Applied Analysis (ACM30020)

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Exercises #2

1. For the linear ODE
$$\sum_{j=0}^{n} p_j(x) \frac{\mathrm{d}^{(j)} y}{\mathrm{d}x^{(j)}} = 0,$$
prove that the Lipschitz constant K in the L^∞ norm can be written as:
$$K = (n-1) + \sum_{i=0}^{\infty} \max_{x \in [a,b]} \left| \frac{p_i(x)}{p_n(x)} \right|.$$
Here, all of the usual assumptions on the $p_j's$ apply: each $p_j(x)$ is continuous on the interval $[a,b]$, and $p_n(x)$ is never zero on $[a,b]$.

2. In class we looked at a comparison theorem and we prematurely named it Gronwall's Inequality. However, Gronwall's Inequality is in reality a bit more general than the particular instance we describe in the notes. In fact, Gronwall's inequality says that if $\sigma(x)$ is a differentiable function satisfying:
$$\sigma'(x) \leq g(x)\sigma(x), \qquad x \in I, \qquad I = (a,b),$$
where $g(x)$ is a continuous function on I, then
$$\sigma(x) \leq \sigma(a) \mathrm{e}^{\int_a^x g(x')\mathrm{d}x'}, \tag{1}$$
for all $x \in I$.

   Prove the differential inequality (1).

3. Consider the following SEIR model from Mathematical Epidemiology:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mu N - \mu S - \frac{\beta I S}{N} \tag{2a}$$
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\beta I S}{N} - (\mu + a)E \tag{2b}$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = aE - (\gamma + \mu)I \tag{2c}$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I - \mu R. \tag{2d}$$

It is not necessary to go into the meaning of the variables S, E, I, and R here, suffice to say, they are dynamic sub-populations which add up to $N = S + E + I + R$, the total populatioin. The other symbols μ, β, a, and γ are constant rates which govern the rate at which individuals leave one sub-population for another. Equation (2) is an IVP, valid at $t > 0$. Initial conditions are expected at $t = 0$.

(a) Let $S + E + I + R = N$. Show that N is constant.

Remark: N is constant in this model because the rate of natural births is the same as the rate of natural deaths.

(b) Show that Equation (2) has two constant solutions, a disease-free equilibrium

$$DFE = (N, 0, 0, 0),$$

and an endemic equilibrium

$$EE = (S_*, E_*, I_*, R_*),$$

where all of the values here are non-zero.

(c) Compute the coefficients of the endemic equilibrium in terms of a, β, γ, μ, and N.

(d) Show that, for given initial conditions $S(0) > 0$, $E(0) = 0$, $I(0) > 0$, and $R(0) = 0$, the solution of Equation (2) remains inside the hypercube $[0, N]^4$ for all time.

Hint: Assume for contradiction that $I(t_*) = 0$. By continuity, there is an interval of time $[0, t_*)$ where $I(t) > 0$. On this interval, show:

- Use the integrating-factor method of ordinary-differential equations to write the solution of Equation (2)(a) as:

$$S(t) = S(0)\mathrm{e}^{-\int_0^t [\mu + (\beta/N)I]\mathrm{d}t} + \cdots ,$$

where the $\cdots$ are to be filled in, hence conclude that $S(t) > 0$ for $t \in [0, t_*)$.
- Use a similar approach to show that $E(t) > 0$ for $t \in [0, t_*)$.
- Use

$$\frac{\mathrm{d}I}{\mathrm{d}t} = aE - (\gamma + \mu) > -(\gamma + \mu)I.$$

From this, use Gronwall's inequality to conclude that

$$I(t) > I(0)\mathrm{e}^{-(\gamma+\mu)t}, \qquad t \in [0, t_*).$$

In particular, $I(t) > I(0)\mathrm{e}^{-(\gamma+\mu)t} > 0$ as $t \to t_*$, which is a contradiction, since $I(t_*) = 0$.

- Fill in the remaining missing pieces yourselves.