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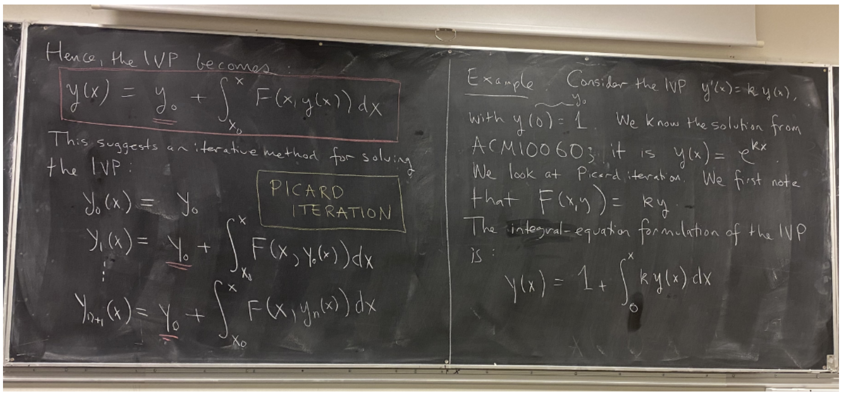
Scoil na nEolaíochtaí Matamaitice

Applied Analysis (ACM 30020)



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Lecture notes in Applied Analysis, January 2025

Applied Analysis (ACM30020)

The purpose of this course is to learn a variety of mathematical methods for deriving useful approximate solutions of the differential equations and integrals found in the Mathematical Sciences. The course will be structured as:

1. Existence and uniqueness results for ordinary differential equations: The Lipschitz condition and Picard's theorem. Comparison theorems.
2. Integral Equations: The Volterra integral equation and initial value problems, the Fredholm integral equation and boundary value problems.
3. Sturm-Liouville Theory: The adjoint differential operator, the Sturm-Liouville problem, basic properties of a Sturm-Liouville eigenvalue problem, unboundedness of the eigenvalues, completeness in the appropriate sense of the set of eigenfunctions.
4. Theory of Infinite-dimensional vector spaces: Inner product spaces, complete metric spaces, Hilbert spaces, square summable series and square integrable functions, Least squares approximation, projection theorem, generalized Fourier coefficients, Bessel's inequality, Parseval's equality and completeness.
5. Introduction to generalised functions.

What will I learn?

On completion of this module students should be able to

1. Understand conditions guaranteeing existence and uniqueness results for ordinary differential equations and recognize examples where those conditions do not hold;
2. State and prove Picard's theorem;
3. Transform between an initial value problem and the corresponding Volterra integral equation;
4. Transform between a boundary value problem and the corresponding Fredholm integral equation;
5. State the axiomatic properties of the Green function for a second order initial value problem and boundary value problem;
6. Understand the concept of the adjoint differential operator;
7. Recognise a Sturm-Liouville eigenvalue problem and prove the basic properties of eigenvalues and eigenfunctions;
8. Understand the relationship between the Dirac delta function and the Fourier integral;
9. Understand the fundamental properties of infinite dimensional vectors spaces.
10. Prove key results such as Bessel's inequality, Parseval's equality and its relationship to completeness.

Statement of authorship

Professor Adrian Ottewill created the original version of these notes. Dr Malbor Asllani contributed to updating the notes in 2021. Professor Neil O'Connell made key contributions on the functional analysis of integral equations in 2022. Dr Lennon Ó Náraigh made major updates to the notes in 2024. The chapter on the application of Fredholm Theory to the Schrödinger equation is due entirely to him. The credit for the photograph on the front cover goes to Emily Gallagher (BSc Theoretical Physics 2024).

Editions

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Contents

Chapter 1

Integral and Differential Equations

Overview

In this chapter we formulate a general theory of ordinary differential equations. We introduce the Initial Value Problem (IVP). We describe in detail the sufficient conditions for the IVP to be well-posed, that is, for a solution to exist, to be unique, and to dependent continuously on the initial conditions.

1.1 Introduction

To date we have derived differential equations (usually linear) such as the planetary orbit equation and then solved them in closed form to derive exact physical solutions.

Unfortunately the real world is usually not so kind and although we may be able to derive differential equations to model a system we may not be able to solve it analytically – indeed this is the rule not the exception. Even very simple looking equations may be impossible to solve in terms of elementary functions. Important example include;

$$y'(x) = 1 + xy^2(x) \tag{1.1}$$

$$y''(x) = xy(x). \tag{1.2}$$

Equation (1.1) is called the Riccati equation, it is non-linear in y and is important in control theory. Equation (1.2) is called the Airy equation, it is linear in y and is important in the theory of rainbows and in WKB theory.

If we can not find an analytical solution (and even if we can) then various questions arise:

- how do we know a solution exists?
- if it does, is it unique?

- can we still determine properties of the solution?
- can we get useful approximate analytical solutions to it?

1.1.1 A heuristic approach to constructing a solution

Start with the simplest example of a first order differential equation so

$$y'(x) = F\big(x, y(x)\big)$$

then to find a solution near $x = x_0$ we may imagine trying to construct the Taylor series:

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \tfrac{1}{2}y''(x_0)(x - x_0)^2 + \dots .$$

Suppose for now we know $y(x_0) = y_0$ then we can also determine

$$y'(x_0) = F\big(x_0, y(x_0)\big).$$

Also by the chain rule

$$\begin{aligned} y''(x) = \frac{\mathrm{d}}{\mathrm{d}x}y'(x) = \frac{\mathrm{d}}{\mathrm{d}x}F\big(x, y(x)\big) &= \frac{\partial F}{\partial x}\big(x, y(x)\big) + \frac{\partial F}{\partial y}\big(x, y(x)\big)\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \frac{\partial F}{\partial x}\big(x, y(x)\big) + \frac{\partial F}{\partial y}\big(x, y(x)\big)F\big(x, y(x)\big) \end{aligned}$$

and hence we can also determine

$$y''(x_0) = \frac{\partial F}{\partial x}\big(x_0, y_0\big) + \frac{\partial F}{\partial y}\big(x_0, y(x_0)\big)F\big(x_0, y(x_0)\big).$$

We may clearly now use the chain rule again (assuming suitable differentiability) to obtain $y'''(x_0)$ etc. but, even assuming infinite differentiability, we have no indication of the radius of convergence of the resulting series.

Example: Let $y'(x) = \mathrm{e}^{-1/x^2}$, with $x_0 = 0$, $y(0) = 0$ for $x \in \mathbb{R}$, then this process yields $y^{(n)}(0) = 0$ for every n.

Notwithstanding this, it can be checked by hand that the following is a solution to the IVP:

$$y(x) = x\mathrm{e}^{-1/x^2} + \sqrt{\pi}\,(\mathrm{erf}\,(1/x) - 1)$$

(see Wikipedia or DLMF – The Digital Library of Mathematical Functions) but the failure of the process suggests we should be cautious.

1.1.2 The Initial Value Problem

To see what we might be able to prove start with a very simple example

Example: Let $y'(x) = 0$ for $x \in [-1, 1]$, i.e. take $F(x, y) = 0$.

The solution is clearly $y(x) = c$, (constant) for $x \in [-1, 1]$. At this stage we have infinitely many solutions. To reduce these we may give the value of the solution at a single point $x \in [-1, 1]$, e.g. we may ask for the solution such that $y(0) = 1$, then we a left with the *unique* solution $y(x) = 1$ for $x \in [-1, 1]$.

The sensible way to phrase the problem is then is through a definition.

Definition 1.1 (The Initial Value Problem (IVP)) *Given the ODE $y'(x) = F\big(x, y(x)\big)$ over an interval I, find a solution $y(x)$ (a differentiable function, by definition) over an interval $J \subset I$ such that $y(x_0) = y_0$ for given $x_0 \in J$ and $y_0 \in \mathbb{R}$.*

Furthermore, we will say that the IVP is 'well-posed' if

- a solution exists;
- the solution is unique;
- the solution depends continuously on y_0 (in a way to be made precise).

To understand how problems may arise, let's have a look at a couple of examples where issues arise:

Example: Suppose we want to solve the IVP

$$y'(x) = \begin{cases} 0 & -1 \le x \le 0 \\ 1 & 0 < x \le 1 \end{cases} \qquad y(0) = 0.$$

Integrating in each subdomain separately we easily find

$$y(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 < x \le 1 \end{cases}.$$

The problem is made clear if we plot $y(x)$ (Figure 1.1). The function is not differentiable at $x = 0$ and correspondingly in no interval J containing $x_0 = 0$. We are forced to conclude that, in this case, *the IVP has no solution.*

Example: Suppose we want to solve the IVP

$$y'(x) = 3y^{2/3}(x), \qquad \text{for } -1 \le x \le 1, \qquad y(0) = 0.$$

Dividing across by $3y^{2/3}$ we immediately find

$$\tfrac{1}{3} y^{-2/3}(x) y'(x) = \frac{\mathrm{d}}{\mathrm{d}x} y^{1/3}(x) = 1$$

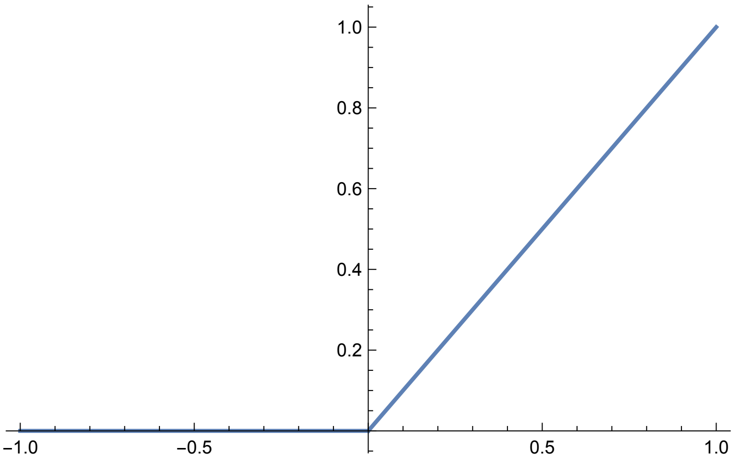


Figure 1.1:

which we can integrate immediately, imposing the initial value, as

$$y^{1/3}(x) = x \qquad \Longrightarrow \quad y(x) = x^3,$$

and it is indeed easy to see that this is a solution to the IVP.

The trouble only arises when we realise that we divided across by $y(x)$ and so should check the case $y(x) = 0$ separately. When we do it is immediately clear that $y(x) = 0$ for $-1 \leq x \leq 1$ also solves the IVP. In fact, we soon realise it's even worse: suppose we pick any $0 \leq x_1 \leq 1$, then the function

$$y(x) = \begin{cases} 0 & -1 \leq x \leq x_1 \\ (x - x_1)^3 & x_1 < x \leq 1 \end{cases}$$

is both differentiable and solves the IVP: *we have an (uncountably) infinite number of solutions of the IVP*.

1.2 The Lipschitz condition

It turns out that continuity of $F(x, y)$ is sufficient to guarantee the existence of at least one solution through each point but is not sufficient to guarantee uniqueness.

$F(x, y)$ continuously differentiable (in both arguments) is sufficient to guarantee the existence of a unique solution but, in fact, a considerably weaker condition suffices.

Definition 1.2 (The Lipschitz Condition) *$F(x, y)$ satisfies a Lipschitz Condition with respect to y in a region $D \subset \mathbb{R}^2$ if there exists a constant $K \geq 0$ such that*

$$\big|F(x, y_1) - F(x, y_2)\big| \leq K|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$. K is called the corresponding Lipschitz constant.

Remark: If $F(x, y)$ is continuously differentiable and D is closed, bounded and convex then, a simple argument evoking the mean value theorem shows that it satisfies a Lipschitz Condition taking

$$K = \sup_{(x,y)\in D} \left|\frac{\partial F}{\partial y}\right|.$$

(Obviously any larger constant would also work and a smaller one might work.)

A previous example revisited: We have $F(x, y) = 3y^{2/3}$ so taking $y_2 = 0$

$$\big|F(x, y_1) - F(x, y_2)\big| = 3|y_1|^{2/3} = \frac{3}{y_1^{1/3}}|y_1 - y_2|.$$

Clearly as $y_1 \to 0$ the prefactor increases without limit so it is not possible to find a suitable Lipschitz constant in any domain D containing a point with $y = 0$ (i.e. any point on the x-axis).

1.3 Re-expressing the IVP as an integral equation

Let us assume from now on that $F(x, y)$ is continuous in both its arguments then if we integrate the ODE $y'(x) = F\big(x, y(x)\big)$ from x_0 (where $y(x_0) = y_0$) to an arbitrary point x we have

$$\int_{x_0}^{x} y'(s)\,\mathrm{d}s = y(x) - y(x_0) = \int_{x_0}^{x} F\big(s, y(s)\big)\,\mathrm{d}s,$$

or

$$y(x) = y_0 + \int_{x_0}^{x} F\big(s, y(s)\big)\,\mathrm{d}s.$$

Note that $y(x)$ appears on both sides, we have not solved for it we've just re-expressed the IVP as an integral equation.

The integral equation formula has numerous advantages:

- it encapsulates both the ODE and the IC;
- it allows us to deal with continuous functions - the Fundamental Theorem of Calculus then guarantees differentiability;

- it suggests a natural iterative scheme

$$
\begin{aligned}
y_0(x) &= y_0 \\
y_1(x) &= y_0 + \int_{x_0}^{x} F\big(s, y_0(s)\big)\,\mathrm{d}s \\
&\cdots \\
y_{n+1}(x) &= y_0 + \int_{x_0}^{x} F\big(s, y_n(s)\big)\,\mathrm{d}s
\end{aligned}
$$

Example: The IVP $y'(x) = ky(x)$ with $y(0) = 1$ clearly has solution $y(x) = \mathrm{e}^{kx}$. The integral equation reformulation is

$$
y(x) = 1 + \int_{x_0}^{x} k\,y(s)\,\mathrm{d}s,
$$

so our integration scheme gives

$$
\begin{aligned}
y_0(x) &= 1 \\
y_1(x) &= 1 + \int_{x_0}^{x} k\,1\,\mathrm{d}s = 1 + kx \\
y_2(x) &= 1 + \int_{x_0}^{x} k\,(1 + ks)\,\mathrm{d}s = 1 + kx + \tfrac{1}{2}(kx)^2
\end{aligned}
$$

and so on, clearly generating the Taylor series expansion of our solution $y(x) = \mathrm{e}^{kx}$ about our initial point $x = 0$ which, in this case, we know converges for all x.

1.4 Picard's Theorem

Let us start by stating the theorem that we shall prove in this section:

Theorem 1.1 (Picard's Theorem (First Form)) *Let $F(x,y)$ be continuous and for all $x \in [a,b]$ and for all $y \in \mathbb{R}$ satisfy a Lipschitz condition, with Lipschitz constant K say. Then the IVP $y'(x) = F\big(x, y(x)\big)$ with $y(x_0) = y_0$ for some $x_0 \in [a,b]$ has a unique solution defined for all $x \in [a,b]$.*

The idea of the proof is very simple: to show that the interative sequence converges to a solution to the IVP. This will be done in three steps:

1. show that the sequence converges *uniformly* on $[a,b]$ to a function $y(x)$;
2. show that $y(x)$ satisfies the IVP;
3. show that $y(x)$ is the unique solution.

1.4.1 Reminder of uniform convergence

Definition 1.3 (Pointwise convergence) *A series $\sum u_n(x)$ of functions $u_n : I \to \mathbb{R}$ converges pointwise to $u : I \to \mathbb{R}$ if for each $x \in I$, given $\epsilon > 0$ there exits N such that*

$$\sum_{r=0}^{n} |u_r(x) - u(x)| < \epsilon,$$

for all $n \geq N$.

Critically here N is only chosen after x is picked so can depend on N. This can lead to consequences that would be disastrous for our approach: a series of continuous functions can sum pointwise to a limit that is not continuous.

Example: Take $u_0(x) = 1$, $u_n(x) = x^n - x^{n-1}$ on $[0, 1]$ then

$$\sum_{r=0}^{n} u_r(x) = x^n,$$

tends pointwise to

$$u(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}.$$

Since continuity is crucial to us we need a stronger form of convergence:

Definition 1.4 (Uniform convergence) *A series $\sum u_n(x)$ of functions $u_n : I \to \mathbb{R}$ converges uniformly to $u : I \to \mathbb{R}$ if, given $\epsilon > 0$ there exits N such that*

$$\sum_{r=0}^{n} |u_r(x) - u(x)| < \epsilon,$$

for all $n \geq N$ and for all $x \in I$.

A fundamental theorem in analysis tells us that the limit function of a series of functions that converges uniformly is continuous.

We need one more result:

Theorem 1.2 (Weierstrass M-test) *Let M_n be numbers such that $\left|u_n(x)\right| \leq M_n$ for $n = 0, 1, 2, \ldots$ and for all $x \in I$ then if the sum of numbers$\sum M_n$ converges so also the sum of functions $\sum u_n(x)$ converges uniformly to a limit function on I.*

1.4.2 Step 1 of the proof: uniform convergence

As before, we have:

$$\begin{aligned} y_0(x) &= y_0 \\ y_1(x) &= y_0 + \int_{x_0}^{x} F\big(s, y_0(s)\big)\,\mathrm{d}s \\ &\dots \\ y_{n+1}(x) &= y_0 + \int_{x_0}^{x} F\big(s, y_n(s)\big)\,\mathrm{d}s. \end{aligned}$$

We define $u_0(x) = y_0$ and $u_n(x) = y_n(x) - y_{n-1}(x)$. Thus:

$$\begin{aligned} \sum_{r=0}^{n} u_r(x) &= (y_n - y_{n-1}) + (y_{n-1} - y_{n-2}) + \cdots + (y_1 - y_0) + u_0(x), \\ &= y_n - \cancel{y_0} + \cancel{y_0}, \\ &= y_n \end{aligned}$$

Here, we have suppressed the x-dependence momentarily, to make the equations neater.

Next, we let M denote a bound for the continuous function $F(x, y_0)$ for $x \in [a, b]$:

$$|F(x, y_0)| \leq M, \qquad x \in [a, b].$$

Thus, we have:

$$\begin{aligned} |u_1(x)| &= \left| \int_{x_0}^{x} F\big(s, y_0\big)\,\mathrm{d}s \right| \\ &\leq \left| \int_{x_0}^{x} \left|F\big(s, y_0\big)\right|\,\mathrm{d}s \right| \\ &\leq \left| \int_{x_0}^{x} M\,\mathrm{d}s \right| \\ &= M\,|x - x_0|\,. \end{aligned}$$

Next (assuming for clarity that $x > x_0$, otherwise we just flip the limits on the second line)

$$\begin{aligned}
|u_2(x)| &= \left| \int_{x_0}^{x} \big(F\big(s, y_1(s)\big) - F\big(s, y_0(s)\big)\big)\ \mathrm{d}s \right| \\
&\leq \int_{x_0}^{x} \left|F\big(s, y_1(s)\big) - F\big(s, y_0(s)\big)\right|\ \mathrm{d}s \\
&\leq \int_{x_0}^{x} K\left|y_1(s) - y_0(s)\right|\ \mathrm{d}s \\
&\leq \int_{x_0}^{x} MK\left|s - x_0\right|\ \mathrm{d}s \\
&= \tfrac{1}{2} MK \left|x - x_0\right|^2
\end{aligned}$$

The pattern is clear and suggests that

$$|u_n(x)| \leq \tfrac{1}{n!} MK^{n-1} \left|x - x_0\right|^n$$

which follows easily by induction.

Since $|x - x_0| \leq |b - a|$ we may take

$$M_n = \tfrac{1}{n!} MK^{n-1} \left|b - a\right|^n$$

and

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \tfrac{1}{n!} MK^{n-1} \left|b - a\right|^n = (M/K) \exp\big(K|b - a|\big).$$

Combining the results, we have:

- $|u_n(x)| \leq M_n$
- $\sum_{n=0}^{\infty} M_n < \infty$.

Hence, by the Weierstrass M-test that the sum $y_n(x) = \sum_{r=0}^{n} u_r(x)$ converges uniformly to some continuous function $y(x)$ on $[a, b]$.

1.4.3 Step 2 of the proof: solution

Now we must show that $y(x)$ satisfies the ODE. We have:

$$y_n(x) \to y(x) \implies \sum_{r=0}^{n} u_r(x) \to y(x).$$

Thus, for a given $\epsilon > 0$ there exists an N such that $\left|\sum^n u_r(x) - y(x)\right| = \left|y_n(x) - y(x)\right| < \epsilon$ for all $x \in [a,b]$ and for all $n \geq N$. We have:

$$\begin{aligned}
&\left|y(x) - y_0 - \int_{x_0}^{x} F\big(s, y(s)\big)\,\mathrm{d}s\right| \\
&\qquad = \left|y(x) - y_0 - \int_{x_0}^{x} F\big(s, y(s)\big)\,\mathrm{d}s - \underbrace{\left(y_{N+1}(x) - y_0 - \int_{x_0}^{x} F\big(s, y_N(s)\big)\,\mathrm{d}s\right)}_{=0}\right| \\
&\qquad = \left|\big(y(x) - y_{N+1}(x)\big) - \int_{x_0}^{x} \big[F\big(s, y(s)\big) - F\big(s, y_N(s)\big)\big]\,\mathrm{d}s\right| \\
&\qquad \leq |y(x) - y_{N+1}(x)| + \left|\int_{x_0}^{x} \big[F\big(s, y(s)\big) - F\big(s, y_N(s)\big)\big]\,\mathrm{d}s\right| \\
&\qquad \leq \epsilon + \int_{x_0}^{x} \left|F\big(s, y(s)\big) - F\big(s, y_N(s)\big)\right|\,\mathrm{d}s \\
&\qquad \leq \epsilon + \int_{x_0}^{x} K\,|y(s) - y_N(s)|\,\mathrm{d}s \\
&\qquad \leq \epsilon + K\epsilon|x - x_0| \leq \epsilon(1 + K|b - a|)
\end{aligned}$$

which can be taken arbitrarily small so the left hand side must equal 0, i.e., we have a solution.

1.4.4 Step 3 of the proof: uniqueness

Suppose $Y(x)$ is also a solution of the integral equation. Then as both $y(x)$ and $Y(x)$ are continuous there exists a number, C, say, such that $\left|Y(x) - y(x)\right| < C$ for all $x \in [a,b]$.

Next

$$\begin{aligned}
\left|Y(x) - y(x)\right| &= \left|y_0 + \int_{x_0}^{x} F\big(s, Y(s)\big)\,\mathrm{d}s - y_0 - \int_{x_0}^{x} F\big(s, y(s)\big)\,\mathrm{d}s\right| \\
&\leq \int_{x_0}^{x} \left|F\big(s, Y(s)\big) - F\big(s, y(s)\big)\right|\,\mathrm{d}s \\
&\leq \int_{x_0}^{x} K\,|Y(s) - y(s)|\,\mathrm{d}s \qquad\qquad (\dagger) \\
&\leq \int_{x_0}^{x} KC\,\mathrm{d}s = KC|x - x_0|.
\end{aligned}$$

Inserting this result back into (†)

$$\left|Y(x) - y(x)\right| \leq \int_{x_0}^{x} K^2 C\,|s - x_0|\,\mathrm{d}s = \frac{1}{2}K^2 C|x - x_0|^2$$

and iterating

$$\left|Y(x) - y(x)\right| \le \frac{1}{n!}K^n C|x - x_0|^n \le \frac{1}{n!}K^n C|b - a|^n.$$

Taking the limit as $n \to \infty$ the right hand side tends to zero (it is the n^{th} term in the Taylor series of the exponential which we know converges for any argument). We conclude the left hand side vanishes, i.e. $Y(x) = y(x)$ and the solution is unique. ∎

1.5 The dependence on y_0

We can use very similar arguments to show that our solution depends continuously on the initial condition in the following sense:

Theorem 1.3 *Under the assumptions of Picard's Theorem (First Form), suppose that $y^{(\delta)}(x)$ denotes the solution for which $y^{(\delta)}(x_0) = y_0 + \delta$ then*

$$\left|y^{(\delta)}(x) - y(x)\right| \le |\delta| \exp\bigl(K|x - x_0|\bigr).$$

Remark: Of course, $y(x) \equiv y^{(0)}(x)$.

Proof: We may clearly construct the corresponding sequence starting with $y_0^{(\delta)}(x) = y_0 + \delta$. Then $\left|y_0^{(\delta)}(x) - y_0(x)\right| = \delta$ and

$$\begin{aligned}
\left|y_1^{(\delta)}(x) - y_1(x)\right| &= \left|y_0 + \delta + \int_{x_0}^{x} F\bigl(s, y_0^{(\delta)}(s)\bigr)\,\mathrm{d}s - y_0 - \int_{x_0}^{x} F\bigl(s, y_0(s)\bigr)\,\mathrm{d}s\right| \\
&\le |\delta| + \int_{x_0}^{x} \left|F\bigl(s, y_0^{(\delta)}(s)\bigr) - F\bigl(s, y_0(s)\bigr)\right|\,\mathrm{d}s \\
&\le |\delta| + \int_{x_0}^{x} K\left|y_0^{(\delta)}(s) - y_0(s)\right|\,\mathrm{d}s \\
&\le |\delta|\left(1 + K|x - x_0|\right).
\end{aligned}$$

It soon becomes clear, and can be proved by induction that

$$\left|y_N^{(\delta)}(x) - y_N(x)\right| \le |\delta| \sum_{n=0}^{N} \frac{1}{n!}(K|x - x_0|)^n,$$

and taking $N \to \infty$ the theorem is proved. ∎

Note that although this provides some sense of continuity and so well-posedness of the IVP problem we should not be lulled into a false sense of security. According to our theorem the difference the solutions can still grow exponentially, so even if the y start very close they can still diverge rapidly – which of course gives rise to 'chaos'.

1.6 Picard's Theorem (Local version)

There are certain circumstances where the assumptions of our First Form of Picard's theorem are not satisfied but we can still prove a different form (by essentially the same argument). We start with an example:

Example: Consider the IVP $y'(x) = -y^2(x)$ with $y(1) = 1$ for $x \in [-1, 3]$.

This may, of course, be written as $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{y(x)} = 1$ and so 'solved' as $y(x) = \frac{1}{x}$. However, it is clear that our 'solution' blows up at $x = 0$ so is only a true solution on the restricted domain $x \in (0, 3]$.

The problem is that $\big|F(x, y_1) - F(x, y_2)\big| = |y_1^2 - y_2^2| = |y_1 + y_2||y_1 - y_2|$ and as y is unbounded we cannot bound $|y_1 + y_2|$ and no Lipschitz condition holds in the whole domain.

Yet still we found a solution valid in a neighbourhood of the initial point, so we expect to be able to formulate a local variant of Picard's theorem:

Theorem 1.4 (Picard's Theorem (Second Form)) *Let $F(x, y)$ be continuous and for all $x \in [x_0 - \alpha, x_0 + \alpha]$ and $y \in [y_0 - \beta, y_0 + \beta]$ satisfy a Lipschitz condition, with Lipschitz constant K say. Further let $M = \sup\big|F(x, y)\big|$ over the domain. Then the IVP $y'(x) = F\big(x, y(x)\big)$ with $y(x_0) = y_0$ has a unique solution defined for in the interval*

$$|x - x_0| \leq \min(\alpha, \beta/M).$$

Remark: This obviously reduces to the First Form in the case where x_0 lies at the centre of the interval in the limit when we can take $\beta \to \infty$.

Example: We took $\alpha = 2$, but now restrict to $y \in [1 - \beta, 1 + \beta]$. Then $|y_1 + y_2| \leq 2(1 + \beta)$ so we have a Lipschitz condition with $K = 2(1 + \beta)$. In addition $\sup\big|F(x, y)\big| = \sup|y^2| = (1 + \beta)^2$ so our theorem proves existence in the interval

$$|x - x_0| \leq \min\left(2, \frac{\beta}{(1 + \beta)^2}\right).$$

Elementary calculus shows that the second term reaches a maximum value of $\frac{1}{4}$ when $\beta = 1$, consistent with our exact solution.

Proof: All the steps above go through if the iterations stay in the domain

$$\{(x, y) : |x - x_0| \leq \min(\alpha, \beta/M), |y - y_0| \leq \beta\}.$$

and the conditions ensure that this is the case. ∎

1.7 Extensions: systems of equations and higher order equations

There are two natural ways in which we might extend this theorem: (1) more dependent variables, (2) higher order equations. In fact the two turn out to be the same but we start with the former.

1.7.1 System of equations/Vector equations

In this subsection we consider a system of differential equations

$$y_i'(x) = F_i(\big(x, y_1(x), y_2(x), \ldots, y_n(x)\big) \qquad i = 1, \ldots, n$$

or equivalently taking $\boldsymbol{y}(x)$ to be a vector-valued function taking values in $\mathbb{R}^n$ with components $y_i(x)$

$$\boldsymbol{y}'(x) = \boldsymbol{F}\big(x, \boldsymbol{y}(x)\big).$$

Definition 1.5 (The Vector Initial Value Problem (IVP)) *Given the ODE $\boldsymbol{y}'(x) = \boldsymbol{F}\big(x, \boldsymbol{y}(x)\big)$ over an interval I, find a solution $\boldsymbol{y}(x)$ (a differentiable function, by definition) over an interval $J \subset I$ such that $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$ for given $x_0 \in J$ and $\boldsymbol{y}_0 \in \mathbb{R}^n$.*

In defining a Lipschitz Condition we need to be aware that there are many different norms on $\mathbb{R}^n$, for example, for any $p \in \mathbb{N}$:

$$\|\boldsymbol{y}\|_p = \left|\sum_{i=1}^{n} |y_i|^p\right|^{1/p}$$

which in the limit $p \to \infty$ yields

$$\|\boldsymbol{y}\|_\infty = \max |y_i|.$$

Although these are all equivalent, (for example, $\|\boldsymbol{y}\|_2 \leq \|\boldsymbol{y}\|_1 \leq \sqrt{n}\|\boldsymbol{y}\|_2$) one can be more convenient to work with and in generalizing Picard's theorem the ∞-norm is particularly convenient.

Definition 1.6 (The Vector Lipschitz Condition) *$\boldsymbol{F}\big(x, \boldsymbol{y}\big)$ satisfies a Lipschitz Condition with respect to $\boldsymbol{y}$ in a region $D \subset \mathbb{R}^{n+1}$ in the p-norm if there exists a constant $K \geq 0$ such that*

$$\|\boldsymbol{F}(x, \boldsymbol{y}_1) - \boldsymbol{F}(x, \boldsymbol{y}_2)\|_p \leq K\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_p$$

for all $(x, \boldsymbol{y}_1), (x, \boldsymbol{y}_2) \in D$. K is called the corresponding Lipschitz constant in the p-norm.

Theorem 1.5 (Vector Picard's Theorem (Second Form)) *Let $\boldsymbol{F}\big(x, \boldsymbol{y}(x)\big)$ be continuous and for all $x \in [x_0 - \alpha, x_0 + \alpha]$ and $\|\boldsymbol{y} - \boldsymbol{y}_0\|_\infty < \beta$ satisfy a Lipschitz condition with respect to the ∞-norm, with Lipschitz constant K say. Further let $M = \sup \|\boldsymbol{F}\big(x, \boldsymbol{y}\big)\|_\infty$ over the domain. Then the IVP $\boldsymbol{y}'(x) = \boldsymbol{F}\big(x, \boldsymbol{y}(x)\big)$ with $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$ has a unique solution defined for in the interval*

$$|x - x_0| \leq \min(\alpha, \beta/M).$$

1.7.2 Higher-order equations

Next consider an equation relating derivatives up to the nth

$$y^{(n)}(x) = F\big(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\big)$$

with initial conditions on $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$.

We can reduce this to the problem of a system of equations by defining

$$y_1(x) = y(x), \quad y_2(x) = y'(x), \ldots, y_n(x) = y^{(n-1)}(x)$$

and so

$$\begin{aligned} y_i'(x) &= y_{i+1}(x) \equiv F_i\big(x, y_1(x), y_2(x), \ldots, y_n(x)\big) \qquad\qquad i = 1, \ldots, n-1 \\ y_n'(x) &= y^{(n)}(x) = F\big(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\big) \\ &\equiv F_n\big(x, y_1(x), y_2(x), \ldots, y_n(x)\big), \end{aligned}$$

with $\equiv$ denoting that this serves as a definition of the function on the right hand side.

Chapter 2

Comparison Theorems

Overview

In this chapter we look at so-called comparison theorems. We show how these can be used to address some of the well-posedness questions from the previous chapter – but in a greatly simplified manner. This is a short chapter, which would seem to suggest that such comparison theorems are of limited use. However, in the accompanying exercises we showcase the power and applicability of comparison theorems in analyzing important systems of equations in Applied Mathematics.

2.1 Gronwall's Inequality

Lemma 2.1 (Gronwall's Inequality) *Let $\sigma(x)$ be a differentiable function satisfying the differential inequality*

$$\sigma'(x) \le c\,\sigma(x), \qquad x_0 \le x \le b,$$

for some constant c then $\sigma(x) \le \sigma(x_0)\mathrm{e}^{c(x-x_0)}$ for $x_0 \le x \le b$.

Proof: Multiply our differential inequality by e^{-cx} and rearrange to get

$$0 \ge \mathrm{e}^{-cx}\sigma'(x) - c\,\mathrm{e}^{-cx}\sigma(x) = \left[\mathrm{e}^{-cx}\sigma(x)\right]'$$

so $\mathrm{e}^{-cx}\sigma(x)$ is a decreasing function and correspondingly

$$\mathrm{e}^{-cx}\sigma(x) \le \mathrm{e}^{-cx_0}\sigma(x_0)$$

which may be rearranged to the form given. ∎

2.1.1 Continuity / Uniqueness Theorems Revisited

Let $y(x)$ and $Y(x)$ be solutions to $y'(x) = F\big(x, y(x)\big)$, with $y(x_0) = y_0$ and $y^{(\delta)}(x_0) = y_0 + \delta$ and F satisfies a Lipschitz condition in y as before.

Consider $\sigma(x) = \big(y(x) - y^{(\delta)}(x)\big)^2$, then

$$\begin{aligned}
\sigma'(x) &= 2\big(y(x) - y^{(\delta)}(x)\big)\big(y'(x) - {y^{(\delta)}}'(x)\big), \\
&= 2\big(y(x) - y^{(\delta)}(x)\big)\Big(F\big(x, y(x)\big) - F\big(x, y^{(\delta)}(x)\big)\Big), \\
&\leq 2\left|y(x) - y^{(\delta)}(x)\right|\left|F\big(x, y(x)\big) - F\big(x, y^{(\delta)}(x)\right|.
\end{aligned}$$

By the Lipschitz condition for F, we have:

$$\begin{aligned}
\sigma'(x) &\leq 2K\left|y(x) - y^{(\delta)}(x)\right|^2, \\
&= K\sigma.
\end{aligned}$$

Thus

$$\sigma'(x) \leq 2K\sigma(x),$$

and our theorem tells us that

$$\big(y(x) - y^{(\delta)}(x)\big)^2 \leq \mathrm{e}^{2K(x-x_0)}\big(y(x_0) - y^{(\delta)}(x_0)\big)^2,$$

or

$$\big|y(x) - y^{(\delta)}(x)\big| \leq \mathrm{e}^{K(x-x_0)}\big|y(x_0) - y^{(\delta)}(x_0)\big|,$$

which is our continuity result. Equally for $y^{(0)}(x_0) = y_0$ we get our uniqueness result, as we then have $y^{(0)}(x) = y(x)$ for $x_0 \leq x \leq b$. .

2.2 More advanced results

Theorem 2.2 *Let $F(x, y)$ satisfy a Lipschitz condition in $[a, b]$. Suppose that $y(x)$ satisfies $y'(x) = F\big(x, y(x)\big)$ while $z(x)$ satisfies the differential inequality (and also the Lipschitz condition above)*

$$z'(x) \leq F\big(x, z(x)\big)$$

with the same initial condition $y(x_0) = z(x_0)$ then

$$z(x) \leq y(x) \qquad \textit{for} \quad x \geq x_0.$$

Proof: Let $\sigma(x) = z(x) - y(x)$ then

$$\sigma'(x) = z'(x) - y'(x) \leq F\big(x, z(x)\big) - F\big(x, y(x)\big).$$

Now the Lipschitz condition $\big|F(x,z) - F(x,y)\big| \leq K|z-y|$ can be rewritten as

$$-K\big|z(x) - y(x)\big| \leq F\big(x, z(x)\big) - F\big(x, y(x)\big) \leq K\big|z(x) - y(x)\big|,$$

so

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x} \leq K|\sigma(x)|. \tag{2.1}$$

We have $\sigma(x_0) = 0$. Let x_2 be the first x-value greater than x_0 such that $\sigma(x) > 0$. From Equation (2.1), we have:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x} \leq -K\sigma, \qquad x_0 \leq x \leq x_2.$$

By using the integrating-factor technique, we have:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{Kx}\sigma) \leq 0,$$

so $\mathrm{e}^{Kx}\sigma(x)$ is a decreasing function, hence:

$$\mathrm{e}^{Kx_2}\sigma(x_2) \leq \mathrm{e}^{Kx_0}\sigma(x_0).$$

But $\sigma(x_0) = 0$, hence $\sigma(x_2) \leq 0$, which is a contradiction. Hence, so $z(x_2) > y(x_2)$ is impossible.
∎

Corollary 2.3 *Suppose $y'(x) = F\big(x, y(x)\big)$ and $z'(x) = G\big(x, z(x)\big)$ where $G(x,y) \leq F(x,y)$ for $x \in [a,b]$ and that G satisfies a Lipschitz condition in y. Then if $y(x_0) = z(x_0)$ it follow that*

$$z(x) \leq y(x) \qquad \textit{for} \quad x \geq x_0.$$

2.2.1 Worked example

Consider the horrible differential equation

$$y'(x) = \cosh\big(x + y(x)\big) + y(x)^2 \equiv F\big(x, y(x)\big)$$

with $y(0) = 0$. We know that $\cosh z \geq 1$ for all $z \in \mathbb{R}$ so we may take $G(x,y) = 1 + y^2$ and the theorem will apply.

Now the solution to $z'(x) = 1 + z(x)^2$ with $z(0) = 0$ is easily seen to be $z(x) = \tan x$. Since this diverges at $x = \pi/2$ we deduce that the solution of original equation must diverge at or before this point.

This is illustrated below (Figure 2.1) where the F solution is plotted in blue and the G solution in orange.

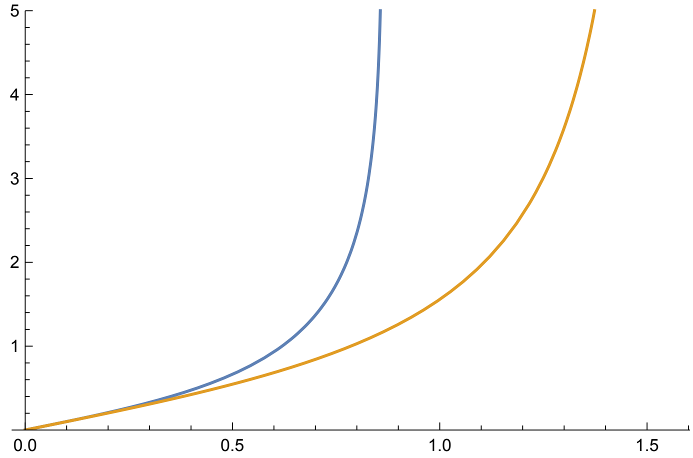


Figure 2.1:

Chapter 3

Linear equations: Homogeneous Case

Overview

In this chapter we look at linear ODEs and their properties.

3.1 Introduction

An ODE is said to be linear if it is linear in $y(x)$ and all of its derivatives. Thus an n^{th} order ODE has the general form

$$p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_0(x)y(x) = r(x)$$

where if $r(x) = 0$ the equation is said to be homogeneous and otherwise is said to be inhomogeneous. The focus of this chapter is on the homogeneous case.

If we assume that $p_i(x)$ and $r(x)$ are continuous on $[a, b]$, and $p_n(x) \neq 0$ for any $x \in [a, b]$ (so the equation truly determines $y^{(n)}(x)$) then the first (global) version of Picard's theorem holds since continuity gives us a Lipschitz condition in the bounded interval $[a, b]$ with

$$K = (n-1) + \sum_{i=0}^{n-1} \max_{x \in [a,b]} \left| \frac{p_i(x)}{p_n(x)} \right|,$$

using the ∞-norm (maximum-norm) on $\mathbb{R}^n$.

3.2 Linear Independence of Solutions

We will start by considering the homogeneous linear ODE and introduce the linear operator L defined by

$$L = p_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \cdots + p_0(x)$$

so our homogeneous linear ODE may be written compactly as $L[y] = 0$.

Now it is clear from linearity that if $L[u] = 0$ and $L[v] = 0$ then $L[u+v] = 0$ and $L[c\,u] = 0$ for any constant $c \in \mathbb{R}$. Hence we have a vector space of solutions and the first natural question is 'What is its dimension?'

Recall that we say that a set of functions $\{u_1(x), u_2(x), \ldots, u_n(x)\}$ are linearly dependent on the interval $[a,b]$ if there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\,u_1(x) + c_2\,u_2(x) + \cdots + c_n\,u_n(x) = 0.$$

Theorem 3.1 *There exist at most n linearly independent solutions of an n^{th} order homogeneous linear ODE.*

Proof: Suppose we have $n+1$ solutions $\{u_1(x), u_2(x), \ldots, u_n(x), u_{n+1}(x)\}$. Consider the problem of finding constants $c_1, c_2, \ldots, c_n, c_{n+1}$ such that, for some $x_0 \in [a,b]$:

$$\begin{aligned}
c_1\,u_1(x_0) + c_2\,u_2(x_0) + \cdots + c_n\,u_n(x_0) + c_{n+1}\,u_{n+1}(x_0) &= 0 \\
c_1\,u_1'(x_0) + c_2\,u_2'(x_0) + \cdots + c_n\,u_n'(x_0) + c_{n+1}\,u_{n+1}'(x_0) &= 0 \\
&\cdots \\
c_1\,u_1^{(n-1)}(x_0) + c_2\,u_2^{(n-1)}(x_0) + \cdots + c_n\,u_n^{(n-1)}(x_0) + c_{n+1}\,u_{n+1}^{(n-1)}(x_0) &= 0.
\end{aligned}$$

In other words, we seek a solution $(c_1, \cdots, c_n, c_{n+1})$ to the following n planar equations:

$$\begin{aligned}
(c_1, \cdots, c_n, c_{n+1})\cdot\boldsymbol{n}_1 &= 0, \qquad \boldsymbol{n}_1 = (u_1(x_0), \cdots, u_n(x_0), u_{n+1}(x_0)) \\
(c_1, \cdots, c_n, c_{n+1})\cdot\boldsymbol{n}_2 &= 0, \qquad \boldsymbol{n}_2 = (u_1'(x_0), \cdots, u_n'(x_0), u_{n+1}'(x_0)) \\
&\vdots \\
(c_1, \cdots, c_n, c_{n+1})\cdot\boldsymbol{n}_n &= 0, \qquad \boldsymbol{n}_n = (u_1^{(n-1)}(x_0), \cdots, u_n'(x_0), u_{n+1}^{(n-1)}(x_0)).
\end{aligned}$$

A constant vector $(c_1, \cdots, c_n, c_{n+1})^T$ solving these equations can always be found: it lies in the vector subspace orthogonal to that spanned by $\{\boldsymbol{n}_1, \cdots, \boldsymbol{n}_n\}$. As we are dealing with $\mathbb{R}^{n+1}$, this vector space is at least one-dimensional.

Next, we introduce:

$$v(x) = c_1\,u_1(x) + c_2\,u_2(x) + \cdots + c_n\,u_n(x) + c_{n+1}\,u_{n+1}(x).$$

This clearly satisfies $L[v] = 0$ and by construction,

$$v(x_0) = v'(x_0) = \cdots = v^{(n-1)}(x_0) = 0.$$

But the zero function also clearly satisfies this IVP and as the solution is unique we must have $v(x) = 0$ for all $x \in [a, b]$, that is, $\{u_1(x), u_2(x), \ldots, u_n(x), u_{n+1}(x)\}$ are linearly dependent. ∎

Definition 3.1 *The Wronskian of a set of $r-1$-times differentiable functions*

$$\{u_1(x), u_2(x), \ldots, u_r(x)\}$$

is defined as the r-dimensional determinant

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) & \ldots & u_r(x) \\ u_1'(x) & u_2'(x) & \ldots & u_r'(x) \\ \vdots & \vdots & \vdots & \vdots \\ u_1^{(r-1)}(x) & u_2^{(r-1)}(x) & \ldots & u_r^{(r-1)}(x) \end{vmatrix}$$

Theorem 3.2 *If $\{u_1(x), u_2(x), \ldots, u_{r-1}(x), u_r(x)\}$ are linearly dependent then their Wronskian vanishes identically.*

Proof: As the functions are linearly dependent there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\,u_1(x) + c_2\,u_2(x) + \cdots + c_r\,u_r(x) = 0.$$

and so also

$$\begin{aligned} c_1\,u_1(x) + c_2\,u_2(x) + \cdots + c_r\,u_r(x) &= 0 \\ &\vdots \\ c_1 u_1^{(r-1)}(x) + u_2^{(r-1)}(x) + \cdots + c_r u_r^{(r-1)}(x) &= 0 \end{aligned}$$

which may be combined in matrix form as

$$\begin{pmatrix} u_1(x) & u_2(x) & \ldots & u_r(x) \\ u_1'(x) & u_2'(x) & \ldots & u_r'(x) \\ \vdots & \vdots & \vdots & \vdots \\ u_1^{(r-1)}(x) & u_2^{(r-1)}(x) & \ldots & u_r^{(r-1)}(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and, of course, the condition for a non-trivial solution is precisely that the determinant of the matrix vanishes but that is nothing other than the Wronskian. ∎

The converse is not, in general, true as the following example shows:

Example: Define the differentiable functions

$$u_1(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \geq 0 \end{cases} \qquad u_2(x) = \begin{cases} x^2 & x < 0 \\ 0 & x \geq 0 \end{cases}.$$

Then if $c_1 u_1(x) + c_2 u_2(x) = 0$, taking $x = 1$ we find $c_1 = 0$ and taking $x = -1$ we find $c_2 = 0$ so the functions are linearly independent. On the other hand

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{cases} \begin{vmatrix} 0 & x^2 \\ 0 & 2x \end{vmatrix} = 0 & x < 0 \\ \\ \begin{vmatrix} x^2 & 0 \\ 2x & 0 \end{vmatrix} = 0 & x \geq 0 \end{cases},$$

so the Wronskian vanishes identically.

Lemma 3.3 (Abel's Identity) *Suppose that* $\{u_1(x), u_2(x), \ldots, u_{n-1}(x), u_n(x)\}$ *are solutions to a linear ODE. Then the Wronskian satisfies the relation for any value of* $x_0 \in [a, b]$

$$W(x) = W(x_0) \exp\left(- \int_{x_0}^{x} \frac{p_{n-1}(s)}{p_n(s)} \, \mathrm{d}s \right). \tag{3.1}$$

Proof: We recall Leibniz's formula for determinants:

$$\det(A) = \sum_{\tau \in S_n} \operatorname{sgn}(\tau) \prod_{i=1}^{n} a_{i,\tau(i)} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$

where sgn is the sign function of permutations in the permuation group: $\operatorname{sgn} = +1$ if τ (σ) is an even permulation, and -1 if τ (σ) is an odd permutation. This can be written more compactly using the Levi–Cevita symbol:

$$\det(A) = \epsilon_{i_1 \cdots i_n} a_{1,i_1} \cdots a_{n,i_n}.$$

Thus, if the matrix A is a function of a single variable x, then

$$\frac{\mathrm{d}}{\mathrm{d}x} \det(A) = \epsilon_{i_1 \, i_2 \cdots i_n} \frac{\mathrm{d}a_{1,i_1}}{\mathrm{d}x} a_{2,i_2} \cdots a_{n,i_n} + \cdots \epsilon_{i_1 \cdots i_{n-1} \, i_n} a_{1,i_1} \cdots a_{n-1,i_{n-1}} \frac{\mathrm{d}a_{n,i_n}}{\mathrm{d}x},$$

This can be written more compactly as:

$$\frac{\mathrm{d}}{\mathrm{d}x}\det(A) = \sum_{i=1}^{n} \det[A_i(x)]$$

where $A_i(x)$ is the matrix $A(x)$ whose $i-$th row has been substituted by the derivative of each entry of that row. We apply this result to the Wronskian $W(x)$:

$$W'(x) = \begin{vmatrix} u_1'(x) & u_2'(x) & \cdots & u_n'(x) \\ u_1'(x) & u_2'(x) & \cdots & u_n'(x) \\ u_1''(x) & u_2''(x) & \cdots & u_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)}(x) & u_2^{(n-1)}(x) & \cdots & u_n^{(n-1)}(x) \end{vmatrix} + \begin{vmatrix} u_1(x) & u_2(x) & \cdots & u_n(x) \\ u_1''(x) & u_2''(x) & \cdots & u_n''(x) \\ u_1''(x) & u_2''(x) & \cdots & u_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)}(x) & u_2^{(n-1)}(x) & \cdots & u_n^{(n-1)}(x) \end{vmatrix}$$
$$+ \cdots + \begin{vmatrix} u_1(x) & u_2(x) & \cdots & u_n(x) \\ u_1'(x) & u_2'(x) & \cdots & u_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-2)}(x) & u_2^{(n-2)}(x) & \cdots & u_n^{(n-2)}(x) \\ u_1^{(n)}(x) & u_2^{(n)}(x) & \cdots & u_n^{(n)}(x) \end{vmatrix}.$$

However, note that every determinant from the expansion above contains a pair of identical rows, except the last one. Since determinants with linearly dependent rows are equal to 0, the only one left with the last one.

On the other hand every u_i solves the ordinary differential equation, thus we have

$$u_i^{(n)}(x) + \frac{1}{p_n(x)}\left[p_{n-2}\, u_i^{(n-2)}(x) + \cdots + p_1\, u_i'(x) + p_0\, u_i\right] = -\frac{p_{n-1}}{p_n}\, u_i^{(n-1)}(x)$$

for every $i \in \{1, \ldots, n\}$. Hence, adding to the last row of the above determinant p_0 times its first row, p_1 times its second row, and so on until p_{n-2} times its next to last row, the value of the determinant for the derivative of W is unchanged and we get

$$W'(x) = \begin{vmatrix} u_1(x) & u_2(x) & \cdots & u_n(x) \\ u_1'(x) & u_2'(x) & \cdots & u_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-2)} & u_2^{(n-2)} & \cdots & u_n^{(n-2)} \\ -\frac{p_{n-1}}{p_n}\, u_1^{(n-1)}(x) & -\frac{p_{n-1}}{p_n}\, u_2^{(n-1)}(x) & \cdots & -\frac{p_{n-1}}{p_n}\, u_n^{(n-1)}(x) \end{vmatrix} = -\frac{p_{n-1}}{p_n} W(x).$$

Hence

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} \frac{p_{n-1}(s)}{p_n(s)}\,\mathrm{d}s\right).$$

so $W(x)$ either vanishes identically or else never vanishes. ∎

Theorem 3.4 *If $\{u_1(x), u_2(x), \ldots, u_{n-1}(x), u_n(x)\}$ are solutions to a linear ODE $L[u] = 0$ and $W(x) = 0$ then the functions are linearly dependent.*

Proof: If $W(x) = 0$ then, in particular, $W(x_0) = 0$ so there exists a non-zero solution to the linear system

$$\begin{pmatrix} u_1(x_0) & u_2(x_0) & \ldots & u_n(x_0) \\ u_1'(x_0) & u_2'({}_0x) & \ldots & u_n'(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ u_1^{(n-1)}(x_0) & u_2^{(n-1)}(x_0) & \ldots & u_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now define $v(x) = c_1u_1(x) + c_2u_2(x) + \ldots c_nu_n(x)$ the $L[v] = 0$ and $v(x_0) = v'(x_0) = \cdots = v^{(n-1)}(x_0) = 0$ as does the zero function so by our uniqueness theorem $v(x) = 0$ and our functions are linearly dependent. ∎

We can now complete the proof that the space of solutions to $L[u] = 0$ is a vector space of dimension n.

Theorem 3.5 *We may find an n-dimensional set of linearly independent solutions.*

Proof: Choose solutions $\{u_1(x), u_2(x), \ldots, u_n(x)\}$ with

$$u_i^{(j-1)}(x_0) = \delta_i^{j-1}.$$

Here, δ_l^k is the Kronecker-δ function, $\delta_l^k = 1, \text{iff } l = k$). Then $W(x_0) = 1$ and so $\{u_1(x), u_2(x), \ldots, u_n(x)\}$ are linearly independent.

Remark: We could clearly use $u_i^{(j-1)}(x_0) = a_{ij}$ where A is any invertible matrix.

Chapter 4

Linear equations: Inhomogeneous Case

Overview

Second order inhomogeneous linear ODE play a central role in Applied & Computational Mathematics, for example, in the case of a forced damped pendulum

$$ml^2\ddot{\theta}(t) + m\nu\dot{\theta}(t) + mgl\theta(t) = F(t).$$

Such equations will be the focus of this Chapter; while some the results, such as variation of parameters, will be familiar we will build on it later in this Chapter and the following. Throughout this course, we use the following notation for the generic 2^{nd} order inhomogeneous linear ODE:

$$L[y] = p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x) = r(x).$$

4.1 Variation of Parameters

Suppose we have two linearly independent solutions $u(x)$, $v(x)$ of the *homogeneous* problem $L[u] = L[v] = 0$. The general solution is $\alpha u(x) + \beta v(x)$ where α and β are constants. Now try to find a solution to solve the inhomogeneous problem $L[y] = r$ by elevating α and β to functions $a(x)$ and $b(x)$, which we are free to choose, so we try to find a solution in the form

$$y(x) = a(x)u(x) + b(x)v(x).$$

Then

$$y'(x) = a'(x)u(x) + b'(x)v(x) + a(x)u'(x) + b(x)v'(x).$$

We now use the freedom in $a(x)$ and $b(x)$ to ask that $a'(x)u(x) + b'(x)v(x) = 0$ and consequently

$$y'(x) = a(x)u'(x) + b(x)v'(x)$$

and so

$$y''(x) = a'(x)u'(x) + b'(x)v'(x) + a(x)u''(x) + b(x)v''(x).$$

Then rewriting using the previous terms we have

$$L[y] = p_2(x)\big(a'(x)u'(x) + b'(x)v'(x)\big) + a(x)L[u] + b(x)L[v]$$

and since $L[u] = L[v] = 0$ finally $L[y] = p_2(x)\big(a'(x)u'(x) + b'(x)v'(x)\big)$.

Since we want $L[y] = r$ we have a pair of simultaneous equations

$$\begin{aligned} a'(x)u(x) + b'(x)v(x) &= 0 \\ a'(x)u'(x) + b'(x)v'(x) &= r(x)/p_2(x), \end{aligned}$$

or equivalently,

$$\begin{pmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{pmatrix} \begin{pmatrix} a'(x) \\ b'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ r(x)/p_2(x) \end{pmatrix}.$$

This is easily solved as

$$\begin{pmatrix} a'(x) \\ b'(x) \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} v'(x) & -v(x) \\ -u'(x) & u(x) \end{pmatrix} \begin{pmatrix} 0 \\ r(x)/p_2(x) \end{pmatrix} = \frac{r(x)}{p_2(x)W(x)} \begin{pmatrix} -v(x) \\ u(x) \end{pmatrix}.$$

So

$$\begin{aligned} a(x) &= \alpha - \int_{x_\alpha}^{x} \frac{v(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s \\ b(x) &= \beta + \int_{x_\beta}^{x} \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s \end{aligned}$$

where x_α and x_β are arbitrary and may be chosen at our convenience, and recalling that $W(s) \neq 0$ for $s \in [a, b]$ since $u(x)$, $v(x)$ are linearly independent functions. So

$$y(x) = \alpha u(x) + \beta v(x) - u(x) \int_{x_\alpha}^{x} \frac{v(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s + v(x) \int_{x_\beta}^{x} \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s. \tag{4.1}$$

Remark: There appears to be a redundancy of constants in the formulation above (in a 2nd-order inhomogeneous ODE, we would expect only 2 constants and not 4). However, once the integral expressions are defined, α and β become functions of x_α and x_β, meaning there are only two independent constants of integration.

4.2 The Initial-Value Problem

First suppose we are looking for the particular solution with $y_P(x_0) = y_P'(x_0) = 0$; we can show that is corresponds to taking $x_\alpha = x_\beta = x_0$ and $\alpha = \beta = 0$. In fact, proceeding with the calculations we have

$$y_P(x) = -u(x) \int_{x_0}^{x} \frac{v(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s + v(x) \int_{x_0}^{x} \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s. \tag{4.2}$$

which clearly satisfies $y_P(x_0) = 0$ while

$$\begin{aligned} y_P'(x) &= -u'(x) \int_{x_0}^{x} \frac{v(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s - \cancel{u(x)\frac{v(x)r(x)}{p_2(x)W(x)}} + v'(x) \int_{x_0}^{x} \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s + \cancel{v(x)\frac{u(x)r(x)}{p_2(x)W(x)}} \\ &= -u'(x) \int_{x_0}^{x} \frac{v(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s + v'(x) \int_{x_0}^{x} \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s, \end{aligned}$$

so also $y_P'(x_0) = 0$, as required. Notice that Equation (4.2) gives also the definition of the *particular solution* of the inhomogeneous ODE.

Now suppose we want $y(x_0) = y_0$ and $y'(x_0) = y_0'$, then $y(x) = y_P(x) + \alpha u(x) + \beta v(x)$ is a solution of the ODE and

$$\begin{aligned} y(x_0) &= y_P(x_0) + \alpha u(x_0) + \beta v(x_0) = \alpha u(x_0) + \beta v(x_0) \\ y'(x_0) &= y_P'(x_0) + \alpha u'(x_0) + \beta v'(x_0) = \alpha u'(x_0) + \beta v'(x_0) \end{aligned}$$

since from the definition (4.2), the particular solution will vanish at the initial conditions. In compact form we have the linear system

$$\begin{pmatrix} u(x_0) & v(x_0) \\ u'(x_0) & v'(x_0) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

which may be readily solved as the determinant of the matrix is $W(x_0)$ which by assumption is non-zero (as $u(x)$, $v(x)$ are linearly independent).

Example: $L[y] = y''(x) + 2y'(x) + 2y(x) = r(x)$ with $y(0) = 1$ and $y'(0) = 0$.

Linearly independent solutions are $u(x) = \mathrm{e}^{-x}\cos x$ and $v(x) = \mathrm{e}^{-x}\sin x$. Hence,

$$\begin{aligned} W(x) &= \mathrm{e}^{-x}\cos x\big(-\mathrm{e}^{-x}\sin x + \mathrm{e}^{-x}\cos x\big) - \mathrm{e}^{-x}\sin x\big(-\mathrm{e}^{-x}\cos x - \mathrm{e}^{-x}\sin x\big) \\ &= \mathrm{e}^{-2x}. \end{aligned}$$

Note that this is consistent with Abel's Theorem, with $W' = -[p_1(x)/p_2(x)]W = -2W$. Then

$$\begin{aligned} y_P(x) &= -\mathrm{e}^{-x}\cos x \int_0^x \mathrm{e}^s \sin s\, r(s)\,\mathrm{d}s + \mathrm{e}^{-x}\sin x \int_0^x \mathrm{e}^s\cos s\, r(s)\,\mathrm{d}s \\ &= \int_0^x \mathrm{e}^{s-x}\sin(x-s) r(s)\,\mathrm{d}s. \end{aligned}$$

Now we write $y(x) = y_P(x) + \alpha u(x) + \beta v(x)$ and find we want

$$\begin{aligned} y(x_0) &= 1 = \alpha \\ y'(x_0) &= 0 = -\alpha + \beta \end{aligned}$$

so the required solution is

$$y(x) = y_P(x) + \mathrm{e}^{-x}(\cos x + \sin x).$$

4.3 The Initial-Value Green's Function

Definition 4.1 *We can write our particular solution with initial condition $y_P(x_0) = y_P'(x_0) = 0$ compactly as*

$$y_P(x) = \int_{x_0}^x G(x,s) r(s)\,\mathrm{d}s,$$

where

$$G(x,s) = \frac{-u(x)v(s) + v(x)u(s)}{p_2(s)W(s)}.$$

$G(x,s)$ is called the Green function for the initial value problem.

Theorem 4.1 *The Green's Function satisfies the following properties:*

1. *Defined for $x_0 \le s \le x$;*
2. *$L_x[G] = 0$;*
3. *$G(x, x-) = 0$;*

4. $\dfrac{\partial G}{\partial x}(x, x-) = \dfrac{-u'(x)v(x) + v'(x)u(x)}{p_2(x)W(x)} = \dfrac{1}{p_2(x)}$.

Here we use L_x to denote we are thinking of L as a differential operator in x with s held fixed, and $x-$ to denote the (one-sided) limit s tends up to x (it has to be one-sided as $G(x, s)$ is not defined for $s > x$).

In fact, we can turn things around and use these properties to define $G(x, s)$:

- Property 2 says $G(x, s) = a(s)u(x) + b(s)v(x)$, then
- Property 3 says $G(x, x) = 0 = a(x)u(x) + b(x)v(x)$, so $a(x) = c(x)v(x)$, $b(x) = -c(x)u(x)$ for some $c(x)$ and $G(x, s) = c(s)\big(v(s)u(x) - u(s)v(x)\big)$,
  Then
  $$\frac{\partial G}{\partial x}(x, s) = c(s)\big(v(s)u'(x) - u(s)v'(x)\big)$$
  so
- Property 4 says $c(x)W(x) = 1/p_2(x)$.

4.3.1 The adjoint operator

Definition 4.2 *If $L[y] = p_2y'' + p_1y' + p_0y$ then the operator*

$$M[y] = (p_2y)'' - (p_1y)' + p_0y$$

satsifies

$$\int u\big(L[v]\big) = \int v\big(M[u]\big) + \textit{boundary terms.}$$

M is called the adjoint of L. (Note that we have omitted the variable x for simplicity).

In fact, this is just our standard concept of the adjoint:

Definition 4.3 *If we define an inner product by*

$$\langle u, v\rangle = \int uv\,\mathrm{d}x$$

then the adjoint operator $M = L^\dagger$ is given by

$$\langle u, Lv\rangle = \langle L^\dagger u, v\rangle = \langle Mu, v\rangle.$$

Definition 4.4 *A differential operator is said to be self-adjoint if $M = L$.*

If we write out M and L explicitly

$$M[y] = p_2 y'' + (2p_2' - p_1)y' + (p_2'' - p_1' + p_0)y$$
$$L[y] = p_2 y'' + p_1 y' + p_0 y$$

we see that that we need the y' coefficient requires $p_2' = p_1$ and this then guarantees the equality of the y' coefficient and consequently for the y one, too. Thus the general self-adjoint equation has the form

$$L[y] = p_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}p_2}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + p_0 y = \frac{\mathrm{d}}{\mathrm{d}x}\left(p_2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) + p_0 y.$$

In this case Abel's Theorem (Equation (3.1)) becomes

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} \frac{p_2'(s)}{p_2(s)}\,\mathrm{d}s\right) = W(x_0)\exp\left(-\ln p_2(x) + \ln p_2(x_0)\right) = W(x_0)\frac{p_2(x_0)}{p_2(x)},$$

in other words

$$p_2(x)W(x) = p_2(x_0)W(x_0) \qquad \text{(constant)}.$$

That is, the denominator of our Green function is constant.

Note that any linear equation may be cast in self-adjoint form by using an appropriate integrating factor $I(x)$:

$$Ip_2 y'' + Ip_1 y' + Ip_0 y = Ir$$

and try to choose I so that $(Ip_2)' = Ip_1$ which gives

$$\frac{I'}{I} = \frac{p_1 - p_2'}{p_2} \qquad \Longrightarrow \qquad I(x) = \frac{1}{p_2(x)}\exp\left(\int_{x_0}^{x} \frac{p_1(s)}{p_2(s)}\,\mathrm{d}s\right).$$

4.4 The IVP as an integral equation

Given the remarks at the end of the last section let us assume that we have a self-adjoint IVP:

$$L[y] = (p_2 y')' + p_0 y = r,$$

with $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Then rewriting as $(p_2 y')' = r - p_0 y$ and integrating we have

$$p_2(x)y'(x) - p_2(x_0)y'(x_0) = \int_{x_0}^{x} \big(r(s) - p_0(s)y(s)\big)\,\mathrm{d}s$$

or

$$y'(x) = p_2(x_0)y_0'\frac{1}{p_2(x)} + \frac{1}{p_2(x)}\int_{x_0}^{x} \big(r(s) - p_0(s)y(s)\big)\,\mathrm{d}s.$$

If we define

$$P(x) = \int_{x_0}^{x} \frac{\mathrm{d}s}{p_2(s)},$$

we may integrate again to get

$$y(x) - y_0 = p_2(x_0)y_0'P(x) + \int_{x_0}^{x} \frac{1}{p_2(s)}\left(\int_{x_0}^{s} \big(r(t) - p_0(t)y(t)\big)\,\mathrm{d}t\right)\,\mathrm{d}s.$$

Interchanging the order of integration in the last term (Figure 4.1), we arrive at

$$\begin{aligned} y(x) &= y_0 + p_2(x_0)y_0'P(x) + \int_{x_0}^{x} \big(r(t) - p_0(t)y(t)\big)\left(\int_{t}^{x} \frac{1}{p_2(s)}\mathrm{d}s\right)\,\mathrm{d}t \\ &= y_0 + p_2(x_0)y_0'P(x) + \int_{x_0}^{x} \big(r(t) - p_0(t)y(t)\big)\left(P(x) - P(t)\right)\,\mathrm{d}t \end{aligned} \tag{4.3}$$

Alternatively, this can be written as:

$$\begin{aligned} y(x) = &\left[y_0 + p_2(x_0)y_0'P(x) + \int_{x_0}^{x} r(t)\left(P(x) - P(t)\right)\,\mathrm{d}t\right] \\ &+ \int_{x_0}^{x} p_0(t)\left(P(t) - P(x)\right)y(t)\,\mathrm{d}t. \end{aligned} \tag{4.4}$$

This has the form

$$y(x) = F(x) + \int_{x_0}^{x} K(x,t)y(t)\,\mathrm{d}t. \tag{4.5}$$

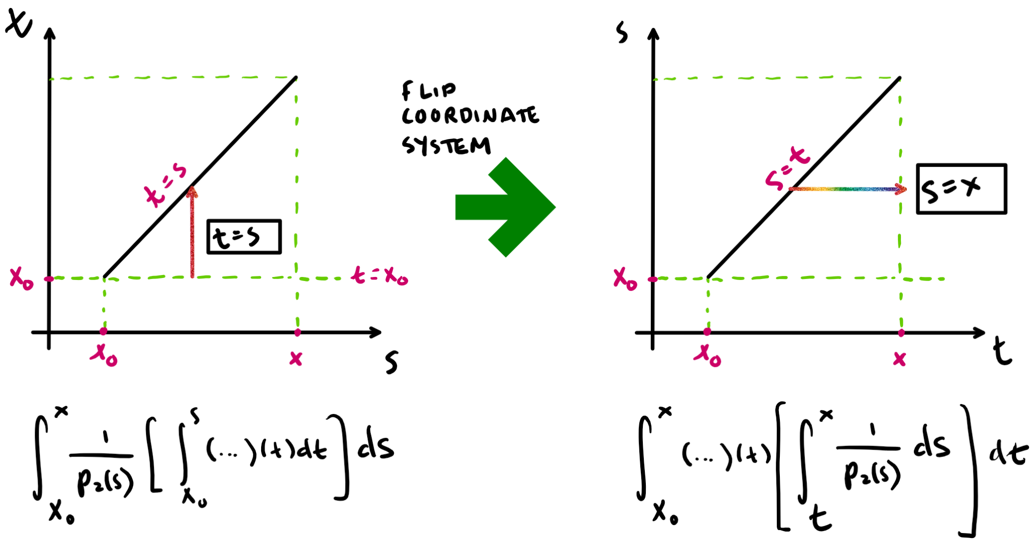


Figure 4.1: A graphical description showing how the limits of integration can be flipped in Equation (4.3)

which is referred to as a *Volterra* integral equation – that the upper limit of the integral is x is the key feature here.

4.4.1 Alternative derivation

We can derive Equation (4.3) in a different manner: we introduce a linear operator $L_0[y] = (p_2 y')'$, and we look for a particular integral of:

$$L_0[y] = r - p_0 y.$$

The homogeneous problem $L_0[y] = (p_2 y')' = 0$ has solutions with $p_2(x)y'(x) = A$, or $y'(x) = A/p_2(x)$ so

$$\begin{aligned} y(x) &= A \int^x \frac{\mathrm{d}s}{p_2(s)} \\ &= AP(x) + B, \end{aligned}$$

that is, we have independent solutions $u(x) = P(x)$ and $v(x) = 1$ with Wronskian

$$W(x) = P(x).0 - P'(x).1 = -\frac{1}{p_2(x)}.$$

Correspondingly we have $p_2(t)W(t) = -1$ and

$$y_P(x) = \int_{x_0}^{x} \left(u(x)v(t) - v(x)u(t)\right)\left(r(t) - p_0(t)y(t)\right)\,\mathrm{d}t$$
$$= \int_{x_0}^{x} \left(P(x) - P(t)\right)\left(r(t) - p_0(t)y(t)\right)\,\mathrm{d}t$$

Thus, the general solution of $L_0[y] = r - p_0 y$ is:

$$y = AP(x) + B + \int_{x_0}^{x} \left(P(x) - P(t)\right)\left(r(t) - p_0(t)y(t)\right)\,\mathrm{d}t$$

Since A and B are arbitrary constants, this can be matched up exactly with Equation (4.3)

4.4.2 Iterative Solution

The Volterra form Equation (4.5) suggests an iterative solution

$$y_0(x) = F(x),$$

and

$$y_{n+1}(x) = F(x) + \int_{x_0}^{x} K(x,t)y_n(t)\,\mathrm{d}t,$$

which as for the first order case we may show converges under suitable conditions on $K(x,t)$. As we have already proved existence for the IVP we will not pursue this further but just note that the interation is important for example in perturbation theory in quantum field theory where it is known as the Dyson series (Freeman Dyson, one of the pioneers of QED not the vacuum manufacturer!)

Chapter 5

Solution of ODEs via Power Series

Overview

In this chapter we look at some constructive methods for generating solutions of second-order linear ODEs, as the previous chapters have been a bit theoretical, indeed, maybe even frustrating, as the construction of Green's functions etc. relies on pulling solutions $u(x)$ and $v(x)$ out of thin air, as it were. In contrast to your previous studies of such ODEs, we now look at cases where the coefficeints are not constant. Hence, we look at solutions, in the neighbourhood of $z = 0$, of the ODE

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0. \tag{5.1}$$

Obviously we can investigate the behaviour of an arbitrary point $z = z_0$ by a linear shift $z \to z - z_0$. We use independent variable z in this Chapter to emphasise that the analysis naturally belongs in the complex plane, $z \in \mathbb{Z}$.

5.1 Ordinary Points and Regular Singular Points

Refer to Equation (5.1). Now $z = 0$ is called an *ordinary* point of the equation if both $p(z)$ and $q(z)$ are analytic at $z = 0$ and that otherwise $z = 0$ is a *singular* point of the equation. In the case of an ordinary we can use standard argument to generate the Taylor series for $y(z)$ about $z = 0$ (developed in the following); for a singular point we can not expand both $p(z)$ and $q(z)$ about 0 (in general one of them is singular at $z = 0$ so the procedure breaks down).

Some singular points are more benign than others; in particular, if both $zp(z)$ and $z^2q(z)$ are analytic at $z = 0$, then $z = 0$ is called a *regular* singular point of the equation. Otherwise it is called an *irregular* singular point.

Example: The function $y(z) = e^{-1/z^2}$ satisfies the ODE

$$y''(z) + (3z^{-1} + 2z^{-3})y'(z) = 0.$$

As $zp(z) = 3 + 2z^{-2}$ is not analytic at $z = 0$, this corresponds to an irregular singular point. The corresponding solution $y(z) = \mathrm{e}^{-1/z^2}$ has the property that all derivatives vanish at $z = 0$ so the Taylor series vanishes identically and has 0 radius of convergence.

By contrast near a regular singular point we may find solutions of the ODE as Taylor series solutions with just minor modification..

5.2 Solutions near an ordinary point

We have the following theorem:

Theorem 5.1 *If $p(z)$ and $q(z)$ are analytic in the disc $|z| < R$, then there exist two linearly independent solutions of Equation 5.1, $y_1(z)$ and $y_2(z)$, such that:*

- *$y_1(z)$ and $y_2(z)$ are analytic in $|z| < R$ (and possibly in a larger disc);*
- *$y_1(0) \neq 0$; if $y_2(0) = 0$ and $y_2'(0) \neq 0$.*

Remark: We can understand the theorem by noting that, near $z = 0$ the ODE is approximately

$$y''(z) + p(0)y'(z) + q(0)y(z) = 0,$$

for which solutions are of the form either $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ when the roots of the quadratic $\alpha^2 + p(0)\alpha + q(0) = 0$ are distinct or $e^{\alpha_1 z}$ and $ze^{\alpha_1 z}$ when there is a repeated root. We may then take $y_1(z) = e^{\alpha_1 z}$ and $y_2(z) = e^{\alpha_2 z} - e^{\alpha_1 z}$ in the distinct root case, while for the repeated root case we may simply take $y_1(z) = e^{\alpha_1 z}$ and $y_2(z) = ze^{\alpha_1 z}$. (Note, that no solution near an ordinary point can then behave, for example, as z^2 or $z^{1/2}$).

5.3 Euler Linear Equation

We look at

$$y''(z) + \frac{p_0}{z}y'(z) + \frac{q_0}{z^2}y(z) = 0, \tag{5.2}$$

known also as the *Euler linear equation*, for which we know to use a trial function or ansatz[1] $y(z) = z^\alpha$. If we substitute $y(z) = z^\alpha$ in the above ODE it yields

$$\big[\alpha(\alpha - 1) + p_0\alpha + q_0\big]z^{\alpha-2} = 0.$$

[1] It is a german word for "guess" or "hypothesis". In mathematics it is referred to an assumed condition or guess solution which is verified later being true.

Hence, the allowed values of α solve:

$$\alpha(\alpha-1)+p_0\alpha+q_0=0 \implies \alpha_{1,2}=\tfrac{1}{2}\left[(1-p_0)\pm\sqrt{(1-p_0)^2-q_0}\right].$$

Correspondingly the equation has independent solutions z^{α_1} and z^{α_2} when α_1 and α_2 are distinct. When the roots are the same, we have:

$$\alpha_{1,2}=\tfrac{1}{2}(1-p_0)=\alpha.$$

In this case, we write $y_1(z)=z^\alpha$ as the first solution, and attempt a second solution $y_2(z)=y_1(z)u(z)$. We substitute back into Equation (5.2) and obtain a reduced ODE for $u(z)$:

$$u''+\frac{1}{x}u'=0.$$

We let $u'=v$, and solve $\mathrm{d}v/\mathrm{d}x=-v/x$, with solution $v=1/x$. Hence, $u'=1/x$, and $u=\ln(x)$. Thus, the second solution is:

$$y_2(x)=x^\alpha\,\ln x.$$

5.4 Solutions near a regular singular point

Suppose now the weaker assumption that occurs frequently in practice that $zp(z)$ and $z^2q(z)$ are analytic in the disc $|z|<R$ and write

$$zp(z)=\sum_{n=0}^{\infty}p_nz^n \qquad z^2q(z)=\sum_{n=0}^{\infty}q_nz^n. \tag{5.3}$$

Near $z=0$ we now have the approximate equation

$$y''(z)+\frac{p_0}{z}y'(z)+\frac{q_0}{z^2}y(z)=0$$

which is Euler's equation again. Based on this to find a solution near an arbitrary singular point we write

$$y(z)=z^\alpha\sum_{n=0}^{\infty}a_nz^n, \tag{5.4}$$

where the arbitrariness in α is removed by demanding that $a_0\neq 0$. This type of solution (with $\alpha\neq 0$) is known as a *Frobenius series*.

5.4.1 Recurrence Relations

We start by substituting the Frobenius expansion (5.4) into the second order homogeneous ODE Equation (5.1), with $p(z) = z^{-1}\sum_{n=0}^{\infty} p_n z^n$ and $q(z) = z^{-2}\sum_{n=0} \infty q_n z^n$. We have

$$\begin{aligned} L[y] &= \sum_{n=0}^{\infty} a_n(n+\alpha)(n+\alpha-1)z^{n+\alpha-2} + \left(\sum_{n=0}^{\infty} p_n z^n\right)\left(\sum_{n=0}^{\infty} a_n(n+\alpha)z^n\right)\frac{z^\alpha}{z^2} \\ &\quad + \left(\sum_{n=0}^{\infty} q_n z^n\right)\left(\sum_{n=0}^{\infty} a_n z^n\right)\frac{z^\alpha}{z^2}, \\ L[y] &= 0. \end{aligned} \tag{5.5}$$

We use the Cauchy product of series:

$$\left(\sum_{n=0}^{\infty} \beta_n z^n\right)\left(\sum_{n=0}^{\infty} \gamma_n z^n\right) = \sum_{n=0}^{\infty} z^n \left(\sum_{j=0}^{n} \beta_j \gamma_{n-j}\right).$$

Hence, Equation (5.5) becomes:

$$\frac{\cancel{z^\alpha}}{\cancel{z^2}}\sum_{n=0}^{\infty} a_n(n+\alpha)(n+\alpha-1)z^n = -\frac{\cancel{z^\alpha}}{\cancel{z^2}}\sum_{n=0}^{\infty} z^n \sum_{n=0}^{n} a_j\left[(j+\alpha)p_{n-j} + q_{n-j}\right].$$

We break up the sum on the RHS into $j \in \{0, \cdots, n\}$ and $j = n$:

$$\sum_{n=0}^{\infty} a_n(n+\alpha)(n+\alpha-1)z^n = -\sum_{n=0}^{\infty} z^n \left\{\sum_{j=0}^{n-1} a_j\left[(j+\alpha)p_{n-j} + q_{n-j}\right] + a_n\left[(n+\alpha)p_0 + q_0\right]\right\}.$$

We equate the coefficients of z^0 to get:

$$a_0\alpha(\alpha-1) = -a_0\left(\alpha p_0 + q_0\right).$$

Or,

$$a_0 \underbrace{\left[\alpha(\alpha-1) + \alpha p_0 + q_0\right]}_{=F(\alpha)} = 0.$$

Thus, since $a_0 \neq 0$, we have the **indicial equation:**

$$F(\alpha) = 0.$$

The roots α_1 and α_2 of this quadratic are the exponents of the leading powers in our solution and satisfy

$$\alpha_1 + \alpha_2 = 1 - p_0, \qquad \alpha_1\alpha_2 = q_0.$$

For coefficeints of z^n with $n \geq 1$, we have:

$$a_n(n+\alpha)(n+\alpha-1) + a_n\left[(n+\alpha)p_0 + q_0\right] = -\sum_{j=0}^{n-1} a_j\left[(j+\alpha)p_{n-j} + q_{n-j}\right].$$

This can also be written as:

$$a_n F(n+\alpha) = -\sum_{j=0}^{n-1} a_j\left[(j+\alpha)p_{n-j} + q_{n-j}\right]$$

Summarizing, we have:

$$\begin{aligned} a_0 F(\alpha) &= 0 \\ a_n F(n+\alpha) &= -\sum_{j=0}^{n-1} a_j\big[(j+\alpha)p_{n-j} + q_{n-j}\big] \qquad (n>0). \end{aligned} \tag{5.6}$$

We can now solve sequentially with

$$a_n = -\frac{1}{(\alpha-\alpha_1+n)(\alpha-\alpha_2+n)}\sum_{k=0}^{n-1} a_k\big[(k+\alpha)p_{n-k} + q_{n-k}\big] \qquad (n>0), \tag{5.7}$$

with $\alpha = \alpha_1$ and $\alpha = \alpha_2$.

Equation (5.7) is our final answer, and gives a power-series solution to the ODE. Indeed, with $\alpha = \alpha_1$ and $\alpha = \alpha_2$, we get two distinct power series and hence, two linearly indepenent solutions to the ODE. However, there are two cases where this approach generates only one linearly indepenent solution. We outline these cases carefully in what follows.

5.4.2 Equal Roots

In cases where the indicial equation has repeated roots, with $\alpha = \alpha_1$, setting $\alpha = \alpha_1$ in Equation (5.7) gives only one recurrence:

$$a_n = -\frac{1}{n^2}\sum_{k=0}^{n-1} a_k\big[(k+\alpha)p_{n-k} + q_{n-k}\big] \qquad (n>0), \tag{5.8}$$

This gives a perfectly good **first solution** to the ODE, but the linearly independent second solution cannot be generated in this way.

5.4.3 Roots differing by an integer

If $\alpha_1 = \alpha_2 + N$ for some $n = N$ (where $n = 0, 1, 2, \dots$), the recurrence relations will give one solution corresponding to α_1 but will usually break down for the smaller root α_2 when the denominator in the expression for a_N in Equation (5.7) vanishes (because $\alpha_2 - \alpha_1 + N = 0$).

In both these scenarios, Frobenius's Method can be used to generate the second linearly independent solution.

5.5 Frobenius's Method

It consists in finding a second solution for the case when the roots of the indicial equation differ from an integer N. If we solve the recursion relations Eq. (5.6) *without* solving the indicial equation we find that the corresponding series

$$y_\alpha(z) = z^\alpha \sum_{n=0}^{\infty} a_n(\alpha) z^n,$$

satisfies

$$y''_\alpha(z) + p(z)y'_\alpha(z) + q(z)y_\alpha(z) = a_0(\alpha - \alpha_1)(\alpha - \alpha_2)z^{\alpha-2},$$

where the terms associated to the coefficients a_n for $n > 0$ vanish (since we are satisfying all the recurrence relations, but $n = 0$!). Clearly consistent with our results for our non-special cases.

5.5.1 Equal roots

Since this is true for all α we may now differentiate with respect to α to find

$$\begin{aligned}\frac{\partial y''_\alpha}{\partial\alpha}(z) + p(z)\frac{\partial y'_\alpha}{\partial\alpha}(z) + q(z)\frac{\partial y_\alpha}{\partial\alpha}(z) = \\ a_0(\alpha-\alpha_1)z^{\alpha-2} + a_0(\alpha-\alpha_2)z^{\alpha-2} + a_0(\alpha-\alpha_1)(\alpha-\alpha_2)\ln z\, z^{\alpha-2},\end{aligned}$$

since

$$\frac{\partial}{\partial\alpha}z^\alpha = \frac{\partial}{\partial\alpha}e^{\alpha\ln z} = \ln z e^{\alpha \ln z} = \ln z\, z^\alpha.$$

Thus, in the case $\alpha_2 = \alpha_1$ it is clear that $\left.\frac{\partial y_\alpha}{\partial\alpha}\right|_{\alpha=\alpha_1}(z)$ satisfies the equation as well as $y_{\alpha_1}(x)$. Cf. If a polynomial $p(\alpha)$ has a repeated root α_1 then $p(\alpha_1) = 0$.

Writing this in terms of our series we have our second solution of the form

$$\begin{aligned}y_2(z) = \left.\frac{\partial y_\alpha}{\partial\alpha}\right|_{\alpha=\alpha_1}(z) = z^{\alpha_1}\sum_{n=0}^{\infty}\left.\frac{\partial a_n(\alpha)}{\partial\alpha}\right|_{\alpha=\alpha_1} z^n + \ln z\, z^{\alpha_1}\sum_{n=0}^{\infty} a_n(\alpha_1)z^n \\ y_2(z) = z^{\alpha_1}\sum_{n=0}^{\infty}\left.\frac{\partial a_n(\alpha)}{\partial\alpha}\right|_{\alpha=\alpha_1} z^n + \ln z\, y_1(z).\end{aligned}$$

5.5.2 Roots differing by an integer

In the case $\alpha_1 = \alpha_2 + N$ where $N = 1, 2, \ldots$ our problem is that the denominator, which can now be written as $(\alpha - \alpha_2 + n - N)(\alpha - \alpha_2 + n)$ will contain a factor $(\alpha - \alpha_2)$ when we set $n = N$ and this factor will propagate into all higher terms a_n $(n \geq N)$. To counter this we keep α free and multiply by $(\alpha - \alpha_2)$ before differentiating

$$\frac{\partial(\alpha-\alpha_2){y_\alpha}''}{\partial\alpha}(z) + p(z)\frac{\partial(\alpha-\alpha_2){y_\alpha}'}{\partial\alpha}(z) + q(z)\frac{\partial(\alpha-\alpha_2)y_\alpha}{\partial\alpha}(z) =$$
$$a_0(\alpha-\alpha_2)^2 z^{\alpha-2} + 2a_0(\alpha-\alpha_1)(\alpha-\alpha_2)z^{\alpha-2} + a_0(\alpha-\alpha_1)(\alpha-\alpha_2)^2\ln z\, z^{\alpha-2}.$$

Hence, together with our first solution $y_{\alpha_1}(z)$, we have now constructed a second solution

$$y_2 = \left.\frac{\partial(\alpha-\alpha_2)y_\alpha}{\partial\alpha}\right|_{\alpha=\alpha_2}(z).$$

We write this in terms of our series:

$$y_2(z) = \left.\frac{\partial(\alpha-\alpha_2)y_\alpha}{\partial\alpha}\right|_{\alpha=\alpha_2}(z) = z^{\alpha_2}\sum_{n=0}^{\infty}\left.\frac{\partial(\alpha-\alpha_2)a_n(\alpha)}{\partial\alpha}\right|_{\alpha=\alpha_2} z^n + \ln z\, z^{\alpha_2}\sum_{n=0}^{\infty}\left[\lim_{\alpha\to\alpha_2}(\alpha-\alpha_2)a_n(\alpha)\right]z^n$$

For $n < N$, the term $a_n(\alpha)$ is regular at α_2, hence $\lim_{\alpha\to\alpha_2} a_n(\alpha)$ is well defined, and hence also, $\lim_{\alpha\to\alpha_2}(\alpha-\alpha_2)a_n(\alpha) = 0$. Thus, not all of the terms contribute in the second series in the above. Hence, we have:

$$y_2(z) = z^{\alpha_2}\sum_{n=0}^{\infty}\left.\frac{\partial(\alpha-\alpha_2)a_n(\alpha)}{\partial\alpha}\right|_{\alpha=\alpha_2} z^n + \ln z\, z^{\alpha_2}\sum_{n=N}^{\infty}\left[\lim_{\alpha\to\alpha_2}(\alpha-\alpha_2)a_n(\alpha)\right]z^n.$$

We re-index with $n' = n - N$:

$$y_2(z) = z^{\alpha_2}\sum_{n=0}^{\infty}\left.\frac{\partial(\alpha-\alpha_2)a_n(\alpha)}{\partial\alpha}\right|_{\alpha=\alpha_2} z^n + \ln z\, z^{\alpha_2}\sum_{n'=0}^{\infty}\left[\lim_{\alpha\to\alpha_2}(\alpha-\alpha_2)a_{N+n'}(\alpha)\right]z^{n'}z^N.$$

We gather up the powers $z^{\alpha_2+N} = z^{\alpha_1}$. Hence, we have:

$$y_2(z) = z^{\alpha_2}\sum_{n=0}^{\infty}\left.\frac{\partial(\alpha-\alpha_2)a_n(\alpha)}{\partial\alpha}\right|_{\alpha=\alpha_2} z^n + C\ln z\, z^{\alpha_1}u_1(z).$$

Here, $u_1(z)$ is a standard power series – it has a similar form to $y_1(z)$ with with coefficients changed and related to some constant C to still get defined. This method and the resulting forms are most easily understood with an example.

5.6 Bessel's Equation

Bessel's equation can be written as

$$x^2y'' + xy' + \left(x^2 - \nu^2\right)y = 0$$

where, without loss of generality, we may assume that $\nu \geq 0$, or

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0.$$

Comparing to the form $y'' + py' + qy = 0$ we see that we have Taylor series for $xp(x)$ and $x^2q(x)$ possess the following Taylor series about $x = 0$:

$$\begin{aligned} xp(x) &= 1 & p_0 &= 1, \text{ all others 0} \\ x^2q(x) &= -\nu^2 + x^2 & q_0 &= -\nu^2,\ q_2 = 1, \text{ all others 0.} \end{aligned}$$

Inserting the Frobenius expansion $y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n$ we have

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2} + x\sum_{n=0}^{\infty}(n+\alpha)a_n x^{n+\alpha-1} + (x^2-\nu^2)\sum_{n=0}^{\infty}a_n x^{n+\alpha} = 0.$$

We re-index with $p = n - 2$ where appropriate. We also cancel out the factor of x^α. This gives:

$$\sum_{p=-2}^{\infty} a_{p+2}(p+\alpha+2)(p+\alpha+1)x^p + \sum_{p=-2}^{\infty} a_{p+2}(p+\alpha+2)x^p + \sum_{p=0}^{\infty} a_p x^p - \nu^2 \sum_{p=-2}^{\infty} a_{p+2}x^p.$$

We equate coeffcients of like powers of x. At $p = -2$, we have:

$$\alpha^2 - \nu^2 = 0, \tag{5.9}$$

which is the indicial equation, with solution

$$\alpha = \pm\nu.$$

At $p = -1$ we have:

$$a_1\left[\alpha(\alpha+1) + (\alpha+1) - \nu^2\right] = 0 \implies a_1\left[\pm 2\nu + 1\right] = 0. \tag{5.10}$$

Otherwise, we have $p \geq 0$:

$$a_{p+2}\left[(p+\alpha+2)(p+\alpha+1) + (p+\alpha+2) - \nu^2\right] = -a_p.$$

We let $n = p + 2$. This gives:

$$a_n \left[n^2 + 2n\alpha + (\alpha^2 - \nu^2)\right] = -a_{n-2}. \tag{5.11}$$

Notice that this can also be written as:

$$a_n \left[(n + \alpha)^2 - \nu^2\right] = -a_{n-2}, \tag{5.12}$$

if necessary. We sub in for the solution of the indicial equation, $\alpha^2 = \nu^2$:

$$a_n n(n \pm 2\nu) = -a_{n-2}. \tag{5.13}$$

Provided $n(n \pm 2\nu) \neq 0$, this recurrence gives:

$$a_n = -\frac{a_{n-2}}{n(n \pm 2\nu)}. \tag{5.14}$$

There are different cases of ν which we must go through systematically. In some cases, the recurrence works, in others it does not, and Frobenius's method has to be used.

5.6.1 $2\nu \notin \mathbb{N}$

In particular, $2\nu \neq 1$ so Equation (5.10) implies $a_1 = 0$ and the denominator in Equation (5.13) never vanishes. Hence by Equation (5.13) all odd coefficients must vanish: $a_{2n-1} = 0$, $n \in \mathbb{N}$. By the same argument (and using Equation (5.13) again), the even coefficients give 2 independent solutions:

$$a_{2n} = -\frac{1}{2n(2n \pm 2\nu)} a_{2n-2} = \cdots = \frac{(-1)^n}{2^{2n} n!(n \pm \nu)(n - 1 \pm \nu) \cdots (1 \pm \nu)} a_0 \ . \tag{5.15}$$

Convention dictates that we take the normalization to be given by $a_0 = 1/2^{\pm\nu}$ and that the corresponding solutions are denoted by

$$J_{\pm\nu}(x) = \left(\frac{x}{2}\right)^{\pm\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n \pm \nu)(n - 1 \pm \nu) \cdots (1 \pm \nu)} \left(\frac{x}{2}\right)^{2n} \tag{5.16}$$

known as the *Bessel's function* of the first kind.

5.6.2 $\nu = 0$

From above we have indicial equation $\alpha^2 = 0$ so $\alpha_1 = \alpha_2 = 0$. The recurrence relations then give $a_1 = 0$ and

$$a_{2n} = -\frac{1}{4n^2} a_{2(n-1)} = \frac{(-1)^n}{2^{2n}(n!)^2} a_0 \tag{5.17}$$

Taking $a_0 = 1$ we have a solution

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

To obtain the second solution we must use Frobenius's method for equal roots, with:

$$y_2 = \left.\frac{\partial y_1}{\partial \alpha}\right|_{\alpha=0}.$$

To do this computation, we take $\nu = 0$ and a_0 in Equation (5.12). We furthermore keep α arbitrary. This gives:

$$\begin{aligned} a_{2n}(\alpha) &= -\frac{1}{(2n+\alpha)^2} a_{2(n-1)}(\alpha) = \frac{(-1)^n}{(2n+\alpha)^2(2(n-1)+\alpha)^2 \cdots (2+\alpha)^2}, \\ &= (-1)^n \frac{1}{F(\alpha)}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\alpha} a_{2n}(\alpha) &= -(-1)^n \frac{1}{F^2} \frac{\mathrm{d}F}{\mathrm{d}\alpha}, \\ &= -(-1)^n \frac{1}{F} \frac{1}{F} \frac{\mathrm{d}F}{\mathrm{d}\alpha}, \\ &= -a_{2n} \frac{\mathrm{d}\log F}{\mathrm{d}\alpha}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \log F &= \log \prod_j (2j+\alpha)^2, \\ &= \sum_j \log(2j+\alpha)^2, \\ &= \sum_j 2\log(2j+\alpha), \end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \log F = \sum_j \frac{2}{2j+\alpha}.$$

and so

$$\frac{\mathrm{d}a_{2n}}{\mathrm{d}\alpha}(\alpha) = -\left(\frac{-2}{2n+\alpha} + \frac{-2}{2(n-1)+\alpha} + \cdots + \frac{-2}{2\pm\alpha}\right) a_{2n}(\alpha).$$

Hence, the second solution is given by

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(x^{\alpha}\sum_{n=0}^{\infty} a_{2n}(\alpha)x^{2n}\right)_{\alpha=0} = \ln x \sum_{n=0}^{\infty} a_{2n}(0)x^{2n} + \sum_{n=0}^{\infty}\frac{\mathrm{d}a_{2n}}{\mathrm{d}\alpha}(0)x^{2n}$$

$$Y_0(x) = J_0(x)\ln x + \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{2^{2n}(n!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)x^{2n}.$$

This solution is denoted $Y_0(x)$ and it known as the *Bessel's function of the second kind*. It clearly diverges like $\ln x$ as $x \to 0$.

5.6.3 $\nu = 1$

From above we have indicial equation $\alpha^2 = 1$, so $\alpha_1 = 1$, $\alpha_2 = -1$. The recurrence relations again give $(\pm 2 + 1)a_1 = 0$ so $a_1 = 0$ and

$$a_{2n} = -\tfrac{1}{(2n+\alpha)^2-1}a_{2(n-1)},$$

$$= \frac{(-1)^n}{((2n+\alpha)^2-1)((2(n-1)+\alpha)^2-1)\cdots((4+\alpha)^2-1)((2+\alpha)^2-1)}a_0.$$

Hence,

$$a_{2n}\frac{(-1)^n}{((2n+\alpha)^2-1)((2(n-1)+\alpha)^2-1)\cdots((4+\alpha)^2-1)(\alpha^2+4\alpha+3)}a_0, \tag{5.18}$$

For $\alpha = \alpha_1 = 1$ this gives a well-defined solution

$$J_1(x) = \left(\frac{x}{2}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}n!(n+1)!}x^{2n}$$

Second solution: From Equation (5.18) for $n = 1$ we have

$$a_2 = -\frac{1}{(2+\alpha)^2-1}a_0 = -\frac{1}{(\alpha+1)(\alpha+3)}a_0$$

so it is clear that we have problems taking $\alpha = \alpha_2 = -1$. Following the Frobenius method we multiply our coefficients by $(\alpha - \alpha_2) = (\alpha + 1)$:

$$\begin{aligned}(\alpha+1)a_{2n}(\alpha) &= \frac{(-1)^n}{((2n+\alpha)^2-1)((2(n-1)+\alpha)^2-1)\cdots((4+\alpha)^2-1)(\alpha^2+4\alpha+3)}(\alpha+1)a_0, \\ &= \frac{(-1)^n}{((2n+\alpha)^2-1)((2(n-1)+\alpha)^2-1)\cdots((4+\alpha)^2-1)(\alpha+3)\cancel{(\alpha+1)})}\cancel{(\alpha+1)}a_0.\end{aligned}$$

Hence,

$$(\alpha+1)a_{2n}(\alpha) = \frac{(-1)^n}{((2n+\alpha)^2-1)((2(n-1)+\alpha)^2-1)\cdots((4+\alpha)^2-1)(\alpha+3)}a_0 \tag{5.19}$$

It will also be convenient to write this compactly as:

$$(\alpha+1)a_{2n}(\alpha) = (-1)^n\frac{1}{F(\alpha)}.$$

Hence, by the same reasoning as before,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha+1)a_{2n}(\alpha)\right] = -a_{2n}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\log((2n+\alpha)^2-1)+\log((2(n-1)+\alpha)^2-1)+\cdots+\log(\alpha+3)\right].$$

Hence,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha+1)a_{2n}(\alpha)\right] \\ &= \left[\frac{-2(2n+\alpha)}{((2n+\alpha)^2-1)}+\frac{-2(2(n-1)+\alpha)}{((2(n-1)+\alpha)^2-1)}+\cdots+\frac{-2(4+\alpha)}{((4+\alpha)^2-1)}-\frac{1}{(\alpha+3)}\right](\alpha+1)a_{2n}(\alpha).\end{aligned} \tag{5.20}$$

We now look at evaluating this expression at $\alpha = -1$. We start with the easy case:

$$\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha+1)a_0\right]\right)_{\alpha=-1} = a_0$$

Otherwise, we have $n \neq 1$. The term inside the square bracket in Equation (5.20) safely evaluates to:

$$\left[\frac{-2(2n-1)}{(2n-2)(2n)}+\frac{-2(2n-3)}{(2n-4)(2n-2)}+\cdots+\frac{-2(3)}{(2)(4)}-\frac{1}{2}\right]$$

Also, from Equation (5.19), we have:

$$\begin{aligned}\lim_{\alpha\to -1}\left[(\alpha+1)a_{2n}(\alpha)\right] &= (-1)^n\frac{1}{((2n-1)^2-1)c\cdots(3^2-1)\times 2},\\ &= (-1)^n\frac{1}{((2n-1)^2-1)\cdots(3^2-1)\times 2},\\ &= (-1)^n\frac{1}{(4n^2-4n)\cdots(8)\times 2},\\ &= (-1)^n\frac{1}{(2^2n(n-1))\cdots(2^2 2(1))\times 2},\\ &= (-1)^n\frac{1}{2^{2n-1}(n-1)!n!}.\end{aligned}$$

Putting it all together, we have:

$$\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha+1)a_{2n}\right]\right)_{\alpha=-1} = \left[\frac{-2(2n-1)}{(2n-2)(2n)}+\frac{-2(2n-3)}{(2n-4)(2n-2)}+\cdots+\frac{-2(3)}{(2)(4)}-\frac{1}{2}\right] \times(-1)^n\frac{1}{2^{2n-1}(n-1)!n!}. \quad (5.21)$$

Now according to the Frobenius method the second solution is given by

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(z^\alpha\sum_{n=0}^{\infty}(\alpha+1)a_{2n}(\alpha)z^{2n}\right)_{\alpha=-1} = \ln z.z^{-1}\sum_{n=0}^{\infty}\left((\alpha+1)a_{2n}(\alpha)\right)_{\alpha=-1}z^{2n}+ z^{-1}\sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha+1)a_{2n}(\alpha)\right]_{\alpha=-1}z^{2n}$$

and from above $((\alpha+1)a_0(\alpha))_{\alpha=-1}=0$ while for $n\geq 1$,

$$((\alpha+1)a_{2n}(\alpha))_{\alpha=-1}=\frac{(-1)^n}{2^{2n-1}(n-1)!n!}a_0$$

with the convention $0!=1$ (the term is not present in that case). In addition,

$$\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha+1)a_0\right]\right)_{\alpha=-1}=a_0$$

while for $n \geq 1$

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[(\alpha+1)a_{2n}\right]\right)_{\alpha=-1} &= \left[\frac{-2(2n-1)}{(2n-2)(2n)} + \frac{-2(2n-3)}{(2n-4)(2n-2)} + \cdots + \frac{-2(3)}{(2)(4)} - \frac{1}{2}\right] \\ &\quad \times (-1)^n \frac{1}{2^{2n-1}(n-1)!n!} \\ &= -\frac{1}{2}\left[\left(\frac{1}{n}+\frac{1}{n-1}\right) + \left(\frac{1}{n-1}+\frac{1}{n-2}\right) + \cdots + \left(\frac{1}{2}+\frac{1}{1}\right) + 1\right] \frac{(-1)^n}{2^{2n-1}(n-1)!n!} a_0 \\ &= \left[\frac{1}{n} + 2\sum_{k=1} \frac{1}{k}\right] \frac{(-1)^{n+1}}{2^{2n}(n-1)!n!} a_0.\end{aligned}$$

Putting this together our second solution, taking $a_0 = 1$ is

$$\begin{aligned}&= \ln z . z^{-1} \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n-1}(n-1)!n!} z^{2n} + z^{-1} + z^{-1} \sum_{n=1}^{\infty} \left[\frac{1}{n} + 2\sum_{k=1} \frac{1}{k}\right] \frac{(-1)^{n+1}}{2^{2n}(n-1)!n!} z^{2n} \\ &= -\ln(z).J_1(z) + z^{-1} + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{n} + 2\sum_{k=1}\frac{1}{k}\right] \frac{(-1)^{n+1}}{(n-1)!n!}\left(\frac{z}{2}\right)^{2n-1} .\end{aligned}$$

(relabelling $n \to n-1$ in the first sum).

5.6.4 $\nu = \frac{1}{2}$

This case is addressed in the homeworks.

5.7 Infinity as a Singular Point

There is one important cases where the change of variable $z \to z - z_0$ is not sufficient namely when the singular point is at infinity. In this case the appropriate action is to transform it to zero with the transformation $z \to 1/w$. Then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}z} &= -w^2 \frac{\mathrm{d}y}{\mathrm{d}w} \\ \frac{\mathrm{d}^2y}{\mathrm{d}z^2} &= w^4 \frac{\mathrm{d}^2y}{\mathrm{d}w^2} + 2w^3 \frac{\mathrm{d}y}{\mathrm{d}w}\end{aligned}$$

so our equation becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}w^2} + \left[\frac{2}{w} - \frac{1}{w^2} p\left(\frac{1}{w}\right)\right] \frac{\mathrm{d}y}{\mathrm{d}w} + \frac{1}{w^4} q\left(\frac{1}{w}\right) y = 0$$

We then have an ordinary point at $w = 0$ if

$$\frac{2}{w} - \frac{1}{w^2}p\left(\frac{1}{w}\right) \qquad \text{and} \qquad \frac{1}{w^4}q\left(\frac{1}{w}\right)$$

are regular there, that is, have an ordinary point at $z = \infty$ if

$$2z - z^2p\left(z\right) \qquad \text{and} \qquad z^4q\left(z\right)$$

are regular at $z = \infty$. Correspondingly, we have a regular singular point at $w = 0$ if

$$2 - \frac{1}{w}p\left(\frac{1}{w}\right) \qquad \text{and} \qquad \frac{1}{w^2}q\left(\frac{1}{w}\right)$$

are regular there, that is, have a regular singular point at $z = \infty$ if

$$2 - zp\left(z\right) \qquad \text{and} \qquad z^2q\left(z\right)$$

are regular at $z = \infty$ (and, of course, we may drop the 2 in the first condition!).

Chapter 6

Boundary-Value Problems

Overview

When we have a second-order equation there is an alternative problem to the IVP which is when we seek a solution on an interval $[a, b]$ and we specify **boundary conditions** on $y(x)$ or $y'(x)$ (or a combination) at a and b. We describe solving the equation $L[y] = r$ subject to such boundary conditions the Boundary Value Problem (BVP). In this section we outline some solution techniques for the standard (linear) BVP.

6.1 Boundary Conditions

Standard **homogeneous** boundary conditions include:

- The Dirichlet Problem: $y(a) = y(b) = 0$
- The Neumann Problem: $y'(a) = y'(b) = 0$
- The Robin Problem:

$$\alpha_a y(a) + \beta_a y'(a) = \alpha_b y(b) + \beta_b y'(b) = 0. \tag{6.1}$$

Clearly, the Robin Problem includes Dirichlet and Neumann problems as special cases.

Remark: In many cases we may also take either $a = -\infty$ or $b = \infty$ subject to suitable limits on y or y' and, of course, taking the corresponding end to be open.

6.2 The Solution

If we return to our general solution of the form Equation (4.1) it is now natural to take $x_\alpha = b$, $x_\beta = a$ and $\alpha = \beta = 0$ so

$$y_P(x) = y(x) = -u(x)\int_b^x \frac{v(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s + v(x)\int_a^x \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s,$$

and correspondingly

$$y'(x) = -u'(x)\int_b^x \frac{v(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s - \cancel{u(x)\frac{v(x)r(x)}{p_2(x)W(x)}} + v'(x)\int_a^x \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s + \cancel{v(x)\frac{u(x)r(x)}{p_2(x)W(x)}},$$

so that

$$0 = \alpha_a y(a) + \beta_a y'(a) = \big(\alpha_a u(a) + \beta_a u'(a)\big)\int_a^b \frac{v(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s$$

$$0 = \alpha_b y(b) + \beta_b y'(b) = \big(\alpha_b v(b) + \beta_b v'(b)\big)\int_a^b \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s$$

Hence we may satisfy the boundary conditions on $y(x)$ by choosing the homogeneous solutions such that $u(x)$ satisfies the boundary condition at a (e.g. $u(a) = \beta_a$, $u'(a) = -\alpha_a$) and $v(x)$ the boundary condition at b.

Remark: A potential problem arises here in that we need $u(x)$ and $v(x)$ to be linearly independent and this is not ensured by our construction – we will return to this point later.

Definition 6.1 *Assume that $u(x)$ and $v(x)$ are linearly independent. Then we may write our solution as*

$$y(x) = \int_a^b G(x,s)r(s)\,\mathrm{d}s = \int_a^x G_1(x,s)r(s)\,\mathrm{d}s + \int_x^b G_2(x,s)r(s)\,\mathrm{d}s$$

where

$$G(x,s) = \begin{cases} G_1(x,s) = \dfrac{v(x)u(s)}{p_2(s)W(s)} & a \le s < x \le b \\[2ex] G_2(x,s) = \dfrac{u(x)v(s)}{p_2(s)W(s)} & a \le x < s \le b \end{cases} \tag{6.2}$$

$G(x,s)$ is called the Green's function for the Boundary-Value Problem.

Remark: In the case that L is self-adjoint then $p_2(s)W(s) =$ constant, $G(x, s)$ is symmetric, and Equation (6.2) simplifies greatly.

Theorem 6.1 *The Green's Function has the following properties:*

1. *Defined for* $a \le x \le b$, $a \le s \le b$;
2. $L_x[G] = 0$;
3. $G(x, x-) = G(x, x+)$;
4. $\dfrac{\partial G}{\partial x}(x, x-) - \dfrac{\partial G}{\partial x}(x, x+) = \dfrac{1}{p_2(x)}$;
5. $\alpha_a G(a, s) + \beta_a \dfrac{\partial G}{\partial x}(a, s) = 0$ *and* $\alpha_b G(b, s) + \beta_b \dfrac{\partial G}{\partial x}(b, s) = 0$.

Properties 1-3 and 5 are inherited from $u(x)$ and $v(x)$; property 4 follows from direct differentiation. Here, and as before, we use L_x to denote that we are thinking of L as a differential operator in x with s held fixed, and $x-$ to denote the (one-sided) limit s tends up to x and likewise for $x+$.

Remark: These properties may instead be used as axioms to define $G(x, s)$.

6.2.1 Worked Example

We solve the BVP

$$y''(x) + y(x) = r(x)$$

subject to the boundary condition $y(0) = 0$ and $y(\frac{\pi}{2}) = 0$.

Clearly $u(x) = \sin x$ and $v(x) = \cos x$ are solutions of the homogeneous equations satisfying the appropriate boundary conditions and $W(x) = -1$ (constant since L is self-adjoint and $p_2(x) = 1$). Our solution to the inhomogeneous problem is then,

$$y(x) = -\sin x \int_x^{\frac{\pi}{2}} \cos(s)\, r(s)\, \mathrm{d}s - \cos x \int_0^x \sin(s)\, r(s)\, \mathrm{d}s$$

As an example, if $r(x) = \sin x$

$$\begin{aligned} y(x) &= -\sin x \left[-\tfrac{1}{2}\cos^2 s\right]_x^{\frac{\pi}{2}} - \cos x \left[\tfrac{1}{2}s - \tfrac{1}{4}\sin 2s\right]_0^x \\ &= -\tfrac{1}{2}\sin x \cos^2 x - \tfrac{1}{2}x\cos x + \tfrac{1}{2}\sin x\cos^2 x \\ &= -\tfrac{1}{2}x\cos x, \end{aligned}$$

which can be checked by direct substitution.

6.3 The BVP alternative

Let us return to the question of our method when $u(x)$ and $v(x)$ are linearly dependent. That it does not can be seen by a slight modification of our last example:

6.3.1 Worked Example

Solve the BVP $y''(x) + y(x) = r(x)$ subject to the boundary condition $y(0) = 0$ and $y(\pi) = 0$.

Clearly, the general solution of the homogeneous ODE is $y(x) = A\sin x + B\cos x$, then for $u(0) = 0$ we require $0 = B$, while for $v(\pi) = 0$ we require $0 = -B$ leaving us with (up to irrelevant normalization) $u(x) = \sin x = v(x)$. Since the Wronskian which appears in the denominator vanishes our previous solution does not work.

The problem is that $u(x)$ is a solution to the *homogeneous* BVP. Clearly if we can find any solution of the inhomogeneous BVP we could add an arbitrary multiple of $u(x)$ and still have a solution.

6.3.2 Progress

To make progress assume instead that $u(x)$ is a solution that satisfies the ODE and *both* the corresponding boundary condition at a and that at b and let $v(x)$ be any solution linearly independent of $u(x)$. Note that this implies that $\alpha_b v(b) + \beta_b v'(b) \neq 0$. To see this, note that because

$$0 \neq W(b) = u(b)v'(b) - u'(b)v(b)$$

Hence,

$$\begin{aligned} 0 \neq \alpha_b W(b) &= [\alpha_b u(b)]\, v'(b) - [\alpha_b u'(b)]\, v(b), \\ &\overset{\text{BC}}{=} [-\beta_b u'(b)]\, v'(b) - [\alpha_b u'(b)]\, v(b), \\ &= -u'(b)\, [\beta_b v'(b) + \alpha_b v(b)]\, . \end{aligned}$$

As $u'(b) \neq 0$ in general, we have $\beta_b v'(b) + \alpha_b v(b) \neq 0$ also.

Now we want

$$0 = \alpha_a y(a) + \beta_a y'(a) = \big(\alpha_a u(a) + \beta_a u'(a)\big) \int_a^b \frac{v(s)r(s)}{p_2(s)W(s)}\, \mathrm{d}s$$

$$0 = \alpha_b y(b) + \beta_b y'(b) = -\big(\alpha_b v(b) + \beta_b v'(b)\big) \int_a^b \frac{u(s)r(s)}{p_2(s)W(s)}\, \mathrm{d}s.$$

The first of these holds true by the choice of $u(x)$ while we have just shown that the prefactor of the integral in the second is non-zero and so it can only hold if

$$0 = \int_a^b \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s.$$

which is a restriction on the 'source' $r(x)$. This is therefore a necessary condition for the existence of a solution and even if it holds, as noted above, the solution is not uniquely determined as we can add an arbitrary multiple of $u(x)$. We have therefore proved the following theorem called the **BVP Alternative.**

Theorem 6.2 *Take the ODE $L[y] = r$, with BCs* (6.1). *Suppose that u and v are solutions of the homogeneous problem $L[y] = 0$; u satisfies both BCs, and v is a second linearly independent solution. Then, either*

- $\displaystyle\int_a^b \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s \neq 0$ *and we have* no solutions*;*
- *Or,* $\displaystyle\int_a^b \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s = 0$ *and we have* infinitely many solutions *differing by multiples of the homogeneous solution of the ODE $u(x)$.*

6.3.3 Another Worked Example

We look at the BVP $y''(x)+y(x) = r(x)$ subject to the boundary condition $y(0) = 0$ and $y(\pi) = 0$, where (a) $r(x) = 1$ and (b) $r(x) = \sin 2x$.

As shown above we have $u(x) = \sin x$, which satisfies the homogeneous ODE and both BCs. A second linearly independent solution of the homogeneous problem is then $v(x) = \cos x$, yielding $W(x) = -1$.

We proceed by the 'traditional' method of guessing a particular solution:

- Case (a) Clearly, $y_P(x) = 1$ is a particular solution so the general solution is $y(x) = 1 + A\sin x + B\cos x$ and imposing the boundary conditions gives

$$0 = y(0) = 1 + B$$
$$0 = y(\pi) = 1 - B$$

  which are clearly inconsistent so no solution exists.

- Case (b) Only slightly less clearly, $y_P(x) = -\frac{1}{3}\sin 2x$ is a particular solution so the general solution is $y(x) = -\frac{1}{3}\sin 2x + A\sin x + B\cos x$ and imposing the boundary conditions gives

$$\begin{aligned} 0 &= y(0) = B \\ 0 &= y(\pi) = -B \end{aligned}$$

giving $B = 0$ but leaving A undetermined in line with our previous observations.

If we now compare to our BVP Alternative:

- Case (a) $\displaystyle\int_0^\pi \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s = -\int_0^\pi \sin s.1\,\mathrm{d}s = 2$ so we are in the 'no solutions' case.
- Case (b) $\displaystyle\int_0^\pi \frac{u(s)r(s)}{p_2(s)W(s)}\,\mathrm{d}s = -\int_0^\pi \sin s.\sin 2s\,\mathrm{d}s = -\left[\tfrac{2}{3}\sin^3 s\right]_0^\pi = 0$ so we are in the 'infinitely many solutions' case.

Chapter 7

Fredholm Integral Equations: Introduction

In Chapter 4 we transformed the IVP into a Volterra integral equation. Now we will consider a similar transformation in the case of a BVP, looking specifically at the **self-adjoint case**: For this purpose, as noted in the IVP case, it is convenient to view this as the inhomogeneous equation

$$\big(p_2(x)y'(x)\big)' = r(x) - p_0(x)y(x).$$

The corresponding homogeneous case is therefore $[p_2(x)y'(x)]' = 0$, with solutions 1 and $P(x) = \int^x \mathrm{d}s/p_2(s)$. We will show how the solution of this problem can be written as:

$$y(x) = F(x) + \lambda \int_a^b K(x,s)y(s)\,\mathrm{d}s, \tag{7.1}$$

which is called the Fredholm Integral Equation. We will show how the integral equation can be solved in special cases where the kernel function $K(x,s)$ has a simple form.

7.1 The Fredholm Integral Equation

Using the same idea and notation as in Chapters 4 and 6 we may rewrite our BVP as the integral equation

$$y(x) = \int_a^b G(x,s)\,[r(s) - p_0(s)y(s)]\,\mathrm{d}s,$$

with $u(x) = P(x)$ and $v(x) = 1$ with Wronskian $p_2(s)W(s) = -1$ so

$$G(x,s) = \begin{cases} -P(s) & a \leq s < x \leq b \\ \\ -P(x) & a \leq x < s \leq b \end{cases}$$

which is of the form

$$y(x) = F(x) + \int_a^b K(x,s)y(s)\,\mathrm{d}s, \tag{7.2}$$

However, in applications, p_0 is usually multiplied by a free parameter λ, so we let $p_0(s) \to \lambda p_0(s)$ and hence, we consider:

$$y(x) = F(x) + \lambda \int_a^b K(x,s)y(s)\,\mathrm{d}s. \tag{7.3}$$

This is precisely the Fredholm Integral Equation (FIE).

7.2 Separable kernels

We first of all look at the special case where the kernel $K(x,s)$ is separable, and takes the form:

$$K(x,s) = \sum_{j=1}^{n} u_j(x)v_j(s).$$

An example would be $K(x,s) = \sin(x+s) = \sin x \cos s + \cos x \sin s$.

In this case, the FIE becomes:

$$y(x) = F(x) + \lambda \sum_{j=1}^{n} u_j(x) \underbrace{\int_a^b v_j(s)y(s)\,\mathrm{d}s}_{=c_j}. \tag{7.4}$$

Hence:

$$y(x) = F(x) + \lambda \sum_{j=1}^{n} c_j u_j(x).$$

We test both sides of the equation with $v_(x)$:

$$\underbrace{\int_a^b v_i(x)y(x)\,\mathrm{d}x}_{=c_i} = \underbrace{\int_a^b v_i(x)F(x)\,\mathrm{d}s}_{=b_i} + \lambda \sum_{j=1}^{n} c_j \underbrace{\int_0^1 v_i(x)u_j(x)\mathrm{d}x}_{A_{ij}}.$$

Here, we have defined:

$$b_i = \int_a^b v_i(s)F(s)\,\mathrm{d}s, \qquad A_{ij} = \int_a^b v_i(s)u_j(s)\,\mathrm{d}s.$$

This may be written in matrix form as $\mathbf{c} = \mathbf{b} - \lambda\mathbf{A}\mathbf{c}$ or

$$(\mathbb{I} - \lambda\mathbf{A})\mathbf{c} = \mathbf{b}.$$

The **homogeneous case** has $\mathbf{b} = 0$. In this case, standard (or nearly standard) theory from Linear Algebra applies. In the homogeneous case, non-trivial solutions to the FIE are possible if, and only if,

$$\det(\mathbb{I} - \lambda\mathbf{A}) = 0. \tag{7.5}$$

Notice:

- The roots of the characteristic equation (7.5) are the inverse-eigenvalues of the matrix $\mathbf{A}$.
- The roots of the characteristic equation (7.5) are called the **eigenvalues** of the FIE.

It is possible to theorize what the solutions of the FIE look like in the inhomogeneous case, but it is more helpful to do this by way of a worked example in the first instance.

7.3 Worked Example

We consider the kernel

$$K(x,s) = u_1(x)v_1(s) + u_2(x)v_2(s),$$

where $u_1(x) = 1$, $u_2(x) = -3x$, $v_1(s) = 1$, and $v_2(s) = s$, hence

$$K(x,s) = 1 - 3xs.$$

The Fredholm integral which we look at is therefore:

$$
\begin{aligned}
y(x) &= F(x) + \lambda \int_0^1 (1 - 3xs) y(s)\,\mathrm{d}s \\
&= F(x) + \lambda(c_1 - 3c_2 x)
\end{aligned}
$$

with

$$
c_1 = \int_0^1 y(s)\,\mathrm{d}s, \qquad c_2 = \int_0^1 s y(s)\,\mathrm{d}s
$$

We follow the procedure outlined previously and obtain:

$$
\begin{aligned}
c_1 &= b_1 + \lambda(c_1 - \tfrac{3}{2}c_2) \\
c_2 &= b_2 + \lambda(\tfrac{1}{2}c_1 - c_2)
\end{aligned}
$$

where

$$
b_1 = \int_0^1 F(s)\,\mathrm{d}s, \qquad b_2 = \int_0^1 sF(s)\,\mathrm{d}s.
$$

In matrix form

$$
\begin{pmatrix} 1-\lambda & \frac{3}{2}\lambda \\ -\frac{1}{2}\lambda & 1+\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{7.6}
$$

Correspondingly

$$
\det(\mathbb{I} - \lambda \mathbf{A}) = 1 - \lambda^2 + \tfrac{3}{4}\lambda^2 = 1 - \tfrac{1}{4}\lambda^2,
$$

and the eigenvalues are $\lambda = 2$ and $\lambda = -2$.

7.3.1 Homogeneous Case

We look at the homogeneous case first, with $b_1 = b_2 = 0$. If $\lambda \neq \pm 2$, then Equation (7.6) has only the trivial solution $c_1 = c_2 = 0$, hence, the FIE has the solution $y(x) = F(x)$. We therefore look at the cases with $\lambda = \pm 2$.

Case 1: In the case $\lambda = 2$, we have:

$$
\begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0.
$$

Hence, $c_1 = 3c_2$. We subtitute back into:

$$\begin{aligned} y(x) &= \lambda\left[c_1u_1(x) + c_2u_2(x)\right], \\ &= \lambda\left[3c_2(1) + c_2(-3x)\right], \\ &= \lambda(3c_2)\left(1 - x\right), \\ &= \alpha y_2(x). \end{aligned}$$

Here, $\alpha = \lambda(3c_2) = 6c_2$ is a free parameter, and $y_2(x) = 1 - x$ is an eigenfunction of the FIE:

$$y_2(x) = 2\int_0^1 K(x,s)y_2(s)\mathrm{d}s.$$

Case 2: In the case $\lambda = -2$, we have:

$$\begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0.$$

Hence, $c_1 = c_2$, and:

$$\begin{aligned} y(x) &= \lambda\left[c_1u_1(x) + c_2u_2(x)\right], \\ &= \lambda\left[c_2(1) + c_2(-3x)\right], \\ &= \lambda(c_2)\left(1 - 3x\right), \\ &= \alpha y_{-2}(x). \end{aligned}$$

Here, $\alpha = \lambda(c_2) = -2c_2$ is a free parameter, and $y_{-2}(x) = 1 - 3x$ is an eigenfunction of the FIE:

$$y_{-2}(x) = -2\int_0^1 K(x,s)y_{-2}(s)\mathrm{d}s.$$

7.3.2 Inhomogeneous Case

If $\lambda \neq \pm 2$, then Equation (7.6) has a unique solution:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1-\lambda & \frac{3}{2}\lambda \\ -\frac{1}{2}\lambda & 1+\lambda \end{pmatrix}^{-1}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{7.7}$$

and correspondingly, the FIE has a unique solution also. This case is obvious. So we next look at the case when $\lambda = \pm 2$.

Case 1: In the case $\lambda = 2$, we have:

$$\begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

so the system is incompatible (no solution) unless $b_1 = b_2$, that is unless

$$\int_0^1 F(s)\,\mathrm{d}s = \int_0^1 sF(s)\,\mathrm{d}s \qquad \text{or, equivalently} \qquad \int_0^1 (1-s)F(s)\,\mathrm{d}s = 0.$$

In other words, the compatibility condition is:

$$\int_0^1 F(s)y_2(s)\mathrm{d}s = 0.$$

Under this condition, we have:

$$-c_1 + 3c_2 = b_1 = b_2 \implies c_1 = 3c_2 - b_1. \tag{7.8}$$

We now construct the solution $y(x)$:

$$\begin{aligned} y(x) \quad &= \quad F(x) + \underbrace{\lambda}_{=2}\left[c_1 + c_2(-3x)\right], \\ &\overset{\text{Eq. (7.8)}}{=} \quad F(x) + \lambda\left[(3c_2 - b_1) + c_2(-3x)\right], \\ &= \quad F(x) - \lambda b_1 + \lambda\left[3c_2 - 3c_2 x\right], \\ &= \quad F(x) - \lambda b_1 + \lambda(3c_2)\left(1 - x\right), \\ &= \quad \underbrace{F(x) - \lambda b_1}_{=\widetilde{F(x)}} + \alpha y_2(x). \end{aligned}$$

Thus, the solution is:

$$y(x) = \underbrace{F(x) - \lambda b_1}_{=\widetilde{F(x)}} + \alpha y_2(x), \tag{7.9}$$

that is, a function $\widetilde{F(x)}$, plus an arbitrary multiple of an eigenfunction.

It is illustrative to validate the solution (7.9). Hence, with $\lambda = 2$ and $y(x) = F(x) - \lambda b_1 + \alpha y_2(x)$, we want to show that:

$$\lambda \int_0^1 K(x,s)y(s) = y(x) - F(x).$$

We have:

$$
\begin{aligned}
\text{LHS} &= \lambda \int_0^1 K(x,s)y(s)\mathrm{d}s, \\
&= 2\int_0^1 K(x,s)\left[\widetilde{F(s)} + \alpha y_2(s)\right]\mathrm{d}s, \\
&= 2\int_0^1 K(x,s)\widetilde{F(s)}\mathrm{d}s + 2\left[\alpha \int_0^1 K(x,s)y_2(s)\mathrm{d}s\mathrm{d}s\right], \\
&\overset{\text{Eigenfunction}}{=} 2\int_0^1 K(x,s)\widetilde{F(s)}\mathrm{d}s + \alpha y_2(s), \\
&\overset{\text{Eq. (7.9)}}{=} 2\int_0^1 K(x,s)\widetilde{F(s)}\mathrm{d}s + y(x) - \widetilde{F(x)}, \\
&= 2\int_0^1 K(x,s)\widetilde{F(s)}\mathrm{d}s + \lambda b_1 + [y(x) - F(x)]\,.
\end{aligned}
$$

So, if we can show that $2\int_0^1 K(x,s)\widetilde{F(s)}\mathrm{d}s + \lambda b_1 = 0$, we are done. We have:

$$
\begin{aligned}
2\int_0^1 K(x,s)\widetilde{F(s)}\mathrm{d}s + \lambda b_1 &= 2\int_0^1 (1-3xs)\left[F(s) - 2b_1\right] + \lambda b_1, \\
&= 2\left(b_1 - 2b_1 - 3xb_2 + 3xb_1\right] + 2b_1, \\
&= 0.
\end{aligned}
$$

Here, we have used compatibility, $b_1 = b_2$, and the eigenvalue condition $\lambda = 2$. Thus,

$$
\begin{aligned}
\text{LHS} &= 0 + y(x) - F(x), \\
&= \text{RHS}.
\end{aligned}
$$

This completes the proof.

Case 2: In the case $\lambda = -2$

$$
\begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.
$$

so the system is incompatible (no solution) unless $b_1 = 3b_2$, that is unless

$$
\int_0^1 F(s)\,\mathrm{d}s = 3\int_0^1 sF(s)\,\mathrm{d}s \qquad \text{or, equivalently} \qquad \int_0^1 (1-3s)F(s)\,\mathrm{d}s = 0
$$

In other words, the compatibility condition is:

$$
\int_0^1 F(s)y_{-2}(s)\mathrm{d}s = 0.
$$

Under this condition, we have:

$$c_1 = c_2 + b_2. \tag{7.10}$$

We now construct the solution $y(x)$:

$$\begin{aligned} y(x) &= F(x) + \underbrace{\lambda}_{=-2} \left[c_1 + c_2(-3x)\right], \\ &\overset{\text{Eq. (7.10)}}{=} F(x) + \lambda \left[(c_2 + b_2) + c_2(-3x)\right], \\ &= F(x) + \lambda b_2 + \lambda c_2 \left(1 - 3x\right), \\ &= F(x) + \lambda b_2 + \lambda(3c_2) \left(1 - x\right), \\ &= \underbrace{F(x) + \lambda b_2}_{=\widetilde{F(x)}} + \alpha y_{-2}(x). \end{aligned}$$

Again, the solution is a function $\widetilde{F(x)}$, plus an arbitrary multiple of an eigenfunction.

7.4 General Results

It is possible now to put some results together for a general separable kernel

$$K(x, s) = \sum_{j=1}^{n} u_j(x) v_j(s). \tag{7.11}$$

- The FIE is:

$$y(x) = F(x) + \lambda \int_a^b K(x, s) y(s) \mathrm{d}s. \tag{7.12a}$$

- By projecting on to the $v_j(s)$ functions, this becomes:

$$\mathbf{c} = \mathbf{b} + \lambda \mathbf{A} \mathbf{c}. \tag{7.12b}$$

- The eigenvalues of the FIE solve the characteristic polynomial

$$\det\left(\mathbb{I} - \lambda \mathbf{A}\right) = 0. \tag{7.12c}$$

  Or,

$$\lambda \in \{\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(n)}\} = \mathcal{S}. \tag{7.12d}$$

This gives rise to a number of cases.

1. Homogeneous case.
   - If $\lambda \notin \mathcal{S}$, then the FIE has a unique solution – the **trivial solution** $y(x) = F(x)$.

- If $\lambda \in \mathcal{S}$, then the FIE has infinitely many solutions:

$$y(x) = \alpha \sum_{j=1}^{n} u_j(x) c_j^{(p)}.$$

Here, the p labels an eigenvector of Equation (7.12b). In other words,

$$y(x) = \alpha y^{(p)}(x),$$

where

$$y^{(p)}(x) = \lambda^{(p)} \int_a^b K(x,s) y^{(p)}(s) \mathrm{d}s.$$

2. General homogeneous case.

- If $\lambda \notin \mathcal{S}$, then the FIE has a unique solution.
- If $\lambda \in \mathcal{S}$, **and a compatibility condition is met**, then the FIE has infinitely many solutions:

$$y(x) = [F(x) - \cdots] + \alpha y^{(p)}(x), \tag{7.13}$$

where the compatibility condition is given by:

$$\int_a^b F(s) y^{(p)}(s) \mathrm{d}s = 0. \tag{7.14}$$

3. Special homogeneous case. In this case, $F(x) \neq 0$, but $\int_a^b F(x) v_i(x) \mathrm{d}x = 0$ for each $i \in \{1, 2, \cdots\}$. This case is similar to before:

- If $\lambda \notin \mathcal{S}$, then the FIE has a unique solution.
- If $\lambda \in \mathcal{S}$, and the compatibility condition (7.14) is met, then the FIE has infinitely many solutions:

$$y(x) = F(x) + \alpha y^{(p)}(x),$$

This is a nice, general theory. Of course, the $\cdots$ in Equation (7.13) is not satisfying, however, these extra terms can be written down precisely using eigenfunctions of the FIE. We look at this idea in the next chapter.

Chapter 8

Fredholm Integral Equations: General Case

Overview

In the last section, we looked at solutions of the Fredholm Integral Equation,

$$y(x) = F(x) + \lambda \int_a^b K(x,s) y(s) \, \mathrm{d}s, \tag{8.1}$$

in cases where the kernel function $K(x,s)$ was separable. Now we look at non-separable kernels. In this case, the integral equation has infinitely many eigenvalues. This general case is very relevant when $K(x,s)$ is given by different analytic expressions in the intervals $x < s$ and $s > x$. This theory for non-separable kernels also goes by the name of Hilbert–Schmidt Theory.

8.1 Worked Example

We consider the BVP

$$x^2 y''(x) + xy'(x) + (\lambda x^2 - 1)y(x) = 0, \qquad y(0) = 0, \quad y(1) = 0$$

We start by putting the equation into self-adjoint form and move the λ-term to the right hand side

$$\big(xy'(x)\big)' - \frac{1}{x} y(x) = -\lambda x y(x).$$

Denoting the left hand side as $L[y]$, the general solution of the homogeneous problem $L[y] = 0$ is $y(x) = Ax + Bx^{-1}$ so we may take $u(x) = x$ and $v(x) = x - x^{-1}$ as solutions satisfying the boundary conditions at either end. The Wronskian is then $2/x$ so the Green function is

$$G(x,s) = \begin{cases} \frac{1}{2}s(x - x^{-1}) & 0 \le s < x \le 1 \\ \\ \frac{1}{2}x(s - s^{-1}) & 0 \le x < s \le 1 \end{cases}$$

and the BVP is equivalent to

$$y(x) = \lambda \int_0^1 G(x,s)sy(s)\,\mathrm{d}s.$$

On the other hand with the change of variable $z = \sqrt{\lambda}x$ we may recognise this as Bessel's equation of order 1. The solution regular at $z = x = 0$ is then $J_1(z) = J_1(\sqrt{\lambda}x)$ which is plotted in Figure 8.1. We can see that we will have non-trivial solution if and only if $J_1(\sqrt{\lambda})$. $J_1(z)$ has infinitely many

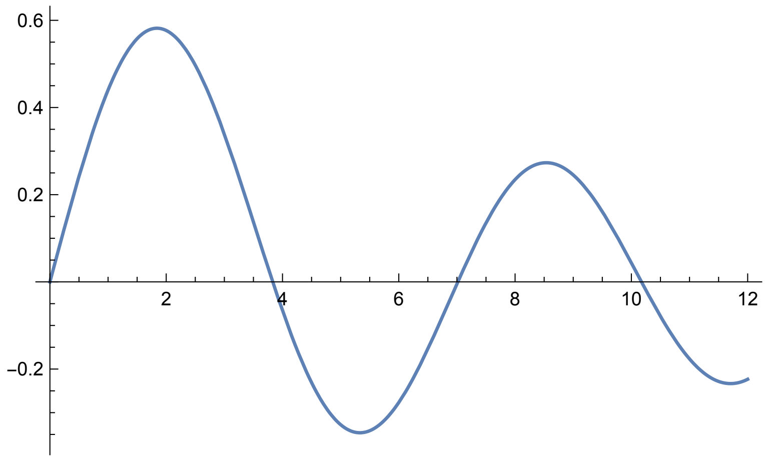


Figure 8.1: The regular Bessel function of order 1, $J_1(z)$.

zeroes typically denoted by $j_{1,n}$ and correspondingly

$$\lambda_n = j_{1,n}^2, \qquad n = 1, 2, 3, \dots.$$

8.2 Symmetric Kernels

We noted previously that for a self-adjoint 2nd order BVP the Green function is symmetric if the corresponding ODE is self-adjoint. Correspondingly, we start by considering the important case which appears naturally in the formulation of physically motivated problems that the kernel $K(x,s)$ is symmetric:

$$K(x,s) = K(s,x).$$

We start by considering solutions of the homogeneous Fredholm problem $f(x) = 0$ so

$$y(x) = \lambda \int_a^b K(x,s)y(s)\,\mathrm{d}s.$$

The equation has special solutions for particular values of λ – eigenfunctions and eigenvalues:

$$y_i(x) = \lambda_i \int_a^b K(x,s)y_i(s)\,\mathrm{d}s, \tag{8.2}$$

Remark: We assume that eigensolutions of the form (8.2) exist. In a later chapter we look at the existence theory. But for now we assume that such eigensolutions exist, and just do computations.

To simplify the notation we introduce the integral operator $\mathcal{K} : C^0[a,b] \to C^0[a,b]$ defined by

$$\mathcal{K}f(x) = \int_a^b K(x,s)f(s)\,\mathrm{d}s.$$

This is also written as a 'convolution', $K * f = \int_a^b K(x,s)f(s)\,\mathrm{d}s$.

8.2.1 Analogy with linear algebra

In linear algebra, a linear operator A is a matrix,

$$\begin{aligned} A : \mathbb{R}^n &\to \mathbb{R}^n, \\ \boldsymbol{x} &\mapsto A\boldsymbol{x}. \end{aligned}$$

The linear operator A on the vector $\boldsymbol{x}$ can be worked out using summation:

$$(A\boldsymbol{x})_i = \sum_{j=1}^{n} A_{ij}x_i.$$

Now, imagine approximating a function $f(x)$ on an interval $[a,b]$ by its value at equally-spaced discrete points $\boldsymbol{x} = \{x_1, \cdots, x_n\}$, with each $x_i \in [a,b]$, and $\Delta x = x_{i+1} - x_i$. Thus, we would have a vector $\{f(x_1), \cdots, f(x_n)\}$. We could then look at a kernel function $K(x,y)$ and correspondingly,

$$(K\boldsymbol{x})_i = \sum_{j=1}^{n} K(x_i, x_j)f(x_j)\Delta x.$$

By taking the limit as $\Delta x \to 0$ and $n \to \infty$, we would have:

$$\mathcal{K}f(x) = \int_a^b K(x,s)f(s)\mathrm{d}s.$$

Thus, the kernel function acting on a function is analogous to a matrix acting on a vector in $\mathbb{R}^n$. Just as a matrix has eigenvalues and eigenvectors, a kernel function will have eigenvalues and eigenfunctions. We just have to be a bit careful, as the eigenvalue problem for integral operators is written as $f = \lambda \mathcal{K} f$ whereas the eigenvalue problem for linear algebra is written as $\lambda \boldsymbol{x} = A\boldsymbol{x}$.

8.2.2 Properties of the Eigenfunctions for a Symmetric Kernel

We look at two distinct eigensolutions $y_i(x)$ and $y_j(x)$ corresponding to different eigenvalues λ_i and λ_j:

$$y_i(x) = \lambda_i \int_a^b K(x,s) y_i(s)\,\mathrm{d}s,$$
$$y_j(x) = \lambda_j \int_a^b K(x,s) y_j(s)\,\mathrm{d}s.$$

We may then prove a number of important (and familiar results).

Theorem 8.1 *Eigenfunctions corresponding to different eigenvalues are orthogonal, in the sense that:*

$$\int_a^b y_i(x) y_j(x)\,\mathrm{d}x = 0, \qquad i \neq j.$$

Proof: Consider, focusing on $y_j(x)$

$$\begin{aligned}\int_a^b y_i(x) y_j(x)\,\mathrm{d}x &= \int_a^b y_i(x) \left[\lambda_j \int_a^b K(x,s) y_j(s)\,\mathrm{d}s\right] \mathrm{d}x \\ &= \lambda_j \int_a^b \int_a^b y_i(x) K(x,s) y_j(s)\,\mathrm{d}s\,\mathrm{d}x\end{aligned}$$

so, recalling that $\lambda \neq 0$:

$$\int_a^b \int_a^b K(x,s) y_i(x) y_j(s)\,\mathrm{d}s\,\mathrm{d}x = \frac{1}{\lambda_j} \int_a^b y_i(x) y_j(x)\,\mathrm{d}s.$$

But the same argument works equally as well focusing on $y_i(x)$ so

$$\int_a^b \int_a^b K(x,s) y_i(s) y_j(x)\,\mathrm{d}s\,\mathrm{d}x = \frac{1}{\lambda_i} \int_a^b y_i(x) y_j(x)\,\mathrm{d}s.$$

Now since the kernel is symmetric we may interchange x and s on the left hand side so subtracting

$$0 = \left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j}\right) \int_a^b y_i(x) y_j(x)\, \mathrm{d}s,$$

and since by assumption $\lambda_i \neq \lambda_j$ we conclude that

$$\int_a^b y_i(x) y_j(x)\, \mathrm{d}x = 0,$$

that is, $y_i(x)$ and $y_j(x)$ are orthogonal in the inner product (norm) defined by

$$\langle u, v \rangle = \int_a^b u(x) v(x)\, \mathrm{d}x,$$

we will return to show that this is indeed a norm over an appropriate space of functions, and discuss its properties later.

If two or more linearly independent eigenfunctions correspond to the same eigenvalue then, using the standard Gram-Schmidt process we may construct a orthogonal linear combinations of them and henceforth we shall assume that this process has been completed when such exceptional cases arise.

Theorem 8.2 *The eigenvalues of a Fredholm equation with a real symmetric kernel are all real.*

Proof: Suppose that λ_i is an complex eigenvalue corresponding to a complex eigenfunction $y_i(x)$ then correspondingly the complex conjugate number λ_i^* would necessarily also be an eigenvalue with eigenfunction $y_i^*(x)$. Hence taking $\lambda_j = \lambda_i^*$ in our previous result

$$\begin{aligned} 0 &= \left(\frac{1}{\lambda_i} - \frac{1}{\lambda_i^*}\right) \int_a^b y_i(x) y_i^*(x)\, \mathrm{d}x \\ &= \left(\frac{\lambda_i^* - \lambda_i}{|\lambda_i|^2}\right) \int_a^b \left|y_i(x)\right|^2 \mathrm{d}x. \end{aligned}$$

As an eigenfunction, by definition, cannot be zero we conclude that $\lambda_i^* = \lambda_i$, i.e., that the eigenvalues are real. ∎

8.3 Expansion in terms of eigenfunctions

In the later chapters of these notes, we will show that any function $g(x)$ which can be written as a convolution,

$$g(x) = \int_a^b K(x,s)r(s)\,\mathrm{d}s,$$

where $K(x,s)$ is continuous, real and symmetric for some continuous function $r(s)$ can be represented over the interval (a,b) by a linear sum of the eigenfunctions of the homogeneous Fredholm integral equation

$$y(x) = \lambda \int_a^b K(x,s)y(s)\,\mathrm{d}s.$$

We take this fact as given for now. Hence, we write

$$g(x) = \sum_i c_i y_i(x) \qquad\qquad x \in (a,b)$$

Because of the orthogonality, the coefficients in this representation of $g(x)$ are determined by the formula

$$c_i \int_a^b \big(y_i(x)\big)^2\,\mathrm{d}x = \int_a^b g(x)y_i(x)\,\mathrm{d}x.$$

In cases where only a finite number of eigenvalues exist the functions generated by this process are very restricted. For example, if $K(x,s) = 1 - 3xs$ then

$$\begin{aligned}\int_a^b K(x,s)r(s)\,\mathrm{d}s &= \int_a^b (1-3xs)r(s)\,\mathrm{d}s \\ &= 1.\int_a^b r(s)\,\mathrm{d}s + x.\int_a^b (3s)r(s)\,\mathrm{d}s\end{aligned}$$

so only functions of the form $A + Bx$ can be generated. In this case it is clear that we can expand such a solution in terms of the eigenfunctions $y_2(x) = (1-x)$ and $y_{-2}(x) = (1-3x)$ we found in Section 7.2.

Even when the number of independent eigenfunctions is infinite, it is not necessarily true that any continuous function defined over (a,b) can be represented over that interval by a series of these functions, that is, eigenfunctions may not form a complete set.

8.4 Solution of the Inhomogeneous Fredholm Integral Equation

In this Section, we will show how knowledge of the eigenvalues and eigenfunctions of the homogeneous Fredholm problem allow a simple determination of the continuous solution of the corresponding inhomogeneous Fredholm Integral Equation:

$$y(x) = f(x) + \lambda \int_a^b K(x,s)y(s)\,\mathrm{d}s. \tag{8.3}$$

where $f(x)$ is a given continuous real function.

We shall assume that the eigenvalues have been ordered with respect to magnitude, and that if any are degenerate then the corresponding eigenfunctions have been orthogonalised. Indeed, to simplify the formula we will also assume that they have been normalised in the sense that

$$1 = \int_a^b (y_i(x))^2\,\mathrm{d}x.$$

Now writing Equation (8.4) as

$$y(x) - f(x) = \int_a^b K(x,s)\lambda y(s)\,\mathrm{d}s, \tag{8.4}$$

we see that $y(x) - f(x)$ is generated from a continuous function by the kernel and so can be expanded as

$$y(x) - f(x) = \sum_i a_i y_i(x) \qquad\qquad x \in (a,b).$$

where

$$a_i = \int_a^b \big(y(x) - f(x)\big) y_i(x)\,\mathrm{d}x = c_i - f_i.$$

Remark: Just be careful here: it is tempting to think that you can expand the solution $y(x)$ in terms of eigenfunctions but that would be wrong. Instead, it is the difference $y(x) - f(x)$ that is expanded in terms of eigenfunctions.

We make the natural definitions

$$c_i = \int_a^b y(x) y_i(x)\,\mathrm{d}x \qquad\qquad f_i = \int_a^b f(x) y_i(x)\,\mathrm{d}x.$$

We now use Equation (8.4) and we proceed with the calculations

$$\begin{aligned}
c_i - f_i &= \lambda \int_a^b y_i(x) \left[\int_a^b K(x,s) y(s)\,\mathrm{d}s \right] \mathrm{d}x \\
&= \lambda \int_a^b y(s) \left[\int_a^b K(x,s) y_i(x)\,\mathrm{d}x \right] \mathrm{d}s \\
&= \lambda \int_a^b y(s) \left[\int_a^b K(s,x) y_i(x)\,\mathrm{d}x \right] \mathrm{d}s \\
&= \lambda \int_a^b y(s) \frac{1}{\lambda_i} y_i(s)\,\mathrm{d}s \\
&= \frac{\lambda}{\lambda_i} c_i,
\end{aligned}$$

where we have used Fubini's theorem (interchange of order of integration) and the symmetry of the kernel. We conclude that

$$(\lambda_i - \lambda) c_i = \lambda_i f_i \qquad \text{and} \qquad (\lambda_i - \lambda) a_i = \lambda f_i \tag{8.5}$$

and so

$$c_i = \frac{\lambda_i}{\lambda_i - \lambda} f_i \qquad \text{and} \qquad a_i = \frac{\lambda}{\lambda_i - \lambda} f_i, \qquad (\lambda \neq \lambda_i) \tag{8.6}$$

that is our solution is given by

$$y(x) = f(x) + \sum_i \frac{\lambda}{\lambda_i - \lambda} f_i y_i(x) \qquad\qquad x \in (a,b), \qquad (\lambda \neq \lambda_i). \tag{8.7}$$

Notice that while the f_i would be coefficients of the expansion of $f(x)$ in terms of the eigenfunctions $y_i(x)$, nowhere in the proof above did we need to assume such an expansion existed.

8.4.1 The Exceptional Cases

We are left with the exceptional cases where $\lambda = \lambda_i$ for some i. Specifically, let $\lambda = \lambda_{i_0}$. Instead of taking $\lambda = \lambda_{i_0}$ directly in Equation (8.7), we look at a limit $\lambda \to \lambda_{i_0}$. Then we have:

$$y(x) = f(x) + \sum_{i \neq i_0} \frac{\lambda_{i_0}}{\lambda_i - \lambda_{i_0}} f_i y_i(x) + \lim_{\lambda \to \lambda_{i_0}} \frac{\lambda_{i_0}}{\lambda_{i_0} - \lambda} f_{i_0} y_{i_0}(x) \tag{8.8}$$

This gives a contribution of the form $1/0$ unless $f_{i_0} = 0$. Therefore, a solvability condition is that $f_{i_0} = 0$ when $\lambda \to \lambda_{i_0}$. In this case, we can write:

$$f_{i_0} = (\lambda_{i_0} - \lambda)C,$$

where C is an arbitrary constant. This guarantees that $f_{i_0} = 0$. Then, Equation (8.8) becomes:

$$y(x) = f(x) + \sum_{i \neq i_0} \frac{\lambda_{i_0}}{\lambda_i - \lambda_{i_0}} f_i y_i(x) + C y_{i_0}(x) \tag{8.9}$$

Notice that eigenvalue λ_{i_0} which appeared in Equation (8.8) has been buried in the definition of the constant C in Equation (8.9). Notice also that the constant C in Equation (8.9) is arbitrary. This gives us the following *Fredholm Alternative:*

- *Either* $f_{i_0} \neq 0$ in which case *no* solution exists,
- *Or* $f_{i_0} = 0$ in which case *infinitely many* solutions exist differing by an arbitrary multiple of the corresponding eigenfunction $y_{i_0}(x)$.

8.5 Worked Example

We look again at $K(x, s) = 1 - 3xs$. The (normalized) eigenfunctions are $y_2(x) = \sqrt{3}(1 - x)$ and $y_{-2}(x) = (1 - 3x)$. As noted earlier, only functions of the form $A + Bx$ can be generated. Nevertheless we can solve

$$y(x) = x^2 + \lambda \int_0^1 (1 - 3xs) y(s) \, \mathrm{d}s.$$

Reading off from Equation (8.6), we have

$$f_1 = \int_a^b x^2 \, y_2(x) \, \mathrm{d}x = \frac{1}{4\sqrt{3}} \qquad f_2 = \int_a^b x^2 \, y_{-2}(x) \, \mathrm{d}x = -\frac{5}{12},$$

Hence,

$$\begin{aligned} y(x) &= x^2 + \sum_{i=2,-2} \frac{\lambda}{\lambda_i - \lambda} f_i y_i(x), \qquad \lambda \neq \lambda_i, \\ &= x^2 + \frac{\lambda}{2-\lambda} \frac{1}{4\cancel{\sqrt{3}}} \cancel{\sqrt{3}}(1-x) + \frac{\lambda}{2+\lambda} \frac{5}{12}(1-3x), \qquad (\lambda \neq \pm 2). \end{aligned}$$

Hence,

$$y(x) = x^2 + \tfrac{1}{4} \frac{\lambda}{2-\lambda}(1-x) + \tfrac{5}{12} \frac{\lambda}{2+\lambda}(1-3x), \qquad (\lambda \neq \pm 2).$$

8.6 A Variant of the Inhomogeneous Fredholm Integral Equation

A variant of the inhomogeneous Fredholm Integral Equation is the problem of finding $y(x)$ such that

$$0 = f(x) + \lambda \int_a^b K(x,s) y(s) \, \mathrm{d}s. \tag{8.10}$$

which is called a Fredholm IE of *the first kind*. From the result quoted in the previous section a solution is only possible if it is possible to expand $f(x)$ in terms of solutions of the homogeneous Fredholm IE

$$y(x) = \lambda \int_a^b K(x,s) y(s) \, \mathrm{d}s.$$

In this case, we will be able to write

$$f(x) = \sum_i f_i y_i(x) \qquad x \in (a,b).$$

Correspondingly,

$$\begin{aligned} 0 &= \sum_i f_i y_i(x) + \lambda \int_a^b K(x,s) y(s) \, \mathrm{d}s \\ &= \sum_i f_i \lambda_i \int_a^b K(x,s) y_i(s) \, \mathrm{d}s + \lambda \int_a^b K(x,s) y(s) \, \mathrm{d}s \\ &= \int_a^b K(x,s) \left[\sum_i \lambda_i f_i y_i(s) + \lambda y(s) \right] \mathrm{d}s \end{aligned}$$

This requires that

$$y(x) = -\sum_i \frac{\lambda_i}{\lambda} f_i y_i(x) + Y(x)$$

where $Y(x)$ is a solution of the homogeneous equation

$$0 = \lambda \int_a^b K(x,s) Y(s)\,\mathrm{d}s.$$

Multiplying by $y_i(x)$ and using Fubini and symmetry in the now familiar way we deduce that any solution $Y(x)$ must satisfy

$$0 = Y_i = \lambda_i \int_a^b y_i(x) Y(x)\,\mathrm{d}x,$$

for all i. If the set of eigenfunctions is finite then infinitely many such solutions exist. On the other hand if the $y_i(x)$ form a *complete* set then no non-trivial solution can exist.

8.6.1 Worked Example

Suppose $K(x,s) = \sin(x+s) = \sin x \cos s + \cos x \sin s$ on $(0, 2\pi)$ then

$$\lambda \int_0^{2\pi} K(x,s) y(s)\,\mathrm{d}s = \sin x\,\lambda \int_0^{2\pi} \cos s\, y(s)\,\mathrm{d}s + \cos x\,\lambda \int_0^{2\pi} \sin s\, y(s)\,\mathrm{d}s$$

we will clearly only have a solution for functions of the form

$$f(x) = A\sin x + B\cos x. \tag{8.11}$$

Eigenfunctions: The eigenfunctions of the Fredholm operator satisfy $y = \lambda \int_0^{2\pi} K(x,s)y(s)\mathrm{d}s$. These must also be of the form (8.11). We have:

$$\alpha \sin x + \beta \cos x = \lambda \sin x \int_0^{2\pi} \cos(s)\left[\alpha \sin s + \beta \cos s\right]\mathrm{d}s + \lambda \cos x \int_0^{2\pi} \sin s \left[\alpha \sin s + \beta \cos s\right]\mathrm{d}s.$$

Hence:

$$\alpha \sin x + \beta \cos x = \lambda\pi\beta \sin x + \lambda\pi\alpha \cos x.$$

Equating coefficients gives two equations in three unknowns:

$$\begin{aligned} \alpha - \lambda\pi\beta &= 0, \\ \beta - \lambda\pi\alpha &= 0. \end{aligned}$$

In matrix form:

$$\begin{pmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$$

There is no non-trivial solution unless the characteristic equation is satisfied. The characteristic equation is:

$$\begin{vmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{vmatrix} = 0.$$

The eigenvalues are $\lambda^2\pi^2 = 1$, hence $\lambda = \pm 1/\pi$. In the case where $\lambda = 1/\pi$ we get $\alpha = \beta$, and the normalized eigenfunction is:

$$y_1(x) = \frac{1}{\sqrt{2\pi}}(\sin x + \cos x).$$

In the case where $\lambda = -1/\pi$, we get $\alpha = -\beta$, and the normalized eigenfunction can be written as:

$$y_2(x) = \frac{1}{\sqrt{2\pi}}(\sin x - \cos x).$$

Notice that $y_1(x)$ and $y_2(x)$ are orthogonal. We also have:

$$f_1 = \int_0^{2\pi} y_1(x) f(x) \mathrm{d}x = \sqrt{\pi/2}(A + B),$$

and

$$f_2 = \int_0^{2\pi} y_2(x) f(x) \mathrm{d}x = \sqrt{\pi/2}(A - B).$$

Now:

$$\begin{aligned} y(x) &= -\sum_i \frac{\lambda_i}{\lambda} f_i y_i(x) + Y(x), \\ &= -\left\{ \frac{1}{\lambda}\frac{1}{\pi}\sqrt{\frac{\pi}{2}}\,(A + B)\,\frac{1}{\sqrt{2\pi}}\,(\sin x + \cos x) \right. \\ &\qquad \left. - \frac{1}{\lambda}\frac{1}{\pi}\sqrt{\frac{\pi}{2}}\,(A - B)\,\frac{1}{\sqrt{2\pi}}\,(\sin x - \cos x) \right\} + Y(x), \\ &= -\frac{1}{\lambda\pi}\,(B \sin x + A \cos x) + Y(x). \end{aligned}$$

Here, $Y(x)$ is any function orthogonal to $y_1(x)$ and $y_2(x)$, or equivalently, $\sin x$ and $\cos x$ over $(0, 2\pi)$. Clearly there are infinitely many such functions, for example, $\sin nx$ or $\cos nx$ for $n \in \mathbb{N}\backslash\{1\}$. Thus, the general solution is:

$$y(x) = -\frac{1}{\lambda\pi}\,(B \sin x + A \cos x) + \sum_{n=2}^{\infty} \left[a_n \sin(nx) + b_n \cos(nx)\right],$$

where the a_n's and b_n's are arbitrary.

Chapter 9

Perturbative Solution of Fredholm Equations

Overview

We look at an iterative method to solve the Fredholm Integral Equation known in the literature as Neumann (or Liouville–Neumann) series. When certain conditions, e.g., on the kernel, the parameter λ, etc., are satisfied, such series will converge to the solution of the integral equation.

9.1 An Iteration Scheme for Fredholm Equations

The form of a Fredholm Integral Equation (IE) of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,s) y(s)\,\mathrm{d}s.$$

might remind us of our approach to Picard's theorem and suggest an iterative scheme where we start with an initial guess at our solution $y_0(x)$ and then iterate through

$$y_{n+1}(x) = f(x) + \lambda \int_a^b K(x,s) y_n(s)\,\mathrm{d}s.$$

Notice that the initial guess might influence convergence speed but not whether the iteration converges. Also, a standard initial guess is $y_0(x) = f(x)$ (otherwise, you should have some better choice through some numerical or alternative methods!).

Explicitly

$$
\begin{aligned}
y_1(x) &= f(x) + \lambda \int_a^b K(x,s) y_0(s)\,\mathrm{d}s \\
y_2(x) &= f(x) + \lambda \int_a^b K(x,s) y_1(s)\,\mathrm{d}s \\
&= f(x) + \lambda \int_a^b K(x,s) f(s)\,\mathrm{d}s + \lambda^2 \int_a^b K(x,s) \left[\int_a^b K(s,t) y_0(t)\,\mathrm{d}t \right] \mathrm{d}s \\
&= f(x) + \lambda \int_a^b K(x,s) f(s)\,\mathrm{d}s + \lambda^2 \int_a^b \int_a^b K(x,s) K(s,t) y_0(t)\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

We see a crucial difference here, in that while in the first order Picard scheme, or indeed the corresponding Volterra IE where the region of integration on the final line would have reduced to a triangle, here the integral covers the whole square – this will affect the convergence proof.

We again use the integral operator notation for $\mathcal{K} : C^0[a,b] \to C^0[a,b]$, where:

$$
\mathcal{K} g(x) = \int_a^b K(x,s) g(s)\,\mathrm{d}s.
$$

Then we may write

$$
y_{n+1}(x) = f(x) + \mathcal{K} y_n(x),
$$

and correspondingly

$$
\begin{aligned}
y_1(x) &= f(x) + \lambda \mathcal{K} y_0(x) \\
y_2(x) &= f(x) + \lambda \mathcal{K} f(x) + \lambda^2 \mathcal{K}\mathcal{K} y_0(x) \\
&= f(x) + \lambda \mathcal{K} f(x) + \lambda^2 \mathcal{K}^2 y_0(x) \\
&\dots \\
y_n(x) &= f(x) + \lambda \mathcal{K} f(x) + \lambda^2 \mathcal{K}^2 f(x) \dots \lambda^{n-1} \mathcal{K}^{n-1} f(x) + \lambda^n \mathcal{K}^n y_0(x)
\end{aligned}
$$

9.1.1 Convergence

The idea here is that under suitable conditions as we continue this process the final 'remainder' term gets smaller and smaller and we will have a convergent series representation

$$y(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \mathcal{K}^n f(x).$$

The proof is similar to the local version of Picard's theorem: first as $f(x)$, $y_0(x)$ and $K(x,s)$ are assumed continuous there exist constants C, D and M such that

$$\left|f(x)\right| \leq C, \qquad \left|y_0(x)\right| \leq D \qquad \text{and} \qquad \left|K(x,s)\right| \leq M$$

on $[a,b]$, $[a,b]$ and $[a,b]\times[a,b]$, respectively. Then

$$\begin{aligned} |\mathcal{K} y_0(x)| &= \left| \int_a^b K(x,s) y_0(s) \, \mathrm{d}s \right| \\ &\leq \int_a^b |K(x,s)| \, |y_0(s)| \, \mathrm{d}s \leq D\, M |b-a| \end{aligned}$$

and so, by induction,

$$|\mathcal{K}^n y_0(x)| \leq D\, M^n |b-a|^n$$

and correspondingly

$$|\mathcal{K}^n f(x)| \leq C\, M^n |b-a|^n.$$

Thus the 'remainder' term in $y_n(x)$ is bounded by

$$D\, \lambda^n M^n |b-a|^n$$

and so tends to zero as $n \to \infty$ provided that

$$\lambda < \frac{1}{M|b-a|}. \tag{9.1}$$

Indeed, for clarity, we will write the requirement (9.1) as:

$$\lambda \leq \frac{r}{M|b-a|}, \qquad 0 \leq r < 1. \tag{9.2}$$

Note that the limit did not depend on our choice of initial guess $y_0(x)$ (only that it was bounded).

To use the Weirstrass M-test, we take $y(x) = f(x) + \sum_{n=1}^{\infty} u_n(x)$, where $u_n(x) = \lambda^n \mathcal{K}^n f$. Each $u_n(x)$ is bounded by $\mathcal{M}_n = C\lambda^n M^n |b-a|^n$. Furthermore, we have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathcal{M}_n &= C \sum_{n=1}^{\infty} \lambda^n M^n |b-a|^n \\
&\leq C \sum_{n=1}^{\infty} r^n \frac{1}{M^n |b-a|^n} \times M^n |b-a|^n, \\
&= \sum_{n=1}^{\infty} r^n, \\
&= C \left[\frac{1}{1-r} - 1 \right] ., \qquad 0 \leq r < 1.
\end{aligned}$$

Thus, the series converges when Equation (9.1) holds. Hence, we may invoke the Weierstrass M-test to conclude that our limit function is continuous. Although the series might converge it is not straightforward to check that the limit function is indeed a solution.

In order to observe that this series not only converges but *converges to the solution* we look at:

$$\begin{aligned}
y(x) &= f(x) + \lambda \int_a^b K(x,s) y(s) \mathrm{d}s, \\
y_{n+1}(x) &= f(x) + \lambda \int_a^b K(x,s) y_n(s) \mathrm{d}s,
\end{aligned}$$

hence

$$y(x) - y_{n+1}(x) = \lambda \int_a^b K(x,s)[y(s) - y_n(s)] \mathrm{d}s.$$

We choose x to maximize $|y(x) - y_{n+1}(x)|$ over $[a,b]$, i.e. $x = \operatorname{argmax}_{[a,b]} |y(x) - y_{n+1}(x)|$. Hence,

$$\begin{aligned}
\|y(x) - y_{n+1}(x)\|_\infty &= |y(x) - y_{n+1}(x)|, \\
&\leq \lambda \int_a^b K(x,s)[y(s) - y_n(s)] \mathrm{d}s, \\
&\leq \lambda M |b-a| \|y(x) - y_n(x)\|_\infty, \\
&\leq r \|y(x) - y_n(x)\|_\infty, \qquad 0 \leq r < 1.
\end{aligned}$$

By telescoping the result, we obtain:

$$\|y(x) - y_{n+1}(x)\|_\infty \leq r^{n+1} \|y(x) - \underbrace{f(x)}_{=y_0(x)}\|_\infty.$$

With $0 \leq r < 1$ as per Equation (9.2) we have:

$$\|y(x) - y_{n+1}(x)\|_\infty \to 0, \qquad n \to \infty,$$

hence

$$y_{n+1}(x) \to y(x), \qquad n \to \infty. \tag{9.3}$$

9.1.2 Formal Manipulations

Another way to see why the iterative solution $y(x) = f(x) + \sum\limits_{n=1}^{\infty} \lambda^n \mathcal{K}^n f(x)$ converges to the solution of the IE is to re-write the IE

$$y(x) = f(x) + \lambda \int\limits_a^b K(x,s)y(s)\,\mathrm{d}s,$$

as

$$y(x) = f(x) + \lambda \mathcal{K} y(x) \qquad \text{or} \qquad (\mathcal{I} - \lambda\mathcal{K})y(x) = f(x).$$

Then, in a formal notation, so the solution is:

$$y(x) = (\mathcal{I} - \lambda\mathcal{K})^{-1} f(x).$$

On the other hand, if we apply the binomial expansion $(A-B)^{-1} = \sum_{k=0}^{\infty}(A^{-1}B)^k A^{-1}$ we obtain our iterative solution

$$y(x) = (\mathcal{I} - \lambda\mathcal{K})^{-1} f(x) = \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n f(x).$$

Note that the fact that we have a bound on $|\lambda|$ is not in hindsight a surprise as we know that when λ is equal to an eigenvalue we cannot find a unique solution so our process must break down at some $|\lambda| < |\lambda_1|$, where λ_1 denotes the eigenvalue with smallest absolute value. One can eventually make rigorous sense of these equations provided $|\lambda| < |\lambda_1|$ through appropriate functional analysis.

9.1.3 Worked Example

We look again at a familiar example:

$$y(x) = 1 + \lambda \int\limits_0^1 (1 - 3xs) y(s)\,\mathrm{d}s,$$

The kernel is separable, so it is easy to determine the solution by direct computation:

$$y(x) = \frac{4(\lambda+1)}{4-\lambda^2} - \frac{6\lambda}{4-\lambda^2}x \qquad\qquad \lambda \neq \pm 2.$$

Here, however, we illustrate the application of the iterative method instead.

$$\mathcal{K}g(x) = \int_0^1 (1-3xs)g(s)\,\mathrm{d}s,$$

and so, as $f(x) = 1$,

$$\begin{aligned}
\mathcal{K}f(x) &= \int_0^1 (1-3xs)\,\mathrm{d}s = 1 - \tfrac{3}{2}x \\
\mathcal{K}^2 f(x) &= \int_0^1 (1-3xs)\left(1-\tfrac{3}{2}s\right)\,\mathrm{d}s = \tfrac{1}{4} \\
\mathcal{K}^3 f(x) &= \int_0^1 (1-3xs)\tfrac{1}{4}\,\mathrm{d}s = \tfrac{1}{4}\left(1-\tfrac{3}{2}x\right).
\end{aligned}$$

The structure is clearly alternating between the two functions. Thus, we furthermore have:

$$\begin{aligned}
\mathcal{K}^4 f(x) &= \tfrac{1}{4^2}, \\
\mathcal{K}^5 f(x) &= \tfrac{1}{4^2}\left(1-\tfrac{3}{2}x\right), \\
\mathcal{K}^6 f(x) &= \tfrac{1}{4^3},
\end{aligned}$$

etc. Thus, we have:

$$\begin{aligned}
y(x) &= 1 + \lambda\mathcal{K}f(x) + \lambda^2\mathcal{K}^2 f(x) + \cdots \\
&= 1 + \lambda\left(1-\tfrac{3}{2}x\right) + \lambda^2\cdot\tfrac{1}{4} + \lambda^3\tfrac{1}{4}\left(1-\tfrac{3}{2}x\right) + \lambda^4\cdot\tfrac{1}{4^2} + \lambda^5\cdot\tfrac{1}{4^2}\left(1-\tfrac{3}{2}x\right) + \lambda^6\cdot\tfrac{1}{4^3} + \cdots.
\end{aligned}$$

We re-group the terms to get:

$$y(x) = 1 + \lambda^2\cdot\tfrac{1}{4} + \lambda^4\cdot\tfrac{1}{4^2} + \lambda^6\tfrac{1}{4^3} + \cdots + \lambda\left(1-\tfrac{3}{2}x\right)\left[1 + \lambda\cdot\tfrac{1}{4} + \lambda^4\tfrac{1}{4^2} + \cdots\right].$$

This can be further simplified to:

$$y(x) = \left(1 + \tfrac{1}{4}\lambda^2 + \tfrac{1}{16}\lambda^4 + \ldots\right)\left[1 + \lambda\left(1-\tfrac{3}{2}x\right)\right].$$

The prefactor is clearly a geometric progression which converges to $1/(1-\frac{1}{4}\lambda^2)$ provided $|\lambda| < 2$, reproducing

$$y(x) = \frac{4}{4-\lambda^2}\left[(1+\lambda) - \tfrac{3}{2}\lambda x\right].$$

9.2 The Resolvent Kernel

In the previous section we defined

$$\mathcal{K}g(x) = \int_a^b K(x,s)g(s)\,\mathrm{d}s.$$

and found that, for sufficiently small $|\lambda|$, we could write

$$y(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \mathcal{K}^n f(x).$$

If we look first at $\mathcal{K}^2 f(x)$ we may write

$$\begin{aligned}
\mathcal{K}^2 f(x) &= \int_a^b K(x,s)\left[\int_a^b K(s,t)f(t)\,\mathrm{d}t\right]\mathrm{d}s \\
&= \int_a^b \left[\int_a^b K(x,s)K(s,t)\,\mathrm{d}s\right] f(t)\,\mathrm{d}t \\
&= \int_a^b K_2(x,t)f(t)\,\mathrm{d}t,
\end{aligned}$$

where $K_2(x,t)$ is defined by the term in square brackets on the previous line.

Repeating the argument it is clear that we can write

$$\mathcal{K}^n f(x) = \int_a^b K_n(x,t)f(t)\,\mathrm{d}t,$$

where

$$K_n(x,t) = \int_a^b K(x,s)K_{n-1}(s,t)\,\mathrm{d}s$$

which also applies to $n = 2$ with the understanding that $K_1(x,t) = K(x,t)$. Indeed it is not hard to see that

$$K_n(x,t) = \int_a^b K_m(x,s)K_{n-m}(s,t)\,\mathrm{d}s$$

for any $1 \le m \le n-1$.

In addition if $|K(x,s)|$ is bounded by M on $[a,b]$ then

$$|K_n(x,t)| \leq M^n |b-a|^{n-1}.$$

We go back to the power-series expansion of the solution of the FIE:

$$y(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \mathcal{K}^n f(x).$$

Using the notation $K_n(x,t)$ for the self-convolution, we can write the solution, for sufficiently small $|\lambda|$, as

$$\begin{aligned} y(x) &= f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b K_n(x,t) f(t)\,\mathrm{d}t \\ &= f(x) + \lambda \int_a^b \left[\sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t) \right] f(t)\,\mathrm{d}t. \end{aligned}$$

This suggests that we introduce the function

$$\Gamma(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t),$$

called the *resolvent kernel*, in terms of which we have

$$y(x) = f(x) + \lambda \int_a^b \Gamma(x,t;\lambda) f(t)\,\mathrm{d}t.$$

Theorem 9.1 *The resolvent kernel satisfies the inhomogeneous Fredholm IE.*

Proof: From its definition

$$\begin{aligned} \Gamma(x,t;\lambda) &= \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t) \\ &= K(x,t) + \sum_{n=1}^{\infty} \lambda^n K_{n+1}(x,t). \end{aligned}$$

Continuing thus, we have:

$$\begin{aligned}
\Gamma(x,t;\lambda) &= K(x,t) + \lambda \sum_{n=1}^{\infty} \lambda^{n-1} K_{n+1}(x,t) \\
&\overset{p=n-1}{=} K(x,t) + \lambda \sum_{p=0}^{\infty} \lambda^{p} K_{p+2}(x,t) \\
&\overset{\text{Dummy Indices}}{=} K(x,t) + \lambda \sum_{n=0}^{\infty} \lambda^{n} K_{n+2}(x,t) \\
&= K(x,t) + \lambda \sum_{n=0}^{\infty} \lambda^{n} \int_a^b K(x,s) K_{n+1}(s,t)\,\mathrm{d}s \\
&= K(x,t) + \lambda \int_a^b K(x,s) \left[\sum_{n=0}^{\infty} \lambda^{n} K_{n+1}(s,t)\right] \mathrm{d}s \\
&= K(x,t) + \lambda \int_a^b K(x,s) \Gamma(s,t;\lambda)\,\mathrm{d}s
\end{aligned}$$

That is, the resolvent kernel Γ, considered as a function of x and t and the parameter λ satisfies the inhomogeneous Fredholm IE when the function $f(x)$ is replaced by the kernel K considered as a function of x and t.

9.2.1 Worked Example

We revisit the FIE

$$y(x) = f(x) + \lambda \int_0^1 (1-3xt) y(t)\,\mathrm{d}t.$$

The kernel is $K(x,t) = 1 - 3xt$, so

$$K_1(x,t) = 1 - 3xt.$$

Then ,

$$\begin{aligned}
K_2(x,t) &= \int_0^1 (1-3xs)(1-3st)\,\mathrm{d}s \\
&= 1 - \tfrac{3}{2}(x+t) + 3xt,
\end{aligned}$$

and so

$$\begin{aligned} K_3(x,t) &= \int_0^1 (1-3xs)\big(1-\tfrac{3}{2}(s+t)+3st\big)\,\mathrm{d}s \\ &= \tfrac{1}{4}(1-3xt). \end{aligned}$$

It is then clear that $K_n(x,t) = \frac{1}{4}K_{n-2}(x,t)$ for $n \geq 3$ and hence

$$\Gamma(x,t;\lambda) = \left(1+\tfrac{1}{4}\lambda^2+\tfrac{1}{16}\lambda^4+\dots\right)K_1(x,t)+\lambda\left(1+\tfrac{1}{4}\lambda^2+\tfrac{1}{16}\lambda^4+\dots\right)K_2(x,t)$$

so

$$\begin{aligned} \Gamma(x,t;\lambda) &= \frac{1}{1-\frac{1}{4}\lambda^2}\left[K_1(x,t)+\lambda K_2(x,t)\right] \\ &= \frac{1}{1-\frac{1}{4}\lambda^2}\big[(1-3xt)+\lambda\big(1-\tfrac{3}{2}(x+t)+3xt\big)\big] \\ &= \frac{1}{1-\frac{1}{4}\lambda^2}\big[(1+\lambda)-\tfrac{3}{2}\lambda(x+t)-3(1-\lambda)xt\big] \end{aligned}$$

for $|\lambda| < 2$. In fact, the summed form may easily be checked to be valid for all λ except the characteristic values $\lambda = \pm 2$ (as may be also argued by analytic continuation).

Remark: In many of these examples, the key step is to derive a difference equation for K_n, such as $K_{n+2} = (\cdots)K_{n+1} + (\cdots)K_n + \cdots$. Typically, such a difference equation provides a means of computing the sum $\sum_{n=0}^{\infty} \lambda^n K_{n+1}$.

Chapter 10

Functional Analysis of the Fredholm Equation

Overview

In this section we explore the theory that guarantees the existence of an eigensolution of the Fredholm Integral Equation. There are a few steps that we have to leave out, but we nevertheless endeavour to outline the key steps involved. This is an important discussion in any serious study of Applied Mathematics, as the theory (e.g. compact operators) can be used quite generally to make statements about the existence and uniqueness of solutions of various problems in ODEs and PDEs.

10.1 Analogy with Linear Algebra

In Linear Algebra, we have the (L^2) operator norm for a linear operator A, where:

$$\begin{aligned} A : \mathbb{R}^n &\rightarrow \mathbb{R}^n, \\ \boldsymbol{x} &\mapsto A\boldsymbol{x}. \end{aligned}$$

In this instance, the L^2 operator norm is defined as:

$$\|A\|_2 = \sup_{\|\boldsymbol{x}\|_2=1} \|A\boldsymbol{x}\|_2.$$

If the matrix A is self-adjoint, the eigenvalues are real and the eigenvectors are orthogonal and moreover, span $\mathbb{R}^n$, so any vector $\boldsymbol{x}$ can be written as a linear combination of eigenvectors:

$$\boldsymbol{x} = \sum_{i=1}^{n} c_i \boldsymbol{x}_i, \tag{10.1}$$

where $A\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i$ and $\langle x_i, x_j \rangle = \delta_{ij}$. Here, the angle brackets are the usual inner product on $\mathbb{R}^n$. Hence, the coefficients c_i in Equation (10.1) are given by

$$c_i = \langle \boldsymbol{x}_i, \boldsymbol{x} \rangle.$$

We assume the eigenvalues are ordered so that λ_1 is the maximum eigenvalue. Thus,

$$\left(\frac{\|A\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} \right)^2 = \frac{\sum_{i=1}^n \lambda_i^2 c_i^2}{\sum_{i=1}^n c_i^2}.$$

The way to maximize this sum is by taking $c_1 = 1$ and corresponding to the λ_1-eigenvalue, and by taking $c_i = 0$ for $i \neq 1$. Thus,

$$\|A\|_2 = |\lambda_1|.$$

Clearly, another way to write this – in case of A self-adjoint and symmetric is:

$$|\lambda_1| = \|A\|_2 = \sup\{\langle \boldsymbol{x}, A\boldsymbol{x} \rangle : \|\boldsymbol{x}\|_2 = 1\}.$$

The identification of the eigenvalue with the supremum of the set is called Rayleigh's Principle, which we give its own heading here because it is so important:

Theorem 10.1 (Rayleigh Principle)

$$|\lambda_1| = \sup\{\langle \boldsymbol{x}, A\boldsymbol{x} \rangle : \|\boldsymbol{x}\|_2 = 1\}.$$

In cases where the eigenvalues are strictly ordered as $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, we can extend Rayleigh's Principle to pick out λ_2:

$$|\lambda_2| = \sup\{\langle \boldsymbol{x}, A\boldsymbol{x} \rangle : \|\boldsymbol{x}\|_2 = 1,\ \langle \boldsymbol{x}_1, \boldsymbol{x} \rangle = 0\}.$$

This can also be got by 'projecting out the λ_1-eigenvalue'. Assuming the eigenvectors are column vectors, we define:

$$A_1 = A - \boldsymbol{x}_1 \boldsymbol{x}_1^T,$$

then

$$|\lambda_2| = \sup\{\langle \boldsymbol{x}, A_1\boldsymbol{x} \rangle : \|\boldsymbol{x}\|_2 = 1\}$$

And so, guessing the pattern,

$$A_k = A - \sum_{j=1}^k \boldsymbol{x}_j \boldsymbol{x}_j^T$$

and

$$|\lambda_{k+1}| = \sup\{\langle \boldsymbol{x}, A_k\boldsymbol{x} \rangle : \|\boldsymbol{x}\|_2 = 1\}, \qquad k < n.$$

10.2 Hilbert spaces

To extend the concept of operator norms to kernel functions, we introduce the concept of Hilbert Space.

Definition 10.1 (Inner Product Space) *A (real) inner product space is a (real) vector space V equipped with an* inner product*, that is, a map $(\cdot,\cdot) : V \times V \to \mathbb{R}$ which is*

- *Symmetic: $(u,v) = (v,u)$ for all $u, v \in V$*
- *Linear: $(au + bv, w) = a(u,w) + b(v,w)$ for all $u, v, w \in V$ and $a, b \in \mathbb{R}$*
- *Positive definite: $(u,u) > 0$ for all $0 \neq u \in V$.*

If V is an inner product space then $\|u\| = \sqrt{(u,u)}$ defines a *norm* on V, that is,

- $\|au\| = |a|\,\|u\|$ for all $a \in \mathbb{R}$ and $u \in V$
- $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$
- $\|u\| > 0$ for all $0 \neq u \in V$.

Definition 10.2 (Convergent Sequence) *Let V be an inner product space with corresponding norm $\|u\| = \sqrt{(u,u)}$. A sequence $u_1, u_2, \ldots$ in V converges to $u \in V$ if $\|u_n - u\| \to 0$ as $n \to \infty$.*

Definition 10.3 (Convergent Sequence; Completeness) *A sequence $u_1, u_2, \ldots$ in V is called a* Cauchy sequence *if, for all $\epsilon > 0$, there exists N such that $\|u_n - u_m\| < \epsilon$ for all $n, m \geq N$. We say that V is* complete *if every Cauchy sequence in V converges to a limit in V.*

Definition 10.4 (Hilbert Space) *A complete (real) inner product space is called a (real) Hilbert space.*

The Euclidean space $\mathbb{R}^n$ with the usual inner (or 'dot') product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ is an example of a Hilbert space. Another important example is the space $L^2([a,b])$ of Lebesgue measurable functions $f:\ [a,b] \to \mathbb{R}$ satisfying

$$\int_a^b f(x)^2 \, dx < \infty,$$

with functions which are equal almost everywhere[1] identified. This is a real vector space and, when equipped with the inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx, \tag{10.2}$$

is a Hilbert space. We denote the corresponding norm by

$$\|f\|_2 = \sqrt{(f,f)} = \left(\int_a^b f(x)^2 dx\right)^{1/2},$$

[1] I.e. equal everywhere except on sets of measure zero

and refer to it as the L^2-norm.

The space $C([a, b])$ of continuous $f : [a, b] \to \mathbb{R}$ may be regarded as a subspace of $L^2([a, b])$ and is also an inner product space in its own right, with inner product defined by (10.2). However it is not complete with respect to this inner product, as the following example shows.

10.2.1 Pathological Example

Without loss of generality, we may assume that $a = 0$ and $b = 2$, and consider the sequence of functions $f_n \in C([0, 2])$ defined by

$$f_n(x) = \begin{cases} x^n & 0 \le x < 1 \\ 1 & 1 \le x \le 2. \end{cases}$$

Then

$$\|f_n - f_m\|_2^2 = \int_0^1 (x^n - x^m)^2 dx = \frac{1}{2n+1} - \frac{2}{n+m+1} + \frac{1}{2m+1} \to 0$$

as $n, m \to \infty$, hence f_n is a Cauchy sequence with respect to the L^2-norm in $C([0, 2])$. But it does not converge to a continuous limit, rather it converges to the discontinuous function

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x \le 2 \end{cases}$$

since, as $n \to \infty$,

$$\|f_n - f\|_2^2 = \int_0^1 x^{2n} dx = \frac{1}{2n+1} \to 0.$$

In fact, $C([a, b])$ is *dense* in $L^2([a, b])$, that is, its closure in $L^2([a, b])$ is $L^2([a, b])$.

10.2.2 Properties of Hilbert Spaces

We record here some important properties of Hilbert spaces. For these purposes, let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|u\| = \sqrt{(u, u)}$.

1. Two vectors $u, v \in H$ are *orthogonal* if $(u, v) = 0$.
2. If M is a vector subspace of H the orthogonal complement of M, denoted $M^\perp$, is the set of $v \in H$ such that $(u, v) = 0$ for all $u \in M$.
3. A (possibly finite) sequence of vectors $u_1, u_2, \ldots$ in H are called *orthonormal* if $(u_i, u_j) = \delta_{ij}$ for all i, j.
4. An orthonormal sequence $u_1, u_2, \ldots$ in H is a (orthonormal) *basis* for H if every $v \in H$ can be written as $v = \sum_i c_i u_i$ where, if the sum is infinite, it is understood to be convergent with respect to the norm on H, that is, writing $v_n = \sum_{i=1}^n c_i u_i$, it holds that $\|v_n - v\| \to 0$ as $n \to \infty$.

5. A set of vectors in H is *complete* if the only vector orthogonal to every member of the set is the zero vector.

We also have:

Theorem 10.2 (Bessel's Inequality) *If $u_1, u_2, \ldots$ is an orthonormal sequence in H then for all $v \in H$,*

$$\sum_i (v, u_i)^2 \leq \|v\|^2.$$

Proof: Writing $c_i = (v, u_i)$,

$$\begin{aligned} 0 \leq \left\| v - \sum_{i=1}^{n} c_i u_i \right\|^2 &= \left(v - \sum_{i=1}^{n} c_i u_i, v - \sum_{i=1}^{n} c_i u_i \right) \\ &= \|v\|^2 - 2\sum_{i=1}^{n} c_i^2 + \sum_{i=1}^{n} c_i^2 = \|v\|^2 - \sum_{i=1}^{n} c_i^2, \end{aligned}$$

hence

$$\sum_{i=1}^{n} c_i^2 \leq \|v\|^2.$$

If $u_1, u_2, \ldots$ is a finite sequence we are done, otherwise let $n \to \infty$ to conclude that $\sum_i c_i^2$ is convergent and not greater than $\|v\|^2$.

Finally, we have the following key results which we simply state here:

- **Parseval's relation:** If $u_1, u_2, \ldots$ is an orthonormal sequence in H then it is a basis if, and only if, for all $v \in H$,

$$\sum_i (v, u_i)^2 = \|v\|^2.$$

- **Completeness:** An orthonormal sequence in H is a basis if, and only if, it is complete.
- **Riesz-Fisher Theorem:** If $u_1, u_2, \ldots$ is an orthonormal basis for H and $\sum_i c_i^2 < \infty$ then $v = \sum_i c_i u_i$ converges (with respect to the norm) in H and $c_i = (v, u_i)$.
- **Orthogonal Projection:** If M is a vector subspace of H then

$$H = M \oplus M^{\perp},$$

that is, every $w \in H$ can be written uniquely as $w = u + v$ where $u \in M$ and $v \in M^{\perp}$.

10.3 Linear Operators on Hilbert Spaces

A linear operator $A : H \to H$ is a linear map such that

$$A(\lambda \boldsymbol{x} + \mu \boldsymbol{y}) = \lambda A\boldsymbol{x} + \mu A\boldsymbol{y},$$

for all real scalars λ and μ and all vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in H. A further key definition, especially relevant for spaces of functions concerns the boundedness of such operators:

Definition 10.5 *A linear operator $A : H \to H$ is* bounded *if there exists a number M such that $\|Au\| \le M\|u\|$ for all $u \in H$.*

The set of bounded linear operators on H is itself a normed vector space, with norm defined by

$$\|A\| = \sup\{\|Au\| : \ u \in S\}$$

where $S = \{u \in H : \ \|u\| = 1\}$. Here the choice of notation is important, as one should think of S as being the unit sphere in the Hilbert space. We note in particular, if $0 \neq u \in H$, then $\|u\| \neq 0$ and $u/\|u\| \in S$. Thus, if $\|A\| = 0$ then $Au = 0$ for all $u \in H$.

In particular, if $K(x, y)$ is a real-valued continuous kernel function valid on $[a, b]^2$, then the convolution $\mathcal{K}y = K * y$ is a linear operator on $L^2([a, b])$. Furthermore, $\mathcal{K}$ is a bounded linear operator, since:

$$\begin{aligned}
\|\mathcal{K}y\|_2^2 &= \int_a^b \left[\int_a^b K(x,y)y(s)\mathrm{d}s\right]^2 \mathrm{d}x, \\
&= \int_a^b \left[\int_a^b K(x,y)y(s)\mathrm{d}s\right]\left[\int_a^b K(x,y)y(s)\mathrm{d}s\right] \mathrm{d}x, \\
&\overset{\text{C.S.}}{\le} \int_a^b \left[\int_a^b K^2(x,s)\mathrm{d}s\right]^{1/2} \left[\int_a^b K^2(x,s)\mathrm{d}s\right]^{1/2} \|y\|_2^2, \\
&= \left[\int_a^b \int_a^b K^2(x,s)\mathrm{d}x\mathrm{d}s\right] \|y\|_2^2, \\
&\le M^2|b-a|^2\|y\|_2^2.
\end{aligned}$$

Furthermore, a linear operator $A : H \to H$ is called *self-adjoint* if $\langle Au, v\rangle = \langle u, Av\rangle$ for all $u, v \in H$.

Theorem 10.3 *If the kernel function $K(x, y)$ is symmetric, then the linear operator $\mathcal{K}$ is self-adjoint.*

This follows by direct computation, e.g.

$$\begin{aligned}
\langle f, \mathcal{K}g\rangle &= \int_a^b \left[f(x)\int_a^b K(x,s)\mathrm{d}s\right] \mathrm{d}x, \\
&\overset{\text{Fubini}}{=} \int_a^b \int_a^b f(x)g(s)K(x,s)\mathrm{d}s\mathrm{d}x, \\
&\overset{\text{Symmetric}}{=} \int_a^b \int_a^b f(x)g(s)K(s,x)\mathrm{d}s\mathrm{d}x, \\
&\overset{\text{Fubini}}{=} \int_a^b \left[\int_a^b K(s,x)f(x)\mathrm{d}x\right] g(s)\mathrm{d}s, \\
&= \langle \mathcal{K}f, g\rangle.
\end{aligned}$$

Like in the case of linear algebra, for self-adjoint operators we are able to identify

$$\|A\|_2 = \sup_{u \in S} \langle Au, u \rangle,$$

which is now a key theorem for us.

Theorem 10.4 *If $A : H \to H$ is bounded and self-adjoint, then its norm may be represented as*

$$\|A\| = \sup\{|(Au, u)| : \; u \in S\}.$$

The strategy of the proof is to show that $\sup_{u \in S}\langle Au, u\rangle \leq \|A\|$ and also that $\sup_{u \in S}\langle Au, u\rangle \geq \|A\|$, which will allow us to conclude that equality holds.

Part 1: For $u \in S$, we have:

$$\begin{aligned} \langle Au, u \rangle &\overset{C.S.}{\leq} \|Au\|_2 \|u\|_2, \\ &\leq \|A\|_2. \end{aligned}$$

Part 2: Consider:

$$\begin{aligned} \langle A(x+y), (x+y) \rangle - \langle A(x-y), (x-y) \rangle = {} & \langle Ax, x \rangle + \langle Ax, y \rangle + \langle Ay, x \rangle + \langle Ay, y \rangle \\ & - \langle Ax, x \rangle + \langle Ax, y \rangle + \langle Ay, x \rangle - \langle Ay, y \rangle. \end{aligned}$$

The terms either cancel out or are equal to $\langle Ax, y \rangle$ by self-adjointness so overall we get:

$$\langle Ax, y \rangle = \tfrac{1}{4} \left[\langle A(x+y), (x+y) \rangle - \langle A(x-y), (x-y) \rangle \right].$$

Hence,

$$\begin{aligned} |\langle Ax, y \rangle| &\leq \tfrac{1}{4} \left[\langle A(x+y), (x+y) \rangle + \langle A(x-y), (x-y) \right], \\ &\leq \sup_{u \in S} \langle Au, u \rangle \left[\|x+y\|^2 + \|x-y\|_2^2 \right]. \end{aligned}$$

We use the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2,$$

and conclude:

$$|\langle Ax, y \rangle| \leq \tfrac{1}{2} \left(\sup_{u \in S} \langle Au, u \rangle \right) \left(\|x\|_2^2 + \|y\|_2^2 \right).$$

This is true for all vectors x and y. Hence, for x and y in S, we have:

$$|\langle Ax, y \rangle| \leq \sup_{u \in S} \langle Au, u \rangle.$$

Take $y = Ax/\|Ax\|_2$:

$$\|Ax\|_2 = \frac{\langle Ax, Ax\rangle}{\|Ax\|_2} \leq \sup_{u \in S} \langle Au, u\rangle.$$

As this is true for all $x \in S$ we have:

$$\|A\|_2 \leq \sup_{u \in S} \langle Au, u\rangle.$$

Combining Parts (1) and (2) we have:

$$\|A\|_2 = \sup_{u \in S} \langle Au, u\rangle.$$

10.4 Compact Operators

For real numbers, given a sequence $\{u_n\}$ with $a \leq u_n \leq b$, it is always possible to extract a convergent subsequence. This is called the Bolzano–Weirstrass theorem. This is called the compactness property of the interval $I = [a, b]$, which is the same thing as saying that the interval contains its own boundary points, or is closed. The aim of this section is to extend these concepts to sets of operators.

Let H be a Hilbert space and let U be a subset of H.

Definition 10.6 *The set U is called* **bounded** *if there exists a positive real number M such that $\|u\| < M$ for all $u \in U$.*

Definition 10.7 *The set U is called* **relatively compact** *if every sequence $\{u_n\}$ in U has a convergent subsequence which converges to a limit in H.*

Theorem 10.5

If the set U is relatively compact then it is bounded.

Remark: If $H = \mathbb{R}^n$, then a set $U \in H$ is relatively compact if and only if it is bounded. This is the Bolzano–Weirstrass theorem, extended to $\mathbb{R}^n$.

We now look at extending the concept of compactness to operators.

Definition 10.8 *A linear operator is sequentially compact if it maps bounded sets to relatively compact sets.*

Suppose that $A : H \to H$ is compact and self-adjoint, and that $\|A\| \neq 0$. Then, by self-adjointness,

$$\|A\| = \sup_{u \in S} \|Au\|_2 = \sup_{u \in S} |\langle Au, u\rangle|. \tag{10.3}$$

Given $\mu = \pm\|A\|$, we want to find $v \in H$ such that v attains the bound in Equation (10.3), that is, $v \in S$ maximizes $|\langle Av, v\rangle|$. This can be achieved if A is a compact operator.

Now, we can always find a point $w_n \in S$ such that $\langle Aw_n, w_n\rangle$ is within $1/n$ of the supremum μ. Using the axiom of choice, we can construct a sequence $\{w_n\}$ such that each $\langle w_n, Aw_n\rangle$ is within $1/n$ of the supremum. The limit of this sequence gives:

$$\langle Aw_n, w_n\rangle \to \mu, \qquad \text{as } n \to \infty. \tag{10.4}$$

Since A is a compact operator, not only is the sequence of real numbers in Equation (10.4) convergent, but Av_n, a sequence of vectors, is also convergent, and we can write:

$$Av_n \to \mu v.$$

We have:

$$\begin{aligned} \|Av_n - \mu v_n\|_2^2 &= \|Av_n\|_2^2 - 2\mu\langle Av_n, v_n\rangle + \mu^2\|v_n\|_2^2, \\ &\leq \|A\|_2^2 - 2\langle Av_n, v_n\rangle + \mu^2. \end{aligned}$$

Hence,

$$\|Av_n - \mu v_n\|_2^2 \to 0, \qquad \text{as } n \to \infty. \tag{10.5}$$

The claim now is that not only does Av_n converge to a limit point, but that v_n does also. We have:

$$\begin{aligned} \mu(v_n - v) &= \mu v_n - \mu V + Av_n - Av_n, \\ &= -(Av_n - \mu v_n) + (Av_n - \mu v), \\ \mu\|v_n - v\|_2 &\overset{\Delta}{\leq} \|Av_n - \mu v_n\| + \|Av_n - \mu v\|_2. \end{aligned}$$

The first term vanishes because of Equation (10.5) while the second term vanishes because $\{Av_n\}$ is the convergent subsequence. Hence, $\|v_n - v\|_2 \to 0$,

$$v_n \to v,$$

and thus, the supremum is attained.

10.5 Continuous Operators

We prove that the linear operator $\mathcal{K}$ is continuous in the following sense:

Theorem 10.6 *If $K(x, s)$ is continuous and $r \in \mathcal{H}$ then $g = \mathcal{K}r$ is continuous.*

Proof: First note that since $r \in \mathcal{H} = L^2([a,b])$, it is also in $L^1([a,b])$, that is:

$$\begin{aligned} \|r\|_1 &= \int_a^b |r(s)|\mathrm{d}s, \\ &= \langle |r|, 1\rangle, \\ &\overset{\text{C.S.}}{\leq} \|r\|_2 \|1\|_2, \\ &= \|r\|_2 |b-a|^{1/2}, \\ &\leq \infty. \end{aligned}$$

We want to show that g is continuous, that is, for each $x \in [a,b]$ and $\epsilon > 0$ there exists $\delta > 0$ such that $|g(x) - g(y)| < \epsilon$ for all y such that $|x - y| < \delta$. To prove this we will use the fact that since K is continuous and its domain $[a,b] \times [a,b]$ is compact, it is also *uniformly continuous*, that is, for each $\epsilon' > 0$, there exists $\delta > 0$ such that

$$|K(x,s) - K(y,t)| < \epsilon' \text{ whenever } |x-y| + |s-t| < \delta. \tag{10.6}$$

Let $\epsilon > 0$ and set $\epsilon' = \epsilon / \max\{\|r\|_1, 1\}$. Then there exists $\delta > 0$ such that $|K(x,s) - K(y,t)| < \epsilon'$ whenever $|x-y| + |s-t| < \delta$, hence for all x, y satisfying $|x - y| < \delta$, we have

$$\begin{aligned} |g(x) - g(y)| &\overset{\Delta}{\leq} \int_a^b \underbrace{|K(x,s) - K(y,s)|}_{s=t,\text{Cf. (10.6)}} |r(s)|\mathrm{d}s, \\ &\leq \epsilon' \|r\|_1 \leq \epsilon, \end{aligned}$$

as required.

10.6 The Totality of Eigenfunctions

We look at the definition

$$\|\mathcal{K}\| = \sup\{|(y, \mathcal{K}y)| : \; y \in S\}. \tag{10.7}$$

The supremum is generated by a sequence of functions $\{y_n\}$ with $y_n \in S$, which get ever closer to realising the least upper bound. But is not true *a priori* that the sequence has a limit.

However, the Fredholm Integral Equation defines a compact operator[2]. As such, the bounded sequence of functions $\{y_n\}$ possesses a convergent subsequence, with limit $y_0 \in S$. Thus, the supremum is realised by $y_0 \in S$, so we can write:

$$\|\mathcal{K}\| = \max\{|(y, \mathcal{K}y)| : \; y \in S\}. \tag{10.8}$$

[2]Unfortunately it is beyond the scope of the course to prove this result

Furthermore, there is a generalisation of Rayleigh's principle[3] which implies that the supremum satisfies the Fredholm Integral Equation $y_0(x) = \lambda_0 \int_a^b K(x,s)y_0(s)\mathrm{d}s$, with eigenvalue $1/|\lambda_0| = \|\mathcal{K}\|$.

Another fact which we rely on is that, under our assumptions, the operator $\mathcal{K}$ is continuous, that is, if $\|v_n - v\|_2 \to 0$ then $\|\mathcal{K}v_n - \mathcal{K}v\|_2 \to 0$.

Note that, by Theorem (10.6), the eigenfunction $y_1(x)$ is continuous. Given this eigenfunction, we may now construct a new symmetric kernel by

$$K_1(x,s) = K(x,s) - \frac{y_1(x)y_1(s)}{\lambda_1}.$$

Note that

$$\begin{aligned}\int_a^b K_1(x,s)y_1(s)\,\mathrm{d}s &= \int_a^b K(x,s)y_1(s)\,\mathrm{d}s - \frac{y_1(x)}{\lambda_1}\int_a^b \big(y_2(s)\big)^2\,\mathrm{d}s \\ &= \frac{1}{\lambda_1}y_1(x) - \frac{1}{\lambda_1}y_1(x) = 0.\end{aligned}$$

Now, either $K_1(x,s)$ is trivial, so $K(x,s) = y_1(x)y_1(s)/\lambda_1$ is separable, or $K_1(x,s)$ is non-trivial and, as above, we conclude that $K_1(x,s)$ possesses an eigenfunction $y_2 \in S$, say, with eigenvalue λ_2, such that

$$\frac{1}{|\lambda_2|} = \max\{|\langle y, \mathcal{K}y\rangle| : \ y \in S \cap Q_1\}, \qquad Q_1 = \{y \in \mathcal{H} : \ (y, y_1) = 0\}.$$

Note also that $\mathcal{K}$ maps Q_1 into itself, since for $y \in Q_1$,

$$\langle \mathcal{K}y, y_1\rangle = \langle y, \mathcal{K}y_1\rangle = \frac{1}{\lambda_1}\langle y, y_1\rangle = 0.$$

Proceeding in this way we see we have two options:

- either the process stops after a finite number of steps with $K_n(x,s) = 0$ and

$$K(x,s) = \sum_{i=1}^{n} \frac{y_i(x)y_i(s)}{\lambda_i}, \tag{10.9}$$

  and the kernel is separable;
- or we have a (countably) infinite number of eigenvalues.

In the latter case we may prove:

Theorem 10.7 *If $K(x,s)$ is not separable then $|\lambda_n| \to \infty$ as $n \to \infty$.*

[3]See Chapter 12, Section 12.4

Proof: The functions $y_i(x)y_j(s)$ are orthonormal in the Hilbert space $L^2([a,b]^2)$ and

$$\int_a^b \int_a^b K(x,s) y_i(x) y_j(s) dx ds = \frac{1}{\lambda_i}\delta_{ij}.$$

It therefore follows from Bessel's inequality that

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i{}^2} \leq \int_a^b \int_a^b (K(x,s))^2 \ \mathsf{d}s\, \mathsf{d}x.$$

In particular, this implies that

$$\frac{1}{\lambda_n{}^2} \to 0 \qquad \text{as} \qquad n \to \infty,$$

hence $|\lambda_n| \to \infty$ as $n \to \infty$.

We are now in a position to show:

Theorem 10.8 *If $g = \mathcal{K}r$, for some $r \in \mathcal{H}$, the g may be expanded as*

$$g(x) = \sum_i a_i y_i(x), \qquad a_i = \frac{1}{\lambda_i} \int_a^b r(s) y_i(s) ds.$$

Proof: In the separable case this follows immediately from the representation (10.9), so let us assume that K is not separable. Now the proof breaks into two parts.

Part 1: Let $M = \mathcal{S}\{y_1, y_2, \cdots\}$, and write $g = \mathcal{K}r$. Since $\mathcal{H}$ is a Hilbert space, we can write r uniquely as:

$$r = u + v, \qquad u \in M, \qquad v \in M^\perp.$$

Now the aim of Part 1 is to show that $\mathcal{K}v = 0$.

If $v = 0$ we are done immediately, so assume that $v \neq 0$ and let $v_1 = v/\|v\|_2$. We introduce the notation

$$Q_n = \{y \in \mathcal{H} : \ (y, y_1) = \cdots = (y, y_n) = 0\}.$$

Hence,

$$\frac{1}{|\lambda_{n+1}|} = \max\{|\langle y, \mathcal{K}y\rangle| : \ y \in S \cap Q_n\},$$

The space $M^\perp$ involves more constraints than Q^n but there is overlap hence $M^\perp \subset Q_n$ for all n. Thus, $M^\perp$ is a subset of Q^n, and hence,

$$|\langle v_1, \mathcal{K}v_1\rangle| \leq 1/|\lambda_{n+1}|$$

or equivalently $|\langle v, \mathcal{K}v\rangle| \leq \|v\|_2^2/|\lambda_{n+1}|$, for all n. We take $n \to \infty$. We have $1/|\lambda_{n+1}| \to 0$ hence

$$\langle v, \mathcal{K}v\rangle = 0, \text{ as } n \to \infty.$$

Now, $\mathcal{K}$ maps the subspace $M^\perp$ into itself so may be considered as a linear operator from $M^\perp$ to $M^\perp$ (homework). Analogous to Equation (10.7), the norm of this restricted operator is the maximum value of $|\langle y, \mathcal{K}y\rangle|$ for $y \in S \cap M^\perp$, which must therefore be zero, and this implies that $\mathcal{K}v = 0$ for all $v \in M^\perp$, as required.

Part 2: We have $\mathcal{K}v = 0$, hence $g = \mathcal{K}(u + v) = \mathcal{K}u$, and since $u \in M$ we may write

$$u(x) = \sum_{i=1}^{\infty} b_i y_i(x).$$

for some sequence of real numbers b_i. Hence, the aim of Part 2 is to establish the values of b_i.

As a closed subspace of the Hilbert space $\mathcal{H}$, the space M is also a Hilbert space and moreover has the sequence $y_1, y_2, \ldots$ as an orthonormal basis. Thus, according the Riesz-Fischer Theorem, the coefficients are given by $b_i = \langle u, y_i\rangle$ or equivalently $b_i = \langle r, y_i\rangle$, since $r = u + v$ and $v \in M^\perp$. Applying $\mathcal{K}$ to both sides yields (recall that $\mathcal{K}$ is continuous) the expansion

$$g(x) = (\mathcal{K}u)(x) = \sum_{i=1}^{\infty} a_i y_i(x),$$

where $a_i = b_i/\lambda_i$, as required.

Remark: The condition that $K(x, s)$ is continuous is not necessary for the above conclusions to hold, for example it can be weakened to the requirement that $K \in L^2([a, b] \times [a, b])$, that is, $K(x, s)$ is a measurable function on $[a, b] \times [a, b]$ which satisfies

$$\int_a^b \int_a^b K(x, s)^2 \, \mathrm{d}x \, \mathrm{d}s < \infty.$$

The condition that $K(x, s)$ is continuous does however ensure that the eigenfunctions $y_i(x)$ are continuous, a fact that will be important later.

Chapter 11

The Sturm-Liouville Problem: Introduction

Overview

In this chapter, we study the eigenvalues of the Sturm–Liouville problem,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right]+q(x)u(x)=-\lambda r(x)u(x), \tag{11.1}$$

where $p(x)>0$ and $r(x)>0$ are continuous functions. As all second-order linear problems can be put into the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right]+q(x)u(x)=0, \tag{11.2}$$

corresponding to $\lambda=0$ in Equation (12.1) the resulting theory is very general and can be used to sole a wide range of problems in Applied Mathematics, Physics, and Engineering.

Key terminology:

- Equation (12.1) with appropriate boundary conditions is called a Sturm–Liouville problem. The eigenvalue λ is TBC.
- Equation (11.2) is called a Sturm–Liouville equation.

11.1 Motivating Example

Consider an elastic string of variable mass density, $\rho(x)$, stretched horizontally between two points positioned in $x=a$ and $x=b$ (as in Figure 11.1). Because of the stretching the string is under variable tension, $T(x)$, along its length. Let $A(x,t)$ be the transverse displacement of the string.

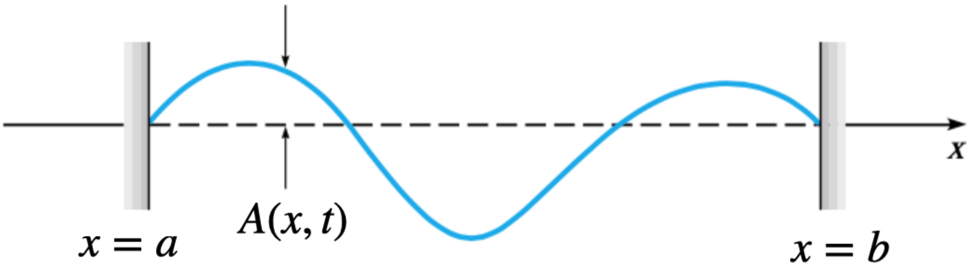


Figure 11.1: Vibrations in a stretched string.

Application of Newton's second law of motion, to an element of length $\mathrm{d}x$ with corresponding mass $\rho(x)\mathrm{d}x$ of each segment leads to the wave equation for $A(x,t)$,

$$\rho(x)\frac{\partial^2 A(x,t)}{\partial t^2} = \frac{\partial}{\partial x}T(x)\frac{\partial A(x,t)}{\partial x}.$$

The force (per unit length) on the right hand side is due to the bending of the string. We may additionally suppose that the string experiences a restoring force $-k(x)A(x,t)$, where $k(x)$ is the position dependent Hooke's constant experienced by the string segment $\mathrm{d}x$. Consequently, the more general version of the wave equation is

$$\rho(x)\frac{\partial^2 A(x,t)}{\partial t^2} = \frac{\partial}{\partial x}T(x)\frac{\partial A(x,t)}{\partial x} - k(x)A(x,t).$$

We look for normal modes, that is, solutions of the pure harmonic form have vibrational frequencies ω and amplitudes:

$$A(x,t) = y(x)\cos\omega(t-t_0).$$

Thus the spatial function $y(x)$ then must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[T(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x}\right] + \big(\omega^2\rho(x) - k(x)\big)y(x) = 0 \qquad x \in (a,b). \tag{11.3}$$

If we change notation using $u(x) = y(x)$, $p(x) = T(x)$, $\rho(x) = r(x)$, $q(x) = -k(x)$ and $\lambda = \omega^2$ we obtain a *Sturm-Liouville problem*,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right] + q(x)u(x) = -\lambda r(x)u(x) \qquad x \in (a,b). \tag{11.4}$$

From the Physics of the example, it is clear that $u(a) = 0$ and $u(b) = 0$ are the appropriate boundary conditions.

Furthermore, it is clear that $p(x) = T(x)$ is positive everywhere except possibly at an end-point (actually the tension must vanish at the bottom of the string in our example), and $r(x) = \rho(x)$ is positive everywhere. To ensure existence of solutions we assume that the functions $q(x)$ and $r(x)$

are continuous and the function $p(x)$ is continuously differentiable.

11.2 Boundary Conditions

To specify a Sturm–Liouville (SL) problem completely, we specify appropriate boundary conditions:

- **Regular SL problem:** $\alpha_a u(a) + \beta_a u'(a) = 0$ and $\alpha_b u(b) + \beta_b u'(b) = 0$.
- **Periodic SL problem:** When the coefficients are periodic with period $b-a$ (so, in particular, $p(a) = p(b)$), we require $u(a) = u(b)$ and $u'(a) = u'(b)$.

We refer to the first set of BCs here as **regular BCs**.

11.2.1 Other possible boundary conditions

We can also look at SL problems with (a,b) replaced with $(-\infty, 0]$ or $[0,\infty)$ or $(-\infty,\infty)$. In this case, the boundary conditions are **behavioural** at the infinite endpoint(s), e.g. u and u' should vanish. Similarly, if (a,b) is finite as before, but if the coefficient functions in the SL equation:

$$u''(x) + \frac{p'(x)}{p(x)}u'(x) + \frac{\lambda r(x) - q(x)}{p(x)}u(x) = 0 \qquad\qquad x \in (a,b).$$

are regular-singular[1] at $x = a$ (say), then we would have a behavioural BC at $x = a$ and a standard BC at $x = b$, such as $\alpha_b u(b) + \beta_b u'(b) = 0$.

11.3 Reduction to Sturm–Liouville form

Theorem 11.1 *Every homogeneous second order ODE can be transformed to a Sturm–Liouville equation.*

Proof: We start with a general second-order ODE

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0. \tag{11.5}$$

We wish to show that this reduces to:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p\frac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u(x) = 0.$$

[1] Refer back to Chapter 5 for the taxonomy of ODEs as either ordinary, regular-singular and irregular-singular.

To do this, we multiply both sides of Equation (11.5) by an as-yet unknown integrating factor:

$$\mu u'' + P\mu u' + \mu Q u = 0.$$

If we take $P\mu = \mu'$, then this becomes:

$$(\mu u')' + \mu Q u = 0.$$

This is in Sturm–Liouville form, with $\mu Q = q$. Summarizing, we have:

$$\mu' = \mu P \implies \mu = \mathrm{e}^{\int P(x)\mathrm{d}x} \text{ and } q = \mu Q.$$

11.3.1 Worked Example

Transform the following Bessel's equation into a SL form

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0\,.$$

First we bring it in the standard form

$$y'' + \frac{y'}{x} + \left(1 - \frac{\nu^2}{x^2}\right) y = 0$$

so that we can notice that $P(x) = \dfrac{1}{x}$. From here our integrating factor is

$$\mu(x) = \mathrm{e}^{\int \mathrm{d}x/x} = \mathrm{e}^{\ln x} = x.$$

Finally Bessel's equation in the SL form becomes

$$[xy']' + \left(x - \frac{\nu^2}{x}\right) y = 0.$$

11.4 The adjoint Sturm-Liouville operator

We have previously been cavalier with the boundary terms in constructing our adjoint. Let us introduce the operator (functional) L such that

$$L\big[u\big](x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right] + q(x)u(x)$$

so our Sturm-Liouville equation is

$$L\big[u\big](x) = -\lambda r(x) u(x) \qquad\qquad x \in (a, b).$$

From this perspective we can now define λ as the *eigenvalue* of the SL equation and $u(x)$ the corresponding *eigenfunction*. The function $r(x)$, (usually different from 0) is instead known as the *weight* or *density* function. In this sense the eigenvalues, respectively, eigenfunctions are defined with respect to the weight function $r(x)$.

Recall now the definition of the inner product $(u,v) = \int_a^b u(x)v(x)dx$. Based on that we can prove an important property for the Sturm-Liouville equation:

Theorem 11.2 (Lagrange Identity) *Let $u(x)$ and $v(x)$ be functions with continuous second derivatives in the interval (a,b). Then,*

$$\int_a^b u(x)\big(L[v](x)\big)\,\mathrm{d}x = \int_a^b \big(L[u](x)\big)v(x)\,\mathrm{d}x + \big[p(x)\big(u(x)v'(x) - v(x)u'(x)\big)\big]_a^b.$$

Proof: After suppressing some of the explicit x-dependencies, we have:

$$\begin{aligned}
\int_a^b u(x)\big(L[v](x)\big)\,\mathrm{d}x &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}v}{\mathrm{d}x}\right)u\,\mathrm{d}x + \int_a^b quv\,\mathrm{d}x, \\
&= \int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}v}{\mathrm{d}x}\cdot u\right) - p\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\right]\mathrm{d}x + \int_a^b quv\,\mathrm{d}x, \\
&= [pv'u]_a^b - \int_a^b p\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} + \int_a^b quv\,\mathrm{d}x, \\
&= [pv'u]_a^b - \int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(pv\frac{\mathrm{d}u}{\mathrm{d}x}\right) - v\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right]\mathrm{d}x + \int_a^b quv\,\mathrm{d}x, \\
&= [p(x)\left(v'(x)u(x) - u'(x)v(x)\right)]_a^b + \int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x}\right) + qu\right]v\,\mathrm{d}x, \\
&= [p(x)\left(v'(x)u(x) - u'(x)v(x)\right)]_a^b + \int_a^b v(x)\big(L[u](x)\big)\,\mathrm{d}x.
\end{aligned}$$

Notice that our regular or periodic boundary conditions above ensure that the second term of the right hand side vanishes so the operator is truly self-adjoint on the space of functions satisfying the boundary conditions, $\langle u, Lv\rangle = \langle Lu, v\rangle$. In fact, for the case of periodic boundary conditions is trivial, but for the regular ones we should check that for only one of the bound, let's say $x = a$, then we have the following relations

$$u'(a)v(a) - v'(a)u(a) = -v(a)\frac{\alpha_a}{\beta_a}u(a) + u(a)\frac{\alpha_a}{\beta_a}v(a) = 0$$

and similarly for $x = b$.

For the case of a singular Sturm-Liouville problem we may need to restrict the class of functions we consider to ensure that the corresponding boundary term vanishes.

11.4.1 Example

Consider the case $p(x) = 1$, $q(x) = 0$, $r(x) = 1$ on $(0, \pi)$ with Dirichlet boundary conditions so:

$$u''(x) + \lambda u(x) = 0, \qquad u(0) = u(\pi) = 0$$

It is easy to find the eigenfunctions (solutions) of this homogeneous ODE with constant coefficients starting from the characteristic equation $m^2 + \lambda = 0$ which roots are $m_1 = -\sqrt{-\lambda}$ and $m_2 = +\sqrt{-\lambda}$. Then the solutions are constituting by the fundamental sets of functions $\mathrm{e}^{m_1 x}$ and $\mathrm{e}^{m_2 x}$.

Assuming for the moment that λ must be real (something we will prove in the next section), there are 3 possibilities:

- **Case 1**, $\lambda < 0$: then we may write $\lambda = -\omega^2$ $(\omega > 0)$, our solutions are $\mathrm{e}^{\pm\omega x}$ so our general solution is $A\mathrm{e}^{\omega x} + B\mathrm{e}^{-\omega x}$ and our boundary conditions require

$$\begin{aligned} A + B &= 0 \\ A\mathrm{e}^{\omega\pi} + B\mathrm{e}^{-\omega\pi} &= 0 \end{aligned}$$

  but this has only trivial solutions for A and B as the determinant is $\mathrm{e}^{-\omega\pi} - \mathrm{e}^{\omega\pi} = -2\sinh\omega\pi$ which does not vanish on $\omega > 0$.
- **Case 2**, $\lambda = 0$: In this case we have repeated roots and our general solution is $A + Bx$ and our boundary conditions require

$$\begin{aligned} A &= 0 \\ A + B\pi &= 0 \end{aligned}$$

  giving again the trivial solution.
- **Case 3** , $\lambda > 0$: In this case, we may write $\lambda = \omega^2$ $(\omega > 0)$ and being the routes purely imaginary our solutions are $\cos\omega x$ and $\sin\omega x$ so our general solution is $A\cos\omega x + B\sin\omega x$ our boundary conditions require

$$\begin{aligned} A &= 0 \\ \cancel{A\cos\omega\pi} + B\sin\omega\pi &= 0 \end{aligned}$$

  so a non-zero solution exists if and only if $\sin\omega\pi = 0$ that is if $\omega = n$ where $n \in \mathbb{N}$ as shown in Figure 11.4.1.

11.5 General Properties

Looking at our simple example we note a number of properties that we shall prove are valid for an arbitrary Sturm–Liouville problem:

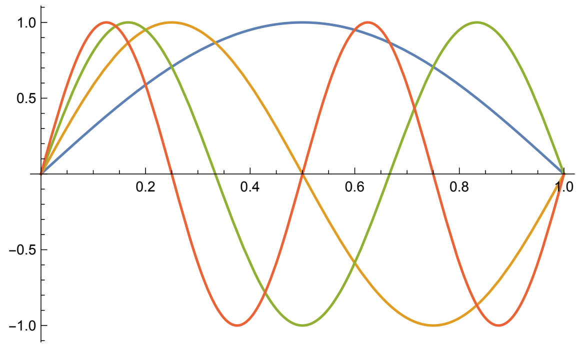


Figure 11.2: The lowest 4 eigenfunctions showing the increased oscillations and number of zeros as n increases (1:blue, 2:yellow, 3:green, 4:red).

1. The eigenvalues are real;
2. Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to weight function $r(x)$;

We have some further results, which are valid for regular SL problems with a finite domain:

3. Eigenvalues of a regular Sturm-Liouville problem are discrete if the domain is finite; furthermore, the eigenfunctions are complate and span the Hilbert space:

$$\mathcal{H} = \left\{ u(x) | \int_a^b r(x)u(x)^2 \, \mathrm{d}x \text{ exists and } \alpha_a u(a) + \beta_a u'(a) = 0, \alpha_b u(b) + \beta_b u'(b) = 0 \right\}.$$

4. The eigenfunctions oscillate;
5. The larger the eigenvalue the faster the eigenfunction oscillates;
6. The modulus of the eigenvalues have a lowest member and increase without limit.

We prove the first two results in this chapter; the rest of the proofs are left to Chapter 12.

To understand the first two results we denote by $G(x, s)$ the Green's function for the operator

$$L[u](x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right] + q(x)u(x), \qquad x \in (a, b).$$

Hence, the solution to

$$L[u](x) = -\lambda r(x)u(x), \qquad x \in (a, b),$$

subject to the corresponding boundary conditions of the Sturm–Liouville problem can be written as a Fredholm Integral Equation

$$u(x) = -\lambda \int_a^b G(x, s)r(s)u(s) \, \mathrm{d}s.$$

Given our assumption that $r(x) > 0$ we may rewrite this in symmetric form as

$$\sqrt{r(x)}u(x) = \lambda \int_a^b \left[-\sqrt{r(x)}G(x,s)\sqrt{r(s)}\right]\sqrt{r(s)}u(s)\,\mathrm{d}s,$$

or

$$y(x) = \lambda \int_a^b K(x,s)y(s)\,\mathrm{d}s,$$

where $y(x) = \sqrt{r(x)}u(x)$ and $K(x,s) = -\sqrt{r(x)}G(x,s)\sqrt{r(s)}$ is symmetric.

It follows from the results of Chapter 8 that the eigenvalues are real and that if $\lambda_i \neq \lambda_j$ then

$$0 = \int_a^b y_i(x)y_j(x)\,\mathrm{d}x = \int_a^b r(x)u_i(x)u_j(x)\,\mathrm{d}x,$$

which is what we mean by being orthogonal with respect to weight function $r(x)$.

Next we prove the following result:

Theorem 11.3 *Suppose $u_1(x)$ and $u_2(x)$ solve the regular Sturm–Liouville problem on a finite domain with the same eigenvalue then they are proportional (i.e. the eigenspace is of dimension 1).*

Proof: Since both $u_1(x)$ and $u_2(x)$ satisfy exactly the same equation (same eigenvalue and same BCs), we have:

$$\begin{aligned} \alpha_a u_1(a) + \beta_a u_1'(a) &= 0, \\ \alpha_a u_2(a) + \beta_a u_2'(a) &= 0. \end{aligned}$$

Thus, u_1 and u_2 are linearly dependent at a, so $W(a) = 0$. By Abel's identity applied to $pu'' + p'u' + (\cdots)u = 0$, we have $p(x)W(x) = p(x_0)W(x_0)$, where x_0 is an arbitrary reference point. If we take $x_0 = a$, we get $p(x)W(x) = 0$, and since $p(x) > 0$ for a regular SL problem, we get $W(x) = 0$ for all $x \in [a,b]$. Hence:

$$u_1 u_2' = u_2 u_1',$$

and

$$\frac{u_2'}{u_2} = \frac{u_1'}{u_1}.$$

This is a simple ODE with solution $u_2 = Cu_1$, where C is a constant. Hence, linear dependence is shown. ∎

11.5.1 Special Case

In the periodic case we don't have enough information to deduce that $W(u_1, u_2)(a)$ vanishes and the eigenvalues can be degenerate. For example, $u'' + \lambda^2 u = 0$ with $u(0) = u(2\pi)$ has solutions $u_n(x) = \sin(2\pi\lambda_n x)$, with $\lambda_n \in \{1/2, 1, 3/2, 2, \cdots\}$ and $v_n(x) = \cos(2\pi\mu_n x)$ with $\mu_n \in \{1, 2, \cdots\}$. But even in this case, the lowest eigenvalue ($= \lambda_1 = 1/2$) is non-degenerate with eigenfunction $\sin(x/2)$.

11.6 Sturm Comparison Theorem

In this section we prove an important result:

Theorem 11.4 (Sturm Comparison Theorem) *Let $u_i(x)$ and $u_j(x)$ be eigenfunctions of Equation* (11.4) *subject to regular boundary conditions where $\lambda_j > \lambda_i$ then between any two zeroes of $u_i(x)$ there exists at least one zero of $u_j(x)$.*

Remark: This is a key result which will enable us to prove properties #3–#6 in Section 11.5. Loosely speaking, what it means is that 'the bigger the eigenvalue, the faster eigenfunction oscillates'.

Proof: We take $i = 1$ and $j = 2$ to simplify the notation. Thus, we are working with:

$$\lambda_2 > \lambda_1 \tag{11.6}$$

Thus, we also have $L[u_1](x) = -\lambda_1 r(x) u_1(x)$ and $L[u_2](x) = -\lambda_2 r(x) u_2(x)$. The boundary conditions are the same for u_1 and u_2.

So now we have:

$$u_1(x)L[u_2](x) - u_2(x)L[u_1](x) = -\underbrace{(\lambda_2 - \lambda_1)}_{>0} r(x)u_1(x)u_2(x).$$

We integrate from a to x and obtain:

$$\int_a^x \left[u_1 \frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u_2}{\mathrm{d}x}\right) - u_2 \frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u_1}{\mathrm{d}x}\right)\right] \mathrm{d}x = -(\lambda_2 - \lambda_1)\int_a^x r u_1 u_2 \,\mathrm{d}x.$$

Hence:

$$\int_a^x \left[-\frac{\mathrm{d}u_1}{\mathrm{d}x} p \frac{\mathrm{d}u_2}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(u_1 p \frac{\mathrm{d}u_2}{\mathrm{d}x}\right)\right] \mathrm{d}x + \int_a^x \left[+\frac{\mathrm{d}u_1}{\mathrm{d}x} p \frac{\mathrm{d}u_2}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(u_1 p \frac{\mathrm{d}u_2}{\mathrm{d}x}\right)\right] \mathrm{d}x$$
$$= -(\lambda_2 - \lambda_1)\int_a^x r u_1 u_2 \,\mathrm{d}x.$$

This gives:

$$\underbrace{\left[p(s)\big(u_1(s)u_2'(s) - u_2(s)u_1'(s)\big)\right]_a^x}_{=0 \text{ at } s=a} = -(\lambda_2 - \lambda_1)\int\limits_a^x r(s)u_1(s)u_2(s)\,\mathrm{d}s.$$

Here, we have switched to using a proper dummy index, s. We apply the BCs at $x = a$ to get:

$$p(x)\big(u_1(x)u_2'(x) - u_2(x)u_1'(x)\big) = -(\lambda_2 - \lambda_1)\int\limits_a^x r(s)u_1(s)u_2(s)\,\mathrm{d}s$$

Now suppose ξ_1 and ξ_2 are successive zeros of $u_1(x)$, with $\xi_2 > \xi_1$, say, then

$$-p(\xi_1)u_2(\xi_1)u_1'(\xi_1) = -(\lambda_2 - \lambda_1)\int\limits_a^{\xi_1} r(s)u_1(s)u_2(s)\,\mathrm{d}s$$

$$-p(\xi_2)u_2(\xi_2)u_1'(\xi_2) = -(\lambda_2 - \lambda_1)\int\limits_a^{\xi_2} r(s)u_1(s)u_2(s)\,\mathrm{d}s$$

and subtracting

$$-p(\xi_2)u_2(\xi_2)u_1'(\xi_2) + p(\xi_1)u_2(\xi_1)u_1'(\xi_1) = -(\lambda_2 - \lambda_1)\int\limits_{\xi_1}^{\xi_2} r(s)u_1(s)u_2(s)\,\mathrm{d}s. \tag{11.7}$$

Here, we have used:

$$\int_a^{\xi_2}(\cdots) - \int_a^{\xi_1}(\cdots) = \int_{\xi_1}^{\xi_2}(\cdots), \qquad \xi_2 > \xi_1.$$

Since ξ_1 and ξ_2 are the successive zeros of $u_1(x)$, we have two cases to consider:

- $u_1(x) > 0$ for $x \in (\xi_1, \xi_2)$ with $u_1'(\xi_1) > 0$... u_1 is $\cap$-shaped;
- $u_2(x) < 0$ for $x \in (\xi_1, \xi_2)$ with $u_1'(\xi_1) < 0$ and $u_1'(\xi_2) > 0$... u_1 is $\cup$-shaped.

For definiteness assume the first (otherwise we just multiply by -1). Now, for contradiction, assume that $u_2(x)$ has no zero between ξ_1 and ξ_2 then, as before, we may assume it is positive over the whole interval so that $u_2(\xi_1) \geq 0$ and $u_2(\xi_2) \geq 0$, then looking just at the sign of terms in Equation (11.7), on the left hand side we have

$$-p(\xi_2)u_2(\xi_2)u_1'(\xi_2) + p(\xi_1)u_2(\xi_1)u_1'(\xi_1) = -(+)(+)(-)_0 + (+)(+)(+)_0 = (+)_0$$

where, $(+)_0$ means positive or 0 while $(+)$ means strictly positive. On the other hand, on right hand side we have

$$-(\lambda_2-\lambda_1)\int\limits_{\xi_1}^{\xi_2} r(s)u_1(s)u_2(s)\,\mathrm{d}s = -(+)\int\limits_{\xi_1}^{\xi_2}(+)(+)(+)\,\mathrm{d}s = (-).$$

This gives us our contradiction and we conclude that as claimed $u_2(x)$ must have a zero between ξ_1 and ξ_2.

There is a more general result than Theorem 11.4, which we state here as follows:

Theorem 11.5 (Sturm–Picone Comparison Theorem) *Let y_i be real-valued continuous functions on the interval $[a,b]$, with $i=1,2$, and let*

$$(p_1(x)y')' + q_1(x)y = 0, \tag{11.8a}$$

$$(p_2(x)y')' + q_2(x)y = 0, \tag{11.8b}$$

$$0 < p_2(x) \le p_1(x)$$

and

$$q_1(x) \le q_2(x).$$

Let u be a non-trivial solution of (11.8a) *with successive roots at z_1 and z_2 and let v be a non-trivial solution of* (11.8b)*. Then one of the following properties holds:*

- *There exists an $x \in (z_1, z_2)$ such that $v(x) = 0$, or*
- *There exists a $\mu \in \mathbb{R}$ such that $v(x) = \mu u(x)$.*

Chapter 12

Theoretical Analysis of the Sturm-Liouville Problem

Overview

In Chapter 11 we showed – using results built up already – that the solution to the SL problem can be exapanded in terms of eigenfunctions. Buried in the detail, this relies on the compactness property if integral operators, and the existence of a solution of the Rayleigh–Ritz variational problem – which we have not proved. This can be a bit unsatisfactory. Therefore, in this Chapter, we look at a more intuitive way of characterizing the eigenfunctions of the SL equation, and proving a lot of important results along the way. As before, we are concerned with the eigenvalue problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u(x) = -\lambda r(x)u(x), \qquad x \in (a,b), \tag{12.1a}$$

where $p(x) > 0$ and $r(x) > 0$ are continuous functions, with boundary conditions

$$\alpha_a u(a) + \beta_a u'(a) = 0, \qquad \alpha_b u(b) + \beta_b u'(b) = 0. \tag{12.1b}$$

12.1 The Prüfer System

To examine the zeros of the eigenfunctions it turns out to be convenient to introduce an argument-phase representation of our solution in the (conventional) form

$$u(x) = \rho(x)\sin\theta(x)$$
$$p(x)u'(x) = \rho(x)\cos\theta(x).$$

Note that

- a non-trivial solution can never pass through the origin $\rho(x_0) = 0$ as that would imply $u(x_0) = 0$ and $u'(x_0) = 0$ and so, by Picard $u(x) = 0$;
- zeroes of $u(x)$ correspond to points where $\theta(x) = 0, \pm\pi, \pm 2\pi, \ldots$

For notational simplicity, and to emphasise the generality of the results, we write our equation as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right] - Q(x)u(x) = 0$$

so the Sturm–Liouville equation corresponds to $Q(x) = q(x) - \lambda r(x)$. Then

$$\begin{aligned} p(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho(x)\sin\theta(x)\right) &= \rho(x)\cos\theta(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\rho(x)\cos\theta(x)\right) &= Q(x)\rho(x)\sin\theta(x) \end{aligned}$$

so

$$\begin{aligned} \rho'(x)\sin\theta(x) + \rho(x)\theta'(x)\cos\theta(x) &= \frac{\rho(x)}{p(x)}\cos\theta(x) \\ \rho'(x)\cos\theta(x) - \rho(x)\theta'(x)\sin\theta(x) &= Q(x)\rho(x)\sin\theta(x) \end{aligned}$$

or

$$\begin{pmatrix} \sin\theta(x) & \rho(x)\cos\theta(x) \\ \cos\theta(x) & -\rho(x)\sin\theta(x) \end{pmatrix} \begin{pmatrix} \rho'(x) \\ \theta'(x) \end{pmatrix} = \begin{pmatrix} \dfrac{\rho(x)}{p(x)}\cos\theta(x) \\ Q(x)\rho(x)\sin\theta(x) \end{pmatrix}.$$

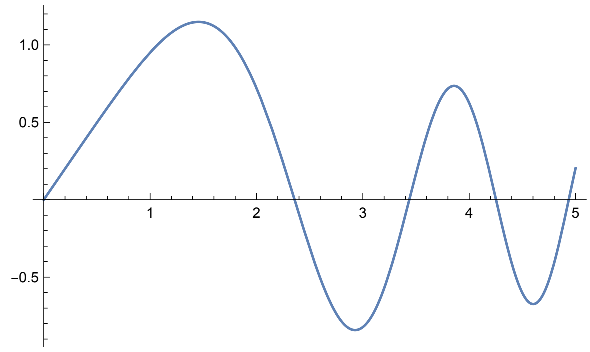
Solving we obtain

$$\begin{aligned} \rho'(x) &= \rho(x)\left(\frac{1}{p(x)} + Q(x)\right)\sin\theta(x)\cos\theta(x) \\ \theta'(x) &= \frac{1}{p(x)}\cos^2\theta(x) - Q(x)\sin^2\theta(x). \end{aligned}$$

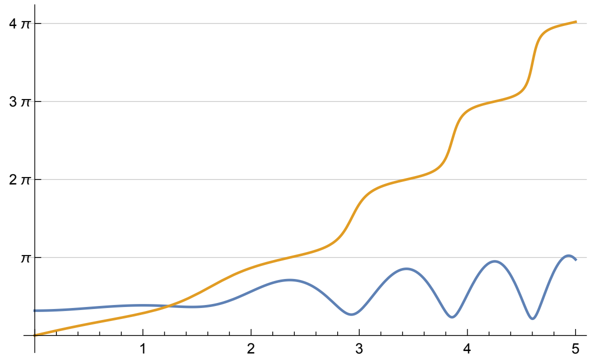
Obvious but important properties follow

- The $\theta'(x)$ equation does not involve $\rho(x)$ and so is just a first order ODE for $\theta(x)$;
- once we have $\theta(x)$ the $\rho'(x)$ equation maybe solved immediately by quadrature:

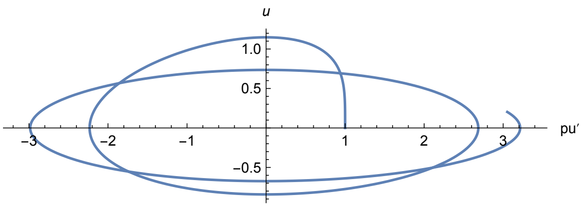
$$\rho(x) = A\exp\left(\int_a^x \left(\frac{1}{p(s)} + Q(s)\right)\sin\theta(s)\cos\theta(s)\,\mathrm{d}s\right);$$



(a) $u(x)$



(b) $\rho(x)$, $\theta(x)$



(c) Phase-space plot

Figure 12.1: Illustration of the Prüfer system with $p(x) = 1$, $Q(x) = -x^2$, $y(0) = 0$, $y'(0) = 1$. Panel (a): the solution as a function of x; Panel (b): $\rho(x)$ (blue) and $\theta(x)$ (orange) Panel (c) Phase-space plot $(\rho(x)\cos\theta(x), \rho(x)\sin\theta(x)) = \big(p(x)u'(x), u(x)\big)$ Top left: the solution as a function of x, top right: $\rho(x)$ (blue) and $\theta(x)$ (orange). In panel (c) the zeros of $u(x)$ correspond to crossings of the horizontal axis or equivalently $\theta(x) = 0, \pm\pi, \pm 2\pi, \ldots$.

- this form emphasises our comment about not passing through the origin and that zeroes of $u(x)$ are determined entirely by the $\theta'(x)$ equation.

Considering then the angular equation

$$\theta'(x) = \frac{1}{p(x)}\cos^2\theta(x) - Q(x)\sin^2\theta(x),$$

we may make an important observation: at any point x_0 where we have a zero of $u(x)$ and so $\sin\theta(x_0)$, we also have $\cos\theta(x_0) = \pm 1$ and so

$$\theta'(x_0) = \frac{1}{p(x_0)} > 0,$$

in other words, $\theta(x)$ may only cross a line $\theta = 0, \pm\pi, \pm 2\pi, \ldots$ in an increasing direction An immediate consequence is that $\sin\theta(x_0) = 0$ cannot have a repeated root as this would require

$$\theta'(x_0)\cos\theta(x_0) = 0,$$

in other words a solution of a self-adjoint differential equation cannot have a repeated zero (a point where it just touches the x-axis).

12.2 Incorporating the boundary conditions into the Prüfer system

Our boundary condition on $\alpha_a u(a) + \beta_a u'(a) = 0$ directly implies

$$0 = \alpha_a p(a)u(a) + \beta_a p(a)u'(a) = r(a)\left(\alpha_a p(a)\cos\theta(a) + \beta_a \sin\theta(a)\right)$$

and so, since $r(a)$ must be non-zero, $\theta(a)$ is determined by

$$\cos\bigl(\theta(a) - \epsilon_a\bigr) = 0$$

where, using the freedom to multiply the boundary condition by -1 so that we may take $\alpha_a \geq 0$, $\epsilon_a \in [-\pi/2, \pi/2)$ is uniquely defined by

$$\cos\epsilon_a = \frac{\alpha_a p(a)}{\Bigl(\bigl(\alpha_a p(a)\bigr)^2 + \beta_a^2\Bigr)^{1/2}}, \qquad \sin\epsilon_a = \frac{\beta_a}{\Bigl(\bigl(\alpha_a p(a)\bigr)^2 + \beta_a^2\Bigr)^{1/2}}.$$

In turn, we may determine a unique value of $\theta(a) = \frac{\pi}{2} + \epsilon_a \in [0, \pi)$ encapsulating the boundary condition.

Since $r(a)$ was an arbitrary normalisation we may then integrate our first order system from $x = a$ to $x = b$. Obviously as we do so, $\theta(x)$ may increase beyond this range and cross the line $\theta = \pi$, then $\theta = 2\pi$ etc but as we have shown it can then never cross back. At $x = b$ we will have a corresponding condition

$$\cos\bigl(\theta(b) - \epsilon_b\bigr) = 0$$

with $\epsilon_b \in (-\pi/2, \pi/2]$ defined in the analogous way (with modified endpoints for later convenience). In the light of our comments at the end of the last paragraph we must seek out the values

$$\theta(b) = \tfrac{\pi}{2} + \epsilon_b + n\pi \qquad n \in \mathbb{N} \cup \{0\}.$$

12.3 The Sturm-Liouville Problem as a Prüfer System

Now specialising to the case of the Sturm-Liouville problem we have $Q(x) = q(x) - \lambda r(x)$. We may consider the one parameter family of solutions $\theta_\lambda(x)$ which all satisfy the same initial $\theta_\lambda(a) = \frac{\pi}{2} + \epsilon_a$, but where for now we do not worry about the boundary condition at $x = b$. Given that we have chosen $\theta_\lambda(a) \in [0, \pi)$, and $\theta_\lambda(x)$ can only increase through the lines $\theta = n\pi$, it is clear that the first zero, if one exists, of $u_\lambda(x)$ in the open interval (a, b) occurs where $\theta_\lambda(x) = \pi$, and by trivially extending the argument, the n^{th} zero of $u_\lambda(x)$, if one exists, occurs where $\theta_\lambda(x) = n\pi$.

The final value $\theta(b)$ will then dependent on the parameter λ. The fundamental question is then: do there exists appropriate values for λ such that we can attain each of the values $\theta(b) = \frac{\pi}{2} + \epsilon_b + n\pi$ for every $n \in \mathbb{N} \cup \{0\}$? To see that the answer is yes, we need to understand how the phase $\theta_\lambda(x)$ with fixed initial condition $\theta_\lambda(a) = \frac{\pi}{2} + \epsilon_a$ behaves as a function of λ.

To see this just requires a minor modification to the Comparison Theorems of Chapter 2:

Theorem 12.1 (Oscillation Theorem) *For $i = 1, 2$ let*

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}x} = \frac{1}{p(x)} \cos^2 \theta_i(x) - Q_i(x) \sin^2 \theta_i(x), \qquad x \in (a, b),$$

where $Q_1(x) > Q_2(x)$ for $x \in (a, b)$. Suppose $\theta_1(a) = \theta_2(a)$ then $\theta_1(x) < \theta_2(x)$ for all $x \in (a, b]$.

Proof: Take

$$F(x, y) = \frac{1}{p(x)} \cos^2 y(x) - Q_2(x) \sin^2 y(x), \qquad G(x, y) = \frac{1}{p(x)} \cos^2 z(x) - Q_1(x) \sin^2 z(x).$$

Then clearly as continuously differentiable functions they satisfy a Lipschitz condition in y. In addition,

$$G(x, y) = F(x, y) - \big(Q_1(x) - Q_2(x)\big) \sin^2 y < F(x, y)$$

and the result follows.

Remark: Here we have applied the result with a strict inequality rather that the 'less than or equal' result of Chapter 2 but it is trivial to modify the proof there to cover the current case.

For the Sturm–Liouville case, writing $Q_i(x) = q(x) - \lambda_i r(x)$ and recalling that $r(x) > 0$ our result says that if $\lambda_1 < \lambda_2$ then $\theta_{\lambda_1}(x) < \theta_{\lambda_2}(x)$ for all $x \in (a, b]$.

We may deduce the following lemma:

Lemma 12.2 *For all $x \in (a, b]$, $\theta_\lambda(x)$ is a strictly increasing function of λ (and clearly depends continuously on λ).*

We are now ready to prove our fundamental result:

Theorem 12.3 *The solution $\theta_\lambda(x)$ satisfies*

- $\lim\limits_{\lambda \to -\infty} \theta_\lambda(x)$ *exists and is equal to 0;*
- $\theta_\lambda(x) \to \infty$ *as* $\lambda \to \infty$.

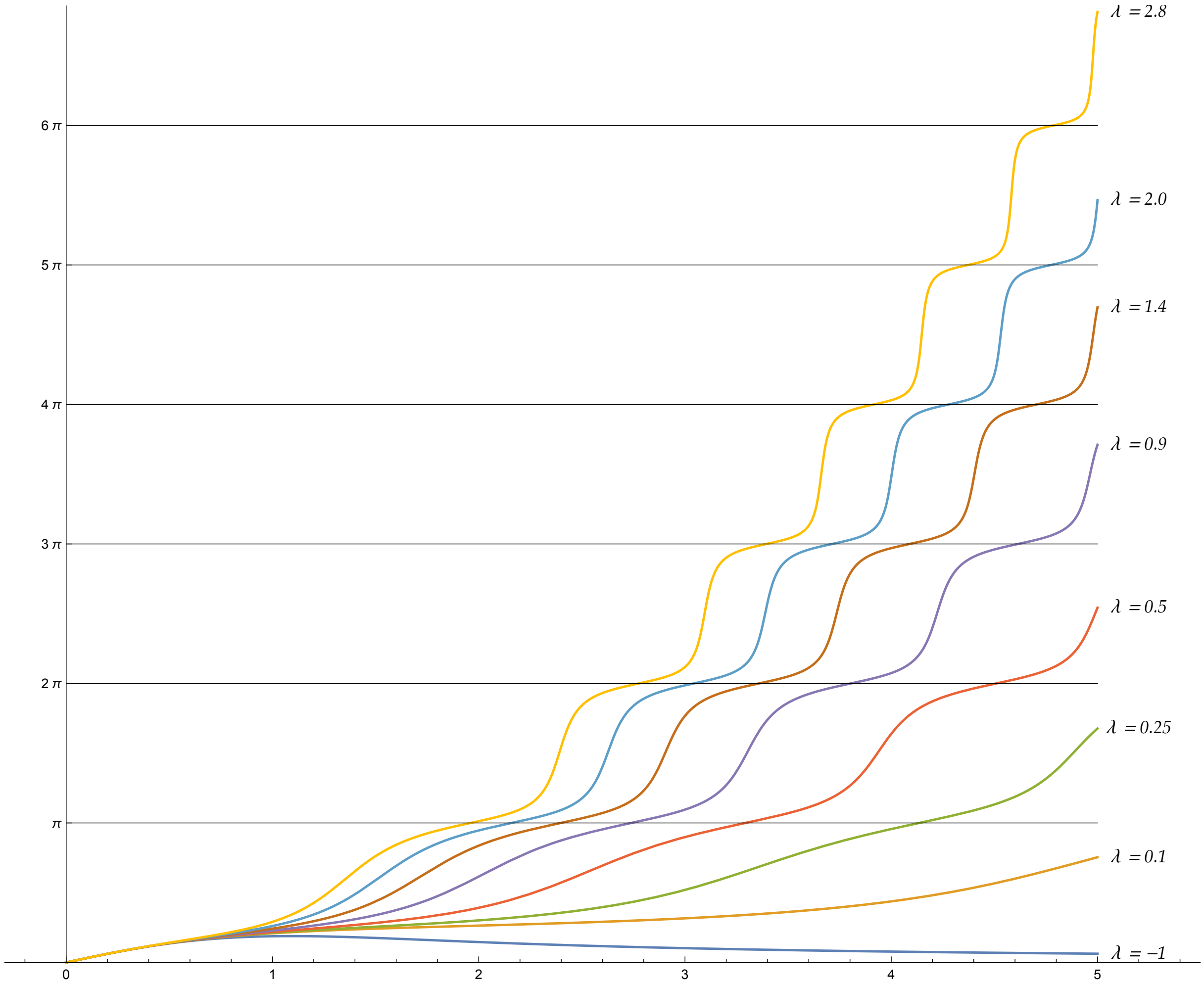


Figure 12.2: Illustration of the Prüfer system for a Sturm-Liouville system with $p(x) = 1$, $Q(x) = q(x) - \lambda r(x) = 1 - \lambda x^2$, $y(0) = 0$, $y'(0) = 1$. $\theta(x)$ as a function of x, for a series of values of λ.

Proof:

Case 1, $\lambda \to -\infty$:

By definition, $\theta_\lambda(a) \in [0, \pi)$, but we know that $\theta(x)$ can only cross $0\,(= 0\,\pi)$ in an upward direction so $\theta_\lambda(b) > 0$ for all λ. Thus $\theta_\lambda(b)$ is a continuous increasing function of λ bounded below by 0 hence the limit (as $\lambda \to -\infty$) exists and is greater than or equal to 0. Next our phase equation

$$\frac{\mathrm{d}\theta_\lambda}{\mathrm{d}x} = \frac{1}{p(x)} \cos^2\theta_\lambda(x) - [q(x) - \lambda r(x)] \sin^2\theta_\lambda(x).$$

implies

$$\frac{\theta_\lambda(x) - \theta_\lambda(a)}{\lambda} = \frac{1}{\lambda} \int_a^x \left[\frac{1}{p(s)} \cos^2\theta_\lambda(s) - q(s) \sin^2\theta_\lambda(s) \right] \mathrm{d}s + \int_a^x r(s) \sin^2\theta_\lambda(s)\,\mathrm{d}s.$$

On the left-hand side, and for $\lambda < 0$, we have that $\theta_\lambda(a)$ and $\theta_\lambda(x)$ are both bounded between 0

and $\max\{\theta_0(a), \theta_0(x)\}$. So in the limit as $\lambda \to -\infty$, the LHS tends to zero. Similarly, the first term on the RHS vanishes as the integrand is easily bounded. We conclude that:

$$0 = 0 + \int_a^x r(s)\sin^2\theta_{-\infty}(s)\,\mathrm{d}s.$$

and since $r(x) > 0$ we deduce that $\sin^2\theta_{-\infty}(s) = 0$, so $\theta_{-\infty}(s) = 0 \mod \pi$ and $\theta_{-\infty}(s) = 0$ is the only possible solution.

Case 2, $\lambda \to \infty$:

We use Gronwall's Inequality:

$$\begin{aligned} \frac{\mathrm{d}\theta_\lambda}{\mathrm{d}x} &= \frac{1}{p(x)}\cos^2\theta(x) - [q(x) - \lambda r(x)]\sin^2\theta(x), \\ &\geq (\lambda r_{min} - q_{max})\sin^2\theta. \end{aligned}$$

Hence

$$\frac{\mathrm{d}\theta_\lambda}{\sin^2\theta_\lambda} \geq (\lambda r_{min} - q_{max})\mathrm{d}x.$$

Hence

$$-\cot\theta_\lambda(x) \geq \cot(\theta_0) + (\lambda r_{min} - q_{max})(x - x_0).$$

Hence,

$$\cot\theta_\lambda(x) \leq -\cot(\theta_0) - (\lambda r_{min} - q_{max})(x - x_0),$$

As $\lambda \to \infty$, we get $\cot\theta_\lambda(x) \leq -\infty$, which forces $\theta(x) = n\pi$, where $n\pi$ is an integer. But the only way for this to be true for all x is either $n = 0$ or $n = \infty$. The value $n = 0$ is ruled out as being inconsistent with the phase equation, so the only other possibility is $n = \infty$, hence $\theta_\lambda(x) \to \infty$ as $\lambda_{\to}\infty$.

We summarize our results so far as follows:

Theorem 12.4 *Any regular Sturm-Liouville problem has an infinite number of solutions $u_n(x)$ which belong to the real eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ with $\lim_{n\to\infty}\lambda_n = \infty$.*

Furthermore, each eigenfunction $u_n(x)$

1. *has exactly n zeroes in the interval $a < x < b$,*
2. *is unique up to a constant multiplicative factor.*

12.4 Completeness of Sturm-Liouville Eigenfunctions

We are now in a position to show – in a quite direct fashion – that the set of eigenfunctions of the regular Sturm-Liouville problem is a generalized Fourier basis. That is, that the eigenfunctions form a complete set. We introduce the Hilbert Space $\mathcal{H}$, where

$$\mathcal{H} = \left\{ u(x) | \int_a^b r(x)u(x)^2 \,\mathrm{d}x \text{ exists and } \alpha_a u(a) + \beta_a u'(a) = 0, \alpha_b u(b) + \beta_b u'(b) = 0 \right\}.$$

We show: We have the eigenfunctions

$$L[u_n](x) = -\lambda_n r(x) u_n(x), \qquad n \in \{0, 1, 2, \cdots\}.$$

As $r(x) > 0$ strictly, we further write this eigenvalue problem as $L_r[u_n](x) = -\lambda_n u_n(x)$. Now, given eigenfunctions $u_0(x), u_1(x), \ldots$ we introduce

$$T_N = \text{Span}\{u_0, u_1, \cdots, u_N\}$$

and following Chapter 9 we write:

$$\mathcal{H} = T_N \oplus T_N^{\perp},$$

The key to proving completeness is to consider the Rayleigh quotient defined by

$$\mathcal{Q}[u] = \frac{(u, -L_r u)}{(u, u)}.$$

Theorem 12.5 (The Rayleigh Variational Principle)

$$\min_{u \in \mathcal{H}} \mathcal{Q}[u] = \lambda_0$$

and as a generalization

$$\min_{u \in \mathcal{T}_N^{\perp}} \mathcal{Q}[u] = \lambda_{N+1}$$

The Rayleigh quotient is a functional mapping functions $u(x) \in \mathcal{H}$ to $\mathbb{R}$. Extreme 'points' (actually functions) must be stationary with respect to arbitrary variations $\delta u(x)$ to first order, so that dropping term of order δu^2:

$$\begin{aligned}
0 = \mathcal{Q}[u + \delta u] - \mathcal{Q}[u] &= \frac{\langle u + \delta u, -L_r u - L_r \delta u\rangle}{\langle u + \delta u, u + \delta u\rangle} - \frac{\langle u, -L_r u\rangle}{\langle u, u\rangle} \\
&= \frac{\langle u, -L_r u\rangle + 2\langle \delta u, -L_r u\rangle}{\langle u, u\rangle\big(1 + 2\langle \delta u, u\rangle / \langle u, u\rangle\big)} - \frac{\langle u, -L_r u\rangle}{\langle u, u\rangle} \\
&= \frac{2}{\langle u, u\rangle}\left[\langle \delta u, -L_r u\rangle - \frac{\langle u, -L_r u\rangle}{\langle u, u\rangle}\langle \delta u, u\rangle\right] \\
&= \frac{2}{\langle u, u\rangle}\langle \delta u, -L_r u - \frac{\langle u, -L_r u\rangle}{\langle u, u\rangle} u\rangle.
\end{aligned}$$

For this to hold true for arbitrary $\delta u(x)$ we conclude that the extreme points must satisfy

$$L_r u(x) = \frac{\langle u, L_r u\rangle}{\langle u, u\rangle} u(x),$$

that is $u(x)$ must correspond to an eigenfunction of our Sturm-Liouville problem. Since

$$\mathcal{Q}[u_i] = \lambda_i$$

the result is now immediate.

Theorem 12.6 *The set of eigenfunctions of the regular Sturm–Liouville problem form a complete basis set for $\mathcal{H}$.*

Proof: We consider the error function

$$d_N(x) = u(x) - \sum_{i=0}^{N} c_i u_i(x)$$

which, of course, satisfies

$$\langle u_i, d_N\rangle = 0, \qquad\qquad i = 0, \ldots, N$$

that is, $d_N \in T_N^{\perp}$ so

$$\mathcal{Q}[d_N] \geq \lambda_{N+1}.$$

If we write this out in full it says

$$\langle d_N, -L_r d_N\rangle \geq \lambda_{N+1}\langle d_N, d_N\rangle.$$

or

$$\langle d_N, d_N\rangle \leq \frac{\langle d_N, -L_r d_N\rangle}{\lambda_{N+1}}.$$

Now as before

$$\begin{aligned}
\langle d_N, -L_r d_N\rangle &= \langle u - \sum_{i=0}^{N} c_i u_i, -L_r u - \sum_{j=0}^{N} c_j(-L_r u_j)\rangle \\
&= \langle u, -L_r u\rangle - 2\sum_{i=0}^{N} c_i\lambda_i\langle u, u_i\rangle, - \sum_{i=0}^{N}\sum_{j=0}^{N} c_i c_j \lambda_j \langle u_i, u_j\rangle \\
&= \langle u, -L_r u\rangle - \sum_{i=0}^{N} c_i^2 \lambda_i,
\end{aligned}$$

where in the second line we have used the self-adjointness of L_r. Now since the eigenvalues tend to infinity only a finite number n_0, say of them can be negative. Correspondingly every subsequent term in the sum is negative, so we have

$$(d_N, -L_r d_N) = \langle u, -L_r u\rangle - \sum_{i=0}^{N} c_i^2 \lambda_i \leq \langle u, -L_r u\rangle - \sum_{i=0}^{n_0} c_i^2 \lambda_i$$

independent of N. Hence

$$\|d_N\|^2 = \langle d_N, d_N\rangle \leq \frac{\langle u, -L_r u\rangle - \sum_{i=0}^{n_0} c_i^2 \lambda_i}{\lambda_{N+1}}.$$

where the only dependence on N on the right hand side is in the denominator. But we know that $\lambda_{N+1} \to \infty$ as $N \to \infty$ so correspondingly

$$\|d_N\| = \left\| u(x) - \sum_{i=0}^{N} c_i u_i(x) \right\| \to 0 \qquad \text{as } N \to \infty.$$

Hence, the result is shown.

12.5 Expansions of Green functions

Suppose we want to solve

$$L[u] + \lambda r(x) u(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right] - q(x)u + \lambda r(x) u(x) = f(x), \tag{12.2}$$

subject to appropriate Sturm-Liouville boundary conditions, where we include the λ term for generality.

Let $G_\lambda(x, s)$ be the Green's Function of the linear operator

$$L + \lambda r.$$

Thus, the solution of Equation (12.2) can be written as:

$$u(x) = \int_a^b G_\lambda(x, s) f(s) \mathrm{d}s.$$

However, by completeness of the eigenfunctions of the SL operator $L[u]$, $u(x)$ has the expansion:

$$u(x) = \sum_{n=0}^{\infty} a_n u_n(x).$$

Substitution into Equation (12.2) gives:

$$a_n = \frac{\langle f, u_n \rangle}{\lambda - \lambda_n}, \qquad \lambda \neq \lambda_n.$$

Thus,

$$\begin{aligned} u(x) &= \sum_{n=0}^{\infty} a_n u_n(x), \\ &= \sum_{n=0}^{\infty} \langle f, u_n \rangle \frac{u_n(x)}{\lambda - \lambda_n}, \\ &= \sum_{n=0}^{\infty} \int_a^b f(s) u_n(s) \frac{u_n(x)}{\lambda - \lambda_n}, \\ &= \int_a^b \left(\sum_{n=0}^{\infty} \frac{u_n(x) u_n(s)}{\lambda - \lambda_n} \right) f(s) \mathrm{d}s. \end{aligned}$$

Thus, we have the following spectral representation of the Green's function:

$$G_\lambda(x, s) = \sum_{n=0}^{\infty} \frac{u_n(x) u_n(s)}{\lambda - \lambda_n}, \qquad \lambda \neq \lambda_n.$$

12.5.1 Example

Consider the boundary value problem $u'' + \lambda u = f$ with $u(0) = u(\pi) = 0$. The normalized eigenfunctions and respective eigenvalues are given, for $n \geq 1$, by

$$u_n(x) = \sqrt{\frac{2}{\pi}} \sin nx, \qquad \lambda_n = n^2.$$

Using the above Green function expansion, the solution to the BVP is given, for $\lambda \notin \{\lambda_n\}$, by

$$u(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda - n^2} \sqrt{\frac{2}{\pi}} \sin nx,$$

where

$$c_n = \sqrt{\frac{2}{\pi}} \int_0^\pi f(x) \sin(nx) dx.$$

For example, if $\lambda = 0$ and $f(x) = -x$, the solution is

$$u(x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nx.$$

Note that in this case we may also solve the BVP directly using the Greens' function formula

$$G(x,s) = \begin{cases} s(x-\pi)/\pi & s < x \\ x(s-\pi)/\pi & x < s, \end{cases}$$

which yields

$$u(x) = \int_0^\pi G(x,s)(-s)ds = \frac{1}{6}x(\pi^2 - x^2).$$

12.6 Fourier series

Consider the boundary value problem $u'' + \lambda u = 0$ with periodic boundary conditions $u(-\pi) = u(\pi)$ and $u'(-\pi) = u'(\pi)$. In this case, 0 is an eigenvalue with corresponding eigenfunction given by the constant function $1/(2\pi)$. The other eigenvalues are again n^2 for $n \geq 1$, but these are not simple: there are two eigenfunctions with eigenvalue n^2, namely $\sin(nx)/\sqrt{\pi}$ and $\cos(nx)/\sqrt{\pi}$.

From the Sturm-Liouville theory, these eigenfunctions form a complete basis for the Hilbert space $L^2([-\pi,\pi])$. In other words, any $f \in L^2([-\pi,\pi])$ may be expanded as

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$$

Note that in this expansion, the convergence of the series is in $L^2([-\pi,\pi])$ and the limit equals $f(x)$ almost everywhere. In this context, Parseval's identity may be expressed as

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = \tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

The above Fourier series may also be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

In this notation, Parseval's identity may be written as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

12.6.1 Example

Letting $f(x) = x$, we have $c_0 = 0$ and, for $n \neq 0$, $c_n = i(-1)^n/n$. Now

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = 2\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3},$$

so in this example Parseval's identity yields Euler's remarkable formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The Fourier series is given by

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Recall that this identity is in an L^2 sense. In the next chapter we will see that, in fact, it holds pointwise for $-\pi < x < \pi$. So for example when $x = \pi/2$ this yields the identity

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}.$$

On the other hand, it is easy to see that the Fourier series equals zero at $x = \pi$ and $x = -\pi$ and in fact this is also the value of $(f(\pi) + f(-\pi))/2$, as we shall explain in the next chapter.

Chapter 13

Fourier Series and their Convergence

Overview

As a special case of a Sturm-Liouville system, we have shown that Fourier series converge in a mean-square sense. However, given the simplicity of the eigenfunctions (which are, of course, just the trigonometric functions) we are able to prove stronger results that we summarise in this Chapter.

13.1 Formal Definition

Recall the formal definition: if $f\colon [-\pi,\pi] \to \mathbb{R}$ is integrable on $[-\pi,\pi]$ or in other words that $f \in L^1[-\pi,\pi]$:

$$\int_{-\pi}^{\pi} |f(x)|\,\mathrm{d}x < \infty$$

then we may define for $n \in \mathbb{N} \cup \{0\}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \quad ,$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}x.$$

Clearly, $a_n, b_n \in \mathbb{R}$, $b_0 = 0$, and

$$c_n = \frac{1}{2}\left(a_n - ib_n\right) \qquad \text{and} \qquad c_{-n} = \frac{1}{2}\left(a_n + ib_n\right).$$

Definition 13.1 *With these 'Fourier coefficients' defined we may define the formal real Fourier series as:*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

and the formal complex Fourier series as

$$\sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx}.$$

The partial sums of these two series, interpreted as the sum up to N including the $n = 0$ term in the real case and the sum from $-N$ to N in the complex case, are clearly equal so we may work with whichever is more convenient. We choose the complex form, and define

$$s_N(x) = \sum_{n=-N}^{N} c_n \mathrm{e}^{\mathrm{i}nx}. \tag{13.1}$$

Remark: As Equation (13.1) involves a finite sum, $s_N(x)$ is perfectly well defined unlike the formal infinite sums where we must address the issue of convergence.

Remark: The integrals above could have been defined for a general interval $[x_0, x_0 + T]$ where T is the period $f(x + T) = f(x)$ and where the coefficients becomes e.g.

$$a_n = \frac{2}{T}\int_{x_0}^{x_0+T} f(x) \cos \frac{2\pi nx}{T} \,\mathrm{d}x.$$

One could transform variables $T \to 2\pi$ to resort to the previous notation.

There are two central question we address below:

1. To what extent do the formal Fourier series determine the function pointwise (rather than just in a mean-square sense)?
2. If it does, how can we recover the function under the least restrictive conditions?

13.2 Some Properties of the Fourier Coefficients

Theorem 13.1 (Riemann–Lebesgue Lemma) *Let $f \in L^2[-\pi, \pi]$ then*

$$c_n \to 0 \qquad \textit{as} \quad |n| \to \infty.$$

Proof: This is an immediate consequence of (the obvious complex extension of) Bessel's inequality as

$$\sum_{n=-N}^{N} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|f(x)\right|^2 \mathrm{d}x,$$

since e^{-inx} are orthogonal eigenfunctions and also here the weight function $r(x) = 1/2\pi$ being also the normalization term. The only way for the sum above to be upper bounded for any N is that $\lim_{n\to\infty} c_n = 0$.

Theorem 13.2 (Bounds related to smoothness) *Let f be periodic with period 2π and C^k then*

$$|n^k c_n| \qquad \text{is bounded as} \quad n \to \infty.$$

Remark: For those familiar with the 'Big-oh', O, notation, this may be rephrased as $c_n = O(n^{-k})$ as $n \to \infty$.

Proof: We simply integrate by parts (for $n \neq 0$):

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}x \\ &= \frac{1}{2\pi} \left(\left[f(x) \frac{\mathrm{e}^{-\mathrm{i}nx}}{-\mathrm{i}n} \right]_{-\pi}^{\pi} - \frac{1}{-\mathrm{i}n} \int_{-\pi}^{\pi} f'(x)\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}x \right) \\ &= \frac{1}{2\pi(\mathrm{i}n)} \int_{-\pi}^{\pi} f'(x)\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}x \end{aligned}$$

as the boundary term vanishes by periodicity. Recall that the T-periodicity of f induce also the T-periodicity of the higher order derivatives $f^{(k)}$. Repeating k times

$$c_n = \frac{1}{2\pi(\mathrm{i}n)^k} \int_{-\pi}^{\pi} f^{(k)}(x)\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}x$$

and as

$$\left|2\pi(\mathrm{i}n)^k c_n\right| = \left| \int_{-\pi}^{\pi} f^{(k)}(x)\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}x \right| \leq \int_{-\pi}^{\pi} \left|f^{(k)}(x)\right|\left|\mathrm{e}^{-\mathrm{i}nx}\right| \mathrm{d}x \leq 2\pi \sup_{[-\pi,\pi]} \left|f^{(k)}(x)\right|,$$

recalling the norm of complex numbers involved the result follows.

Remark: We emphasize that here we are requiring the corresponding degree of smoothness even at $\pm\pi$. A common problem involves a function $f \in C^k(-\pi, \pi)$ with a weaker definition of periodicity, so only $f(-\pi+) = f(\pi-)$, ..., $f^{(j-1)}(-\pi+) = f^{(j-1)}(\pi-)$ $(j < k)$. In this case we may integrate by parts $j+1$ times to find

$$c_n = \frac{1}{2\pi(in)^j}\left(\left[f^{(j)}(x)\frac{\mathrm{e}^{-\mathrm{i}nx}}{-\mathrm{i}n}\right]_{-\pi}^{\pi} - \frac{1}{-\mathrm{i}n}\int_{-\pi}^{\pi} f^{(j+1)}(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x\right)$$
$$= \frac{1}{2\pi(in)^{j+1}}\left((-1)^n\left[f^{(j)}(-\pi+) - f^{(j)}(\pi-)\right] + \int_{-\pi}^{\pi} f^{(j+1)}(x)\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}xx\right).$$

As $f^{(j+1)}(x)$ is assumed integrable we may in general expect $c_n = O(n^{-(j+1)})$.

13.2.1 Example

The periodic extension of x^2 from $(-\pi, \pi]$ to $\mathbb{R}$ is $f \in C^0(-\pi, \pi)$ and $f(-\pi+) = f(\pi-)$, but $f'(-\pi+) = -2\pi$ while $f'(\pi-) = 2\pi$, so taking $j = 1$

$$c_n = \frac{1}{2\pi(in)^2}\left((-1)^n(-4\pi) + \int_{-\pi}^{\pi} 2\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}x\right)$$
$$= 2\frac{(-1)^n}{n^2} + \frac{1}{\pi\mathrm{i}n^3}\left[\mathrm{e}^{-\mathrm{i}nx}\right]_{-\pi}^{\pi} = 2\frac{(-1)^n}{n^2} \qquad (n \neq 0).$$

13.3 Uniform convergence for C^1 functions

We show now that not only do Fourier series converge to the generating function in the norm, but in the case of C^1 functions, that a much stronger result holds:

Theorem 13.3 (Absolute and Uniform Convergence) *If the periodic function f is continuous on $[-\pi, \pi]$, with continuous first derivative on the same interval, then its Fourier series converges uniformly to $f(x)$, i.e.* $\lim_{N\to\infty} s_N(x) = f(x)$.

Proof: We have:

$$S_N(x) = \sum_{n=-N}^{N} c_n \mathrm{e}^{\mathrm{i}nx},$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x.$$

Here, f is 2π-periodic, it's continuous on $[-\pi, \pi]$ and so is its derivative, so we can do integration by parts and re-write c_n as:

$$c_n = \frac{1}{2\pi}\frac{1}{\mathrm{i}n}\int_{-\pi}^{\pi} f'(x)\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}x.$$

Hence,

$$|c_n| \leq \frac{1}{2\pi|n|} d_n, \qquad d_n = \left| \int_{-\pi}^{\pi} f'(x) \mathrm{e}^{-\mathrm{i}nx} \mathrm{d}x \right|.$$

Then by our proof of the Riemann-Lebesgue $\sum\limits_{n=-\infty}^{\infty} |d_n|^2$ converges and $|d_n| \to 0$ as $n \to \infty$, the sum being bounded by:

$$\sum_{n=-\infty}^{\infty} |d_n|^2 \leq \int_{-\pi}^{\pi} |f'(x)|^2 \mathrm{d}x$$

(The bound here follows by applying Bessel's Inequality and / or Parseveal's Identity to $f'(x)$).

Thus, if we identify $M_n = |d_n|/|n|$, we see that $M_n \to 0$ as $n \to \infty$. Furthermore, we look at $\sum_n M_n$, but we have to avoid the $n = 0$ term. Thus, letting $\sum'$ denote the sum excluding the $n = 0$ term, we have:

$$\begin{aligned} \sum_{n=-N}^{N}{}' M_n &= \sum_{n=-N}^{N}{}' |d_n||n|, \\ &\overset{\text{CS}}{\leq} \left(\sum_{n=-N}^{N}{}' |d_n|^2 \right)^{1/2} \left(\sum_{n=-N}^{N}{}' \frac{1}{n^2} \right)^{1/2}. \end{aligned}$$

The sum $\sum_{n=-\infty}^{\prime\infty} n^{-2}$ is clearly bounded (by B^2, say), thus:

$$\begin{aligned} \sum_{n=-N}^{N}{}' M_n &\leq B \left(\sum_{n=-N}^{N}{}' |d_n|^2 \right)^{1/2}, \\ &\leq B \left(\int_{-\pi}^{\pi} |f'(x)|^2 \mathrm{d}x \right)^{1/2}, \\ &< \infty. \end{aligned}$$

Thus, each function $c_n \mathrm{e}^{\mathrm{i}nx}$ is bounded by M_n, and $\sum_n M_n < \infty$, so by the Weirstrass M-test, $s_N(x)$ converges uniformly (to something). Due to the completeness of the trigonometric functions that something is $f(x)$, establishing the result that $\lim_{N\to\infty} s_N(x) = f(x)$.

Remark: The completeness of the Fourier eigenfunctions or equivalently the validity of the Parseval's identity might seem sufficient for the convergence of the Fourier series. However, such results ensure the mean-square convergence rather than the uniform one that we obtained in the previous Theorem. The latter is a much stronger convergence condition.

Now that we established the condition for the convergence let see an example where Fourier expansion is used.

13.3.1 Example

Find the Fourier series of the following 2π-periodic function as shown in Figure 13.1 (upper panel)

$$f(x) = \begin{cases} -x, & -\pi \le x < 0, \\ x, & 0 \le x < \pi. \end{cases}$$

We first notice that such function satisfies the conditions to have a convergent Fourier expansion being periodic, continuous and also picewise continuously differentiable. We look first at the trigonometric series by finding the corresponding coefficients

$$a_0 = -\frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{[x^2]_0^\pi - [x^2]_{-\pi}^0}{2\pi} = \pi$$

and for $n \neq 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\,\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,\cos nx\,dx = -\frac{nx\sin(nx) + \cos(nx)}{n^2}\Bigg|_{-\pi}^{0} + \frac{nx\sin(nx) + \cos(nx)}{n^2}\Bigg|_{0}^{\pi}$$
$$= 2\frac{(-1)^n - 1}{n^2} = \begin{cases} -\frac{4}{n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

where we have integrated by parts. Similarly it can be shown that $b_n = 0$. The Fourier series is then

$$f(x) = \frac{\pi}{2} - 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

One can see from Figure 13.1 (lower panel) how the first two term of the Fourier series approximate the original triangular function. Notice that being an even function will have only cosine eigenfunctions and the same for an odd one that will be a combination of only sine eigenfunctions.

13.4 Uniform Convergence for Piecewise C^1 functions; The Dirichlet Kernel

We can extend these ideas to a larger class of functions, namely those that are piecewise continuously differentiable.

Definition 13.2 *A function f defined on an interval $[a, b]$ is piecewise continuously differentiable if there exists a finite subdivision $a = x_0 < x_1 < \cdots < x_s = b$ such that*

- *f is continuously differentiable on (x_{i-1}, x_i) for $i = 1, 2, \ldots, s$;*
- *the one-sided limits $f'(x_{i-1}+) = \lim_{x \searrow x_{i-1}} f'(x)$ and $f'(x_i-) = \lim_{x \nearrow x_i} f'(x)$ for $i = 1, 2, \ldots, s$.*

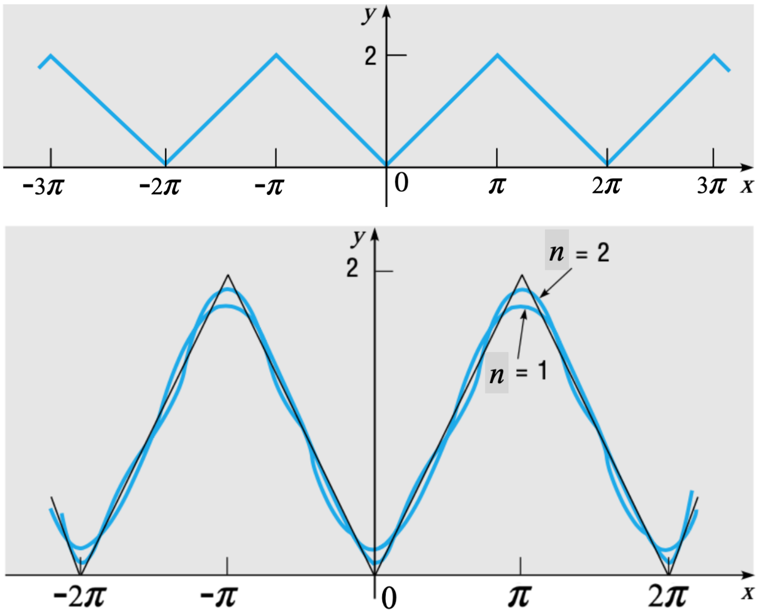


Figure 13.1:

13.4.1 Basel Problem

The previous results mean that the graphs of of f and f' may have only finitely many jumps. The existence of the left and right derivatives an each x_i also means that the corresponding limits of the function itself must exist.

Theorem 13.4 (Basel Problem) *The following sum is true:*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Proof: We will this result using Parseval's identity. Let us start considering the (Fourier) coefficients of $f(x) = x$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\mathrm{e}^{-\mathrm{i}nx} \mathrm{d}x = \frac{(inx+1)\,\mathrm{e}^{-inx}}{2\pi n^2}\bigg|_{-\pi}^{\pi} = i\frac{\cos(n\pi)}{n} = i\frac{(-1)^n}{n} \qquad (n \neq 0 \text{ and } c_0 = 0).$$

This means that $|c_n|^2 = \dfrac{1}{n^2}$ for $n \neq 0$. Now using the Parseval's identity we finally have

$$\tfrac{1}{2} \sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4\pi} \int_{-\pi}^{\pi} x^2 \, \mathrm{d}x = \frac{\pi^2}{6}.$$

Notice here that for the Parseval identity we don't require the periodicity of $f(x)$ but just the completeness of the eigenfunctions e^{-inx}.

13.4.2 Partial Sums

We look at $s_N(x)$ in cases where the generating function $f(x)$ is:

- Continuous on $[-\pi, \pi]$;
- The first derivative possesses a finite number of jump discontinuities but is otherwise piecewise differentiable.

We have:

$$s_N(x) = \sum_{n=-N}^{N} c_n \mathrm{e}^{\mathrm{i}nx},$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x.$$

Here, f is 2π-periodic. We perform integration by parts here, remembering that $f'(x)$ is defined on sub-intervals $[x_{i-1}, x_i]$, where $i = 1, 2, \cdots, s$, and $x_0 = -\pi$, and $x_s = \pi$. We have:

$$c_n = \frac{1}{2\pi}\sum_{i=1}^{s}\left(\left[f(x)\frac{\mathrm{e}^{-\mathrm{i}nx}}{-\mathrm{i}n}\right]_{x_{i-1}}^{x_i} - \frac{1}{-\mathrm{i}n}\int\limits_{x_{i-1}}^{x_i} f'(x)\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}x\right).$$

Also,

$$\begin{aligned}\sum_{i=1}^{s}\left[-f(x)\mathrm{e}^{-\mathrm{i}nx}\right]_{x_{i-1}}^{x_i} &= \left(f(x_0)\mathrm{e}^{-\mathrm{i}nx_0} - f(x_1)\mathrm{e}^{-\mathrm{i}nx_1}\right) + \left(f(x_1)\mathrm{e}^{-\mathrm{i}nx_1} - f(x_2)\mathrm{e}^{-\mathrm{i}nx_2}\right) + \cdots \\ &\quad + \left(f(x_{s-2})\mathrm{e}^{-\mathrm{i}nx_{s-2}} - f(x_{s-1})\mathrm{e}^{-\mathrm{i}nx_{s-1}}\right) + \left(f(x_{s-1})\mathrm{e}^{-\mathrm{i}nx_{s-1}} - f(x_s)\mathrm{e}^{-\mathrm{i}nx_s}\right) \\ &= f(-\pi)\mathrm{e}^{-\mathrm{i}n\pi} - f(\pi)\mathrm{e}^{\mathrm{i}n\pi} = 0.\end{aligned}$$

Hence,

$$c_n = -\frac{1}{2\pi\mathrm{i}n}\sum_{i=1}^{s}\int\limits_{x_{i-1}}^{x_i} f'(x)\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}x$$

and

$$|c_n| \le \frac{1}{2\pi|n|}\sum_{i=1}^{s}\left|\int\limits_{x_{i-1}}^{x_i} f'(x)\mathrm{e}^{-\mathrm{i}nx}\mathrm{d}x\right| \le \frac{1}{2\pi|n|}s\,d_n,$$

where we have defined

$$d_n = \max_{1\le i\le s} I_i, \qquad I_i = \left|\int\limits_{x_{i-1}}^{x_i} f'(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x\right|.$$

Let the maximum in this equation be realised at $i = M$. Thus, for the full range of n, the values $\pm d_n$ can themselves be viewed as a set of Fourier coefficients of $f'(x)$ over the $[x_{M-1}, x_M]$ Then by our proof of the Riemann-Lebesgue Lemma, $\sum_{n=-\infty}^{\infty} |d_n|^2$ converges and $|d_n| \to 0$ as $n \to \infty$, the sum being bounded by

$$\sum_{n=-\infty}^{\infty} |d_n|^2 \leq \max_{1\leq i\leq s} \left| \int_{x_{i-1}}^{x_i} |f'(x)|^2 \mathrm{d}x \right| \leq \int_{-\pi}^{\pi} |f'(x)|^2 \mathrm{d}x.$$

Following the same procedure as before, we have:

$$\begin{aligned} \frac{2\pi}{s} \sum_{n=-N}^{N}{}' |c_n| \leq \sum_{n=-N}^{N}{}' \frac{d_n}{|n|} &\leq \left(\sum_{n=-N}^{N}{}' |d_n|^2 \right)^{1/2} \left(\sum_{n=-N}^{N}{}' \frac{1}{n^2} \right)^{1/2} \\ &\leq \frac{\pi}{\sqrt{3}} \left(\int_{-\pi}^{\pi} \left| f'(x) \right|^2 \mathrm{d}x \right)^{1/2}, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality and the result of the Basel problem. Now, if $f(x)$ is continuous at x, then all of the previous calculations involving the M-test go through as before, and we have

$$s_N(x) \to f(x), \qquad N \to \infty.$$

In cases where $f(x)$ has a jump discontinuity, more work needs to be done, as below.

13.4.3 The Dirichlet Kernel

Consider the partial sum

$$\begin{aligned} s_N(x) &= \sum_{n=-N}^{N} c_n \mathrm{e}^{\mathrm{i}nx} \\ &= \frac{1}{2\pi} \sum_{n=-N}^{N} \mathrm{e}^{\mathrm{i}nx} \int_{-\pi}^{\pi} f(x') \mathrm{e}^{-\mathrm{i}nx'} \mathrm{d}x' \end{aligned}$$

Continuing thus, we have:

$$\begin{aligned} s_N(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-N}^{N} \mathrm{e}^{\mathrm{i}n(x-x')} \right) f(x') \mathrm{d}x' \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-N}^{N} \mathrm{e}^{-\mathrm{i}nu} \right) f(x+u) \mathrm{d}u \end{aligned}$$

on letting $u = x' - x$ and using the periodicity of the integrand. This looks like an Fredholm integral equation of the First Kind. We have:

$$\sum_{n=-N}^{N} \mathrm{e}^{-\mathrm{i}nu} \overset{\text{GP}}{=} \frac{\mathrm{e}^{\mathrm{i}Nu}(1-\mathrm{e}^{-\mathrm{i}(2N+1)u})}{1-\mathrm{e}^{-\mathrm{i}u}}, \\ = \frac{\mathrm{e}^{\mathrm{i}(N+\frac{1}{2})u} - \mathrm{e}^{-\mathrm{i}(N+\frac{1}{2})u})}{\mathrm{e}^{\mathrm{i}\frac{1}{2}u} - \mathrm{e}^{-\mathrm{i}\frac{1}{2}u}}.$$

Continuing thus, we have:

$$\sum_{n=-N}^{N} \mathrm{e}^{-\mathrm{i}nu} = \begin{cases} \dfrac{\sin(N+\frac{1}{2})u}{\sin\frac{1}{2}u} & u \neq 0 \\ 2N+1 & u = 0, \end{cases} = D_N(u).$$

Here, we have identified the Dirichlet kernel, $D_N(u)$, a plot of which is shown in Figure 13.2.

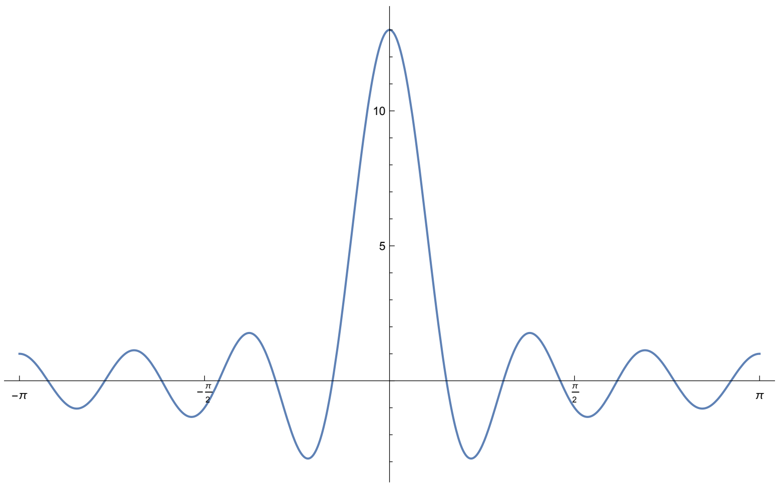


Figure 13.2: Illustration of the Dirichlet kernel for $N = 6$.

Theorem 13.5 (Properties of the Dirichlet Kernel) *The following properties are true:*

1. $D_N(-\pi) = D_N(\pi) = (-1)^N$.
2. $\int_{-\pi}^{\pi} D_N(u)\,\mathrm{d}u = 2\pi$.

Proof: For (i), we have:

$$\begin{aligned}
D_N(\pm\pi) &= \frac{\pm\sin(N+\frac{1}{2})\pi}{\pm\sin\frac{1}{2}\pi}, \\
&= \frac{\sin N\pi\cos\frac{1}{2}\pi + \cos N\pi \sin\frac{1}{2}\pi}{\sin\frac{1}{2}\pi}, \\
&= \cos N\pi, \\
&= (-1)^N.
\end{aligned}$$

For (ii) we have:

$$\begin{aligned}
\int_{-\pi}^{\pi} D_N(u)\,\mathrm{d}u &= \int_{-\pi}^{\pi} \sum_{n=-N}^{N} \mathrm{e}^{-inu}\mathrm{d}u, \\
&= {\sum_{n=-N}^{N}}' 2\pi\frac{1}{-in}\delta_{n0} + 2\pi \\
&= 2\pi.
\end{aligned}$$

Remark: As $D_N(u)$ is manifestly even $\int_{-\pi}^{0} D_N(u)\mathrm{d}u = \int_{0}^{\pi} D_N(u)\mathrm{d}u = \pi.$

We are now able to show our key result regarding the convergence of the partial sum $s_N(x)$ at points of discontinuity.

Theorem 13.6 *Let $f(x)$ be an absolutely integrable $2\pi-$periodic function, i.e., a piecewise C^0 and C^1 in the interval $[-\pi,\pi]$. Then at every point where $f(x)$ has a right-hand and left-hand derivative, the Fourier series of $f(x)$ converges with*

$$\lim_{N\to\infty} s_N(x) = \tfrac{1}{2}\left[f(x+) + f(x-)\right].$$

Proof: Let $x_0 \in [-\pi,\pi]$ be a point where $f(x)$ is continous. We have:

$$\begin{aligned}
S_N(x_0) - f(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0-y)D_N(y)\mathrm{d}y - f(x_0)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y)\mathrm{d}y\right], \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x_0-y) - f(x_0)\right] D_M(y)\mathrm{d}y, \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x_0-y)-f(x_0)}{y}\frac{y}{\sin\frac{1}{2}y}\sin\left[(N+\tfrac{1}{2})y\right]\mathrm{d}z, \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)\sin\left[(N+\tfrac{1}{2})y\right]\mathrm{d}y.
\end{aligned}$$

As $y\to 0$, the expression

$$g(y) = \frac{f(x_0-y)-f(x_0)}{y}\frac{y}{\sin\frac{1}{2}y}$$

tends to

$$2f'(x_0 \pm 0),$$

where $x_0 \pm 0$ indicates the left- and right-hand limits. By assumption these are bounded, and so the function $g(y)$ is bounded on $[-\pi, \pi]$. The function $g(y)$ therefore lives in the set $L^2([-\pi, \pi])$ and thus, by the Riemann–Lebesgue Lemma,

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} g(y) \sin\left[(M + \tfrac{1}{2})y\right] \mathrm{d}y = 0.$$

Thus, we have shown that if $f(x)$ is continuous at x_0, then the partial sum $S_N(x_0)$ converges to $f(x_0)$ as $N \to \infty$.

On the other hand, if $x_0$is a point of discontinuity, we consider $x_0 - \epsilon$ and $x_0 + \epsilon$, where $\epsilon > 0$. Then,

$$S_N(x_0 - \epsilon) \to f(x_0 - \epsilon),$$

and

$$S_N(x_0 + \epsilon) \to f(x_0 + \epsilon).$$

Since the partial sums are continuous (they are sines and cosines0, we have:

$$S_N(x_0) \to \tfrac{1}{2} \lim_{\epsilon \downarrow 0} \left[f(x_0 - \epsilon) + f(x_0 + \epsilon)\right].$$

13.5 Uniform Convergence and the Fejér Kernel

To broaden the class of functions for which we can obtain pointwise convergence we introduce the concept of arithmetic mean and Cesàro summability.

13.5.1 Cesàro summability

Let $a_0, a_1, \ldots$ be a series, and let

$$s_n = \sum_{k=0}^{n} a_k.$$

be its n^{th} partial sum (so if the series is summable in the usual sense if the limit $\lim_{n\to\infty} s_n$ exists). Next construct the arithmetic mean of the first N partial sums by

$$\sigma_N = \frac{1}{N+1} \sum_{n=0}^{N} s_n.$$

Then the sequence is called 'Cesàro summable', with Cesàro sum σ, if σ_N converges in the standard sense to σ.

13.5.2 Properties

If the series is $a_1, a_2, \ldots$ convergent in the standard sense, then it is also Cesàro summable and its Cesàro sum is the usual sum (as s_n is essentially s for all sufficiently large n).

To see that the converse is not true consider the alternating sequence $a_k = (-1)^k$ then $s_n = (1+(-1)^n)/2$, i.e. the repeating sequence of $1, 0, 1, 0, \ldots$ which clearly does not converge. On the other hand

$$\begin{aligned}
\sigma_N &= \frac{1}{N+1}\sum_{n=0}^{N}, \\
&= \frac{1}{N+1}\left[\tfrac{1}{2}(N+1) + \tfrac{1}{2}\sum_{n=0}^{N}(-1)^n\right], \\
&= \frac{1}{N+1}\left[\tfrac{1}{2}(N+1) + s_n\right].
\end{aligned}$$

Thus, $\sigma_N \to 1/2$ as $N \to \infty$, so $a_k = (-1)^k$ $(k = 0, 1, \ldots)$ is Cesàro summable with Cesàro sum $\frac{1}{2}$.

In this sense Cesàro summablility is a non-trivial extension of the standard concept of convergence.

13.5.3 Arithmetic Means for Fourier Series

Applying this concept to partial sums of Fourier Series we arrive at the sequence of arithmetic means

$$\begin{aligned}
\sigma_N(x) &= \frac{1}{N+1}\sum_{n=0}^{N} s_n(x) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{N+1}\sum_{n=0}^{N} D_n(u) f(x+u)\,\mathrm{d}u.
\end{aligned}$$

Correspondingly, we define the Fejér kernel $K_N(u)$ by

$$\begin{aligned}
K_N(u) &= \frac{1}{N+1}\sum_{n=0}^{N} D_n(u) \\
&= \frac{1}{N+1}\sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u} \qquad (u \neq 0) \\
&= \frac{1}{N+1}\sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})u\sin\frac{1}{2}u}{\sin^2\frac{1}{2}u} \\
&= \frac{1}{N+1}\frac{1}{2\sin^2\frac{1}{2}u}\left[(1-\cos u)+(\cos u-\cos 2u)+\cdots(\cos Nu-\cos(N+1)u)\right], \\
&= \frac{1}{N+1}\sum_{n=0}^{N} \frac{\cos nu-\cos(n+1)u}{2\sin^2\frac{1}{2}u} \\
&= \frac{1}{N+1}\frac{1-\cos(N+1)u}{2\sin^2\frac{1}{2}u} \\
&= \frac{1}{N+1}\frac{\sin^2\frac{N+1}{2}u}{\sin^2\frac{1}{2}u}
\end{aligned}$$

The value $K_N(0) = N+1$ follows from direct computation:

$$\begin{aligned}
\frac{1}{N+1}\sum_{n=0}^{N} D_n(0) &= \frac{1}{N+1}\sum_{n=0}^{N}(2n+1), \\
&= \frac{1}{N+1}\left[\tfrac{1}{2}\times 2\times N(N+1)+(N+1)\right], \\
&= N+1.
\end{aligned}$$

Remark: From the explicit expression here it is clear that the Fejér kernel is positive definite; this property means it has much nicer properties than the Dirichlet kernel.

The kernel is plotted in Figure 13.5.3.

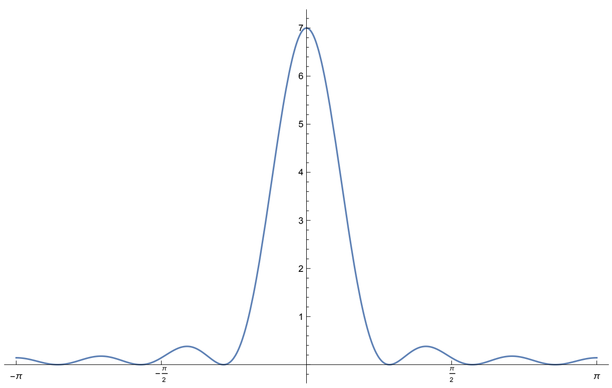


Figure 13.3: Illustration of the Fejér kernel for $N = 6$.

Notice that, for the Fejer kernel,

$$\begin{aligned}\int_{-\pi}^{\pi} K_N(u)\mathrm{d}u &= \frac{1}{N+1}\sum_{n=0}^{N}\int_{-\pi}^{\pi} D_n(u)\mathrm{d}u,\\ &= \frac{1}{N+1}\sum_{n=0}^{N} 2\pi\\ &= 2\pi.\end{aligned}$$

Notice also that $K_N(0) = N+1$. Thus, the height of the central lobe increases (without limit) as $N \to \infty$ while its width, $2\pi/(N+1)$, decreases but overall in such a way that the integral stays constant.

We are now in a position to prove the main result of this section:

Theorem 13.7 (Pointwise convergence (without derivatives)) *Let $f(x)$ be an absolutely integrable function of period 2π. Then at every point where $f(x)$ has a right-hand and left-hand limit, the Fourier series of $f(x)$ converges in the Cesàro sense with*

$$\lim_{N\to\infty} \sigma_N(x) = \tfrac{1}{2}\left[f(x+) + f(x-)\right].$$

Remark: The assumptions of the theorem immediately apply to a function that is piecewise continuous on $[-\pi, \pi]$.

Proof: We split the integrand into the two regions $(-\pi, 0)$ and $(0, \pi)$. As such, we aim to show:

$$\lim_{N\to\infty} \frac{1}{2\pi}\int_{0}^{\pi} K_N(u)\left(f(x+u) - f(x+)\right)\mathrm{d}u = 0.$$

A similar result for $x-$ will establish the result. Here, we are using the notation $x+$ and $x-$ to denote points just to the right of x and just to the left of x, respectively.

We focus on the interval $(0, \pi)$. We split the integrand into a region around 0 and the remainder away from 0. First given any $\epsilon > 0$ as the limit $f(x+)$ exists we can find a $\delta > 0$ such that

$$\left|f(x+u) - f(x+)\right| < \epsilon \text{ whenever } x \in (0, \delta).$$

Correspondingly

$$\begin{aligned}\left|\int_{0}^{\delta} K_N(u)\left(f(x+u) - f(x+)\right)\mathrm{d}u\right| &\leq \epsilon\int_{0}^{\delta} K_N(u)\mathrm{d}u\\ &\leq \epsilon\, 2\pi\end{aligned}$$

where we have used the positivity of $K_N(u)$ (twice). Then on $[\delta, \pi]$:

$$\begin{aligned} K_N(u) &= \frac{1}{N+1} \frac{\sin^2 \frac{N+1}{2} u}{\sin^2 \frac{1}{2} u} \\ &\leq \frac{1}{N+1} \frac{1}{\sin^2 \frac{1}{2} u} \leq \frac{1}{N+1} \frac{1}{\sin^2 \frac{1}{2} \delta} \end{aligned}$$

and so

$$\left| \int_\delta^\pi K_N(u) \left(f(x+u) - f(x+)\right) \mathrm{d}u \right| \leq \frac{1}{N+1} \frac{1}{\sin^2 \frac{1}{2}\delta} \int_\delta^\pi |f(x+u) - f(x+)| \, \mathrm{d}u$$

which clearly tends to 0 as $N \to \infty$ as, by assumption, the integral exists. Putting both parts of the integral together (i.e. $[0, \delta]$ and $[\delta, \pi]$, ew have

$$\int_0^\pi K_N(u) \left(f(x+u) - f(x+)\right) \mathrm{d}u \to 0.$$

A similar result applies for $x-$:

$$\int_{-\pi}^0 K_N(u) \left(f(x+u) - f(x-)\right) \mathrm{d}u \to 0.$$

and so:

$$\int_{-\pi}^\pi K_N(u) f(x+u) \mathrm{d}u \to \tfrac{1}{2} \left[f(x-) + f(x+)\right],$$

and the result is shown.

Remark: If we tried the same argument with the Dirichlet kernel we would need to consider $\int_0^\delta \big|D_N(u)\big| \mathrm{d}u$ but this tends to ∞ as $N \to \infty$.

Remark: Furthermore, if $f \in C^0[-\pi, \pi]$ we may choose a δ that works for all x $\left(\min_{x\in[-\pi,\pi]} \delta(x)\right)$ and bound our integrals above by a suitable multiple of $\sup_{x\in[-\pi,\pi]} \big|f(x)\big|$, to deduce that $\sigma_N(x)$ converges *uniformly* to $f(x)$ on $[-\pi, \pi]$.

Chapter 14

Application of Fredholm Integral Equations

Overview

In this chapter we look at an application of Fredholm Integral Equations in Quantum Mechanics – in particular, in solving the Schródinger equation in case of a time-independent Hamiltonian. It is not necessary to understand very much about Quantum Mechanics to follow the derivations.

14.1 The Schrödinger Equation

In this chapter we are concerned with the Schrödinger equation in one spatial dimension:

$$\mathrm{i}\frac{\partial \psi}{\partial t} = \hat{H}\psi, \qquad \psi(x, t=0) = \psi_0(x), \qquad x \in (-\infty, \infty), \tag{14.1a}$$

We focus on systems with time-translation symmetry, such that the Hamiltonian operator $\hat{H}$ can be written as:

$$\hat{H} = -\frac{\partial^2}{\partial x^2} + \mathcal{U}(x). \tag{14.1b}$$

The potential $\mathcal{U}(x)$ is assumed to have the following properties:

- The analytical continuation of $\mathcal{U}(x)$ to the complex plane produces a holomorphic function;
- We have

$$\mathcal{U}(x) \sim \begin{cases} Cx^{2p}, \ |x| \to \infty, & \text{case 1,} \\ C, \ |x| \to \infty, & \text{case 2,} \end{cases} \tag{14.1c}$$

  where p is a positive integer and C is a positive real number. In the second case, it is also assumed that $\mathcal{U}'(x) \to 0$ as $|x| \to 0$.
- Without loss of generality, it is further assumed that $\mathcal{U}(x)$ is even in x.

No conditions are as yet placed on the initial data $\psi_0(x)$, except that it should be a continuous function. It is not necessarily square integrable. Solving Equation (14.1a) forward in time can be regarded as a **signalling problem**. Such problems are often encountered in Engineering and Physics. A first step in solving a signalling problem is often to carry out a Laplace Transform.

14.1.1 The Laplace Transform

We multiply both sides of Equation (14.1a) by $\mathrm{e}^{-\lambda t}$, where $t > 0$ and λ is an arbitrary (possibly complex-valued) parameter:

$$\mathrm{e}^{-\lambda t}\left(\mathrm{i}\frac{\partial\psi}{\partial t}\right) = \left[-\partial_{xx} + \mathcal{U}(x)\right]\psi(x,t)\mathrm{e}^{-\lambda t}.$$

We then integrate with respect to t from $t = 0$ to $t = \infty$, applying the boundary conditions at $t = 0$:

$$-\mathrm{i}\psi(x,0) + \lambda\int_0^\infty \psi(x,t)\mathrm{e}^{-\lambda t}\mathrm{d}t = \left[-\partial_{xx} + \mathcal{U}(x)\right]\int_0^\infty \psi(x,t)\mathrm{e}^{-\lambda t}\mathrm{d}t. \tag{14.2}$$

We identify the **Laplace transform** of $\psi(x,t)$:

$$\psi_\lambda(x) = \int_0^\infty \psi(x,t)\mathrm{e}^{-\lambda t}\mathrm{d}t. \tag{14.3}$$

We assume that $\psi(x,t)$ is at worst exponentially divergent in t:

$$|\psi(x,t)| \le M\mathrm{e}^{\lambda_0 t}, \qquad M, \lambda_0 \in \mathbb{R}^+.$$

Hence, in order for Equation (14.3) to make sense, require:

$$\Re(\lambda) > \lambda_0.$$

Thus, Equation (14.2) can be written as:

$$-\mathrm{i}\psi_0(x) + \mathrm{i}\lambda\psi_\lambda(x) = \hat{H}\psi_\lambda(x), \tag{14.4}$$

Notice that this equation is of Sturm–Liouville type, albeit that now the interval on which the problem is posed is infinite.

From now on, we write

$$\lambda = -\mathrm{i}\omega, \tag{14.5}$$

and we consider the equation

$$-\mathrm{i}\psi_0(x) + \omega\psi_\omega(x) = \hat{H}\psi_\omega(x), \tag{14.6}$$

Remark: Properly, we have $\psi_\lambda(x) = \psi_{-\mathrm{i}\omega}(x)$. Thus, we should have:

$$\psi_\lambda(x) = \psi_{-\mathrm{i}\omega}(x) = \Psi_\omega(x),$$

where Ψ and ψ are different functions. However, we use an *abuse of notation* here and write $\psi_\omega(x)$ for $\Psi_\omega(x)$, it being understood that this really means the Laplace-transform of $\psi(x,t)$, evaluated at $\lambda = -\mathrm{i}\omega$.

It now remains to find the response $\psi_\omega(x)$ to the initial configuration $\psi_0(x)$. Formally, we have

$$\psi_\omega(x) = \left(\hat{H} - \omega\mathbb{I}\right)^{-1}\left[-\mathrm{i}\psi_0(x)\right]. \tag{14.7}$$

The aim of this chapter is to give meaning to the expression (14.7).

14.2 Green's functions

For definiteness, we consider Case 1 in the classification of the potential function. We investigate the inhomogeneous equation

$$\left[-\partial_{xx} + \mathcal{U}(x)\right]\psi_\omega(x) = \underbrace{\omega\psi_\omega(x) - \mathrm{i}\psi_0(x)}_{=s(x)}, \tag{14.8a}$$

as well as its homogeneous counterpart,

$$\left[-\partial_{xx} + \mathcal{U}(x)\right]\psi_\omega(x) = 0. \tag{14.8b}$$

Notation alert: We now drop the subscript ω from $\psi_\omega(x)$, it being understood that we are working with the Laplace transform of $\psi(x,t)$.

As $|x| \to \infty$, the homogeneous equation (14.8b) becomes

$$\partial_{xx}\psi(x) \sim Cx^{2p}\psi(x),$$

and is is straightforward to show that the asymptotic solution contains two linearly independent solutions, one exponentially growing as $|x| \to \infty$, and the other exponentially decaying. These asymptotic solutions must connect continuously to the solution at finite x, leading to the conclusion that the solution of the homogeneous equation (14.8b) comes in two linearly independent parts,

$$\psi(x) = A\psi_1(x) + B\psi_2(x), \tag{14.9a}$$

where

$$\lim_{|x|\to\infty}\psi_1(x) = 0, \qquad \lim_{|x|\to\infty}\psi_2(x) = \infty. \tag{14.9b}$$

On the other hand, given the symmetry of the potential, $\mathcal{U}(x) = \mathcal{U}(-x)$, we must take $\psi(x)$ to be odd or even. Using the method of Frobenius to generate power-series solutions of Equation (14.8b) near $x = 0$, it is possible to construct two linearly independent solutions, $\psi(x) = C\phi_1(x) + D\phi_2(x)$, where

$$\begin{aligned} \phi_1(x) &= 0, \qquad \phi_1'(x) = 1, \\ \phi_2(x) &= 1, \qquad \phi_2'(x) = 0, \end{aligned}$$

corresponding to odd and even solutions respectively. In this way, continuous, square-integrable Green's functions for Equation (14.8) can be constructed as follows:

$$G_{\text{odd}}(x,y) = \begin{cases} -\phi_1(x)\psi_1(y)/W_{\text{odd}}, & 0 \le x < y, \\ -\phi_1(y)\psi_1(x)/W_{\text{odd}}, & y < x < \infty, \end{cases} \tag{14.10a}$$

where

$$W_{\text{odd}} = \begin{vmatrix} \psi_1(y) & \phi_1(y) \\ \psi_1'(y) & \phi_1'(y) \end{vmatrix} \tag{14.10b}$$

is the Wronskian. For the Schrödinger equation in canonical form $p_2(x)\psi''(x) + p_1(x)\psi'(x) + p_0(x)\psi(x) = 0$, we have $p_2(x) = -1$ and $p_1(x) = 0$. Thus, by Abel's Theorem, the Wronskian is a constant. Hence, in the odd case, $W_{\text{odd}} = \psi_1(0)$.

The even case is constructed in a similar manner. We obtain:

$$G_{\text{even}}(x,t) = \begin{cases} -\phi_2(x)\psi_1(y)/W_{\text{even}}, & 0 \le x < y, \\ -\phi_2(y)\psi_1(x)/W_{\text{even}}, & y < x < \infty, \end{cases}, \tag{14.11a}$$

where now

$$W_{\text{even}} = \begin{vmatrix} \psi_1(y) & \phi_2(y) \\ \psi_1'(y) & \phi_2'(y) \end{vmatrix} = -\psi_1'(0). \tag{14.11b}$$

From now on, we let $G(x,y)$ denote either one of the Green's functions, Equation (14.10) or (14.11). We have:

Theorem 14.1 *The Green's Function $G(x,y)$ is symmetric.*

Proof: Focus on the even case without loss of generality. Look first at $x < y$. Since the Wronskian is constant and equal to C, say, we have:

$$G(x,y) = -\frac{1}{C}\phi_1(x)\psi_1(y), \qquad x < y.$$

Hence,

$$G(y,x) = -\frac{1}{C}\phi_1(y)\psi_1(x).$$

The case $x > y$ is similar. This shows that $G(x,y) = G(y,x)$. ∎

Continuing thus, the solution to the inhomogeneous equation (14.8a) is now available as

$$\psi(x) = \int_0^\infty G(x,y)s(y)\mathrm{d}y. \tag{14.12}$$

Filling in for $s(x) = \omega\psi(x) - \mathrm{i}\psi_0(x)$, this is:

$$\psi(x) = \underbrace{\int_0^\infty G(x,y)\left[-\mathrm{i}\psi_0(y)\right]\mathrm{d}y}_{=\widehat{\psi}_0(x)} + \omega\int_0^\infty G(x,y)\psi(y)\mathrm{d}y. \tag{14.13}$$

This is exactly the Fredholm Integral Equation, so well studied in previous chapters. The corresponding eigenvalue problem can be obtained by setting $\psi_0(x) = 0$ in Equation (14.13):

$$\psi(x) = \omega\int_0^\infty G(x,y)\psi(y)\,\mathrm{d}y.$$

Going back to the original ordinary differential equations, the eigenfunction solves:

$$\left[-\partial_{xx} + \mathcal{U}(x)\right]\psi(x) = \omega\psi(x)$$

for a particular value of ω (the eigenvalue).

14.3 The Integral Operator

We combine aspects of the eigenvalue problem with aspects of the inhomogeneous problem in order to solve Equation (14.6), viz.

$$-\mathrm{i}\psi_0(x) + \omega\psi(x) = \hat{H}\psi(x).$$

We identify $s(x) = \omega\psi(x) - \mathrm{i}\psi_0(x)$. Again, we have Equation (14.13):

$$\psi(x) = \underbrace{\int_0^\infty G(x,y)\left[-\mathrm{i}\psi_0(y)\right]\mathrm{d}y}_{=\widehat{\psi}_0(x)} + \omega\int_0^\infty G(x,y)\psi(y)\mathrm{d}y.$$

We re-write this equation as follows by defining a number of quantities:

$$\mathcal{K}\psi(x) := \int_0^\infty G(x,y)\psi(y)\mathrm{d}y.$$

$$\widehat{\psi}_0(x) := \int_0^\infty G(x,y)\left[-\mathrm{i}\psi_0(y)\right]\mathrm{d}y = -\mathrm{i}\mathcal{K}\psi_0(x).$$

We also have the following key theorem:

Theorem 14.2 *The following identity holds, for all $\psi \in C([0,\infty)) \bigcap L^2([0,\infty))$:*

$$\hat{H}\mathcal{K}\psi(x) = \psi(x), \qquad x \in (0,\infty).$$

Proof: We have:

$$\begin{aligned}
\hat{H}\mathcal{K}\psi(y) &= \hat{H}\int_0^\infty G(x,y)\psi(y)\mathrm{d}y, \\
&= \int_0^\infty \hat{H}_x G(x,y)\psi(y)\mathrm{d}y, \\
&\overset{\text{Theorem 6.1, \#2}}{=} \int_{x-\epsilon}^{x+\epsilon} \psi(y)\mathrm{d}y, \\
&= \int_{x-\epsilon}^{x+\epsilon} \frac{\partial}{\partial x}\frac{\partial G(x,y)}{\partial x}\psi(y)\mathrm{d}y, \\
&= \frac{\partial G(x,y)}{\partial x}\Big|_{y=x-\epsilon}^{y=x+\epsilon}\psi(x), \\
&\overset{\text{Theorem 6.1, \#4}}{=} \psi(x), \\
&= \int_0^\infty \delta(x-y)\psi(y)\mathrm{d}y.
\end{aligned}$$

∎

Remark: Formally, we may write

$$\hat{H}\mathcal{K} = \delta, \tag{14.14}$$

where δ is the Dirac delta function.

14.4 Hilbert–Schmidt Theory

Based on the definition of the integral operator $\mathcal{K}$ and its corresponding Green's Function $G(x,y)$, we are led to consider the following integral equation:

$$\psi(x) = \widehat{\psi}_0(x) + \omega\mathcal{K}\psi(x) \iff [\mathbb{I} - \omega\mathcal{K}]\,\psi(x) = \widehat{\psi}_0(x). \tag{14.15}$$

We observe that, by construction

- $G(x,y)$ is symmetric in x and y;
- $G(x,y)$ is continuous in x and y;
- $G(x,y)$ is square integrable in x and y,

and thus, the operator problem (14.15) satisfies the **Fredholm alternative**, which was developed in Chapter 8:

Either the FIE (14.15) has a unique solution for each $\widehat{\psi}_0(x)$, or the corresponding homogeneous problem

$$[\mathbb{I} - \omega\mathcal{K}]\,\psi(x) = 0$$

has a non-trivial solution.

To make progress in the solution of Equation (14.1a), we consider again the eigenvalue problem

$$[\mathbb{I} - \omega\mathcal{K}]\,\psi(x) = 0. \tag{14.16}$$

Since $G(x, y)$ is a symmetric kernel, $\mathcal{K}$ is self-adjoint:

$$\langle f, \mathcal{K}g\rangle = \langle \mathcal{K}f, g\rangle,$$

where f and g are square-integrable complex-valued functions of a single real-variable, and where

$$\langle f, g\rangle = \int_0^\infty f^*(x)g(x)\mathrm{d}x. \tag{14.17}$$

From Chapter 8, we have:

- The eigenvalues of Equation (14.16) are real;
- Eigenfunctions corresponding to distinct eigenvalues are orthogonal in the inner product (14.17).

We can further look back at the Functional Analysis of FIEs (Chapter 10). These results were for FIEs in a finite interval $[a, b]$. However, after some more background reading, we find that some of these results apply just as well to the interval $[0, \infty)$[1], in particular:

1. Let $\mathcal{K} \neq 0$ be self-adjoint. Then $\mathcal{K}$ has at least one eigenvalue.
2. Let $\mathcal{K} \neq$ be self-adjoint. Then $\mathcal{K}$ has an eigenvalue of smallest modulus, say λ_1, and

$$\frac{1}{\lambda_1^2} = \sup_{f\neq 0}\frac{\langle f, \mathcal{K}f\rangle}{\langle f, f, \rangle} = \|\mathcal{K}\|_2.$$

In a similar manner, we may refer back to Chapter 10 on the Functional Analysis of FIEs and Chapters 11–12 on Sturm–Liouville theory, and generalize from the interval $[a, b]$ to $[0, \infty)$:

3. Let $\mathcal{K} \neq 0$ be the solution operator for the SL problem (14.4). Let $\mathcal{U}(x)$ be in Case 1: $\mathcal{U}(x) \sim Cx^{2p}$ as $x \to \infty$. Then,

    - $\mathcal{K}$ has countably infinitely many eigenvalue-eigenfunction pairs, $\{\omega_n, v_n(x)\}_{n=0}^\infty$;

[1] E.g. Chapter 7, Partial Differential Equations of Mathematical Physics and Integral Equations, R. B. Guenther and J. W. Lee

- For any $f(x)$ square-integrable on $[0,\infty)$ and in range of the integral operator (i.e. $f=\mathcal{K}g$, for some $g\in L^2([0,\infty))$, $f(x)$ converges uniformly and absolutely to

$$f(x)=\sum_{n=0}^{\infty}\langle f,v_n\rangle v_n(x), \tag{14.18}$$

almost everywhere on $[0,\infty)$.

Equation (14.18) is **Hilbert–Schmidt theory**, applied to the Schrödinger equation.

14.5 Solving the signalling problem using Hilbert–Schmidt Theory

We now move forward in solving Equation (14.15), recalled here to be

$$\left[\mathbb{I}-\omega\mathcal{K}\right]\psi(x)=\widehat{\psi}_0(x). \tag{14.19}$$

We now focus on source functions $\psi_0(x)$ that have the following decomposition:

$$\psi_0(x)=\psi_{01}(x)+\psi_{02}(x),$$

where

$$\psi_{01}(x)\in L^2([0,\infty)),\qquad |\psi_{02}(x)|\leq M,\qquad x\to\infty,$$

where M is a positive constant. This gives rise to the following differential equation to be solved:

$$\left[\mathbb{I}-\omega\mathcal{K}\right]\psi(x)=\widehat{\psi}_{0,1}(x)+\widehat{\psi}_{0,2}(x), \tag{14.20a}$$

where

$$\widehat{\psi}_{0,1}=-\mathrm{i}\mathcal{K}\psi_{01},\qquad \widehat{\psi}_{0,2}=-\mathrm{i}\mathcal{K}\psi_{02}.$$

We break up the solution as follows:

$$\psi(x)=\chi_1(x)+\chi_2(x), \tag{14.20b}$$

where

$$\left[\mathbb{I}-\omega\mathcal{K}\right]\chi_1(x)=\widehat{\psi}_{0,1}(x), \tag{14.20c}$$

and

$$\left[\mathbb{I}-\omega\mathcal{K}\right]\chi_2(x)=\widehat{\psi}_{0,2}(x). \tag{14.20d}$$

We consider first of all Equation (14.20c), which is the part of the solution that is square-integrable. Since $\widehat{\psi}_{0,1}=-\mathrm{i}\mathcal{K}\psi_{01}$ is in the range of the operator, we can apply Hilbert–Schmidt theory. Therefore, we can write:

$$\chi_1(x) = \sum_{n=0}^{\infty} a_n v_n(x), \qquad \widehat{\psi}_{0,1}(x) = \sum_{n=0}^{\infty} \langle \widehat{\psi}_{0,1}, v_n \rangle v_n(x), \tag{14.21}$$

and we determine the a_n's. We have:

$$\begin{aligned}
[\mathbb{I} - \omega\mathcal{K}]\,\chi_1(x) &= [\mathbb{I} - \omega\mathcal{K}] \sum_{n=0}^{\infty} a_n v_n(x), \\
&= \sum_{n=0}^{\infty} a_n\,[\mathbb{I} - \omega\mathcal{K}]\,v_n(x), \\
&= \sum_{n=0}^{\infty} a_n \left(1 - \frac{\omega}{\omega_n}\right) v_n(x), \\
&= \sum_{n=0}^{\infty} a_n \left(\frac{\omega_n - \omega}{\omega_n}\right) v_n(x), \\
&\overset{\text{Eq. (14.20c)}}{=} \widehat{\psi}_{0,1}(x), \\
&\overset{\text{Eq. (14.21)}}{=} \sum_{n=0}^{\infty} \langle \widehat{\psi}_{0,1}, v_n \rangle v_n(x).
\end{aligned}$$

Hence,

$$a_n = \frac{\omega_n \langle \widehat{\psi}_{0,1}, v_n \rangle}{\omega_n - \omega},$$

and hence finally,

$$\chi_1(x) = \sum_{n=0}^{\infty} \frac{\omega_n \langle \widehat{\psi}_{0,1}, v_n \rangle}{\omega_n - \omega} v_n(x), \qquad \omega \neq \omega_n.$$

We can develop this a bit further as follows (with $\omega \neq \omega_n$):

$$\begin{aligned}
\chi_1(x) &= \sum_{n=0}^{\infty} \frac{\omega_n \langle \widehat{\psi}_{0,1}, v_n \rangle}{\omega_n - \omega} v_n(x), \\
&= \sum_{n=0}^{\infty} \frac{\omega_n \langle -\mathrm{i}\mathcal{K}\psi_{01}, v_n \rangle}{\omega_n - \omega} v_n(x), \\
&= \sum_{n=0}^{\infty} \frac{\omega_n \langle -\mathrm{i}\psi_{01}, \mathcal{K} v_n \rangle}{\omega_n - \omega} v_n(x), \\
&= \sum_{n=0}^{\infty} \frac{\omega_n \frac{1}{\omega_n} \langle -\mathrm{i}\psi_{01}, v_n \rangle}{\omega_n - \omega} v_n(x), \\
&= \sum_{n=0}^{\infty} \frac{\langle -\mathrm{i}\psi_{01}, v_n \rangle}{\omega_n - \omega} v_n(x),
\end{aligned}$$

so our final answer here is:

$$\chi_1(x) = \sum_{n=0}^{\infty} \frac{\langle -\mathrm{i}\psi_{01}, v_n \rangle}{\omega_n - \omega}, \qquad \omega \neq \omega_n.$$

We now focus on the solution component corresponding to non-square-integrable initial forcing. This is Equation (14.20d), re-written here as

$$\left(\hat{H}-\omega\right)\chi_2(x) = -\mathrm{i}\psi_{02}(x).$$

We identify two fundamental solutions of the homogeneous equation $(\hat{H}-\omega)\chi = 0$ with which to work, say $\psi_1(x;\omega)$ and $\psi_2(x;\omega)$. Using the method of variation of parameters, the solution to the inhomogeneous problem reads

$$\begin{aligned}\chi_2(x;\omega) &= \frac{1}{W}\left[-\psi_1(x;\omega)\int_0^x \psi_2(x;\omega)\left[-\mathrm{i}\psi_{02}(x)\right]\mathrm{d}x + \psi_2(x;\omega)\int_0^x \psi_1(x;\omega)\left[-\mathrm{i}\psi_{02}(x)\right]\mathrm{d}x\right],\\ &= \frac{F(x;\omega)}{W}.\end{aligned}$$

where

$$W = \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{vmatrix}$$

is the Wronskian. By linear independence, $W(x) \neq 0$, and given the structure of the Schrödinger equation, $W = \mathrm{const.}$ We set

$$W(x,\omega) = \mathrm{const.} = \lim_{x\to\infty} W(x,\omega) := W(\omega)$$

We therefore have a complete solution to the signalling problem (14.20a), recalled here to be

$$\left[\mathbb{I}-\omega\mathcal{K}\right]\psi(x) = \widehat{\psi}_{0,1}(x) + \widehat{\psi}_{0,2}(x).$$

It is

$$\psi(x) = \chi_1(x) + \chi_2(x),$$

hence

$$\psi(x) = \sum_{n=0}^{\infty}\frac{\langle -\mathrm{i}\psi_{01}, v_n\rangle}{\omega_n-\omega} + \frac{F(x;\omega)}{W(\omega)}. \qquad (14.22)$$

14.6 The inversion

Given the functional form (14.22) for the solution to the signalling problem (14.20a), we perform the ω-inversion. For definiteness, Equation (14.22) is here recalled to be

$$\psi_\omega(x) = \sum_{n=0}^{\infty}\frac{\langle -\mathrm{i}\psi_{01}, v_n\rangle}{\omega_n-\omega}v_n(x) + \frac{F(x;\omega)}{W(\omega)}. \qquad (14.23)$$

Note that we have restored the subscript ω to the $\psi_\omega(x)$, which reminds us that all along we have been solving for the Laplace-transformed version of $\psi(x,t)$:

$$\psi_\lambda(x) = \int_0^\infty \psi(x,t)\mathrm{e}^{-\lambda t}\mathrm{d}t, \qquad \lambda = -\mathrm{i}\omega.$$

As it turns out, there is a way to invert the Laplace transform, and hence, to reconstruct $\psi(x,t)$ from $\psi_\omega(x)$. The following formula can be taken as given for the time being, but it is proved in ACM 40690:

$$\psi(x,t) = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{B}} \psi_\lambda(x)\mathrm{e}^{\lambda t}\mathrm{d}\lambda.$$

where $\mathcal{B}$ is the Bromwich contour that lies to the **right** of all the λ-singularities in $\psi_\lambda(x)$ (Figure 14.1, left panel). However, we perform the rotation $\lambda = -\mathrm{i}\omega \iff \mathrm{i}\lambda = \omega$ (Figure 14.1, right panel) and get:

$$\psi(x,t) = \frac{1}{2\pi}\int_{\mathcal{B}} [-\psi_\omega(x)]\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}\omega.$$

Thus, in the ω-space, the Bromwich contour lies **above** all the ω-singularities of $\psi_\omega(x)$:

$$\mathcal{B} = \{(x,y) \in \mathbb{C}| -\infty < x < \infty, y = \mathrm{i}\Delta\}.$$

Think about this as being loosely equivalent to a Fourier transform and its inverse, except now the integrals are done along a contour in the complex plane.

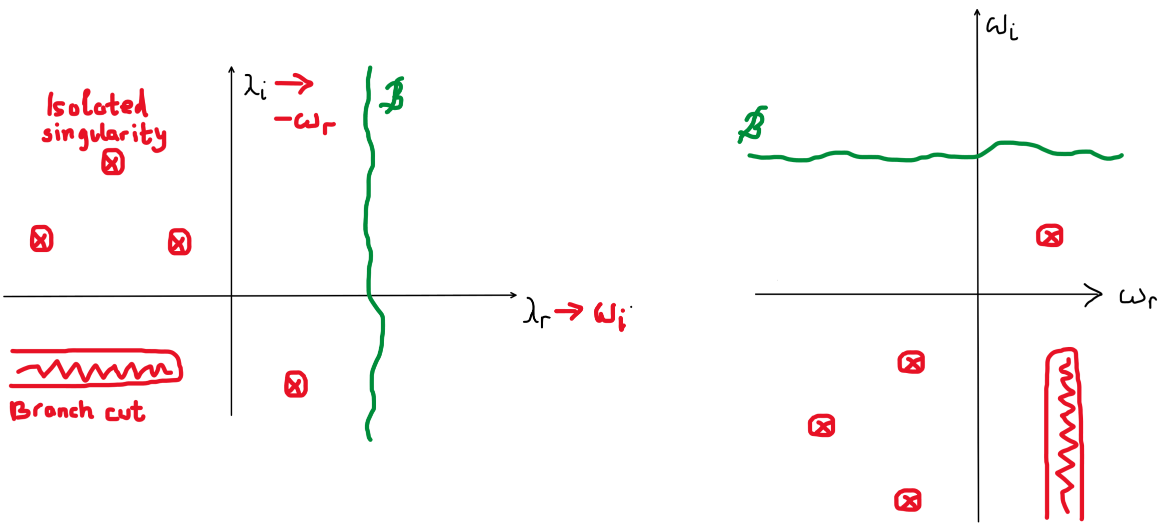


Figure 14.1: Left panel: The Bromwich contour $\mathcal{B}$, expressed in the λ variables. Right panel: the Bromwich contour, transformed into ω variables

It now remains to enumerate the ω-singularities in $\psi_\omega(x)$:

1. The discrete spectrum, where $\omega = \omega_n$. Since the eigenvalues ω_n are real, the discrete spectrum lies along the real axis.

2. Zeros of $W(\omega)$.
3. Singularities in $F(x;\omega)$.

We now rule out the last two possibilities.

1. $W(\omega) \neq 0$. The Wronskian is a constant because it's a self-adjoint problem with $p_2 = -1$ (recall the notation of Chapter 4). In view of Theorem 3.5, it's a non-zero constant.
2. A Frobenius-like series solution of the homogeneous problem $(\hat{H} - \omega)\chi$ yields coefficients of $\chi_{1,2}$ that are analytic in ω. Hence, $F(x;\omega)$ is analytic and therefore has no singularities.

A further possibility is ruled out: contributions to the discrete spectrum contain do not involve double roots, in view of the following theorem:

Theorem 14.3 *The (normalizable) eigenfunctions of the Schrödinger equation in one dimension form one-dimensional eigenspaces.*

Proof: Essentially the same as Theorem 11.3. ∎

It therefore follows that $\Delta = \epsilon$ in the Bromwich contour, where $\epsilon > 0$ is any arbitrary constant. Given that $\lambda = -\mathrm{i}\omega$, the contour is closed in the lower-half-plane for $t > 0$, meaning that residues of the (simple) poles give contributions[2] Thus,

$$\psi(x,t) = \sum_{n=0}^{\infty} \langle \psi_{01}, v_n \rangle v_n(x) \mathrm{e}^{-\mathrm{i}\omega_n t}, \qquad t > 0,$$

which is the solution that one would obtain using PDE theory and separation of variables! **Except** that the time evolution 'filters out' the non-square-integrable part of the initial condition.

14.7 Continuous Spectra

We look at Case 2 of of the classification of the potential function, with $\mathcal{U}(x) \to C$ as $x \to \infty$. For large x, the Schrödinger equation becomes:

$$-\partial_{xx}\psi_\omega + C\psi_\omega = \omega\psi_\omega.$$

The decaying solution is $\psi_\omega(x) \propto \mathrm{e}^{-\sqrt{C-\omega}x}$, valid as $x \to \infty$. Hence, if ω is to be an eigenvalue corresponding to a decaying solution, we require $\omega < C$. Furthermore, by the results in Chapter 10 (in particular, Theorem 10.7) the kernel function for $\mathcal{K}$ is separable, and there are only **finitely many** eigenvalues.

[2] The contour is closed in the lower half-plane for $t > 0$. The reason is that the integrand contains a contribution such as $\mathrm{e}^{\lambda t} = \mathrm{e}^{-\mathrm{i}\omega_r t + \omega_i t}$ In order for the integrand actually to be integrable, we require the $\mathrm{e}^{\omega_i t}$ term to be exponentially decaying. Hence, for $t > 0$, we take $\omega_i < 0$ and hence, the contour is closed in the lower half-plane.

On the other hand, if $\omega > C$, this corresponds to not-square-integrable component of the solution, similar to χ_2 in Section 14.6. Using a similar argument as before (and not filling in details at all), we find that the Wronskian in Equation (14.22) now becomes:

$$W = 2\sqrt{\omega - C}$$

This has a branch cut in the complex plane (Figure 14.2).

Going through all of the previous steps necessary for the inversion (Section 14.6), we obtain:

$$\psi(x,t) = \sum_{n=0}^{N} \langle \psi_{01}, v_n \rangle v_n(x) \mathrm{e}^{-\mathrm{i}\omega_n t} + \frac{1}{2\pi} \int_C^{\infty} (-1) \Delta \left(\frac{F(x;\omega)}{2\sqrt{\omega - C}} \right) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}\omega \tag{14.24}$$

The second contribution here corresponds to **travelling waves** and hence, **unbound states** where the particle can no longer be localized by the potential function (Figure 14.3). Here, Δ denotes evaluation of the function $F(x;\omega)/2\sqrt{\omega - C}$ on either side of the branch cut.

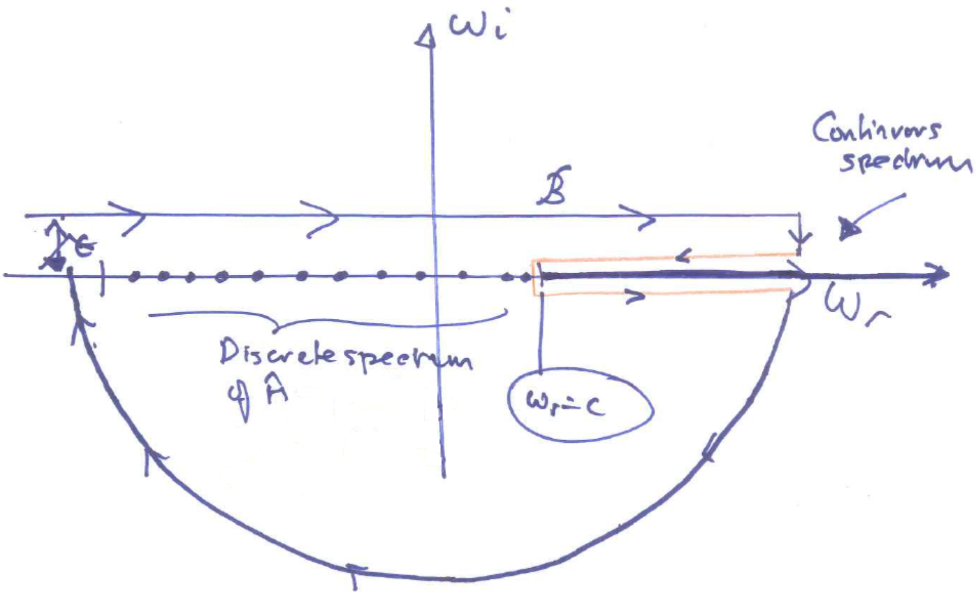


Figure 14.2: Contour of integration for Equation (14.24).

2.1: Bound States and Free States

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A curious feature of wavefunctions in infinite space is that they can have two distinct forms: (i) **bound states** that are localized to one region, and (ii) **free states** that extend over the whole space. Both kinds of states can co-exist in a single system. A simple model exhibiting this is the **1D finite square well**. Consider the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} - U\,\Theta(a - |\hat{x}|), \tag{2.1.1}$$

where $\hat{x}$ and $\hat{p}$ are 1D position and momentum operators, m is the particle mass, U and a are positive real parameters governing the potential function, and Θ denotes the Heaviside step function (1 if the input is positive, and 0 otherwise). As shown below, the potential forms a well of depth U and width $2a$. Outside the well, the potential is zero.

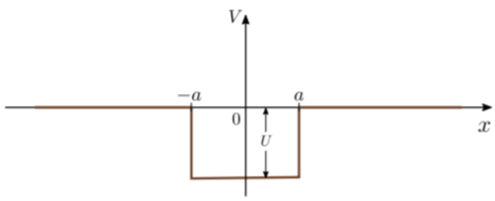

Figure 2.1.1

Figure 14.3: The idea behind unbound states in Quantum Mechanics. Taken from phys.libretexts.org