Applied Analysis (ACM30020)

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Exercises #2

1. For the linear ODE
$$\sum_{j=0}^{n} p_j(x)\frac{\mathrm{d}^{(j)}y}{\mathrm{d}x^{(j)}} = 0,$$
prove that the Lipschitz constant K in the L^∞ norm can be written as:
$$K = (n-1) + \sum_{i=0}^{n-1} \max_{x\in[a,b]} \left|\frac{p_i(x)}{p_n(x)}\right|.$$
Here, all of the usual assumptions on the $p_j's$ apply: each $p_j(x)$ is continuous on the interval $[a,b]$, and $p_n(x)$ is never zero on $[a,b]$.

Solution: Introduce the vector

$$\boldsymbol{y} = \left(y, \frac{\mathrm{d}y}{\mathrm{d}x}, \cdots, \frac{\mathrm{d}^{(n-1)}y}{\mathrm{d}x^{(n-1)}}\right)^T,$$

which is n-dimensional. We start the indexing at zero, so we have $y_i = \mathrm{d}^{(i)}y/\mathrm{d}x^{(i)}$, with $i \in \{0, 1, \cdots, n-1\}$. So from the ODE, we have:

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}x} = \left(y_1, y_2, \cdots, y_{n-1}, -\sum_{j=0}^{n-1}\frac{p_j(x)}{p_n(x)}y_j\right)^T = \boldsymbol{F}(x,\boldsymbol{y}).$$

We are interested in the increment $\boldsymbol{F}(x,\boldsymbol{y}+\boldsymbol{\delta}) - \boldsymbol{F}(x,\boldsymbol{y})$, which by linearity is:

$$\boldsymbol{F}(x,\boldsymbol{y}+\boldsymbol{\delta}) - \boldsymbol{F}(x,\boldsymbol{y}) = \left(\delta_1, \delta_2, \cdots, \delta_{n-1}, -\sum_{j=0}^{n-1}\delta_j p_j(x)/p_n(x)\right)^T.$$

We have:

$$\|\boldsymbol{F}(x,\boldsymbol{y}+\boldsymbol{\delta}) - \boldsymbol{F}(x,\boldsymbol{y})\|_\infty = \left\| \left(\delta_1, \cdots, \delta_{n-1}, -\sum_{j=0}^{n-1}\delta_j p_j(x)/p_n(x)\right)^T \right\|_\infty.$$

Hence:

$$\|\boldsymbol{F}(x, \boldsymbol{y} + \boldsymbol{\delta}) - \boldsymbol{F}(x, \boldsymbol{y})\|_\infty = \max\left(|\delta_1|, \cdots, |\delta_n|, |\sum_{j=0}^{n-1} \delta_j p_j(x)/p_n(x)|\right). \quad (1)$$

We have:

$$\max(|x|, |y|) \le |x| + |y|,$$

for all x and y in $\mathbb{R}$. Apply this to Equation (1) to get:

$$\begin{aligned}
\|\boldsymbol{F}(x, \boldsymbol{y} + \boldsymbol{\delta}) - \boldsymbol{F}(x, \boldsymbol{y})\|_\infty &\le |\delta_1| + \cdots + |\delta_{n-1}| + \max_{j\in\{0,n-1\}} |\delta_j| \sum_{j=0}^{n-1} \max_{x\in[a,b]} |p_j(x)/p_n(x)|, \\
&\le (n-1) \max_{j\in\{1,n-1\}} |\delta_j| + \max_{j\in\{0,n-1\}} |\delta_j| \sum_{j=0}^{n-1} \max_{x\in[a,b]} |p_j(x)/p_n(x)|, \\
&\le (n-1) \max_{j\in\{0,n-1\}} |\delta_j| + \max_{j\in\{0,n-1\}} |\delta_j| \sum_{j=0}^{n-1} \max_{x\in[a,b]} |p_j(x)/p_n(x)|, \\
&\le (n-1)\|\boldsymbol{\delta}\|_\infty + \|\boldsymbol{\delta}\|_\infty \sum_{j=0}^{n-1} \max_{x\in[a,b]} |p_j(x)/p_n(x)|, \\
&= K\|\boldsymbol{\delta}\|_\infty,
\end{aligned}$$

where

$$K = (n-1) + \sum_{j=0}^{n-1} \max_{x\in[a,b]} |p_j(x)/p_n(x)|,$$

as required.

2. In class we looked at a comparison theorem and we prematurely named it Gronwall's Inequality. However, Gronwall's Inequality is in reality a bit more general than the particular instance we describe in the notes. In fact, Gronwall's inequality says that if $\sigma(x)$ is a differentiable function satisfying:

$$\sigma'(x) \leq g(x)\sigma(x), \qquad x \in I, \qquad I = (a,b), \tag{2}$$

where $g(x)$ is a continuous function on I, then

$$\sigma(x) \leq \sigma(a) \mathrm{e}^{\int_a^x g(x')\mathrm{d}x'}, \tag{3}$$

for all $x \in I$.

Prove the differential inequality (3).

Multiply both sides of Equation (2) by the integrating factor

$$\mu(x) = \mathrm{e}^{-\int_a^x g(x')\mathrm{d}x'}.$$

This does not affect the direction of the inequality, since μ is positive or zero. Thus, we have:

$$\mu \frac{\mathrm{d}\sigma}{\mathrm{d}x} \leq \mu g(x)\sigma(x).$$

Or,

$$\mu \frac{\mathrm{d}\sigma}{\mathrm{d}x} - \mu g(x)\sigma(x) \leq 0.$$

Or again,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sigma\mu) \leq 0.$$

Thus, $\sigma\mu$ is a non-increasing function of x:

$$\sigma(x)\mu(x) \leq \sigma(a)\mu(a), \qquad x \in I.$$

We have $\mu(a) = 1$, hence:

$$\sigma(x) \leq \sigma(a)/\mu(x), \qquad x \in I.$$

or finally,

$$\sigma(x) \leq \sigma(a)\mathrm{e}^{\int_a^x g(x')\mathrm{d}x'}, \qquad x \in I.$$

3. Consider the following SEIR model from Mathematical Epidemiology:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mu N - \mu S - \frac{\beta I S}{N} \tag{4a}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\beta I S}{N} - (\mu + a)E \tag{4b}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = aE - (\gamma + \mu)I \tag{4c}$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I - \mu R. \tag{4d}$$

It is not necessary to go into the meaning of the variables S, E, I, and R here, suffice to say, they are dynamic sub-populations which add up to $N = S + E + I + R$, the total population. The other symbols μ, β, a, and γ are constant rates which govern the rate at which individuals leave one sub-population for another. Equation (4) is an IVP, valid at $t > 0$. Initial conditions are expected at $t = 0$.

(a) Let $S + E + I + R = N$. Show that N is constant.

Remark: N is constant in this model because the rate of natural births is the same as the rate of natural deaths.

(b) Show that Equation (4) has two constant solutions, a disease-free equilibrium

$$DFE = (N, 0, 0, 0),$$

and an endemic equilibrium

$$EE = (S_*, E_*, I_*, R_*),$$

where all of the values here are non-zero.

(c) Compute the coefficients of the endemic equilibrium in terms of a, β, γ, μ, and N.

(d) Show that, for given initial conditions $S(0) > 0$, $E(0) = 0$, $I(0) > 0$, and $R(0) = 0$, the solution of Equation (4) remains inside the hypercube $[0, N]^4$ for all time.

Hint: ...

Parts (a) and (b) are very straightforward and are not repeated here. For part (c), we let $\mathrm{d}S/\mathrm{d}t$ etc. all be equal to zero. Then we have:

$$\begin{aligned} \mu N &= \mu S + \frac{\beta I S}{N}, \\ \frac{\beta I S}{N} &= (\mu + a)E. \end{aligned}$$

Hence,

$$\mu N = \mu S + (\mu + a)E. \tag{5}$$

Also,

$$\begin{aligned} aE &= (\gamma + \mu)I, \\ \gamma I &= \mu R. \end{aligned}$$

We solve for everything in terms of R:

$$\begin{aligned} I &= \frac{\mu}{\gamma}R, \\ E &= \frac{\gamma + \mu}{a}\frac{\mu}{\gamma}R. \end{aligned}$$

Sub into Equation (5) to get:

$$S = N - \frac{\gamma + \mu}{a}\frac{\mu + a}{\gamma}R.$$

It now remains to determine R in terms of the model parameters. We start with:

$$\mu N = \mu S + \frac{\beta I S}{N}.$$

Hence,

$$N = \underbrace{\left(N - \frac{\gamma + \mu}{a}\frac{\mu + a}{\gamma}R\right)}_{=S} + \beta\frac{1}{N}\underbrace{\left(N - \frac{\gamma + \mu}{a}\frac{\mu + a}{\gamma}R\right)}_{=S}\underbrace{\left(\frac{\mu}{\gamma}R\right)}_{=I}.$$

We carry out cancellations and solve for R:

$$R = R_* = N\frac{\gamma}{\beta}\left(\frac{a}{\gamma + \mu}\frac{\beta}{\mu + a} - 1\right).$$

Summarizing, we have:

$$\begin{aligned} S_* &= N - \frac{\gamma + \mu}{a}\frac{\mu + a}{\gamma}R, \\ E_* &= \frac{\gamma + \mu}{a}\frac{\mu}{\gamma}R, \\ I_* &= \frac{\mu}{\gamma}R_*, \\ R_* &= N\frac{\gamma}{\beta}\left(\frac{a}{\gamma + \mu}\frac{\beta}{\mu + a} - 1\right). \end{aligned}$$

For part (d), we assume for contradiction that $I(t_*) = 0$. By continuity, there is an interval of time $[0, t_*)$ where $I(t) > 0$. We look at the S-equation on this interval:

$$\frac{\mathrm{d}S}{\mathrm{d}t} + \underbrace{\left(\mu + \frac{\beta I}{N}\right)}_{=P(t)} S = \mu N. \tag{6}$$

This is a standard first-order ODE. We identify the integrating factor

$$\mathcal{I}(t) = \mathrm{e}^{\int_0^t P(t)\mathrm{d}t}.$$

Notice, $\mathcal{I}(t) > 0$. Also,

$$\frac{1}{\mathcal{I}(t)} = \mathrm{e}^{-\int_0^t P(t)\mathrm{d}t}.$$

Then, Equation (7) can be re-written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(S\mathcal{I}) = \mu N \mathcal{I}.$$

The LHS is now a perfect derivative, so this equation can be solved by integration:

$$\mathcal{I}(t)S(t) = \mathcal{I}(0)S(0) + \mu N \int_0^t \mathcal{I}(t')\mathrm{d}t'.$$

Re-arranging, and using $\mathcal{I}(0) = 1$ gives:

$$S(t) = S(0)\mathrm{e}^{-\int_0^t P(t)\mathrm{d}t} + \frac{\mu N}{\mathcal{I}(t)} \int_0^t \mathcal{I}(t')\mathrm{d}t'.$$

All of the terms on the RHS here are positive, hence $S(t) > 0$ for all $t \in [0, t_*)$.

We next look at the E-equation:

$$\frac{\mathrm{d}E}{\mathrm{d}t} + (\mu + a)E = \underbrace{\frac{\beta I S}{N}}_{=Q(t)}. \tag{7}$$

Notice that $Q(t) > 0$ on $[0, t_*)$. We apply the integrating-factor technique and obtain:

$$E(t) = E(0)\mathrm{e}^{-(\mu+a)t} + \mathrm{e}^{-(\mu+a)t} \int_0^t \mathrm{e}^{(\mu+a)t'} Q(t')\mathrm{d}t'.$$

Again, all of the terms on the RHS are positive or zero. In particular,

$$E(t) > 0, \qquad t \in (0, t_*).$$

We now look at the I-equation, on the interval $(0, t_*)$:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = aE - (\gamma + \mu)I.$$

Since $E > 0$ on $(0, t_*)$, we have:

$$\frac{\mathrm{d}I}{\mathrm{d}t} > -(\gamma + \mu)I.$$

We use Gronwall's inequality (strict) to conclude that

$$I(t) > I(0)\mathrm{e}^{-(\gamma+\mu)t}, \qquad t \in (0, t_*).$$

In particular, $I(t_*) > I(0)\mathrm{e}^{-(\gamma+\mu)t_*} > 0$, which is a contradiction, since $I(t_*) = 0$. A similar approach for $R(t)$ yields $R(t) \geq 0$ for all $t \geq 0$, hence:

$$S(t),\ I(t)\, E(t)\, R(t) \geq 0,\ \text{ for all } t \geq 0.$$

Finally, since $S + E + I + R = N$, and since each of S, E, ... are all positive or zero, we must have $0 \leq S(t) \leq N$, etc., hence

$$(S, E, I, R) \in [0, N]^4,$$

for all $t \geq 0$.