Applied Analysis (ACM30020)

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Exercises #1

1. Prove the statement in Chapter 1 of the lecture notes: if $F(x,y)$ is continuously differentiable and D is closed, bounded and convex then, a F satisfies a Lipschitz Condition with respect to y taking:
$$K = \sup_{(x,y)\in D}\left|\frac{\partial F}{\partial y}\right|.$$

2. All finite-dimensional norms are equivalent. That is, if I have two norms $\|\cdot\|_P$ and $\|\cdot\|_Q$, there exist positive constants c_1 and c_2 such that
$$c_1\|\boldsymbol{x}|_P \le \|\boldsymbol{x}\|_Q \le c_2\|\boldsymbol{x}\|_P, \qquad \text{for all } \boldsymbol{x}\in\mathbb{R}^n.$$
Using the Cauchy–Schwarz inequality, show that the L^1 and L^2 norms are equivalent, such that:
$$\|\boldsymbol{x}\|_2 \le \|\boldsymbol{x}\|_1 \le \sqrt{n}\|\boldsymbol{x}\|_2.$$

3. In class, we showed that the IVP
$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x,y), \qquad x>0, \qquad y(x_0)=y_0, \tag{1}$$
has a unique solution, provided $F(x,y)$ is Lipschitz in y. Here, we tackle the uniqueness question, but in a different way. As such, assume that $y(x)$ and $z(x)$ are two solutions of Equation (1). Show that:
$$|y(x)-z(x)| \le K\int_{x_0}^x |y(s)-z(s)|\mathrm{d}s.$$
Hence, and without using Gronwall's inequality, show that $|y(x)-z(x)|=0$.

4. (a) Convert the initial value problem
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - y, \qquad y(0)=1$$
to an integral equation.

   (b) Prove by induction that the Picard iteration method leads to
$$y_n(x) = 1 - x + 3\sum_{k=2}^{n}\frac{(-1)^k}{k!}x^k + 2\frac{(-x)^{n+1}}{(n+1)!}$$

   (c) Deduce that the initial value problem has the unique solution
$$y(x) = -2 + 2x + 3\mathrm{e}^{-x}.$$