Applied Analysis (ACM30020)

Dr Lennon Ó Náraigh

Exercises #1

1. Prove the statement in Chapter 1 of the lecture notes: if $F(x,y)$ is continuously differentiable and D is closed, bounded and convex then, a F satisfies a Lipschitz Condition with respect to y taking:

$$K = \sup_{(x,y)\in D}\left|\frac{\partial F}{\partial y}\right|.$$

Solution: Let $\boldsymbol{x}_1 = (x_1, y_1)$ and $\boldsymbol{x}_2 = (x_2, y_2)$ be points in D. Because D is convex, the line segment

$$L : \boldsymbol{x}(t) = \boldsymbol{x}_1(1-t) + \boldsymbol{x}_2 t, \qquad t \in [0,1],$$

is contained entirely in D. Introduce the function $g(t)$ of a single variable:

$$g(t) = F(\boldsymbol{x}(t)).$$

We have:

$$\begin{aligned} \frac{\mathrm{d}g}{\mathrm{d}t} &= \nabla F \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}, \\ &= \nabla F(\boldsymbol{x}(t)) \cdot (\boldsymbol{x}_2 - \boldsymbol{x}_1). \end{aligned}$$

Hence, $g(t)$ is continuously differentiable on $(0,1)$. Hence, by the Mean-Value Theorem, there is a $t_* \in (0,1)$ such that:

$$\left.\frac{\mathrm{d}g}{\mathrm{d}t}\right|_{t_*} = \frac{g(1) - g(0)}{1} = F(\boldsymbol{x}_2) - F(\boldsymbol{x}_1).$$

Hence,

$$\begin{aligned} |F(\boldsymbol{x}_2) - F(\boldsymbol{x}_1)| &= |g'(t_*)|, \\ &= \left|\nabla F(\boldsymbol{x}(t_*)) \cdot (\boldsymbol{x}_2 - \boldsymbol{x}_1)\right|. \end{aligned}$$

Take $x_2 = x_1 = x$, hence $\boldsymbol{x}_2 - \boldsymbol{x}_1 = (y_2 - y_1)\mathbf{j}$, hence:

$$\begin{aligned} |F(\boldsymbol{x}_2) - F(\boldsymbol{x}_1)| &= \left| \frac{\partial F}{\partial y}\bigg|_{\boldsymbol{x}(t_*)} (y_2 - y_1) \right|, \\ &\leq \sup_{(x,y)\in D} \left| \frac{\partial F}{\partial y} \right| |y_2 - y_1|, \\ &= K|y_2 - y_1|. \end{aligned}$$

2. All finite-dimensional norms are equivalent. That is, if I have two norms $\|\cdot\|_P$ and $\|\cdot\|_Q$, there exist positive constants c_1 and c_2 such that

$$c_1\|\boldsymbol{x}|_P \leq \|\boldsymbol{x}\|_Q \leq c_2\|\boldsymbol{x}\|_P, \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n.$$

Using the Cauchy–Schwarz inequality, show that the L^1 and L^2 norms are equivalent, such that:

$$\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2.$$

Solution: We have

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i|.$$

Then by Cauchy–Schwarz with $\boldsymbol{x} = (|x_1|, \cdots, |x_n|)$ and $\boldsymbol{a} = (1, 1, \cdots, 1)$, we have:

$$\sum_{i=1}^{n} |x_i| \leq n \left(\sum_{i=1}^{n} x_i^2 \right),$$

hence

$$\|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2.$$

But it is immediately obvious that e.g. $\|\boldsymbol{x}\|_1^2 \geq \|\boldsymbol{x}\|_2^2$, hence

$$\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2,$$

establishing the equivalence of the L^1-norm and the L^2-norm.

3. In class, we showed that the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x,y), \qquad x>0, \qquad y(x_0)=y_0, \tag{1}$$

has a unique solution, provided $F(x,y)$ is Lipschitz in y. Here, we tackle the uniqueness question, but in a different way. As such, assume that $y(x)$ and $z(x)$ are two solutions of Equation (1). Show that:

$$|y(x)-z(x)| \le K\int_{x_0}^{x} |y(s)-z(s)|\mathrm{d}s.$$

Hence, and without using Gronwall's inequality, show that $|y(x)-z(x)|=0$.

We have:

$$\begin{aligned} y(x) &= y_0 + \int_0^x F(s,y(s))\mathrm{d}s, \\ z(x) &= y_0 + \int_0^x F(s,z(s))\mathrm{d}s. \end{aligned}$$

Hence:

$$\begin{aligned} |y(x)-z(x)| &= \left|\int_{x_0}^{x} \left[F(s,y(s))-F(s,z(s))\right]\mathrm{d}s\right|, \\ &\overset{\text{Triangle}}{\le} \int_{x_0}^{x} |F(s,y(s))-F(s,z(s))|\mathrm{d}s, \\ &\overset{\text{Lipschitz}}{\le} \int_{x_0}^{x} K|y(s)-z(s)|\mathrm{d}s. \end{aligned}$$

This relationship can now be iterated to produce:

$$|y(x)-z(x)| \le K^n \int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_{n-1}} |y(x_n)-z(x_n)|\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

Hence,

$$|y(x)-z(x)| \le K^n \left(\max_{s\in[x_0,x]} |y(s)-z(s)|\right) \int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_{n-1}} \mathrm{d}x_1\cdots\mathrm{d}x_n.$$

The integral is the volume of a right-angled triangular pyramid in n dimensions of sides of length $|x-x_0|$. Using this geometric insight, or direct calculation, we obtain:

$$|y(x)-z(x)| \le \frac{1}{n!}K^n\left(\max_{s\in[x_0,x]} |y(s)-z(s)|\right)|x_0-x|^n$$

As K and $\max_{s\in[x_0,x]}|y(s)-z(s)|$ are both fixed, we take $n\to\infty$ to obtain:

$$|y(x)-z(x)| \le 0,$$

for all $x \ge x_0$. Hence, $y(x)=z(x)$, and the solution to the IVP is unique.

4. (a) Convert the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - y, \qquad y(0) = 1$$

to an integral equation.

(b) Prove by induction that the Picard iteration method leads to

$$y_n(x) = 1 - x + 3\sum_{k=2}^{n} \frac{(-1)^k}{k!}x^k + 2\frac{(-x)^{n+1}}{(n+1)!}$$

(c) Deduce that the initial value problem has the unique solution

$$y(x) = -2 + 2x + 3\mathrm{e}^{-x}.$$

For part (a), we have:

$$\begin{aligned} y^{(n+1)}(x) &= y_0 + \int_{x_0}^{x} F(x, y^{(n)}(x))\mathrm{d}x, \\ &= y_0 + \int_{x_0}^{x} [2x - y^{(n)}(x)]\mathrm{d}x, \\ &= 1 + x^2 - \int_0^x y^{(n)}(x)\mathrm{d}x. \end{aligned}$$

We also have $y^{(0)}(x) = 1$.

For part (b), we assume that the Inductive Hypothesis (IH) is true for n:

$$y_n(x) = 1 - x + 3\sum_{k=2}^{n} \frac{(-1)^k}{k!}x^k + 2\frac{(-x)^{n+1}}{(n+1)!} \tag{2}$$

From Picard Iteration / Part (a) we have:

$$\begin{aligned} y^{(n+1)}(x) &= 1 + x^2 - \int_0^x y^{(n)}(x)\mathrm{d}x, \\ &\overset{\text{I.H.}}{=} 1 + x^2 - \int_0^x \left[1 - x + 3\sum_{k=2}^{n} \frac{(-1)^k}{k!}x^k + 2\frac{(-x)^{n+1}}{(n+1)!}\right]\mathrm{d}x, \\ &= 1 + x^2 - x + \tfrac{1}{2}x^2 - 3\sum_{k=2}^{n} \frac{(-1)^k}{(k+1)!}x^{k+1} + 2\frac{(-1)^2(-x)^{n+2}}{(n+2)!} \end{aligned}$$

We re-index, with $\ell = k + 1$, hence $\ell_{min} = 3$ and $\ell_{max} = n + 1$:

$$\begin{aligned} y^{(n+1)}(x) &= 1 + x^2 - x + \tfrac{1}{2}x^2 + 3\sum_{\ell=3}^{n+1} \frac{(-1)^\ell}{\ell!}x^\ell + 2\frac{(-x)^{n+2}}{(n+2)!}, \\ &= 1 - x + \tfrac{3}{2}x^2 + 3\sum_{\ell=3}^{n+1} \frac{(-1)^\ell}{\ell!}x^\ell + 2\frac{(-x)^{n+2}}{(n+2)!}. \end{aligned}$$

Hence,

$$y^{(n+1)}(x) = 1 - x + 3 + 3\sum_{\ell=2}^{n+1} \frac{(-1)^\ell}{\ell!}x^\ell + 2\frac{(-x)^{n+2}}{(n+2)!}. \tag{3}$$

Thus, if the IH is true for n, it is true for $n+1$.

We are given $y^{(0)}(x) = 1$, so we look at $y^{(1)}(x)$ using Picard iteration:

$$\begin{aligned} y^{(1)}(x) &= 1 + x^2 - \int_0^x \mathrm{d}x, \\ &= 1 - x + x^2. \end{aligned}$$

If we look at Equation (2) with $n = 1$, it is:

$$y^{(1)}(x) = 1 - x + x^2. \tag{4}$$

Thus, by mathematical induction (combining Equations (3) and (4)), the IH is true for all $n \in \{1, 2, \cdots\}$, as required.

For part (c), we take $n \to \infty$, such that $y^{(n)}(x) \to y(x)$ and $2(-x)^{n+1}/(n+1)! \to 0$ in Equation (2). We have:

$$\begin{aligned} y(x) &= 1 - x + 3\sum_{k=2}^{\infty} \frac{(-1)^k}{k!}x^k, \\ &= 1 - x + 3\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}x^k - 3(1-x), \\ &= -2 + 2x + 3\mathrm{e}^{-x}, \end{aligned}$$

as required.