Applied Analysis (ACM30020)

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Graded Assignment

Instructions:

- This is a graded assignment.
- Worth 20%. A small number of the available marks will be awarded for precision and clarity.
- Open-book format – proper citation of any literature will count towards the marks for precision and clarity.
- Please do not collaborate with friends (or enemies) – this assignment is to be performed under the code of conduct outlined in the module introduction on Brightspace.
- Please submit a hard copy in Latex. Sign and attach the code-of-conduct coversheet to your work.
- Due date: Monday 24th March 09:00

1. **Local Existence Theory for ODEs**

Consider the ODE system:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x,y), \qquad x \geq a, \tag{1}$$

with initial condition $y(a) = y_0$. As before, construct the Picard iterative solution scheme:

$$y_{n+1}(x) = y_0 + \int_a^x F(x, y_n(x))\mathrm{d}x, n \geq 0,$$

with initial guess $y_0(x) = y_0$. Suppose also that F is Lipschitz, with Lipschitz constant K, such that:

$$|F(x,y_2) - F(x,y_1)| \leq K|y_2 - y_1|,$$

for all x in an interval $(a, a+L)$, and all y_1 and $y_2 \in \mathbb{R}$.

(a) Show that
$$\|y_{n+1} - y_n\|_\infty \leq KL\|y_n - y_{n-1}\|_\infty.$$

(b) Fix L such that $KL < 1$. Hence, deduce that $y_n \mapsto y_{n+1}$, which maps continuous functions to continuous functions, is a contraction mapping.

(c) Use the Contraction Mapping Principle to deduce that Equation (1) has a solution, valid for $x \in [a, a+L]$.

The solution constructed in this way is called a local solution as it is valid on the interval $[a, a+L]$. What happens for $x > a+L$ is anyone's guess. If the solution remains valid for all $x > a+L$, the solution is called a global solution.

(d) Consider the ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^3, \qquad \frac{\mathrm{d}}{\mathrm{d}x}y^2 = -2y^4, \qquad x > 0,$$

with $y(0) = y_0$. Say in each case whether or not a global solution exists.

2. **Bessel Functions**

Consider the expression

$$g(x,t) = \mathrm{e}^{(x/2)(t-t^{-1})} = \sum_{n=-\infty}^{\infty} P_n(x) t^n. \tag{2}$$

In this problem we show that the P_n-coefficients are the Bessel functions of integer order.

(a) Compute $\partial g/\partial t$ in two different ways to show that

$$\frac{2n}{x} P_n = P_{n-1} + P_{n+1}, \qquad n \in \mathbb{Z}. \tag{3}$$

(b) Compute $\partial g/\partial x$ in two different ways to show that

$$2P_n' = P_{n-1} - P_{n+1}, \qquad n \in \mathbb{Z}. \tag{4}$$

(c) View Equations (3) and 4 as simultaneous equations to get:

$$\frac{n}{x} P_n + P_n' = P_{n-1}, \tag{5a}$$
$$\frac{n}{x} P_n - P_n' = P_{n+1}. \tag{5b}$$

Hence, show that

$$x^2 P_n'' + x P_n' + (x^2 - n^2) P_n = 0. \tag{6}$$

(d) By uniqueness of solutions, deduce in a couple of lines that $P_n(x)$ is in fact $J_n(x)$, the Bessel function of integer order.

3. **Bessel Functions, again**

(a) From the product of generating functions $g(x,t)g(x,-t)$, show that

$$1 = [J_0(x)]^2 + 2\,[J_1(x)]^2 + 2\,[J_2(x)]^2 + \cdots, \tag{7}$$

and therefore that $|J_0(x)| \le 1$ and $|J_n(x)| \le 1/\sqrt{2}$, $n = 1, 2, 3, \cdots$.

(b) Using a generating function $g(x,t) = g(u+v,t) = g(u,t) \cdot g(v,t)$, show that

$$J_n(u+v) = \sum_{m=-\infty}^{\infty} J_m(u) \cdot J_{n-m}(v). \tag{8}$$

(c) Using only the generating function, show that $J_n(x)$ has odd or even parity according to whether n is even or odd, that is,

$$J_n(x) = (-1)^n J_n(-x). \tag{9}$$

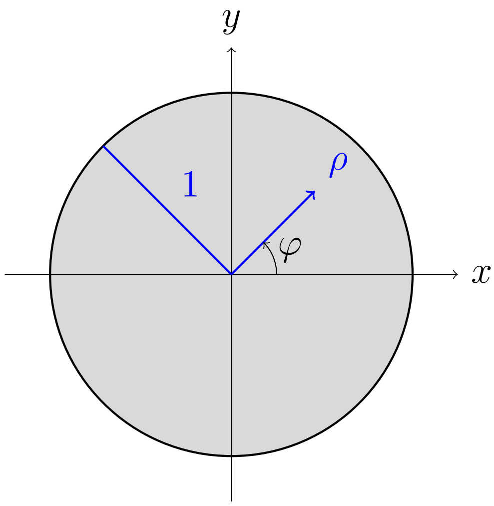

Figure 1: A circular plate of unit radius with its faces insulated

4. **Physical Application of Bessel Functions**

   A circular plate of unit radius has its plane faces insulated (see Figure 1). If the initial temperature is $F(\rho)$ and the rim is kept at temperature zero, find the temperature of the plate at any time.

   Hints: The boundary condition is:

   $$u(1, t) = 0,$$

   and the initial condition is:

   $$u(\rho, 0) = F(\rho).$$

   You may use the following orthogonality property of the Bessel Function J_0:

   $$\int_0^1 \rho J_0(\lambda_m \rho) J_0(\lambda_p \rho) \mathrm{d}\rho = \tfrac{1}{2}\delta_{mp} J_1^2(\lambda_m).$$

   where the λ_m's are the positive roots of $J_0(\lambda) = 0$.

5. **Integral Equations**

   Let

   $$I(x) = \int_{-\infty}^{\infty} \mathrm{e}^{-|x-\xi|} \Phi(\xi) \mathrm{d}\xi. \tag{10}$$

   (a) Verify that $I''(x) = I(x) - 2\Phi(x)$ for any continuous function $\Phi(x)$ which is dominated by $\mathrm{e}^{|x|}$ as $|x| \to \pm\infty$.

   (b) Use this result to show that any continuous solution of the integral equation

   $$y(x) = \lambda \int_{-\infty}^{\infty} \mathrm{e}^{-|x-\xi|} y(\xi) \mathrm{d}\xi + F(x) \tag{11a}$$

   must also satisfy the differential equation

   $$y''(x) - (1 - 2\lambda) y(x) = F''(x) - F(x). \tag{11b}$$

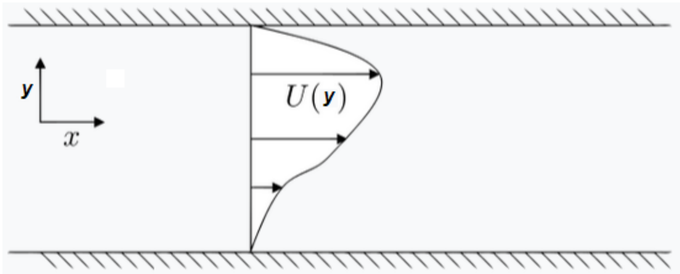

Figure 2: A flow between two plates the x-direction, with spatial variation in the y-direction

6. **A priori analysis of solutions of ODEs**

In Fluid Mechanics, the Taylor–Goldstein equation describes the small-amplitude perturbation of a flow away from its mean value $U(y)$ due to the effect of buoyancy. The idea here is that the flow is in the x-direction but that the flow and the buoyancy vary in the y-direction (hence, $U(y)$, see Figure 2). For the same reason, the variable

$$N^2 = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}, \qquad N^2 > 0$$

encodes the effect of the buoyancy – here g is acceleration due to gravity and ρ_0 is the density.

With this set-up in mind, the Taylor–Goldstein equation reads:

$$v'' + \left[\frac{N^2}{(c-U)^2} + \frac{U''}{c-U} - k^2\right] v = 0. \tag{12}$$

Here, v is the perturbation velocity in the x direction (again though, $v(y)$), k is the wavelength of the perturbation, and c is the wave speed. The flow is bounded between two plates, $-L \le y \le L$, and satisfies the boundary conditions

$$v = 0, \qquad y = \pm L. \tag{13}$$

In this context, both v and c can be complex, and c is an eigenvalue to be determined. The aim of this question is to say something definitive about the eigenvalue, without having to solve Equation (12).

(a) Make the change of variable

$$v = (U-c)^n q, \tag{14}$$

where n is a parameter at our disposal, and re-write the Taylor–Goldstein equation in terms of $q \propto$.

(b) By multiplying both sides of the resulting equation by $q^*(\cdots)$, where the $\cdots$

factor is to be determined, and integrating, show that:

$$\int_{-L}^{L}(U-c)^{2n}\left[|q'|^2+k^2|q|^2\right]\mathrm{d}y$$
$$=\int_{-L}^{L}\left[\{N^2+n(n-1))U'^2\}(U-c)^{2n-2}+(n-1)U''(U-c)^{2n-1}\right]|q|^2\mathrm{d}y. \tag{15}$$

(c) Write $c = c_r + \mathrm{i}c_i$ and, by choosing n suitably, show that c_i must be zero so that the flow is stable, if

$$\frac{N^2}{(\mathrm{d}U/\mathrm{d}y)^2} > \tfrac{1}{4} \qquad \text{in } -L \leq y \leq L. \tag{16}$$