Advanced Mathematical Methods (ACM30020)

Dr Lennon Ó Náraigh

Exercises #3

In this set of exercises we are concerned with the Orr–Sommerfeld equation for Couette flow. The purpose of the exercises is to showcase a very nice application of the method of Variation of Parameters. It is not necessary to go into the Fluid Dynamics; instead, it suffices just to go ahead and look at the equation:

$$\mathrm{i}k\left(z-c\right)\left(\partial_z^2-k^2\right)\Psi=\mathrm{Re}^{-1}\left(\partial_z^2-k^2\right)^2\Psi. \tag{1}$$

Here, k and Re are real parameters, z is the variable, and ∂_z denotes ordinary differentiation with respect to z. The constant c is an eigenvalue to be determined, this can be real or imaginary.

1. Let $v=\left(\partial_z^2-k^2\right)\Psi$. Notice that $v=0$ satisfies Equation (1). Hence, show that

$$\Psi=\cosh(kz),\qquad \Psi=\sinh(kz).$$

   are solutions of Equation (1).

2. Carry out a series of rescalings, $\widetilde{z}=z-c-\mathrm{i}k/\mathrm{Re}$, followed by $\xi=\lambda\widetilde{z}$, where λ is constant. Hence, show that v satisfies Airy's differential equation:

$$\frac{\mathrm{d}^2v}{\mathrm{d}\xi^2}-\xi v=0, \tag{2}$$

   where $\xi=(\mathrm{i}k\mathrm{Re})^{1/3}[z-c-(\mathrm{i}k/\mathrm{Re})]$. We choose the particular cube root of $\mathrm{i}^{1/3}=\mathrm{e}^{\mathrm{i}\pi/6}$.

Equation (2) has solutions

$$v=\mathrm{Ai}(\xi),\qquad v=\mathrm{Bi}(\xi).$$

Here, Ai and Bi are special solutions of Airy's differential equation, these can be looked up.

3. To obtain the remaining two linearly independent solutions of Equation (1), we look at:

$$(\partial_z^2-k^2)\Psi=\mathrm{Ai}(\xi),\qquad (\partial_z^2-k^2)\Psi=\mathrm{Bi}(\xi). \tag{3}$$

   Use the method of variation of parameters to construct the following two solutions:

$$\chi_1(z)=\frac{1}{k}\int_0^z\sinh[k(z-z')]\mathrm{Ai}\left[(\mathrm{i}k\mathrm{Re})^{1/3}\left(z'-c-\frac{\mathrm{i}k}{\mathrm{Re}}\right)\right]\mathrm{d}z', \tag{4a}$$

$$\chi_2(z)=\frac{1}{k}\int_0^z\sinh[k(z-z')]\mathrm{Bi}\left[(\mathrm{i}k\mathrm{Re})^{1/3}\left(z'-c-\frac{\mathrm{i}k}{\mathrm{Re}}\right)\right]\mathrm{d}z', \tag{4b}$$

In the context of Fluid Dynamics, Equation (1) is solved in a channel, with $z \in [0,1]$, with no-slip boundary conditions:

$$\Psi(z) = \Psi'(z) = 0, \qquad z = 0, 1.$$

A general solution of the eigenvalue problem is:

$$\Psi = A\Psi_1(z) + B\Psi_2(z) + C\chi_1(z) + D\chi_2(z).$$

4. Show that the vanishing of the streamfunction at the boundaries implies that

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ \cosh(k) & \sinh(k) & \chi_1(1) & \chi_2(1) \\ k\sinh(k) & k\cosh(k) & \chi_1'(1) & \chi_2'(1) \end{vmatrix} = 0.$$

5. Show that the determinant can be simplified dramatically to:

$$k\left[\chi_1(1)\chi_2'(1) - \chi_2(1)\chi_1'(1)\right] = 0. \tag{5}$$

The right-hand side can be viewed as a complex-valued function of k, Re, and c (the latter a complex variable). We therefore have the generic condition

$$F(k, \mathrm{Re}, c) = 0,$$

which is a root-finding condition, with a set of roots $c_n(k, \mathrm{Re})$ such that $F(k, \mathrm{Re}, c_n) = 0$.