Applied Analysis (ACM30020)

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Graded Assignment

Instructions:

- This is a graded assignment.
- Worth 20%. A number of the available marks will be awarded for precision and clarity.
- Please do not collaborate with friends (or enemies) – this assignment is to be performed under the code of conduct outlined in the module introduction on Brightspace.
- Please submit a hard copy in Latex. Sign and attach the code-of-conduct coversheet to your work.
- Due date: Monday 23rd March 09:00

Materials that may be used to complete the assignment:

- This assignment is an open-book assignment, so the lecture notes, text books, and web resources can be used. Marks will be awarded for proper citation of these, under the 'precision and clarity' rubric.
- This module uses the AI Assessment Scale of the College of Science. AI assistance up to and including level 3 on that scale are acceptable.
- If AI is used in the completion of the assignment, marks will be awarded if this is documented, and the AI-generated answers are interrogated critically (pen-and-paper validations, etc.) – again under the 'precision and clarity rubric'.

1. **Local Existence Theory for ODEs**

Consider the ODE system:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y), \qquad x \geq a, \tag{1}$$

with initial condition $y(a) = y_0$. As before, construct the Picard iterative solution scheme:

$$y_{n+1}(x) = y_0 + \int_a^x F(x, y_n(x))\mathrm{d}x, n \geq 0,$$

with initial guess $y_0(x) = y_0$. Suppose also that F is Lipschitz, with Lipschitz constant K, such that:

$$|F(x, y_2) - F(x, y_1)| \leq K|y_2 - y_1|,$$

for all x in an interval $(a, a + L)$, and all y_1 and $y_2 \in \mathbb{R}$.

(a) Show that

$$\|y_{n+1} - y_n\|_\infty \leq KL\|y_n - y_{n-1}\|_\infty.$$

(b) Fix L such that $KL < 1$. Hence, deduce that $y_n \mapsto y_{n+1}$, which maps continuous functions to continuous functions, is a contraction mapping.

(c) Use the Contraction Mapping Principle to deduce that Equation (1) has a solution, valid for $x \in [a, a + L]$.

The solution constructed in this way is called a local solution as it is valid on the interval $[a, a + L]$. What happens for $x > a + L$ is anyone's guess. If the solution remains valid for all $x > a + L$, the solution is called a global solution.

(d) Consider the ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^3, \qquad \frac{\mathrm{d}}{\mathrm{d}x}y^2 = -2y^4, \qquad x > 0,$$

with $y(0) = y_0$. Say in each case whether or not a global solution exists.

2. **Bessel Functions**

   Take as given the following power-series representation of Bessel functions:

$$J_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{\nu+2r}}{r!\Gamma(n+r+1)}. \tag{2}$$

   (a) Prove:

      i. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$;

      ii. $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$.

   (b) Prove that for all ν:

      i. $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x)$.

      ii. $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)$.

   (c) Hence, prove that:

      i. $J'_\nu(x) = \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right]$;

      ii. $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$.

3. **Bessel Functions, again**

   The Bessel Functions in Equation (2) satisfy Bessel's differential equation, with $n = \nu$ (not necessarily an integer):

$$x^2 J''_n(x) + x J'_n(x) + (x^2 - n^2) J_n(x) = 0. \tag{3}$$

   Re-scale by letting $y_1(x) = J_n(\lambda x)$ and $y_2(x) = J_n(\mu x)$, where $\lambda \neq \mu$ are scalars. Then, $y_1(x)$ and $y_2(x)$ satisfy scaled versions of Bessel's equation:

$$x^2 y''_1(x) + x y'_1(x) + (\lambda^2 x^2 - n^2) y_1(x) = 0, \tag{4a}$$

$$x^2 y''_2(x) + x y'_2(x) + (\mu^2 x^2 - n^2) y_2(x) = 0. \tag{4b}$$

   Hence, prove the orthogonality relation:

$$\int_0^1 x J_n(\lambda x) J_n(\mu x) \mathrm{d}x = \frac{\mu J_n(\lambda) J'_n(\mu) - \lambda J_n(\mu) J'_n(\lambda)}{\lambda^2 - \mu^2}, \qquad \lambda \neq \mu. \tag{5}$$

   Hint: Consider [Eq. (4a)] y_2 − [Eq. (4b)] y_1.

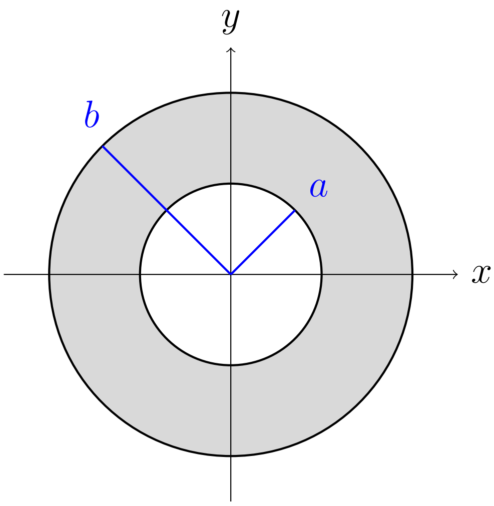

Figure 1: Setup for Question 4

4. **Physical Application of Bessel Functions**

A very long hollow cylinder of inner radius a and outer radius b (whose cross section ios indicated in Figure 1 is made of conducting material of diffusivity κ. If the inner and outer surfaces are kept at temperature zero while the initial temperature is a given function $f(\rho)$, where ρ is the radial distance from the axis, show that the temperature at any later time t is given by:

$$u(\rho, t) = \sum_{m=1}^{\infty} A_m \mathrm{e}^{-\kappa\lambda_m^2 t} u_0(\lambda_m \rho), \tag{6}$$

where

$$u_0(\lambda_m \rho) = Y_0(\lambda_m a) J_0(\lambda_m \rho) - J_0(\lambda_m a) Y_0(\lambda_m \rho). \tag{7}$$

Here also, $f(\rho)$ has been expanded in terms of u_0:

$$f(\rho) = \sum_{m=1}^{\infty} A_m u_0(\lambda_m \rho),$$

where

$$A_m = \frac{\int_a^b \rho f(\rho) u_0(\lambda_m \rho) \mathrm{d}\rho}{\int_a^b \rho [u_0(\lambda_m \rho)]^2 \mathrm{d}\rho}. \tag{8}$$

Key information:

- The heat equation:

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} \right), \qquad t > 0.$$

- Fixed-temperature boundary conditions: $u(a, t) = 0$, $u(b, t) = 0$.
- Bounded solution $|u(\rho, t)| < M$, for all $t \geq 0$.

- Initial condition: $u(\rho, 0) = f(\rho)$.
- Rootfinding condition: The equation

$$Y_0(\lambda a)J_0(\lambda b) - J_0(\lambda)Y_0(\lambda b) = 0$$

  has positive roots λ_1, $\lambda_2, \cdots$.
- Second solution: If $J_0(x)$ is the first solution of Bessel's ODE or order $\nu = 0$, then $Y_0(x)$ is the corresponding second solution, which has a singularity as $x \to 0$.

5. **Analysis of solutions of ODEs**

Consider the following modified SEIR model, which accounts for vaccination in the susceptible class:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\beta IS}{N} - mS + b(1-\nu)N, \quad (9a)$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\beta IS}{N} - fE - mE, \quad (9b)$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = fE - rI - mI, \quad (9c)$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = rI + b\nu N - mR, \quad (9d)$$

$$N(t) = S(t) + E(t) + I(t) + R(t). \quad (9e)$$

The parameters β, m, b, f, and r are all positive parameters with dimensions of $[\text{Time}]^{-1}$. The parameter $\nu \in [0,1]$ is the proportion of susceptibles that are vaccinated.

(a) Show that

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (b-m)N. \quad (10)$$

(b) Show that disease-free and endemic equilibria exist if and only if $b = m$.

(c) Compute the endemic equilibrium explicitly. Show that the endemic equilibrium reverts to the solution discussed in Assignment 2 when $\nu = 0$, once appropriate changes have been made for the symbols representing the rates.

(d) Take as given the initial conditions $S(0) > 0$, $E(0) = 0$, $I(0) > 0$, and $R(0) = 0$. Using an approach similar to that in Assignment 2, show that the solution to Equation (9) is positivity-preserving, in the sense that, $S(t)$, $E(t)$, $I(t)$ and $R(t)$ remain positive for all $t > 0$.