j+1)(j+2)}a_j.$$
   (d) Using $\alpha = 1$, develop a series of odd powers of x ($a_1 = 0$).
   $$y_{odd} = a_0 \left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \cdots\right],$$
   where
   $$a_{j+2} = \frac{(j+1)(j+2) - n(n+1)}{(j+2)(j+3)}a_j.$$

(e) Show that both solutions, y_{even} and y_{odd}, diverge for $x = \pm 1$ if the series continue to infinity.

(f) Finally, show that by an appropriate choice of n, one series at a time may be converted into a polynomial, htereby avoiding the divergence catastrophe.

4. Obtain two series solutions of the confluent hypergeometric equation

$$xy'' + (c - x)y' - ay = 0.$$

Test your solutions for convergence.

5. Bessel's equation can be written as

$$x^2y'' + xy' + \left(x^2 - \nu^2\right) y = 0.$$

Using power series, find the two linearly independent solutions of Bessel's equation with $\nu = 1/2$.