Applied Analysis (ACM30020)

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Exercises #3

In this set of exercises we are concerned with the Orr–Sommerfeld equation for Couette flow. The purpose of the exercises is to showcase a very nice application of the method of Variation of Parameters. The equation reads:

$$\mathrm{i}k\,(z-c)\left(\partial_z^2-k^2\right)\Psi=\mathrm{Re}^{-1}\left(\partial_z^2-k^2\right)^2\Psi. \tag{1}$$

Here, k and Re are real parameters, z is the variable, and ∂_z denotes ordinary differentiation with respect to z. The constant c is an eigenvalue to be determined, this can be real or imaginary.

For context, the equation models the instability of a parallel shear flow – such as the one in the graded assignment, shown schematically here again in Figure 1. The time-independent parallel shear flow $U_0(z)$ is in the x-direction but varies in the z-direction. A perturbation is introduced, which takes the flow to a new state

$$\boldsymbol{u}(x,z,t)=(U_0(z)+\delta u(x,z,t),0,\delta w(x,z,t)),$$

where δu and δw are small perturbations to the basic flow. We fix $U_0(z)\propto z$, which is the equation of Couette flow. By writing δu and δw in a streamfunction representation, $\delta u=\partial\psi/\partial z$ and $\delta w=-\partial\psi/\partial x$ and then by decomposing Ψ into normal models $\psi(x,z,t)=\psi(z)\mathrm{e}^{\mathrm{i}k(x-ct)}$, Equation (1) drops out of the Navier–Stokes equations, these being the basic equations of Fluid Mechanics.

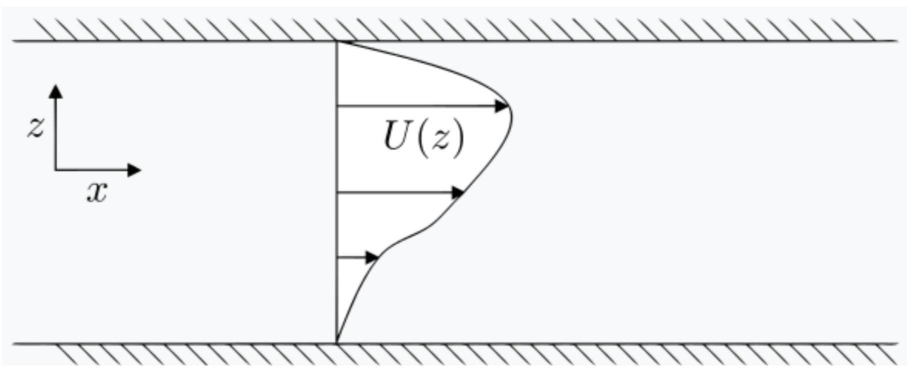

Figure 1: Schematic description of a unidirectional shear flow, similar to the graded assignment. Note the convention here that the wall-normal direction is z (it's y in the graded assignment).

1. Let $v = (\partial_z^2 - k^2)\,\Psi$. Notice that $v = 0$ satisfies Equation (1). Hence, show that

$$\Psi = \cosh(kz), \qquad \Psi = \sinh(kz).$$

are solutions of Equation (1).

We look again at Equation (1):

$$\mathrm{i}k\,(z-c)\underbrace{\left(\partial_z^2 - k^2\right)\Psi}_{=v} = \mathrm{Re}^{-1}\left(\partial_z^2 - k^2\right)\underbrace{\left(\partial_z^2 - k^2\right)\Psi}_{=v}. \tag{2}$$

Hence:

$$\mathrm{i}k\,(z-c)\,v = \mathrm{Re}^{-1}\left(\partial_z^2 - k^2\right)v. \tag{3}$$

This is a homogeneous equation, one solution of which is $v = 0$. Referring back to $v = (\partial_z^2 - k^2)\Psi$, we have:

$$\left(\partial_z^2 - k^2\right)\Psi = 0, \tag{4}$$

with solutions e^{-kz} and e^{kz}. Or, taking linear combinations:

$$\Psi = \cosh(z), \qquad \Psi = \sinh(z). \tag{5}$$

2. Carry out a series of rescalings, $\widetilde{z} = z - c - \mathrm{i}k/\mathrm{Re}$, followed by $\xi = \lambda\widetilde{z}$, where λ is constant. Hence, show that v satisfies Airy's differential equation:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} - \xi v = 0, \tag{6}$$

where $\xi = (\mathrm{i}k\mathrm{Re})^{1/3}[z - c - (\mathrm{i}k/\mathrm{Re})]$. We choose the particular cube root of $\mathrm{i}^{1/3} = \mathrm{e}^{\mathrm{i}\pi/6}$.

We re-write Equation (3) as:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2 - \mathrm{i}k\mathrm{Re}\,(z-c)\right]v = 0, \tag{7}$$

or:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \mathrm{i}k\mathrm{Re}\left(z - c - \frac{\mathrm{i}k}{\mathrm{Re}}\right)\right]v = 0. \tag{8}$$

Let $\widetilde{z} = z - c - \mathrm{i}k/\mathrm{Re}$. We have $\mathrm{d}/\mathrm{d}z = \mathrm{d}/\mathrm{d}\widetilde{z}$, hence:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\widetilde{z}^2} - \mathrm{i}k\mathrm{Re}\widetilde{z}\,v = 0. \tag{9}$$

We make a second change of variable to bring Equation (9) into a standard form. Hence, we let $\xi = \lambda\tilde{z}$. We have:

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{z}} = \frac{\mathrm{d}\xi}{\mathrm{d}\tilde{z}}\frac{\mathrm{d}}{\mathrm{d}\xi} = \lambda\frac{\mathrm{d}}{\mathrm{d}\xi}. \tag{10}$$

Hence, Equation (9) becomes:

$$\lambda^2\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} - \mathrm{i}k\mathrm{Re}\left(\frac{\xi}{\lambda}\right)v = 0. \tag{11}$$

Hence:

$$\lambda^3\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} - \mathrm{i}k\mathrm{Re}\xi\, v = 0. \tag{12}$$

We choose $\lambda^3 = \mathrm{i}k\mathrm{Re}$, hence:

$$\lambda = \left[\mathrm{e}^{\pi\mathrm{i}/2}k\mathrm{Re}\right]^{1/3} \tag{13}$$

hence finally:

$$\lambda = \mathrm{e}^{\pi\mathrm{i}/6}\left(k\mathrm{Re}\right)^{1/3}. \tag{14}$$

Note that other cube roots are possible in Equation (13) but they won't lead to any new linearly independent solutions, since we can have only two of those.

Hence, after these manipulations, Equation (9) becomes:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} - \xi v = 0, \tag{15}$$

which is exactly the form required (*cf.* Equation (15)).

As noted in the question sheet, Equation (6) has solutions

$$v = \mathrm{Ai}(\xi), \qquad v = \mathrm{Bi}(\xi).$$

Here, Ai and Bi are special solutions of Airy's differential equation, these can be looked up. We now use these special properties to construct explicit solutions for $\psi(z)$, i.e. two final linearly independent solutions, apart from those already obtained in Question 1.

3. To obtain the remaining two linearly independent solutions of Equation (1), we look at:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\Psi = \mathrm{Ai}(\xi), \qquad \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\Psi = \mathrm{Bi}(\xi). \tag{16}$$

Use the method of variation of parameters to construct the following two solutions:

$$\chi_1(z) = \frac{1}{k}\int_0^z \sinh[k(z-z')]\mathrm{Ai}\left[(\mathrm{i}k\mathrm{Re})^{1/3}\left(z' - c - \frac{\mathrm{i}k}{\mathrm{Re}}\right)\right]\mathrm{d}z', \tag{17a}$$

$$\chi_2(z) = \frac{1}{k}\int_0^z \sinh[k(z-z')]\mathrm{Bi}\left[(\mathrm{i}k\mathrm{Re})^{1/3}\left(z' - c - \frac{\mathrm{i}k}{\mathrm{Re}}\right)\right]\mathrm{d}z', \tag{17b}$$

We take the first equation of interest:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)\Psi = \mathrm{Ai}(\xi) \tag{18}$$

As this is a linear inhomogeneous ODE, we use the method of variation of parameters. We identify the homogeneous solutions $\Psi_1 = \cosh(kz)$ and $\Psi_2 = \sinh(kz)$. The Wronskian is:

$$W = \begin{vmatrix} \Psi_1 & \Psi_2 \\ \Psi_1' & \Psi_2' \end{vmatrix} = \begin{vmatrix} \cosh(kz) & \sinh(kz) \\ k\sinh(kz) & k\cosh(kz) \end{vmatrix} = k. \tag{19}$$

From class notes, the particular integral is:

$$\begin{aligned} \Psi_{PI}(z) &= \int_0^z G(z,z')\mathrm{Ai}(\xi(z'))\mathrm{d}z', \\ &= \int_0^z \frac{-\Psi_1(z)\Psi_2(z') + \Psi_2(z)\Psi_1'(z')}{k = p_2 W}\mathrm{Ai}(\xi(z'))\mathrm{d}z' \end{aligned} \tag{20}$$

This is:

$$\Psi_{PI}(z) = -\frac{\cosh(kz)}{k}\int_0^z \sinh(kz')\mathrm{Ai}(\xi(z'))\mathrm{d}z' + \frac{\sinh(kz)}{k}\int_0^z \cosh(kz')\mathrm{Ai}(\xi(z'))\mathrm{d}z' \tag{21}$$

Or:

$$\Psi_{PI}(z) = \frac{1}{k}\int_0^z \left[\sinh(kz)\cosh(kz') - \cosh(kz)\sinh(kz')\right]\mathrm{Ai}(\xi(z'))\mathrm{d}z' \tag{22}$$

We use a trigonometric identity to re-write this as:

$$\Psi_{PI}(z) = \frac{1}{k}\int_0^z \sinh[k(z-z')]\mathrm{Ai}(\xi(z'))\mathrm{d}z'. \tag{23}$$

Filling in for the variable ξ, this is:

$$\Psi_{PI}(z) = \frac{1}{k}\int_0^z \sinh[k(z-z')]\mathrm{Ai}\left[(\mathrm{i}k\mathrm{Re})^{1/3}\left(z' - c - \frac{\mathrm{i}k}{\mathrm{Re}}\right)\right]\mathrm{d}z'. \tag{24}$$

We call this solution of the Orr–Sommerfeld Equation (1) $\chi_3(z)$. Hence, we similarly obtain:

$$\chi_4(z) = \frac{1}{k}\int_0^z \sinh[k(z-z')]\mathrm{Bi}\left[(\mathrm{i}k\mathrm{Re})^{1/3}\left(z' - c - \frac{\mathrm{i}k}{\mathrm{Re}}\right)\right]\mathrm{d}z'. \tag{25}$$

As noted in the question sheet, in the context of Fluid Dynamics, Equation (1) is solved in a channel, with $z \in [0,1]$, with no-slip boundary conditions:

$$\Psi(z) = \Psi'(z) = 0, \qquad z = 0, 1.$$

A general solution of the eigenvalue problem is:

$$\Psi = A\Psi_1(z) + B\Psi_2(z) + C\chi_1(z) + D\chi_2(z). \tag{26}$$

4. Show that the vanishing of the streamfunction at the boundaries implies that

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ \cosh(k) & \sinh(k) & \chi_1(1) & \chi_2(1) \\ k\sinh(k) & k\cosh(k) & \chi_1'(1) & \chi_2'(1) \end{vmatrix} = 0.$$

The general solution $\Psi(z)$ in Equation (26) has boundary conditions:

$$\begin{aligned} \Psi(0) = 0 \quad &, \quad \Psi'(0) = 0, \\ \Psi(1) = 0 \quad &, \quad \Psi'(1) = 0. \end{aligned}$$

Note that $\chi_i(0) = (1/k)\int_0^0 (\cdots)\mathrm{d}z' = 0$, for $i = 1, 2$. Similarly,

$$\begin{aligned} \frac{\mathrm{d}\chi_i}{\mathrm{d}z}\Big|_{z=0} &= \frac{1}{k}\sinh[k(z-z)]\left\{\begin{array}{c}\mathrm{Ai}(\cdots)\\ \mathrm{Bi}(\cdots)\end{array}\right\} + \frac{1}{k}\int_0^0 \frac{\partial}{\partial z}\sinh[k(\cdots)]\left\{\begin{array}{c}\mathrm{Ai}(\cdots)\\ \mathrm{Bi}(\cdots)\end{array}\right\}\mathrm{d}z', \\ &= 0. \end{aligned} \tag{27}$$

Hence, $\chi_i(0) = \chi_i'(0) = 0$, for $i = 1, 2$. Hence:

$$\begin{aligned} \Psi_1(0) &= \cosh(0) = 1, \\ \Psi_1'(0) &= k\sinh(0) = 0, \\ \Psi_2(0) &= \sinh(0) = 0, \\ \Psi_2'(0) &= k\cosh(0) = k. \end{aligned} \tag{28}$$

Apply to Equation (26):

$$\begin{aligned} \Psi(0) &= A\cdot 1 + B\cdot 0 + C\cdot 0 + D\cdot 0, \\ \Psi'(0) &= A\cdot 0 + B\cdot k + C\cdot 0 + D\cdot 0. \end{aligned} \tag{29}$$

Also:

$$\begin{aligned} \Psi(1) &= A\cosh(k) + B\sinh(k) + C\chi_1(1) + D\chi_2(1), \\ \Psi'(1) &= kA\sinh(k) + kB\cosh(k) + C\chi_1'(1) + D\chi_2'(1). \end{aligned} \tag{30}$$

Equations (29) and (30) are a linear system:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ \cosh(k) & \sinh(k) & \chi_1(1) & \chi_2(1) \\ k\sinh(k) & k\cosh(k) & \chi_1'(1) & \chi_2'(1) \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0. \tag{31}$$

The linear system has a non-trivial solution for $(A, B, C, D)^T$ provided the determinant in the question vanishes.

5. Show that the determinant can be simplified dramatically to:

$$k\left[\chi_1(1)\chi_2'(1)-\chi_2(1)\chi_1'(1)\right]=0. \tag{32}$$

We evaluate the determinant on Question 4 along the first row to get:

$$\begin{aligned}\left|\cdots\right| &= 1\cdot\left|\begin{array}{ccc} k & 0 & 0\\ \sinh(k) & \chi_1(1) & \chi_2(1)\\ \cosh(k) & \chi_1'(k) & \chi_2'(1)\end{array}\right|+0\cdot\left|\cdots\right|+0\cdot\left|\cdots\right|+0\cdot\left|\cdots\right|,\\ &= k\left|\begin{array}{cc}\chi_1(1) & \chi_2(1)\\ \chi_1'(1) & \chi_2'(1)\end{array}\right|, \end{aligned} \tag{33}$$

hence

$$\left|\cdots\right| = k\left[\chi_1(1)\chi_2'(1)-\chi_2(1)\chi_1'(1)\right], \tag{34}$$

as required.

Interpretation: The determinant can be written as:

$$\left|\cdots\right| = F(k,\mathrm{Re},c). \tag{35}$$

To have a non-trivial solution of the Orr–Sommerfeld equation, we require that

$$F(k,\mathrm{Re},c)=0. \tag{36}$$

For fixed k and Re, this is a root-finding condition for c, with complex roots

$$\left\{c_n(k,\mathrm{Re})\right\})_{n=0}^{\infty}.$$

These are the eigenvalues of the Orr–Sommerfeld equation. Depending on

$$\mathrm{sign}\left[\mathrm{Imag}(c)\right]$$

the travelling-wave solution $\psi(x,z,t)=\mathrm{e}^{\mathrm{i}k(x-ct)}\Psi(z)$ is linearly stable or unstable.

History lesson: Equation (1) is named after William McFadden Orr and Arnold Sommerfeld, who derived it at the beginning of the 20th century. William McFadden Orr (2 May 1866 – 14 August 1934) was a professor in UCD.

Figure 2: Screenshot of the Wikipedia page on William McFadden Orr, professor of mathematics at the Royal College of Science for Ireland (1892) and professor of pure mathematics and applied mathematics when that college merged with University College Dublin in 1926.