Applied Analysis (ACM30020)

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Graded Assignment – Model Answers

Question 1: Local Existence Theory for ODEs

Consider the ODE system:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x,y), \qquad x \geq a, \tag{1}$$

with initial condition $y(a) = y_0$. As before, construct the Picard iterative solution scheme:

$$y_{n+1}(x) = y_0 + \int_a^x F(x, y_n(x))\mathrm{d}x, n \geq 0,$$

with initial guess $y_0(x) = y_0$. Suppose also that F is Lipschitz, with Lipschitz constant K, such that:

$$|F(x,y_2) - F(x,y_1)| \leq K|y_2 - y_1|,$$

for all x in an interval $(a, a+L)$, and all y_1 and $y_2 \in \mathbb{R}$.

(a) Show that

$$\|y_{n+1} - y_n\|_\infty \leq KL\|y_n - y_{n-1}\|_\infty.$$

(b) Fix L such that $KL < 1$. Hence, deduce that $y_n \mapsto y_{n+1}$, which maps continuous functions to continuous functions, is a contraction mapping.

(c) Use the Contraction Mapping Principle to deduce that Equation (1) has a solution, valid for $x \in [a, a+L]$.

The solution constructed in this way is called a local solution as it is valid on the interval $[a, a+L]$. What happens for $x > a + L$ is anyone's guess. If the solution remains valid for all $x > a + L$, the solution is called a global solution.

(d) Consider the ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^3, \qquad \frac{\mathrm{d}}{\mathrm{d}x}y^2 = -2y^4, \qquad x > 0,$$

with $y(0) = y_0$. Say in each case whether or not a global solution exists.

For Part (a), we fix an interval $[a, L+a]$. We consider the function $|y_{n+1}(x) - y_n(x)|$. This is a continuous function, hence, it realises its maximum on the interval. In other words, there exists an $x_* \in [a, L+a]$ such that:

$$\max_{[a,a+L]} |y_{n+1}(x) - y_n(x)| = |y_{n+1}(x_*) - y_n(x_*)|.$$

and hence,

$$|y_{n+1}(x_*) - y_n(x_*)| = \|y_{n+1} - y_n\|_\infty, \qquad x \in [a, L+a].$$

We have:

$$\begin{aligned}
\|y_{n+1} - y_n\|_\infty &= \left| \int_a^{x_*} [F(x, y_n) - F(x, y_{n-1})] \, \mathrm{d}x \right|, \\
&\overset{\Delta}{\leq} \int_a^{a+L} |F(x, y_n) - F(x, y_{n-1})| \, \mathrm{d}x, \\
&\overset{\text{Lipschitz}}{=} \int_a^{a+L} K |y_n(x) - y_{n-1}(x)| \mathrm{d}x, \\
&\leq KL \max_{x \in [a, L+a]} |y_n(x) - y_{n-1}(x)|, \\
&= KL \|y_n - y_{n-1}\|_\infty.
\end{aligned}$$

For Part (b) we consider the map of functions,

$$\begin{aligned}
\varphi : C([a, a+L]) &\to C([a, a+L]) \\
\varphi(y)(x) &\mapsto y_0 + \int_a^x F(s, y(s)) \mathrm{d}s.
\end{aligned}$$

By a similar calculation to Part (a), we have:

$$\|\varphi(y) - \varphi(z)\|_\infty \leq KL \|y - z\|_\infty, \tag{2}$$

for continuous function y and z in $C([a, a+L])$. Choosing $KL < 1$ makes φ into a contraction mapping in $C([a, a+L])$, as required by the Part (b). To set up Part (c), we further note that since

$$\varphi(y_n) = y_{n+1}, \tag{3}$$

it follows that:

$$\|y_{n+1} - y_n\|_\infty \leq KL \|y_n - y_{n-1}\|_\infty. \tag{4}$$

For Part (c), we look at the Contraction Mapping Principle, which states that a Contraction Map must have a fixed point. The distance

$$d(f, g) = \|f - g\|_\infty$$

makes $C([a, a+L])$ into a metric space. By inequality (4),

$$d\left(\varphi(y_n), \varphi(y_{n-1})\right) < d(y_n, y_{n-1}), \qquad \text{provided } KL < 1.$$

Hence, φ is a contraction mapping, provided $KL < 1$. By the contraction mapping principle there exists a fixed point $y_* \in C([a, a+L])$ such that:

$$\varphi(y_*) = y_*,$$

or

$$y_*(x) = y_0 + \int_a^x F(x, y_*(x))\mathrm{d}x, \qquad \text{for all } x \in [a, L+a].$$

This is exactly the solution of the IVP.

For Part (d) we solve the ODEs directly. The first ODE is $\mathrm{d}y/\mathrm{d}x = y^3$. We separate the variables to get:

$$y^{-3}\mathrm{d}y = \mathrm{d}x.$$

We integrate and apply the initial condition $y(0) = y_0$ and get:

$$y = \frac{y_0}{\sqrt{1 - 2xy_0^2}}.$$

Thus, the solution exists on an interval $[0, x_c)$, where x_c is the singularity point such that

$$1 - 2x_c y_0^2 = 0 \implies x_c = \frac{1}{2y_0^2}.$$

Thus, there is only a local solution,

$$y(x) = \frac{y_0}{\sqrt{1 - 2xy_0^2}}, \qquad x \in [0, \frac{1}{2y_0^2}).$$

The second ODE is $\mathrm{d}y^2/\mathrm{d}x = -2y^4$. We let $z = y^2$ and solve $\mathrm{d}z/\mathrm{d}x = -2z^2$. With the given initial condition we get:

$$\frac{1}{z} = \frac{1}{z_0} + 2x, \qquad z_0 = y^2.$$

We solve for $y = z^2$ and get:

$$y(x) = \frac{y_0}{\sqrt{1 + 2xy_0^2}}.$$

Clearly, a globally defined solution exists for $x \in [0, \infty)$. However, if x is allowed to take negative values, there is a singularity at $x = x_c$ where $x_c = -1/(2y_0^2)$.

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Question 2: Bessel Functions

Take as given the following power-series representation of Bessel functions:

$$J_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{\nu+2r}}{r!\Gamma(n+r+1)}. \tag{1}$$

(a) Prove:

(i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x)$;

(ii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos(x)$.

(b) Prove that for all ν:

(i) $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x)$.

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)$.

(c) Hence, prove that:

(i) $J_\nu'(x) = \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right]$;

(ii) $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$.

For (a)(i), we have:

$$J_{1/2}(x) = (x/2)^{1/2} \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r}}{r!\Gamma\left(1+r+\frac{1}{2}\right)}.$$

Consider the denominator

$$r!\,\Gamma\left(1+r+\tfrac{1}{2}\right) = \Gamma(r+1)\Gamma\left(r+1+\tfrac{1}{2}\right).$$

Use Legendre's duplication formula applied to $z = r+1$:

$$\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

We have:

$$\begin{aligned}
\Gamma(r+1)\Gamma\left(r+1+\tfrac{1}{2}\right) &= 2^{1-2(1+r)}\sqrt{\pi}\Gamma\left(2(r+1)\right), \\
&= \tfrac{1}{2}\frac{1}{2^{2r}}\sqrt{\pi}\Gamma\left((2r+1)+1\right), \\
&= \tfrac{1}{2}\frac{1}{2^{2r}}\sqrt{\pi}(2r+1)!.
\end{aligned}$$

Hence:

$$\begin{aligned}
J_{1/2}(x) &= (x/2)^{1/2}\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{\cancel{2^{2r}}}\frac{2\cdot\cancel{2^{2r}}}{(2r+1)!}, \\
&= (x/2)^{1/2}\frac{2}{x\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r+1}}{(2r+1)!}, \\
&= \sqrt{\frac{2}{\pi x}}\sin(x).
\end{aligned}$$

For (a)(ii), we have:

$$J_{-1/2}(x) = (x/2)^{-1/2}\sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{2r}}{r!\Gamma\left(1+r-\frac{1}{2}\right)}.$$

We use Legendre's duplication formula with $z = r + 1/2$ to re-write this as:

$$J_{-1/2}(x) = (x/2)^{-1/2}\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r}\frac{2^{2r}}{(2r)!}$$

Hence:

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos(x).$$

For (b)(i), we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\nu J_\nu(x)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{r=0}^{\infty}\frac{(-1)^r}{2^{\nu+2r}}\frac{x^{2\nu+2r}}{r!\Gamma(\nu+r+1)}, \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{2^{\nu+2r}}\frac{(2\nu+2r)x^{2\nu+2r-1}}{r!\Gamma(\nu+r+1)}, \\
&= x^\nu\sum_{r=0}^{\infty}\frac{(-1)^r}{2^{\nu+2r}2^{-1}}\frac{2(\nu+r)x^{\nu+2r-1}2^{-1}}{r!\Gamma(\nu+r+1)}, \\
&= x^\nu\sum_{r=0}^{\infty}\frac{(-1)^r(\nu+r)(x/2)^{\nu+2r-1}}{r!\,\Gamma(\nu+r+1)}, \\
&= x^\nu\sum_{r=0}^{\infty}\frac{(-1)^r\cancel{(\nu+r)}(x/2)^{\nu+2r-1}}{r!\cancel{(\nu+r)}\Gamma(\nu+r)}, \\
&= x^\nu\sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{\nu-1+2r}}{r!\Gamma\left((\nu-1)+r+1\right)}, \\
&= x^\nu J_{\nu-1}(x).
\end{aligned}$$

For (b)(ii), we have:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu}J_\nu(x)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{r=0}^{\infty}\frac{(-1)^r}{2^{\nu+2r}}\frac{x^{2r}}{r!\Gamma(\nu+r+1)}, \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{r=1}^{\infty}\frac{(-1)^r}{2^{\nu+2r}}\frac{x^{2r}}{r!\Gamma(\nu+r+1)}, \\
&= \sum_{r=1}^{\infty}\frac{(-1)^r}{2^{\nu+2r}}\frac{2rx^{2r-1}}{r!\Gamma(\nu+r+1)}, \\
&= \sum_{r=1}^{\infty}\frac{(-1)^r}{2^{\nu+2r}\cdot 2^{-1}\cdot \cancel{2}}\frac{\cancel{2}rx^{2r-1}}{r!\Gamma(\nu+r+1)}, \\
&= \sum_{r=1}^{\infty}\frac{(-1)^r}{2^{\nu+2r-1}}\frac{x^{2r-1}}{(r-1)!\Gamma(\nu+r+1)}, \\
&= x^{-\nu}\sum_{r=1}^{\infty}\frac{(-1)^r}{2^{\nu+2r-1}}\frac{x^{2r-1+\nu}}{(r-1)!\Gamma(\nu+r+1)}, \\
&= x^{-\nu}\sum_{r=1}^{\infty}(-1)^r\frac{(x/2)^{2r-1+\nu}}{(r-1)!\Gamma(\nu+r+1)}, \\
&\overset{p=r-1}{=} -x^{-\nu}\sum_{p=0}^{\infty}\frac{(-1)^p(x/2)^{2p+\nu+1}}{p!\Gamma\left((\nu+1)+p+1\right)}, \\
&= -x^{-\nu}J_{\nu+1}(x).
\end{aligned}
$$

For (c)(i), we start off by using the results of part (b):

$$
\begin{aligned}
x^\nu J_\nu' + \nu x^{\nu-1}J_\nu &= x^\nu J_{\nu-1}, &\quad \times\ x^{-\nu} \\
x^{-\nu}J_\nu' - \nu x^{-\nu-1}J_\nu &= -x^{-\nu}J_{\nu+1}, &\quad \times\ x^{\nu}.
\end{aligned}
$$

After multiplying as indicated in red, we have:

$$
\begin{aligned}
J_\nu' + \nu x^{-1}J_\nu &= J_{\nu-1}, \\
J_\nu' - \nu x^{-1}J_\nu &= J_{\nu+1}.
\end{aligned}
$$

Adding up gives the required result:

$$J_\nu' = \tfrac{1}{2}\left(J_{\nu-1} - J_{\nu+1}\right). \qquad (2)$$

Subtracting gives the required result for (c)(ii):

$$\frac{2\nu}{x} = J_{\nu-1} + J_{\nu+1}. \qquad (3)$$

Question 4: Bessel Functions, again

The Bessel Functions in Equation (1) satisfy Bessel's differential equation, with $n = \nu$ (not necessarily an integer):

$$x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0. \tag{4}$$

Re-scale by letting $y_1(x) = J_n(\lambda x)$ and $y_2(x) = J_n(\mu x)$, where $\lambda \neq \mu$ are scalars. Then, $y_1(x)$ and $y_2(x)$ satisfy scaled versions of Bessel's equation:

$$x^2 y_1''(x) + x y_1'(x) + (\lambda^2 x^2 - n^2) y_1(x) = 0, \tag{5a}$$

$$x^2 y_2''(x) + x y_2'(x) + (\mu^2 x^2 - n^2) y_2(x) = 0. \tag{5b}$$

Hence, prove the orthogonality relation:

$$\int_0^1 x J_n(\lambda x) J_n(\mu x) \mathrm{d}x = \frac{\mu J_n(\lambda) J_n'(\mu) - \lambda J_n(\mu) J_n'(\lambda)}{\lambda^2 - \mu^2}, \qquad \lambda \neq \mu. \tag{6}$$

Hint: Consider [Eq. (5a)] y_2 − [Eq. (5b)] y_1.

Taking the hint, we have

$$x^2 \left(y_1'' y_2 - y_2'' y_1\right) + x \left(y_1' y_2 - y_1 y_2'\right) + (\lambda^2 - \mu^2) x^2 y_1 y_2 = 0.$$

This can be re-written:

$$x^2 \frac{\mathrm{d}}{\mathrm{d}x} (\underbrace{y_1' y_2 - y_2' y_1}_{=W}) + x W = (\mu^2 - \lambda^2) y_1 y_2.$$

Hence:

$$\frac{\mathrm{d}W}{\mathrm{d}x} + \frac{1}{x} W = (\mu^2 - \lambda^2) y_1 y_2.$$

Identify the integrating factor in the LHS: $\mathcal{I} = \mathrm{e}^{\int (1/x) \mathrm{d}x} = \mathrm{e}^{\log x} = x$. Hence:

$$\frac{\mathrm{d}}{\mathrm{d}x} (W x) = (\mu^2 - \lambda^2) x y_1 y_2.$$

Integrate from $x = 0$ to $x = 1$ to obtain:

$$W(1) = (\mu^2 - \lambda^2) \int_0^1 x \, y_1 \, y_2 \, \mathrm{d}x.$$

Filling in gives:

$$\int_0^1 x J_n(\lambda x) J_n(\mu x) \mathrm{d}x = \frac{\mu J_n(\lambda) J_n'(\mu) - \lambda J_n(\mu) J_n'(\lambda)}{\lambda^2 - \mu^2},$$

as required.

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Question 4: Physical Application of Bessel Functions

A very long hollow cylinder of inner radius a and outer radius b (whose cross section is indicated in Figure 1 is made of conducting material of diffusivity κ. If the inner and outer surfaces are kept at temperature zero while the initial temperature is a given function $f(\rho)$, where ρ is the radial distance from the axis, show that the temperature at any later time t is given by:

$$u(\rho,t)=\sum_{m=1}^{\infty}A_m\mathrm{e}^{-\kappa\lambda_m^2 t}u_0(\lambda_m\rho), \tag{1}$$

where

$$u_0(\lambda_m\rho)=Y_0(\lambda_m a)J_0(\lambda_m\rho)-J_0(\lambda_m a)Y_0(\lambda_m\rho). \tag{2}$$

Here also, $f(\rho)$ has been expanded in terms of u_0:

$$f(\rho)=\sum_{m=1}^{\infty}A_m u_0(\lambda_m\rho),$$

where

$$A_m=\frac{\int_a^b \rho f(\rho)u_0(\lambda_m\rho)\mathrm{d}\rho}{\int_a^b \rho[u_0(\lambda_m\rho)]^2\mathrm{d}\rho}. \tag{3}$$

Key information:

- The heat equation:

$$\frac{\partial u}{\partial t}=\kappa\left(\frac{\partial^2 u}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial\rho}\right),\qquad t>0.$$

- Fixed-temperature boundary conditions: $u(a,t)=0$, $u(b,t)=0$.
- Bounded solution $|u(\rho,t)|<M$, for all $t\geq 0$.
- Initial condition: $u(\rho,0)=f(\rho)$.

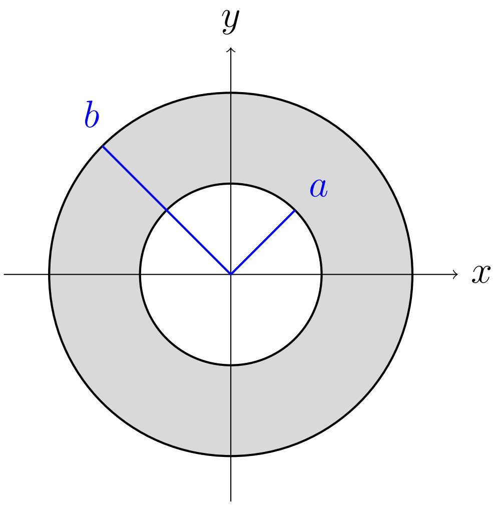

Figure 1: Setup for Question 4

- Rootfinding condition: The equation

$$Y_0(\lambda a)J_0(\lambda b) - J_0(\lambda a)Y_0(\lambda b) = 0$$

has positive roots $\lambda_1,\ \lambda_2, \cdots$.

- Second solution: If $J_0(x)$ is the first solution of Bessel's ODE or order $\nu = 0$, then $Y_0(x)$ is the corresponding second solution, which has a singularity as $x \to 0$.

Solution – we use separation of variables to write $u(\rho, t)$ as $R(\rho)T(t)$. We substitute into the heat equation and get:

$$\frac{T'(t)}{\kappa T} = \frac{R'' + \frac{1}{\rho}R'}{R} = \text{Const.} = -\lambda^2.$$

We take $\lambda \in \mathbb{R}$ for a bounded solution. Thus:

$$\rho R'' + R' + \lambda^2 \rho R = 0.$$

Multiply up by ρ to get:

$$\rho^2 \frac{\mathrm{d}^2 R}{\mathrm{d}\rho^2} + \rho \frac{\mathrm{d}R}{\mathrm{d}\rho} + \lambda^2 \rho^2 R = 0.$$

Introduce a scaled variable $x = \rho\lambda$, with corresponding dependent variable $\widetilde{R}(x) = R(\rho)$. Thus:

$$x^2 \frac{\mathrm{d}^2 \widetilde{R}}{\mathrm{d}x^2} + x \frac{\mathrm{d}\widetilde{R}}{\mathrm{d}x} + x^2 \widetilde{R} = 0.$$

This is Bessel's ODE of order $\nu = 0$. The solutions are:

$$\widetilde{R}(x) = \begin{cases} J_0(x), \\ Y_0(x). \end{cases}$$

The second solution $Y_0(x)$ blows up as $x \to 0$. However, a contribution from this second solution is allowed because $x > a$. Thus, the solution is:

$$R(\rho) = \alpha J_0(\rho\lambda) + \beta Y_0(\rho\lambda).$$

The full solution $u(\rho, t) = T(t)R(\rho)$ is therefore:

$$u(\rho, t) = \mathrm{e}^{-\lambda^2 \kappa t} \left[\alpha J_0(\rho\lambda) + \beta Y_0(\rho\lambda)\right].$$

The boundary conditions:

$$\begin{aligned} \alpha J_0(\lambda a) + \beta Y_0(\lambda a) &= 0, \\ \alpha J_0(\lambda b) + \beta Y_0(\lambda b) &= 0. \end{aligned}$$

In matrix form:

$$\begin{pmatrix} J_0(\lambda a) & Y_0(\lambda a) \\ J_0(\lambda b) & Y_0(\lambda b) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \tag{4}$$

Non-trivial solutions for α and β exist provided the determinant of the matrix in Equation (4) is zero. Therefore:

$$J_0(\lambda a)Y_0(\lambda b) - J_0(\lambda b)Y_0(\lambda a) = 0. \tag{5}$$

The positive roots of Equation (5) are labelled as λ_1, λ_2, etc. Back-substitution into the first of the boundary conditions gives:

$$\beta = -\alpha \frac{J_0(\lambda a)}{Y_0(\lambda a)}.$$

Hence:

$$\begin{aligned} u(\rho, t) &= \mathrm{e}^{-\kappa\lambda^2 t} \left[a J_0(\rho\lambda) - a \frac{J_0(\lambda a)}{Y_0(\lambda a)} Y_0(\rho\lambda) \right], \\ &= A\mathrm{e}^{-\kappa\lambda^2 t} \Bigg[\underbrace{Y_0(\lambda a) J_0(\lambda\rho) - J_0(\lambda a) Y_0(\lambda\rho)}_{=u_0(\lambda\rho)} \Bigg] \end{aligned}$$

For a general solution, we sum over the different eigenvalues λ_1, λ_2, etc.:

$$u(\rho, t) = \sum_{m=1}^{\infty} A_m \mathrm{e}^{-\kappa\lambda_m^2 t} u_0(\lambda_m \rho).$$

Initial conditions:

$$u(\rho, t) = \sum_{m=1}^{\infty} A_m u_0(\lambda_m \rho) = f(\rho).$$

Multiply both sides by $\rho u_0(\lambda_n \rho)$ and integrate:

$$\int_a^b \sum_{m=1}^{\infty} A_m u_0(\lambda_m \rho) u_0(\lambda_n \rho) \rho \,\mathrm{d}\rho = \int_a^b f(\rho) u_0(\lambda_n \rho) \rho \,\mathrm{d}\rho.$$

Use the orthogonality of Bessel functions to obtain:

$$A_m = \frac{\int_a^b \rho f(\rho) u_0(\lambda_m \rho) \mathrm{d}\rho}{\int_a^b \rho [u_0(\lambda_m \rho)]^2 \mathrm{d}\rho}.$$

Putting it all together, we have:

$$u(\rho, t) = \sum_{m=1}^{\infty} A_m \mathrm{e}^{-\kappa \lambda_m^2 t} u_0(\lambda_m \rho). \tag{6a}$$

where

$$A_m = \frac{\int_a^b \rho f(\rho) u_0(\lambda_m \rho) \mathrm{d}\rho}{\int_a^b \rho [u_0(\lambda_m \rho)]^2 \mathrm{d}\rho}. \tag{6b}$$

Equation (6) is the required solution.

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Question 5: Analysis of solutions of ODEs

Consider the following modified SEIR model, which accounts for vaccination in the susceptible class:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\beta IS}{N} - mS + b(1-\nu)N, \tag{1a}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\beta IS}{N} - fE - mE, \tag{1b}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = fE - rI - mI, \tag{1c}$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = rI + b\nu N - mR, \tag{1d}$$

$$N(t) = S(t) + E(t) + I(t) + R(t). \tag{1e}$$

The parameters β, m, b, f, and r are all positive parameters with dimensions of $[\text{Time}]^{-1}$. The parameter $\nu \in [0,1]$ is the proportion of susceptibles that are vaccinated.

(a) Show that

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (b-m)N. \tag{2}$$

(b) Show that disease-free and endemic equilibria exist if and only if $b = m$.

(c) Compute the endemic equilibrium explicitly. Show that the endemic equilibrium reverts to the solution discussed in Assignment 2 when $\nu = 0$, once appropriate changes have been made for the symbols representing the rates.

(d) Take as given the initial conditions $S(0) > 0$, $E(0) = 0$, $I(0) > 0$, and $R(0) = 0$. Using an approach similar to that in Assignment 2, show that the solution to Equation (1) is positivity-preserving, in the sense that, $S(t)$, $E(t)$, $I(t)$ and $R(t)$ remain positive for all $t > 0$.

For part (a), we add up $\mathrm{d}S/\mathrm{d}t + \mathrm{d}E/\mathrm{d}t + \mathrm{d}I/\mathrm{d}t + \mathrm{d}R/\mathrm{d}t$ to get the required equation.

Notice that the solution for $N = S + E + I + R$ is available in closed form:

$$N(t) = N_0 \mathrm{e}^{(b-m)t} \tag{3}$$

where N_0 is the initial population, $N_0 = N(0)$. We make two further observations which will be helpful in answering the remaining question parts:

(i) $N(t) =$ Const. if and only if $b = m$.

(ii) $N(t) > 0$ for all $t \geq 0$.

For part (b), a steady state can exist if and only if $N(t) =$ Const. Hence, in what follows, we can assume that $b = m$. However, we keep the calculations general, and fix $b = m$ only at the end. To compute steady states, we set:

$$\begin{aligned}
0 &= -\frac{\beta IS}{N} - mS + b(1-\nu)N, \\
0 &= \frac{\beta IS}{N} - fE - mE, \\
0 &= fE - rI - mI, \\
0 &= rI + b\nu N - mR.
\end{aligned}$$

The disease-free steady state has $I = 0$. Hence, $E = 0$. The other values are:

$$S = \frac{b(1-\nu)}{m} N, \qquad R = \frac{b\nu}{m} N.$$

Restoring $b = m$ in the steady state, as well as the subscript $*$ for the steady state, we have, in the disease-free state:

$$S_* = (1-\nu)N, \qquad E_* = 0, \qquad I_* = 0, \qquad R_* = \nu N.$$

For the endemic equilibrium, we group together the steady-state equations as follows:

$$\begin{aligned}
\frac{\beta IS}{N} + mS &= b(1-\nu)N, \\
\frac{\beta IS}{N} &= fE + mE, \\
fE &= rI + mI, \\
rI + b\nu &= mR.
\end{aligned}$$

It will be helpful to subtract the first two of these to get:

$$mS = b(1-\nu)N - (f+m)E \tag{4}$$

We have:

$$I = \frac{mR - b\nu uN}{R}. \tag{5}$$

Also:

$$E = \frac{r+m}{f} I = \frac{r+m}{f} \frac{mR - b\nu uN}{R}. \tag{6}$$

Hence, from Equation (4), we have:

$$S = \frac{b(1-\nu)N}{m} - \frac{f+m}{m}\frac{r+m}{f}\frac{mR - b\nu N}{r}. \tag{7}$$

We plug Equation (7) back into $b(1-\nu)N = mS + \beta IS/N$:

$$\begin{aligned} b(1-\nu)N &= b(1-\nu N) - (f+m)\left(\frac{r+m}{f}\right)\left(\frac{mR-b\nu N}{r}\right) \\ &\quad + \frac{\beta}{N}\frac{mR-b\nu N}{r}\left[\frac{b(1-\nu)N}{m} - \left(\frac{f+m}{m}\right)\left(\frac{r+m}{f}\right)\left(\frac{mR-b\nu N}{r}\right)\right]. \end{aligned}$$

Simplifying this gives an expression for R_* – the value of R in the equilibrium state:

$$R_* = N\frac{r}{\beta}\left[\frac{b(1-\nu)}{m}\frac{\beta}{f+m}\frac{f}{r+m} - 1\right] + \frac{b\nu N}{m}. \tag{8}$$

We can double-check that this reverts to the formula in Assignment 2 in the event where $\nu = 0$ and $b = m$:

$$R_*(\nu = 0, b = \mu, m = \mu) = N\frac{r}{\beta}\left[\frac{\beta}{f+\mu}\frac{f}{r+\mu} - 1\right]. \tag{9}$$

By suitable relabelling of the rates f and r, the formula in Assignment 2 is recovered. Lastly, we take $b = m$ to obtain the final expression for R_*, which depends parametrically on N:

$$R_*(N) = N\frac{r}{\beta}\left[(1-\nu)\frac{\beta}{f+\mu}\frac{f}{r+\mu} - 1\right] + \nu N. \tag{10a}$$

Back-substitution then gives:

$$I_* = \frac{mN}{\beta}\left[(1-\nu)\frac{\beta}{f+m}\frac{f}{r+m} - 1\right], \tag{10b}$$

$$E_* = \frac{r+m}{f}\frac{mN}{\beta}\left[(1-\nu)\frac{\beta}{f+m}\frac{f}{r+m} - 1\right], \tag{10c}$$

$$S_* = \frac{f+m}{m}\frac{r+m}{f}\frac{mN}{\beta}. \tag{10d}$$

Equations (10) are the required endemic equilibrium.

For part (d), we reason out the answer in the same way as Assignment 2. The starting-point here is the observation that from Equation (3), we have $N(t) > 0$. Hence, we have:

$$\frac{\mathrm{d}S}{\mathrm{d}t} + \underbrace{\left(m + \frac{\beta I}{N}\right)}_{=P(t)} S = b(1-\nu)N. \tag{11}$$

For contradiction, let there be a time $t_* > 0$ such that $I(t_*) = 0$. Then, on $[0, t_*)$ we have $I(t) > 0$. Following Assignment 2, a solution for S can be extracted:

$$S(t) = \underbrace{\frac{S(0)}{\mathcal{I}(t)} + \frac{b(1-\nu)}{\mathcal{I}(t)}\int_0^t N(t)\mathcal{I}(t)\mathrm{d}t,}_{>0}$$

where

$$\mathcal{I} = \mathrm{e}^{\int_0^t P(t)\mathrm{d}t}.$$

Hence, $S(t) > 0$ for all $t \in [0, t_*)$. A similar calculation yields $E(t) > 0$ for all $t \in [0, t_*)$. Moving on the I-equation, we have:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \underbrace{fE}_{>0} - (r+m)I,$$

where the term in the underbrace is positive for $t \in [0, t_*)$. Hence:

$$\frac{\mathrm{d}I}{\mathrm{d}t} > -(r+m)I, \qquad t \in (0, t_*).$$

By Gronwall's Inequality,

$$I(t) > I(0)\mathrm{e}^{-(r+m)t}, \qquad t \in (0, t_*),$$

and by continuity of the solution of the ODE,

$$I(t_*) > I(0)\mathrm{e}^{-(r+m)t_*},$$

which is a contradiction. Thus, $I(t) > 0$ for all $t \geq 0$. A similar calculation establishes positivity of $R(t)$. Hence, the solution of the ODE is positivity-preserving, for $t \geq 0$.