

Applied Analysis (ACM30020)

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Exercises #1

1. Prove the statement in Chapter 1 of the lecture notes: if $F(x, y)$ is continuously differentiable and D is closed, bounded and convex then, a F satisfies a Lipschitz Condition with respect to y taking:

$$K = \sup_{(x,y) \in D} \left| \frac{\partial F}{\partial y} \right|.$$

Solution: Let $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$ be points in D . Because D is convex, the line segment

$$L : \mathbf{x}(t) = \mathbf{x}_1(1 - t) + \mathbf{x}_2 t, \quad t \in [0, 1],$$

is contained entirely in D . Introduce the function $g(t)$ of a single variable:

$$g(t) = F(\mathbf{x}(t)).$$

We have:

$$\begin{aligned} \frac{dg}{dt} &= \nabla F \cdot \frac{d\mathbf{x}}{dt}, \\ &= \nabla F(\mathbf{x}(t)) \cdot (\mathbf{x}_2 - \mathbf{x}_1). \end{aligned}$$

Hence, $g(t)$ is continuously differentiable on $(0, 1)$. Hence, by the Mean-Value Theorem, there is a $t_* \in (0, 1)$ such that:

$$\left. \frac{dg}{dt} \right|_{t_*} = \frac{g(1) - g(0)}{1} = F(\mathbf{x}_2) - F(\mathbf{x}_1).$$

Hence,

$$\begin{aligned} |F(\mathbf{x}_2) - F(\mathbf{x}_1)| &= |g'(t_*)|, \\ &= |\nabla F(\mathbf{x}(t_*)) \cdot (\mathbf{x}_2 - \mathbf{x}_1)|. \end{aligned}$$

Take $x_2 = x_1 = x$, hence $\mathbf{x}_2 - \mathbf{x}_1 = (y_2 - y_1)\mathbf{j}$, hence:

$$\begin{aligned} |F(\mathbf{x}_2) - F(\mathbf{x}_1)| &= \left| \frac{\partial F}{\partial y} \Big|_{\mathbf{x}(t_*)} (y_2 - y_1) \right|, \\ &\leq \sup_{(x,y) \in D} \left| \frac{\partial F}{\partial y} \right| |y_2 - y_1|, \\ &= K |y_2 - y_1|. \end{aligned}$$

2. All finite-dimensional norms are equivalent. That is, if I have two norms $\|\cdot\|_P$ and $\|\cdot\|_Q$, there exist positive constants c_1 and c_2 such that

$$c_1\|\mathbf{x}\|_P \leq \|\mathbf{x}\|_Q \leq c_2\|\mathbf{x}\|_P, \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Using the Cauchy–Schwarz inequality, show that the L^1 and L^2 norms are equivalent, such that:

$$\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|_2.$$

Solution: We have

$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|.$$

Then by Cauchy–Schwarz with $\mathbf{x} = (|x_1|, \dots, |x_n|)$ and $\mathbf{a} = (1, 1, \dots, 1)$, we have:

$$\sum_{i=1}^n |x_i| \leq n \left(\sum_{i=1}^n x_i^2 \right)^{1/2},$$

hence

$$\|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|_2.$$

But it is immediately obvious that e.g. $\|\mathbf{x}\|_1^2 \geq \|\mathbf{x}\|_2^2$, hence

$$\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|_2,$$

establishing the equivalence of the L^1 -norm and the L^2 -norm.

3. In class, we showed that the IVP

$$\frac{dy}{dx} = F(x, y), \quad x > 0, \quad y(x_0) = y_0, \quad (1)$$

has a unique solution, provided $F(x, y)$ is Lipschitz in y . Here, we tackle the uniqueness question, but in a different way. As such, assume that $y(x)$ and $z(x)$ are two solutions of Equation (1). Show that:

$$|y(x) - z(x)| \leq K \int_{x_0}^x |y(s) - z(s)| ds.$$

Hence, and without using Gronwall's inequality, show that $|y(x) - z(x)| = 0$.

We have:

$$\begin{aligned} y(x) &= y_0 + \int_0^x F(s, y(s)) ds, \\ z(x) &= y_0 + \int_0^x F(s, z(s)) ds. \end{aligned}$$

Hence:

$$\begin{aligned} |y(x) - z(x)| &= \left| \int_{x_0}^x [F(s, y(s)) - F(s, z(s))] ds \right|, \\ &\stackrel{\text{Triangle}}{\leq} \int_{x_0}^x |F(s, y(s)) - F(s, z(s))| ds, \\ &\stackrel{\text{Lipschitz}}{\leq} \int_{x_0}^x K |y(s) - z(s)| ds. \end{aligned}$$

This relationship can now be iterated to produce:

$$|y(x) - z(x)| \leq K^n \int_{x_0}^x \int_{x_0}^{x_1} \cdots \int_{x_0}^{x_{n-1}} |y(x_n) - z(x_n)| dx_1 \cdots dx_n.$$

Hence,

$$|y(x) - z(x)| \leq K^n \left(\max_{s \in [x_0, x]} |y(s) - z(s)| \right) \int_{x_0}^x \int_{x_0}^{x_1} \cdots \int_{x_0}^{x_{n-1}} dx_1 \cdots dx_n.$$

The integral is the volume of a right-angled triangular pyramid in n dimensions of sides of length $|x - x_0|$. Using this geometric insight, or direct calculation, we obtain:

$$|y(x) - z(x)| \leq \frac{1}{n!} K^n \left(\max_{s \in [x_0, x]} |y(s) - z(s)| \right) |x_0 - x|^n$$

As K and $\max_{s \in [x_0, x]} |y(s) - z(s)|$ are both fixed, we take $n \rightarrow \infty$ to obtain:

$$|y(x) - z(x)| \leq 0,$$

for all $x \geq x_0$. Hence, $y(x) = z(x)$, and the solution to the IVP is unique.

4. (a) Convert the initial value problem

$$\frac{dy}{dx} = 2x - y, \quad y(0) = 1$$

to an integral equation.

(b) Prove by induction that the Picard iteration method leads to

$$y_n(x) = 1 - x + 3 \sum_{k=2}^n \frac{(-1)^k}{k!} x^k + 2 \frac{(-x)^{n+1}}{(n+1)!}$$

(c) Deduce that the initial value problem has the unique solution

$$y(x) = -2 + 2x + 3e^{-x}.$$

For part (a), we have:

$$\begin{aligned} y^{(n+1)}(x) &= y_0 + \int_{x_0}^x F(x, y^{(n)}(x)) dx, \\ &= y_0 + \int_{x_0}^x [2x - y^{(n)}(x)] dx, \\ &= 1 + x^2 - \int_0^x y^{(n)}(x) dx. \end{aligned}$$

We also have $y^{(0)}(x) = 1$.

For part (b), we assume that the Inductive Hypothesis (IH) is true for n :

$$y_n(x) = 1 - x + 3 \sum_{k=2}^n \frac{(-1)^k}{k!} x^k + 2 \frac{(-x)^{n+1}}{(n+1)!} \quad (2)$$

From Picard Iteration / Part (a) we have:

$$\begin{aligned} y^{(n+1)}(x) &= 1 + x^2 - \int_0^x y^{(n)}(x) dx, \\ &\stackrel{\text{I.H.}}{=} 1 + x^2 - \int_0^x \left[1 - x + 3 \sum_{k=2}^n \frac{(-1)^k}{k!} x^k + 2 \frac{(-x)^{n+1}}{(n+1)!} \right] dx, \\ &= 1 + x^2 - x + \frac{1}{2}x^2 - 3 \sum_{k=2}^n \frac{(-1)^k}{(k+1)!} x^{k+1} + 2 \frac{(-1)^2(-x)^{n+2}}{(n+2)!} \end{aligned}$$

We re-index, with $\ell = k + 1$, hence $\ell_{\min} = 3$ and $\ell_{\max} = n + 1$:

$$\begin{aligned} y^{(n+1)}(x) &= 1 + x^2 - x + \frac{1}{2}x^2 + 3 \sum_{\ell=3}^{n+1} \frac{(-1)^\ell}{\ell!} x^\ell + 2 \frac{(-x)^{n+2}}{(n+2)!}, \\ &= 1 - x + \frac{3}{2}x^2 + 3 \sum_{\ell=3}^{n+1} \frac{(-1)^\ell}{\ell!} x^\ell + 2 \frac{(-x)^{n+2}}{(n+2)!}. \end{aligned}$$

Hence,

$$y^{(n+1)}(x) = 1 - x + 3 \sum_{\ell=2}^{n+1} \frac{(-1)^\ell}{\ell!} x^\ell + 2 \frac{(-x)^{n+2}}{(n+2)!}. \quad (3)$$

Thus, if the IH is true for n , it is true for $n+1$.

We are given $y^{(0)}(x) = 1$, so we look at $y^{(1)}(x)$ using Picard iteration:

$$\begin{aligned} y^{(1)}(x) &= 1 + x^2 - \int_0^x dx, \\ &= 1 - x + x^2. \end{aligned}$$

If we look at Equation (2) with $n = 1$, it is:

$$y^{(1)}(x) = 1 - x + x^2. \quad (4)$$

Thus, by mathematical induction (combining Equations (3) and (4)), the IH is true for all $n \in \{1, 2, \dots\}$, as required.

For part (c), we take $n \rightarrow \infty$, such that $y^{(n)}(x) \rightarrow y(x)$ and $2(-x)^{n+1}/(n+1)! \rightarrow 0$ in Equation (2). We have:

$$\begin{aligned} y(x) &= 1 - x + 3 \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} x^k, \\ &= 1 - x + 3 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k - 3(1 - x), \\ &= -2 + 2x + 3e^{-x}, \end{aligned}$$

as required.