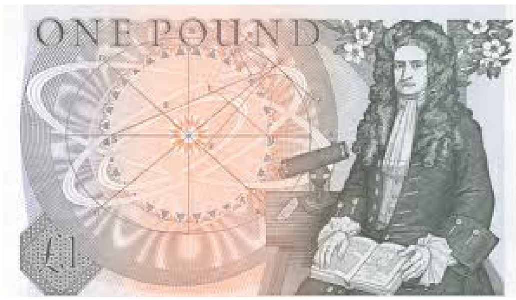
University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

School of Mathematical Sciences
Scoil na nEolaíochtaí Matamaitice
Classical Mechanics and Special Relativity (ACM 20050)



Dr Lennon Ó Náraigh

Lecture notes in Classical Mechanics and Special Relativity, September 2020

Classical Mechanics and Special Relativity (ACM20050)

This course starts by developing the principles of Newtonian Mechanics for particles, systems of particles and rigid bodies. It focuses on applications of Newton's laws with an emphasis on problem solving. It then introduces the fundamental concept of Einstein's theory of Special Relativity and the relativistic extension of Newtonian concepts such as energy relativity including Einstein's famous formula $E = mc^2$.

[Newton's laws of motion] Revision of Newton's laws of motion and the concept of kinetic and potential energy. Conservative force fields and conservation of energy. Equilibrium of a particle and stability of equilibrium.

[Central forces and planetary and satellite motion]: Motion in a plane is discussed with a focus on central forces. Newton's law of gravitation. Determination of the orbit from the central force. Kepler's laws of planetary motion and the motion of satellites.

[Moving coordinate systems]: Non-inertial coordinate systems. Rotating coordinate systems. Velocity and acceleration in a moving frame. Coriolis and centripetal acceleration.

[Systems of Particles]: Centre of mass. Momentum of a system of particles. Motion of the centre of mass. Angular momentum of a system of particles. Total external moment acting on a system and it relation to the angular momentum.

[Einstein's Special Theory of Relativity]: Einstein's basic postulates of special relativity. The Lorentz transformation and the concept of simultaneity. Length contraction. Time dilation. Relative velocity. The concepts of inertial mass, energy and momentum in special relativity including $E = mc^2$.

What will I learn?

On completion of this module students should be able to

1. Analyse the stability of equilibrium positions of a particle;
2. Solve orbit problems in mathematical terms;
3. Explain the concepts of planetary and satellite orbits including Kepler's laws;
4. Derive expressions for the velocity and acceleration in a rotating frame;
5. Explain the concepts of angular velocity, angular momentum for a system of particle;
6. Understand relation to the angular momentum and total external moment acting on a system;
7. Describe Einstein's postulates of special relativity and derive the Lorentz transformation;
8. Explain the geometrical interpretation of the Lorentz transformations in terms of space-time diagrams and the associated concept of 4-vector.
9. Derive the results of special relativity on simultaneity, length contraction, time dilation and relative velocity.
10. Explain the equivalence of mass and energy.

Contents

Chapter 1

Introduction

1.1 Module Outline – Executive Summary

The module has two parts:

1. In the first part, we show that Newton's law, Force=mass×acceleration implies Kepler's Laws, or rather, everything we know about the motion of planets around the sun.
2. In the second part, we show that Newtons laws need to be modified at speeds close to the speed of light. These modifications enable us to solve lots of problems involving particle accelerators, cosmic rays, and radioactive decay.

1.2 Module Outline – Detailed Overview

1. *Advanced Newtonian mechanics:* We will formulate Newton's equation in polar co-ordinates and discuss central forces. Using these two ideas, we will write down, and solve, the equations of motion for two bodies interacting through gravity. This enables us to prove Kepler's empirical laws of planetary motion from first principles.

   As part of this study, we will be motivated to look at the theoretical underpinnings of Newtonian physics, namely the theory of inertial frames and Galilean invariance. This will also motivate us to look at physics in rotating frames of reference.

2. The principle of Galilean invariance says that the laws of physics are the same in all inertial frames. Using this idea, together with Einstein's second postulate of relativity, we will write down the basis of Einstein's special theory of relativity. These postulates allow us to derive the *Lorentz* transformations. As a consequence, we will discuss length contraction, time dilation, and relative velocity. We will also discuss the equation $E = mc^2$.

   Again, we will be motivated to look at the theoretical underpinnings of Special Relativity, namely the theory of four-vectors and Lorentz invariance.

Some notation

We use the notation $\dot{x}$ and $\mathrm{d}x/\mathrm{d}t$ interchangeably, to signify the derivative of the quantity x with respect to time. For the second derivative, we employ two dots.

Chapter 2

Conservative forces in one dimension

2.1 Overview

We look at some general principles of one-dimensional, single-particle motion. This will prepare us for problems in two dimensions. Indeed, the approach to the orbit problem involves a series of clever substitutions to reduce the problem to a one-dimensional one. We assume that we are in an inertial frame so that the equation of motion is

$$m\ddot{x} = F, \tag{2.1}$$

where we follow the standard convention of denoting $\dot{x} = \dfrac{dx}{dt}$ and $\ddot{x} = \dfrac{d^2x}{dt^2}$. Against this backdrop, we consider the following concepts:

- Conservative forces;
- Energy conservation;
- Small oscillations;
- Stability;
- Solution by quadrature.

2.2 Conservative forces

In one dimension, a conservative force is a force that depends only on position.

In particular, this means that the force does not depend on time or velocity. Thus, for example, a drag force of the form $F_D = -k|v|^n v$ is non-conservative. By contrast, gravity depends only on how high you are (i.e. where you are) not how fast you are travelling so is conservative.

Now, associated with the conservative force $F(x)$ is the *potential* $\mathcal{U}$,

$$\mathcal{U}(x) = -\int_a^x F(s)\,\mathrm{d}s. \tag{2.2}$$

Note that the potential $\mathcal{U}$ is defined only up to an arbitrary constant, which depends on the lower limit of integration in Eq. (2.2). This is irrelevant, since the dynamics are governed by the derivative of $\mathcal{U}$, not $\mathcal{U}$ itself.

Two typical conservative forces:

1. The constant-field force $F = -mg$, where m is the particle mass and g is the field strength (positive or negative). Thus,

$$\mathcal{U} = \int^x mg\,\mathrm{d}s = mgx + \text{constant} \tag{2.3}$$

   Now x has the interpretation of the height above the reference level in the constant force field.

2. The spring force ('Hooke's Law') $F = -kx$, where k is the spring constant. Thus,

$$\mathcal{U} = \int^x ks\,\mathrm{d}s = \tfrac{1}{2}kx^2 + \text{constant} \tag{2.4}$$

2.3 Energy

Consider a particle of mass m experiencing the conservative force F, with potential $\mathcal{U}(x) = -\int^x F(s)\,\mathrm{d}s$. The total energy E is the sum

$$E = \tfrac{1}{2}m\dot{x}^2 + \mathcal{U}(x) \tag{2.5}$$

The energy E of a conservative system $\mathcal{U}(x) = -\int^x F(s)\,\mathrm{d}s$ is a constant, $dE/dt = 0$.

Proof: First notice that the force F is the gradient of the potential:

$$F = -\frac{d\mathcal{U}}{dx}.$$

Now differentiate E with respect to time:

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\left[\tfrac{1}{2}m\dot{x}^2 + \mathcal{U}(x)\right],\\
&= \tfrac{1}{2}m\left(2\dot{x}\frac{d\dot{x}}{dt}\right) + \frac{d\mathcal{U}}{dx}\frac{dx}{dt},\\
&= \dot{x}\left(m\ddot{x} + \frac{d\mathcal{U}}{dx}\right),\\
&= \dot{x}\left(m\ddot{x} - F(x)\right),\\
&= 0.
\end{aligned}$$

Note that this result would break down if $\mathcal{U}$ were an explicit function of t, $\mathcal{U} = \mathcal{U}(x,t)$.

This result has many useful corollaries. An immediate one is the so-called *work-energy relation*. Recall the definition of work in one dimension. The work done in moving a particle at x through an infinitesimal distance dx is

$$dW = F(x)\,dx.$$

Thus, the work done in bringing a particle from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x)\,\mathrm{d}x.$$

We have the work-energy relation:

> Let m be a particle experiencing a conservative force. The work done in bringing the particle from $x_1 = x(t_1)$ to $x_2 = x(t_2)$ equals the change in kinetic energy between these two states.

Proof: Since the energy is constant, this can be evaluated at t_1 or t_2, giving the same result:

$$\begin{aligned}
E(t_1) &= E(t_2),\\
\tfrac{1}{2}m\dot{x}^2\big|_{t_1} + \mathcal{U}(x_1) &= \tfrac{1}{2}m\dot{x}^2\big|_{t_2} + \mathcal{U}(x_2).
\end{aligned}$$

Re-arranging,

$$\begin{aligned}
\tfrac{1}{2}m\dot{x}^2\big|_{x_2} - \tfrac{1}{2}m\dot{x}^2\big|_{x_1} &= -\mathcal{U}(x_2) + \mathcal{U}(x_1),\\
&= -\left[\mathcal{U}(x_2) - \mathcal{U}(x_1)\right],\\
&= -\int_{x_1}^{x_2}\frac{d\mathcal{U}}{ds}\mathrm{d}s,\\
&= \int_{x_1}^{x_2} F(s)\,\mathrm{d}s,\\
&= W.
\end{aligned}$$

Conservation of energy is also useful in the construction of *energy diagrams*. Suppose a particle experiences a force whose potential is given by the figure (Fig. 2.1). The force is directed against

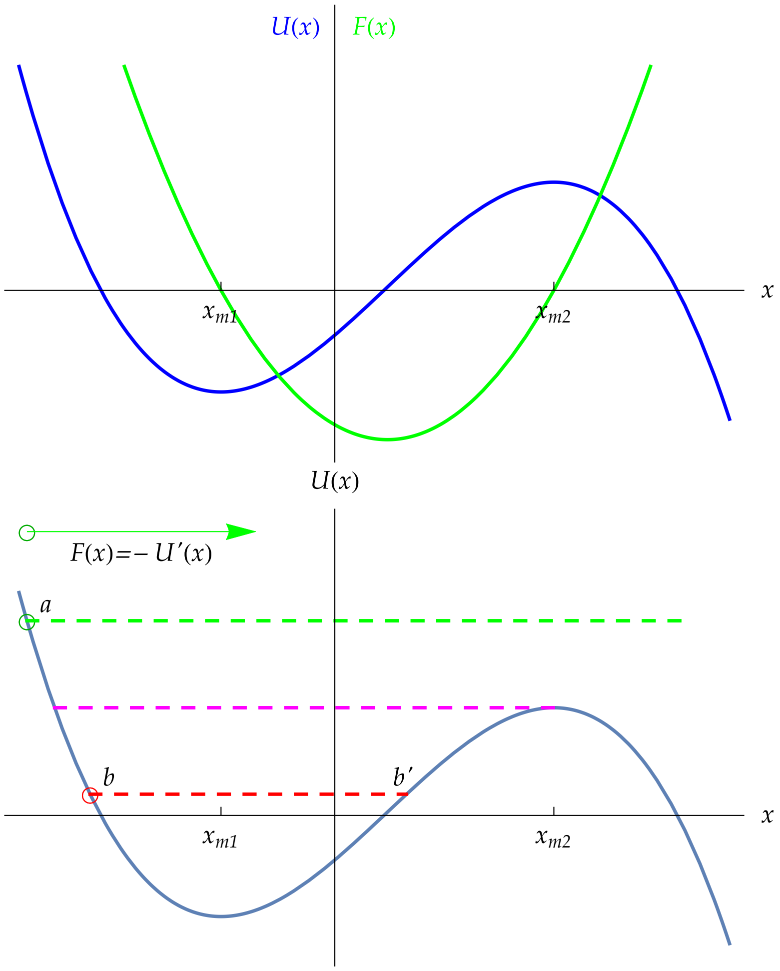


Figure 2.1: Relationship between force and potential.

the gradient of the potential. Thus, a particle will move from areas of high potential, to areas of lower potential.

- Suppose a particle starts from rest at point b. It will move towards the minimum of the potential function and then towards the point b'. At the point b' the particle's kinetic energy vanishes, and the particle cannot move beyond this point. The particle therefore falls back into the well, and continues indefinitely with motion bounded between the states b and b'.

- If, instead, the particle starts from rest at point a, the particle will still possess finite kinetic energy at points b' and x_{m2}. The particle will therefore escape from the neighbourhood of the potential minimum and execute unbounded motion.

2.4 Simple-harmonic motion by energy methods

Recall the equation of motion for a particle experiencing Hooke's force:

$$m\ddot{x} = -kx.$$

Recall that in your ODEs module, you will have learnt that this equation has the solution

$$x = A\cos(\omega t) + B\sin(\omega t), \qquad \omega = \sqrt{k/m},$$

where A and B are constants of integration. The way this solution is usually proposed is by guesswork, which is a little unsatisfactory. It can, however, be derived using energy methods.

The energy of simple-harmonic motion is

$$E = \frac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2,$$

where E is a numerical constant that depends on the initial conditions. We can invert this equation to give

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 &= \frac{2E}{m} - \frac{k}{m}x^2,\\ \frac{dx}{dt} &= \sqrt{\frac{2E}{m} - \frac{k}{m}x^2}.\end{aligned}$$

Hence,

$$dt = \frac{dx}{\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}}.$$

Re-arrange and integrate:

$$\begin{aligned}dt &= \frac{1}{\sqrt{\frac{2E}{m}}}\frac{\mathrm{d}x}{\sqrt{1 - \frac{k}{2E}x^2}},\\ dt &= \frac{1}{\sqrt{\frac{2E}{m}}}\frac{\mathrm{d}\left(\sqrt{k/(2E)}x\right)}{\sqrt{1 - \left(\sqrt{k/(2E)}x\right)^2}} \times \sqrt{\frac{2E}{k}}, \qquad s = \sqrt{k/(2E)}x,\\ dt &= \sqrt{\frac{m}{k}}\frac{\mathrm{d}s}{\sqrt{1-s^2}}, \qquad \omega = \sqrt{k/m}.\end{aligned}$$

$$t = \omega^{-1}\int_{s_0}^{s}\frac{\mathrm{d}s}{\sqrt{1-s^2}}, \qquad \omega = \sqrt{k/m}.$$

This is a standard integral:

$$\int^x \frac{\mathrm{d}s}{\sqrt{1-s^2}} = \sin^{-1} s$$

Hence,

$$\omega t = \sin^{-1} s - \underbrace{\sin^{-1} s_0}_{=\varphi}$$

Or,

$$s = \sin(\omega t + \varphi).$$

But $s = \sqrt{k/(2E)}x$, hence

$$x = \sqrt{\frac{2E}{k}} \sin(\omega t + \varphi). \tag{2.6}$$

The constants φ and $\sqrt{2E/k}$ can be related to A and B by applying the trigonometric addition formula [exercise].

2.5 Small oscillations

Equilibrium corresponds to those points where the force is zero:

$$F(x_{\mathrm{m}}) = 0, \qquad \mathcal{U}'(x_{\mathrm{m}}) = 0$$

(See Fig. 2.2). Hence, equilibrium points correspond to local minima or maxima of the potential

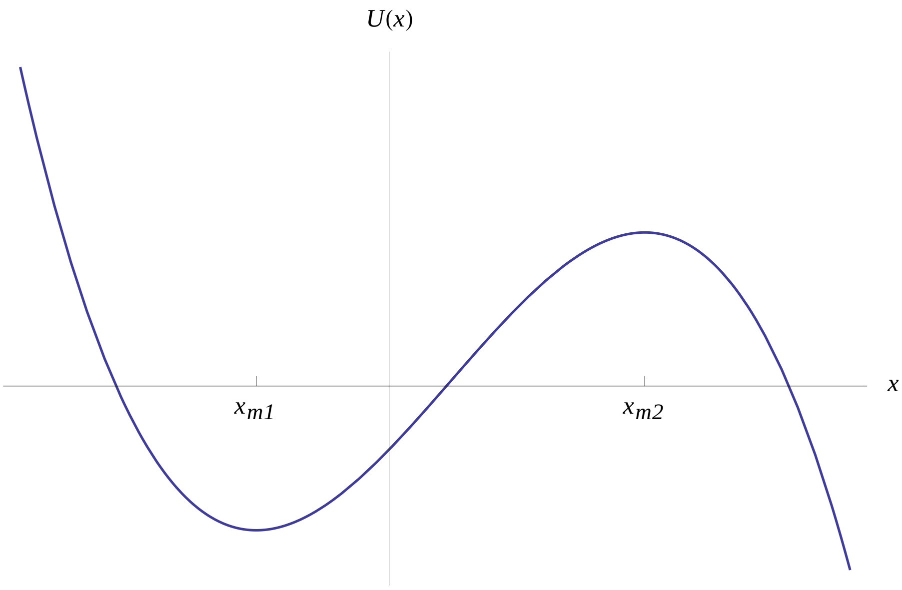


Figure 2.2: Maxima and minima of the potential function.

function. If a particle is at equilibrium and at rest then it will remain at the equilibrium point. What happens is initially close the the equilibrium point x_{m}?

- If a particle initially in the vicinity of the equilibrium stays in the vicinity of the equilibrium, said equilibrium is *stable*. (*Think:* particle at bottom of valley.)

- If a particle initially in the vicinity of the equilibrium moves away from the equilibrium and into another part of the domain, said equilibrium is *unstable*. (*Think:* particle at top of hill.)

Mathematically, we can classify the equilibria as follows: Let us take a particle that is at a position x which is very close to an equilibrium point x_{m}. By Taylor's expansion,

$$\begin{aligned} F(x) &\approx F(x_{\mathrm{m}}) + F'(x_{\mathrm{m}})(x - x_{\mathrm{m}}) + \text{H.O.T.}, \\ &= -\mathcal{U}'(x_{\mathrm{m}}) - \mathcal{U}''(x_{\mathrm{m}})(x - x_{\mathrm{m}}), \\ &= -\mathcal{U}''(x_{\mathrm{m}})(x - x_{\mathrm{m}}). \end{aligned}$$

The equation of motion for the small disturbance $\epsilon(t) = x(t) - x_{\mathrm{m}}$ away from the equilibrium point is given by

$$\frac{d^2\epsilon}{dt^2} = -\frac{\mathcal{U}''(x_{\mathrm{m}})}{m}\epsilon. \tag{2.7}$$

Three possibilities for the motion:

1. $\mathcal{U}''(x_{\mathrm{m}}) > 0$ (local minimum). Identify a frequency

$$\omega = \sqrt{\mathcal{U}''(x_{\mathrm{m}})/m}. \tag{2.8}$$

   Then, using the theory of simple harmonic motion,

$$\epsilon = A\sin(\omega t) + B\cos(\omega t),$$

   where A and B are fixed by the initial conditions. The particle oscillates around the local minimum. The point $x_{\mathrm{m}} = x_{\mathrm{m}1}$ is a *stable* eq^{m}.

2. $\mathcal{U}''(x_{\mathrm{m}}) < 0$ (local maximum). Identify a rate $\sigma = \sqrt{-\mathcal{U}''(x_{\mathrm{m}})/m}$. Doing a similar integral to the one for SHM,

$$\epsilon = Ae^{\sigma t} + Be^{-\sigma t},$$

   where A and B are fixed by the initial conditions. The particle moves exponentially fast away from the local maximum. The point $x_{\mathrm{m}} = x_{\mathrm{m}2}$ is an *unstable* eq^{m}.

3. If $\mathcal{U}''(x_{\mathrm{m}}) = 0$, one has to look at $\mathcal{U}'''(x_{\mathrm{m}})$, or possibly higher-order derivatives.

2.6 Solution by quadrature

So far we have learned some very important concepts in dynamics:

1. Energy conservation;
2. Simple-harmonic motion;
3. Linear stability of equilibria.

One subject we touched on was the *solution of equations of motion by quadrature*, wherein we solved for simple-harmonic motion via the energy method. In this final section, we do this for arbitrary potentials.

When the energy is conserved, the motion can be re-expressed in the following form:

$$\frac{dx}{dt} = \Phi\left(x\right), \tag{2.9}$$

where we take care only to work in an interval I of the potential landscape where the kinetic energy is positive or zero. The points where $\Phi = 0$ are called the *turning points*, and there can be no motion beyond these points. Re-writing Eq. (2.9), we obtain

$$dt = \frac{\mathrm{d}x}{\Phi\left(x\right)}.$$

If the particle begins with a position x_0 at $t = 0$ then

$$t = \int_{x_0}^{x} \frac{\mathrm{d}s}{\Phi\left(s\right)}. \tag{2.10}$$

If we can perform the integration this gives us $t\left(x\right)$ which, *if* we can invert, gives us $x\left(t\right)$. We will solve the orbit problem by reducing it to a one-dimensional system, and then by solving the reduced system by quadrature.

A system of N particles that can be solved by a reduction to a finite number of integrals of type (2.10) is called *integrable*.

Chapter 3

Worked example: Conservative forces in one dimension

A particle with mass m moves in one dimension. The potential-energy function is

$$\mathcal{U}(x) = \alpha x^{-2} - \beta x^{-1},$$

where α and β are positive constants. The particle is released from rest at $x_0 = \alpha/\beta$.

1. Show that $\mathcal{U}(x)$ can be written as

$$\mathcal{U}(x) = \frac{\alpha}{x_0^2}\left[\left(\frac{x_0}{x}\right)^2 - \frac{x_0}{x}\right].$$

   Graph $\mathcal{U}(x)$; calculate $\mathcal{U}(x_0)$ and thereby locate the point x_0 on the graph. If a potential-well minimum exists, calculate the period of small oscillations about that minimum.

2. Calculate $v(x)$, the speed of the particle as a function of position. Graph the result and give a qualitative description of the motion.

3. For what value of x is the speed of the particle maximal? What is the value of that minimum speed?

4. If, instead, the particle is released at $x_1 = 3\alpha/\beta$, compute $v(x)$ and give a qualitative description of the motion. Locate the point x_1 on the graph of $\mathcal{U}$.

5. For each release point (x_0 and x_1), what are the maximum and minimum values of x reached during the motion?

1. We have

$$\mathcal{U}(x) = \frac{\alpha}{x^2} - \frac{\beta}{x}.$$

Get rid of β: $\beta = \alpha/x_0$. Hence,

$$\begin{aligned}\mathcal{U} &= \frac{\alpha}{x^2} - \frac{\alpha}{x_0}\frac{1}{x},\\ &= \frac{\alpha}{x_0^2}\frac{1}{x/x_0}^2 - \frac{\alpha}{x_0^2}\frac{1}{x/x_0},\\ &= \frac{\alpha}{x_0^2}\left[\left(\frac{x_0}{x}\right)^2 - \frac{x_0}{x}\right].\end{aligned}$$

We now analyse $\mathcal{U}(x)$.

$$\begin{aligned}\mathcal{U} &\sim +\infty, \qquad \text{as } x \to 0^+,\\ \mathcal{U} &\sim -0, \qquad \text{as } x \to \infty.\end{aligned}$$

This suggests that $\mathcal{U}$ has a zero. Setting $\mathcal{U} = 0$ gives $(x_0/x)^2 - (x_0/x) = 0$, hence $x = x_0$. We should also try to find the extreme points of $\mathcal{U}$:

$$\begin{aligned}\frac{d\mathcal{U}}{dx} &= \frac{\alpha}{x_0^2}\left[-\frac{2x_0^2}{x^3} + \frac{x_0}{x^2}\right],\\ \frac{d^2\mathcal{U}}{dx^2} &= \frac{\alpha}{x_0^2}\left[\frac{6x_0^2}{x^4} - \frac{2x_0}{x^3}\right].\end{aligned}$$

The extreme point is at $d\mathcal{U}/dx = 0$, i.e. $x = 2x_0$. This is a minimum because

$$\begin{aligned}\left.\frac{d^2\mathcal{U}}{dx^2}\right|_{x=2x_0} &= \frac{2\alpha}{x_0^4}\left[3\left(\frac{x_0}{x}\right)^4 - \left(\frac{x_0}{x}\right)^3\right]_{x=2x_0} = \frac{2\alpha}{x_0^4}\left(\frac{3}{2^4} - \frac{1}{2^3}\right)\\ &= \frac{2\alpha}{x_0^4}\left(\frac{3}{16} - \frac{1}{8}\right) = +\frac{\alpha}{8x_0^4} > 0.\end{aligned}$$

Putting all these facts together, we draw a curve like Fig. 3.1.

Going back to Ch. 2, the frequency of small oscillations around the well minimum is

$$\omega = \sqrt{\frac{\mathcal{U}''(x_{\mathrm{eq}})}{m}} = \sqrt{\frac{\alpha}{8x_0^4 m}}$$

But the period is $T = 2\pi/\omega$, hence

$$T = 2\pi\sqrt{\frac{8mx_0^4}{\alpha}}.$$

2. Calculate $v(x)$, the speed of the particle as a function of position. Graph the result and give a qualitative description of the motion.

We consider the energy associated with the system. Newton's equation is $d^2x/dt = -\mathcal{U}'(x)$,

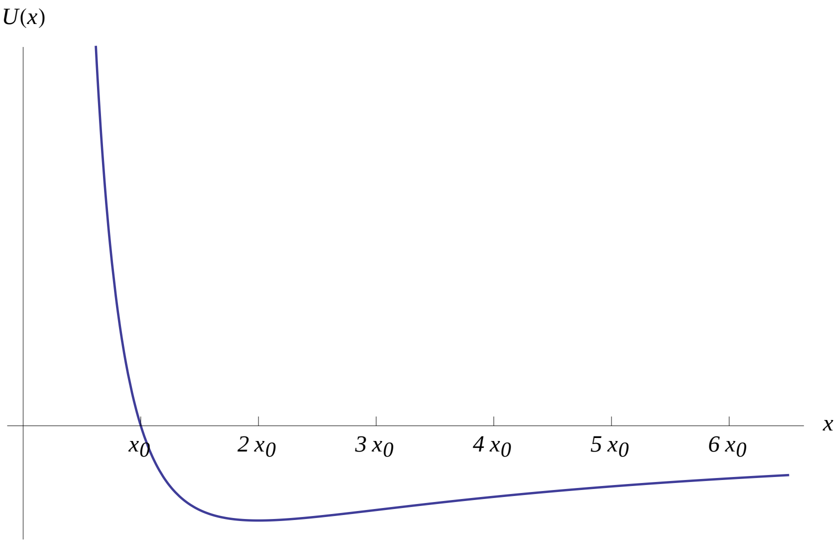


Figure 3.1: Sketch of potential function in worked example

so the energy is

$$E = \tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \mathcal{U}(x) = \tfrac{1}{2}mv^2 + \mathcal{U}(x).$$

The particle starts from rest at $x = x_0$, so $E = 0 + \mathcal{U}(1) = 0$. Therefore, the speed $v(x)$ is

$$v(x) = \sqrt{-2\mathcal{U}(x)/m} = \sqrt{\frac{2\alpha}{mx_0^2}\left[-\left(\frac{x_0}{x}\right)^2 + \frac{x_0}{x}\right]},$$

which is valid for $x_0 \le x < \infty$.

Therefore, in qualitative terms, the particle starts from rest at $x = x_0$ and moves down the gradient of potential towards the minimum at $x = 2x_0$. It overshoots this point because $v(2x_0) = \sqrt{-2\mathcal{U}(2x_0)} = \left(1/\sqrt{2}\right)\sqrt{\alpha/(mx_0^2)}$. It travels towards the second turning point at $x = \infty$, which it never reaches. Thus, the motion is unbounded, but just barely so.

3. For what value of x is the speed of the particle maximal? What is the value of that minimum speed? Since $v = \sqrt{-2\mathcal{U}(x)/m}$, the speed is maximal when the potential is minimal, i.e. $x = 2x_0$, and $v(2x_0) = \left(1/\sqrt{2}\right)\sqrt{\alpha/(mx_0^2)}$. To prove this formally, compute dv/dx and its derivative at $x = 2x_0$, using $d\mathcal{U}/dx = 0$ there:

$$\begin{aligned}
\sqrt{m}\frac{dv}{dx}\bigg|_{x=2x_0} &= -\left(\frac{1}{\sqrt{-2\mathcal{U}(x)}}\frac{d\mathcal{U}}{dx}\right)_{x=2x_0} = 0. \\
\sqrt{m}\frac{d^2v}{dx^2}\bigg|_{x=2x_0} &= \left[-\frac{1}{\sqrt{-2\mathcal{U}(x)}}\frac{d^2\mathcal{U}}{dx^2} - \frac{1}{2\sqrt{2}}\frac{1}{(-\mathcal{U}(x))^{3/2}}\left(\frac{d\mathcal{U}}{dx}\right)^2\right]_{x=2x_0}, \\
&= \left(-\frac{1}{\sqrt{-2\mathcal{U}(x)}}\frac{d^2\mathcal{U}}{dx^2}\right)_{x=2x_0} < 0.
\end{aligned}$$

4. If, instead, the particle is released at $x_1 = 3\alpha/\beta$, compute $v(x)$ and give a qualitative description of the motion. Locate the point x_1 on the graph of $\mathcal{U}$.

   Note: $\alpha/\beta = x_0$, hence the particle starts at $x = 3x_0$, from rest. The energy is $E = \mathcal{U}(3x_0) = -(2/9)(\alpha/x_0^2)$, and

$$v(x) = \sqrt{-\frac{2}{m}\left[\mathcal{U}(x) + \frac{2\alpha}{9x_0^2}\right]}.$$

   The turning points are the the zeros of this function, and lie at

$$\mathcal{U}(x) + \frac{2\alpha}{9x_0^2} = 0,$$

   or

$$\left(\frac{x_0}{x}\right)^2 - \frac{x_0}{x} = -\frac{2}{9}.$$

   Letting $s := x/x_0$, this is

$$\begin{aligned} \frac{1}{s^2} - \frac{1}{s} &= -\tfrac{2}{9}, \\ 1 - s &= -\tfrac{2}{9}s^2, \\ 2s^2 - 9s + 9 &= 0, \\ s &= \frac{9 \pm \sqrt{81-72}}{4} = 3 \text{ or } \frac{3}{2}. \end{aligned}$$

   Qualitatively, the motion is periodic and oscillates between turning points, $\frac{3}{2}x_0 \le x \le 3x_0$.

5. For each release point (x_0 and x_1), what are the maximum and minimum values of x reached during the motion?

   We have done this already:

   - Case 1: $x_0 \le x < \infty$.
   - Case 2: $\frac{3}{2}x_0 \le x \le 3x_0$.

Chapter 4

Motion in a plane

4.1 Overview

In Ch. 2 (Conservative forces in one dimension), we examined one-dimensional single-particle motion. Several important concepts carry over into planar motion:

1. Conservative forces and energy conservation;
2. Using the energy method to reduce the solution to an integral.

To carry out a similar reduction in two dimensions, appropriate coordinates are needed. For the orbit problem, these coordinates are the *plane polar coordinates*. In this chapter, we will

- Define plane polar coordinates;
- Define directional unit vectors in this system;
- Derive the velocity and acceleration along these directions.

The last point is nontrivial because these directions change with time. We will also solve some problems where life is made easier by the use of polar coordinates.

4.2 Plane polar coordinates

In two dimensions, and in an inertial frame, two Cartesian components x and y are necessary and sufficient to specify the position of a particle:

- Position: $\boldsymbol{x} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$.
- Velocity: $\boldsymbol{v} = d\boldsymbol{x}/dt = (dx/dt)\,\hat{\boldsymbol{x}} + (dy/dt)\,\hat{\boldsymbol{y}}$.

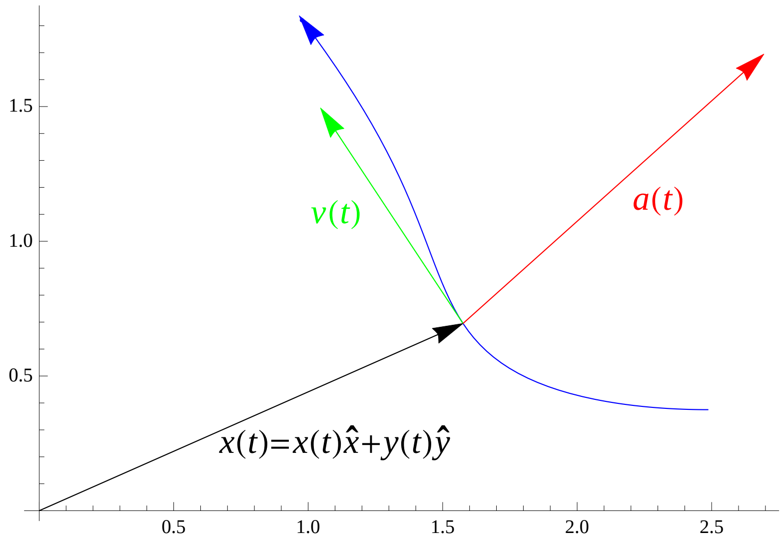


Figure 4.1: Trajectories in the plane.

- Acceleration: $\boldsymbol{a} = d^2\boldsymbol{x}/dt^2 = (d^2x/dt^2)\,\hat{\boldsymbol{x}} + (d^2y/dt^2)\,\hat{\boldsymbol{y}}$.

(See Fig. 12.1.) We could use these coordinates to solve the equations of motion in a plane. They are not always appropriate however (e.g. circular motion). In some situations, it is more appropriate to use two further quantities:

- The distance a particle is away from the origin of the inertial frame, r;
- The angle the displacement vector $\boldsymbol{x}$ makes with the positive sense of the x-axis, θ.

These are the *plane polar coordinates*. From Fig. 4.2,

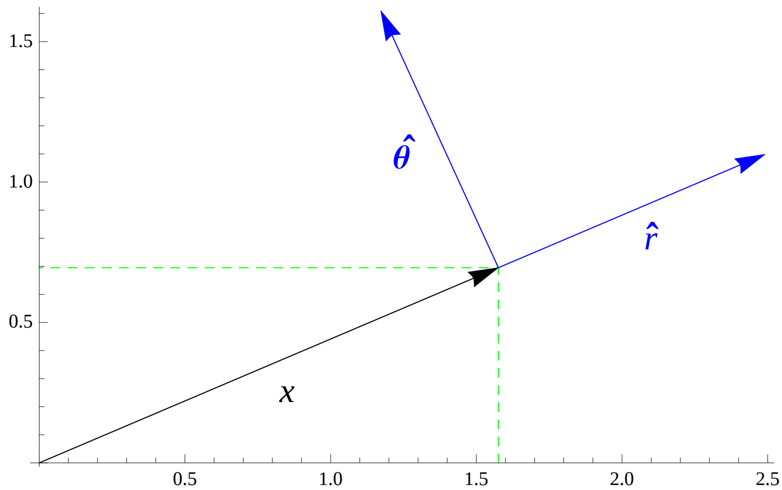


Figure 4.2: Polar coordinates

$$
\begin{aligned}
x &= r\cos\theta, \\
y &= r\sin\theta. \qquad (4.1)
\end{aligned}
$$

[Derive the inverse transformation.] We also introduce a direction vector $\hat{\boldsymbol{r}}$ in the direction of increasing r, and a direction vector $\hat{\boldsymbol{\theta}}$ in the direction of increasing θ. From the figure,

$$
\begin{aligned}
\hat{\boldsymbol{r}} &= \cos\theta\hat{\boldsymbol{x}} + \sin\theta\hat{\boldsymbol{y}}, \\
\hat{\boldsymbol{\theta}} &= -\sin\theta\hat{\boldsymbol{x}} + \cos\theta\hat{\boldsymbol{y}}, \qquad (4.2)
\end{aligned}
$$

and

$$
\boldsymbol{x} = r\hat{\boldsymbol{r}}.
$$

Note: $\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\theta}} = 0$ (orthogonality). Note further the matrix relations

$$
\begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{=R} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{pmatrix},
$$

with inverse

$$
\begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix}. \qquad (4.3)
$$

Exercise: Show that $\det(R) = 1$, and that $R^{-1} = R^T$. Compute R^2 and reduce it to a simple form using trigonometric identities.

Derive the velocity and acceleration in the new directions: We compute the velocity and acceleration in the radial and tangential directions. This is a little complicated, because these directions change with time, along with r and θ. We will need the relations

$$
\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\cos\theta\hat{\boldsymbol{x}} + \sin\theta\hat{\boldsymbol{y}}\right) = -\sin\theta\hat{\boldsymbol{x}} + \cos\theta\hat{\boldsymbol{y}} = \hat{\boldsymbol{\theta}}, \qquad (4.4)
$$

$$
\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\sin\theta\hat{\boldsymbol{x}} + \cos\theta\hat{\boldsymbol{y}}\right) = -\cos\theta\hat{\boldsymbol{x}} - \sin\theta\hat{\boldsymbol{y}} = -\hat{\boldsymbol{r}}. \qquad (4.5)
$$

Derive the velocity:

$$
\begin{aligned}
\boldsymbol{v} &= \frac{d\boldsymbol{x}}{dt} = \frac{d}{dt}\left(r\hat{\boldsymbol{r}}\right), \\
&= \dot{r}\hat{\boldsymbol{r}} + r\frac{d\hat{\boldsymbol{r}}}{dt}.
\end{aligned}
$$

Using Eq. (4.4) and the chain rule,

$$\begin{aligned}\frac{d\hat{\boldsymbol{r}}}{dt} &= \dot{\theta}\frac{\mathsf{d}\hat{\boldsymbol{r}}}{\mathsf{d}\theta}, \\ &= \dot{\theta}\left(-\sin\theta\hat{\boldsymbol{x}} + \cos\theta\hat{\boldsymbol{y}}\right), \\ &= \dot{\theta}\hat{\boldsymbol{\theta}}.\end{aligned}$$

Hence,

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

Identify

- $v_r = \dot{r}$, the velocity in the radial (r-) direction;
- $v_\theta = r\dot{\theta}$, the velocity in the tangential (θ-) direction.

Derive the acceleration:

$$\begin{aligned}\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} &= \frac{d}{dt}\left(\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right), \\ &= \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\theta}\frac{\mathsf{d}\hat{\boldsymbol{r}}}{\mathsf{d}\theta} + \left(r\ddot{\theta} + \dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}} + r\dot{\theta}^2\frac{\mathsf{d}\hat{\boldsymbol{\theta}}}{\mathsf{d}\theta}, \\ &= \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \left(r\ddot{\theta} + \dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\boldsymbol{r}}.\end{aligned}$$

Hence, identify

- $a_r = \ddot{r} - r\dot{\theta}^2$, the acceleration in the radial (r-) direction;
- $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$, the acceleration in the tangential (θ-) direction.

New names for the terms:

1. The *linear acceleration*, $\ddot{r}\hat{\boldsymbol{r}}$, that a particle has when it accelerates radially.
2. The *angular acceleration*, $r\ddot{\theta}\hat{\boldsymbol{\theta}}$, that occurs as a result of a change to the rate at which the particle is rotating.
3. The *centripetal acceleration*, $-r\dot{\theta}^2\hat{\boldsymbol{r}}$, which appears in the context of circular motion.
4. The *Coriolis acceleration*, $2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}}$, which a particle has when both r and θ change with time – even at uniform rates.

4.3 Dynamical situations where polar coordinates are appropriate

Circular motion in the absence of external forces: Consider ordinary circular motion, wherein a particle of mass m, held in tension by a string, undergoes uniform circular motion. We write down the equations of motion in polar coordinates and compute the tension T.

In the absence of constraints (a 'free particle'), the equations of motion are

$$m\left(\ddot{r}-r\dot{\theta}^2\right) = 0, \tag{4.6}$$
$$m\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right) = 0. \tag{4.7}$$

In this system however, r is constant, so a constraint force must enter into the equations. The constraint acts against any change in the radial acceleration $\ddot{r}$, and so we add to Eq. (4.6) a force $-T\hat{\boldsymbol{r}}$, and enforce $\dot{r}=0$:

$$m\left(-r\dot{\theta}^2\right) = -T.$$

Writing $\dot{\theta}=\omega=$ Const., this becomes

$$T = mr\omega^2,$$

while Eq. (4.7) reduces to the trivial expression $0=0$. Thus, in circular motion, the 'centripetal acceleration' and the tension are in balance.

A particle on a rotating groove: Consider a particle on a rotating groove, as shown in Fig. 4.3. The rate of rotation is constant and equal to ω. Calculate its motion as a function of time, and evaluate the constraining forces.

In the absence of any constraining forces (a 'free particle'), the equations of motion are unchanged from Eqs. (4.6) and (4.7). Now, however, there is a constraint in the $\hat{\boldsymbol{\theta}}$ direction, since the acceleration in this direction is forced to zero by the constant rotation. Thus, the constrained equations of motion are

$$m\left(\ddot{r}-r\omega^2\right) = 0, \tag{4.8}$$
$$m\left(r\ddot{\theta}+2\dot{r}\omega\right) = N_\theta. \tag{4.9}$$

We therefore solve Eq. (4.8)

$$\ddot{r} = r\omega^2.$$

It is readily shown (using energy methods) that the solution to this equation is

$$r = Ae^{\omega t}+Be^{-\omega t},$$

where A and B are constants of integration, for which we now solve. Suppose that the wheel starts from rest, so that the initial radial velocity is zero. Thus, $r\left(0\right)=r_0$, and $\dot{r}\left(0\right)=0$. We have a

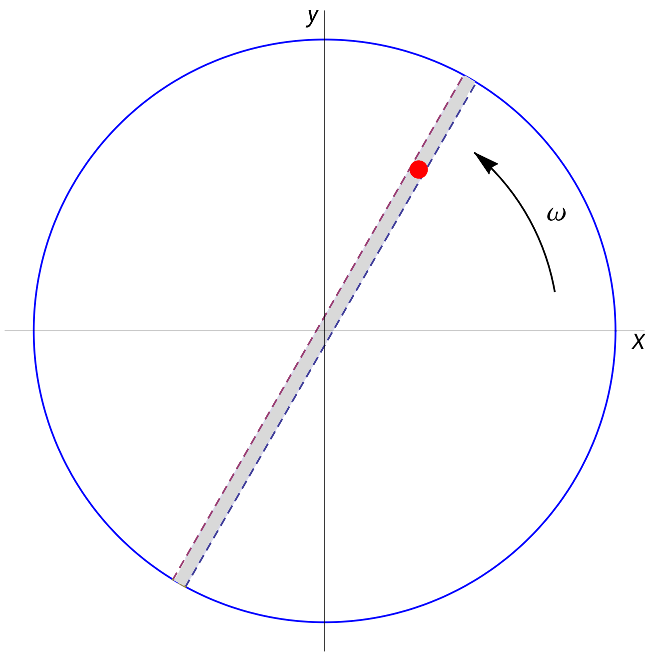


Figure 4.3: A particle in a rotating disk with groove

matrix system:

$$\begin{pmatrix} 1 & 1 \\ \omega & -\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} r_0 \\ 0 \end{pmatrix}.$$

Inverting gives $A = B = r_0/2$. Hence,

$$r = \tfrac{1}{2} r_0 \left(e^{\omega t} + e^{-\omega t} \right) = r_0 \cosh\left(\omega t\right).$$

Note that the derivative is thus $\dot{r} = r_0 \omega \sinh\left(\omega t\right)$. Substitution of this relation into Eq. (4.9) gives the normal force:

$$N_\theta = 2m\omega^2 r_0 \sinh\left(\omega t\right).$$

Circular motion in a gravitational field: Consider the problem of a particle executing circular motion in a gravitational field (See Fig. 4.4). The particle is given an initial 'kick' of velocity v_0 and starts from the bottom of the circular pipe. What is the minimum value of v_0 necessary for the particle not to fall off the pipe? Note: You may have seen this problem before. Here, we will use the polar-coordinate formulation and energy methods to solve the problem.

Step 1: Forces As in problem (1), the particle is constrained to reside on the hoop, such that $\dot{r} = 0$, and such that an additional constraint force N_r is introduced into the equations of motion. Recall the free-space equations of motion:

$$\text{Radial equation :} \qquad m\left(\ddot{r} - r\dot{\theta}^2\right) = 0,$$

$$\text{Tangential equation :} \qquad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0.$$

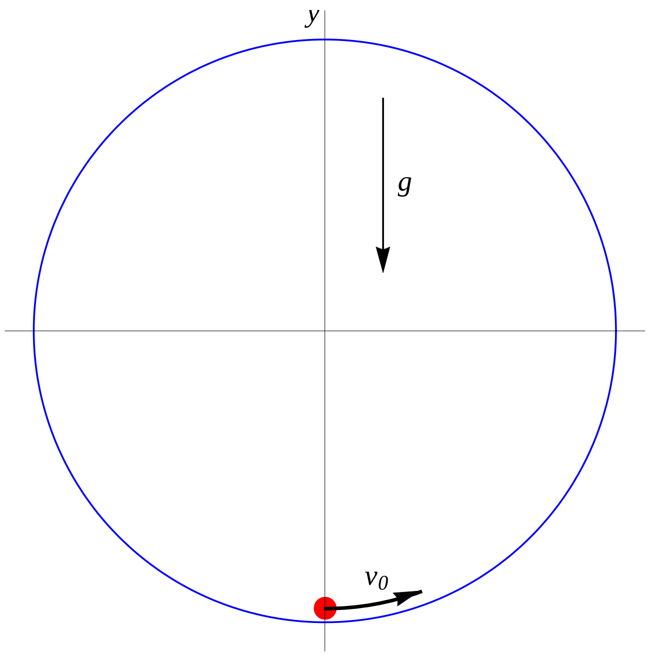


Figure 4.4: Motion inside a circular pipe in a gravitational field

Introduce gravity and N_r. The gravitational force is

$$-g\hat{\boldsymbol{y}}.$$

Using the matrix inversion (4.3), this is

$$-g\left(\sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}}\right).$$

For an unconstrained particle in a gravitational field, the equations are thus

$$\begin{aligned} m\left(\ddot{r} - r\dot{\theta}^2\right) &= -mg\sin\theta, \\ m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) &= -mg\cos\theta. \end{aligned}$$

Impose the constraint: $\dot{r} = 0$, constraint force in radial direction:

$$\begin{aligned} m\left(-r\dot{\theta}^2\right) &= -mg\sin\theta - N_r, \\ m\left(r\ddot{\theta}\right) &= -mg\cos\theta. \end{aligned} \tag{4.10}$$

Hence,

$$N_r = mr\dot{\theta}^2 - mg\sin\theta. \tag{4.11}$$

The particle falls of the pipe when the constraint N_r vanishes. To compute N_r as a function of known quantities, we can use *energy methods*.

Step 2: Energy methods Take the equation of motion in the tangential direction (Eq. (4.10)) and multiply it by $\dot{\theta}$:

$$m\dot{\theta}\left(r\ddot{\theta}\right) = -\dot{\theta}mg\cos\theta.$$

Integrate:

$$\frac{d}{dt}\left(\tfrac{1}{2}mr\dot{\theta}^2\right) = -\frac{d}{dt}mg\sin\theta.$$

Hence,

$$E = \tfrac{1}{2}mr^2\dot{\theta}^2 + mgr\sin\theta = \text{Const.}$$

(We have multipled the conserved quantity by the constant r to obtain an energy). Since E is constant, it must equal its initial value ($\theta = -\pi/2$):

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mr^2\dot{\theta}^2 + mgr\left(1 + \sin\theta\right).$$

Hence,

$$mr\dot{\theta}^2 = \frac{mv_0^2}{r} - 2mg\left(1 + \sin\theta\right). \tag{4.12}$$

Combining Eqs. (4.11) and (4.12),

$$N_r = \frac{mv_0^2}{r} - mg\left(2 + 3\sin\theta\right).$$

The constraint N_r is minimal at $\theta = \pi/2$, $N_{r,\min} = (mv_0^2/r) - 5mg$. The condition for the particle to fall off the pipe is the vanishing of the constraint. Thus, if $N_{r,\min} \geq 0$, then the particle stays in contact with the pipe:

$$\begin{aligned} \left(mv_0^2/r\right) &\geq 5mg, \\ |v_0| &\geq \sqrt{5gr}. \end{aligned}$$

Chapter 5

Angular momentum and central forces

5.1 Overview

Previously, in Ch. 4 (Motion in a plane),

- We studied motion in the plane (two dimensions).
- We were particularly concerned with problems with circular symmetry. Thus, we wrote down the equations of motion in polar coordinates.
- We solved several problems involving motion constrained to a circle.

In this chapter we introduce two further concepts which will enable us to solve far more general problems of motion in a plane: *angular momentum* and *central forces*.

5.2 Angular momentum

Consider a particle of mass m, position vector $\boldsymbol{x}$, and velocity $\boldsymbol{v} = \dot{\boldsymbol{x}}$ relative to an inertial frame. The angular momentum $\boldsymbol{L}$ relative to that frame is

$$\boldsymbol{L} = \boldsymbol{x} \times (m\boldsymbol{v}) \,. \tag{5.1}$$

Let us differentiate this expression and apply Newton's equation:

$$\begin{aligned}
\frac{d\boldsymbol{L}}{dt} &= \frac{d\boldsymbol{x}}{dt} \times (m\boldsymbol{v}) + \boldsymbol{x} \times \left(m\frac{d\boldsymbol{v}}{dt} \right) \\
&= \boldsymbol{v} \times (m\boldsymbol{v}) + \boldsymbol{x} \times (m\boldsymbol{a}) \,, \\
&= 0 + \boldsymbol{x} \times \boldsymbol{F},
\end{aligned}$$

where $\boldsymbol{F}$ is the vector sum of forces acting on the particle. We identify the torque:

$$\boldsymbol{\tau} = \text{Torque} := \boldsymbol{x} \times \boldsymbol{F},$$

the equation of angular momentum:

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{\tau}, \tag{5.2}$$

and the principle of conservation of angular momentum:

If the sum of the torques acting on a particle is zero, then the angular momentum is constant.

Some points about torque:

- Torque depends on the origin we choose, but force does not;
- $\boldsymbol{\tau} = \boldsymbol{x} \times \boldsymbol{F}$: the torque and the force are mutually perpendicular;
- For a system of more than one particle, there can be torque without force, and force without torque, but in general both are present (See Fig. 5.1).

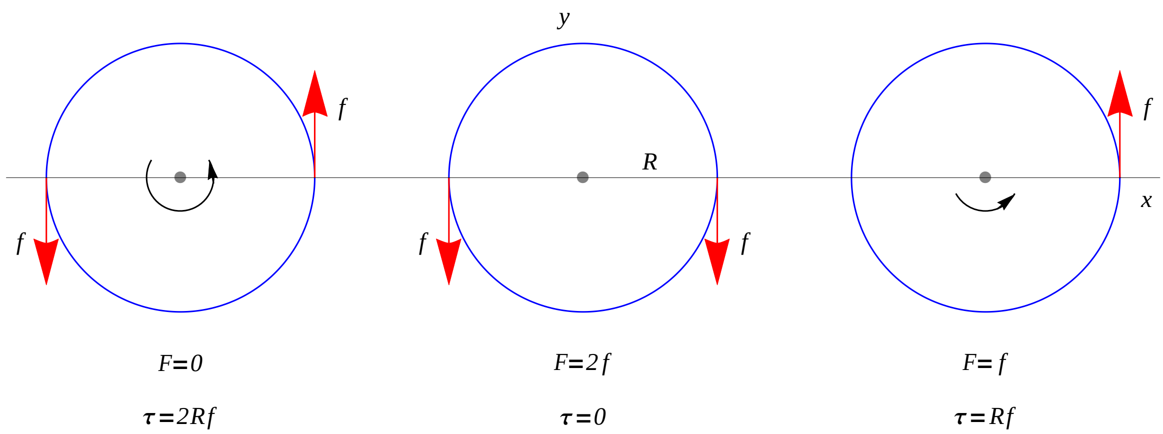


Figure 5.1: Relationship between torque and force. The torques are pointing out of the page.

Example: Computing the angular momentum of the conical pendulum Consider the conical pendulum shown in Fig. 5.2 (a). Calculate the angular momentum about the points $P1$ and $P2$.

1. The point $P1$: We use the polar coordinates in the x-y plane, with unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$. Thus, the triple $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{z}}\}$ form an orthonormal triad. As such, we have:

$$\begin{aligned}
\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{z}}, \\
\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{z}} &= \hat{\boldsymbol{r}}, \\
\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}} &= \hat{\boldsymbol{\theta}}.
\end{aligned}$$

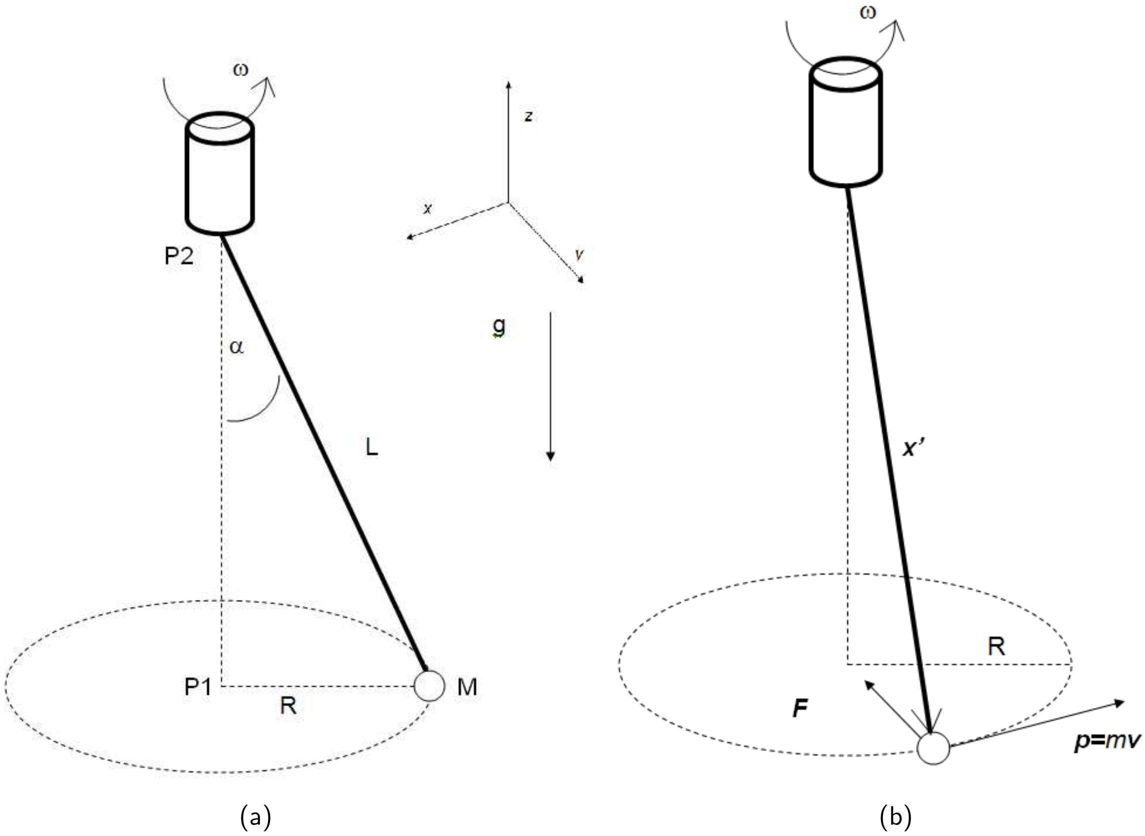


Figure 5.2: The conical pendulum.

These expressions enable us to compute the particle anglular mmentum with respect to the point $P1$. First, we notice that the particle executes circular motion in this plane, with velocity $\boldsymbol{v} = \omega R\hat{\boldsymbol{\theta}}$ in the tangential direction. The radius vector is $R\hat{\boldsymbol{r}}$. Thus,

$$\boldsymbol{L}_{P1} = \omega M R^2 \hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} = \omega M R^2 \hat{\boldsymbol{z}}.$$

2. The point $P2$: Refer to Fig. 5.2. Call the vector from $P1$ to M to $\boldsymbol{x}'$. Clearly,

$$\boldsymbol{x}' = \sin\alpha L\hat{\boldsymbol{r}} - \cos\alpha L\hat{\boldsymbol{z}},$$

and the velocity $\boldsymbol{v}$ is in the $\hat{\boldsymbol{\theta}}$ direction, where $\left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}\right)$ are polar coordinates in the x–y plane. Thus,

$$\begin{aligned}
\boldsymbol{L}_{P2} &= M\boldsymbol{x}' \times \boldsymbol{v}, \\
&= M\left(\sin\alpha L\hat{\boldsymbol{r}} - \cos\alpha L\hat{\boldsymbol{z}}\right) \times \left(v_\theta \hat{\boldsymbol{\theta}}\right), \\
&= ML\sin\alpha v_\theta \left(\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}}\right) - ML\cos\alpha v_\theta \left(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{\theta}}\right), \\
&= ML\sin\alpha v_\theta \hat{\boldsymbol{z}} + ML\cos\alpha v_\theta \hat{\boldsymbol{r}}.
\end{aligned}$$

Taking the absolute magnitude and using $\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}} = 0$, this gives $|\boldsymbol{L}_{P2}| = MLv_\theta = ML\omega R$.

Computing the torque of the conical pendulum In the z-direction, a component of the tension is in balance with gravity:

$$T\cos\alpha = Mg,$$

hence, there is no net force in the z-direction. In the x-y plane, the force is given by a component of the tension, and is radially inwards:

$$\boldsymbol{F} = -T\sin\alpha\hat{\boldsymbol{r}},$$

which balances with the centripetal acceleration.

1. The point $P1$: The total force is $\boldsymbol{F} = -T\sin\alpha\hat{\boldsymbol{r}}$. The torque is therefore

$$\boldsymbol{\tau}_{P1} = R\hat{\boldsymbol{r}} \times (-T\sin\alpha\hat{\boldsymbol{r}}) = 0.$$

   Hence, $d\boldsymbol{L}_{P1}/dt = 0$, consistent with $\boldsymbol{L}_{P1} = \omega MR^2\hat{\boldsymbol{z}} = \text{Const.}$

2. The point $P2$:

$$\begin{aligned}\boldsymbol{\tau}_{P2} &= (\sin\alpha L\hat{\boldsymbol{r}} - \cos\alpha L\hat{\boldsymbol{z}}) \times (-T\sin\alpha\hat{\boldsymbol{r}}),\\ &= +TL\cos\alpha\sin\alpha\,(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}}),\\ &= MgL\sin\alpha\hat{\boldsymbol{\theta}}.\end{aligned}$$

5.3 Central forces in three dimensions

Consider a particle experiencing a force $\boldsymbol{F}$ in a coordinate system (x, y, z). The force $\boldsymbol{F}$ is said to be central with respect to the coordinate system if

$$\boldsymbol{F} = F(r)\,\hat{\boldsymbol{r}}, \tag{5.3}$$

where r is the radial distance and $\hat{\boldsymbol{r}}$ is the radial direction. We call the coordinate origin the *force centre*.

In three dimensions, the radial vector is $\hat{\boldsymbol{r}} = \boldsymbol{x}/|\boldsymbol{x}|$, where $|\boldsymbol{x}| = \sqrt{x^2+y^2+z^2} = r$.

In three dimensions, a force $\boldsymbol{F}$ is called *conservative* if it can be written as the gradient of a function $\mathcal{U}(x, y, z)$:

$$\boldsymbol{F} = -\nabla\mathcal{U}(x, y, z). \tag{5.4}$$

Here ∇ is the *gradient* operator,

$$\nabla = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}.$$

Theorem: All central forces are conservative.

Proof: We write down what we think to be the potential:

$$\mathcal{U} = -\int^{|\boldsymbol{x}|} F(s)\,\mathrm{d}s.$$

Now

$$\begin{aligned}
\nabla\mathcal{U} &= \frac{\partial\mathcal{U}}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\mathcal{U}}{\partial x}\hat{\boldsymbol{y}} + \frac{\partial\mathcal{U}}{\partial x}\hat{\boldsymbol{z}}, \\
&= \frac{\partial\mathcal{U}}{\partial|\boldsymbol{x}|}\frac{\partial|\boldsymbol{x}|}{\partial x}\hat{\boldsymbol{x}} + \cdots, \\
&= -\hat{\boldsymbol{x}}F(|\boldsymbol{x}|)\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{1/2} + \cdots, \\
&= -\hat{\boldsymbol{x}}F(|\boldsymbol{x}|)\frac{x}{\sqrt{x^2+y^2+z^2}} + \cdots, \\
&= -F(|\boldsymbol{x}|)\frac{\boldsymbol{x}}{|\boldsymbol{x}|}.
\end{aligned}$$

Hence, $\boldsymbol{F} = -\nabla\mathcal{U}$, as required.

Theorem: For a particle experiencing a conservative force in three dimensions, the energy

$$E = \tfrac{1}{2}m\dot{\boldsymbol{x}}^2 + \mathcal{U}(|\boldsymbol{x}|)$$

is conserved, provided the motion satisfies Newton's equation, $m\ddot{\boldsymbol{x}} = -\nabla\mathcal{U}$.

Proof: Left as an exercise.

5.3.1 Central forces for a system of two particles

Consider a system of two particles of masses m_1 and m_2, interacting via a force $\boldsymbol{F}$, in any spatial dimension. For such a system, the force is central if the force on particle 1 due to particle 2 is

$$\boldsymbol{F}_{12} = F(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)\frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}.$$

By Newton's Third Law,

$$\boldsymbol{F}_{21} = -\boldsymbol{F}_{12} = F\left(|\boldsymbol{x}_1 - \boldsymbol{x}_2|\right)\frac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}.$$

The most important example is gravity (See Fig. 5.3):

$$\boldsymbol{F}_{12} = -\frac{Gm_1m_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^2}\left(\frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\right), \tag{5.5}$$

where G is the gravitational constant. When one of the particles is much more massive than the

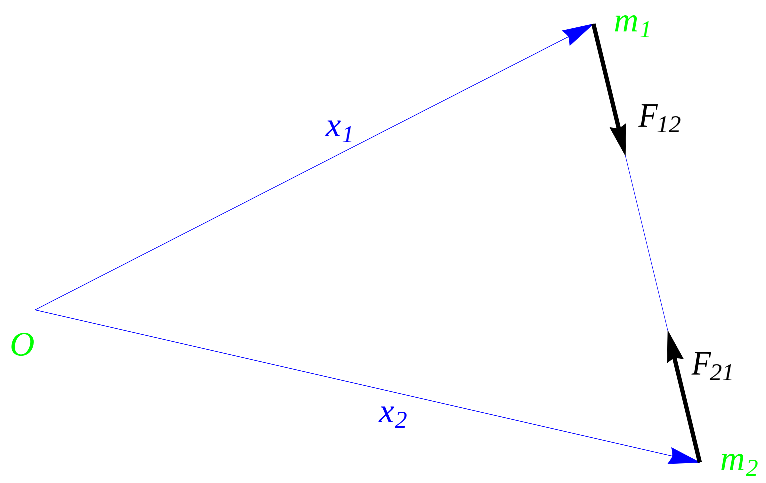


Figure 5.3: Force diagram for a two-particle gravitational interaction.

other ($m_2 \gg m_1$, say), then m_2 can be regarded as having infinite inertia, and to be unmoved by the interaction force. The location of m_2 can then be treated as a force centre, and we may regard m_1 as experiencing a central force field relative to this centre. An example of such a scenario is satellite motion around the earth. Later on we shall formulate this approximation in a rigorous way.

Coulomb's law for interactions between charged particles is also a central force

$$\boldsymbol{F}_{12} = \frac{e_1e_2}{4\pi\varepsilon_0|\boldsymbol{x}_1 - \boldsymbol{x}_2|^2}\left(\frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\right). \tag{5.6}$$

Unlike Newton's law, wherein m_1 and m_2 must be positive, the charges e_1 and e_2 can have either sign. If both charges have the same sign, then the force (5.6) is *repulsive* (Fig. 5.4).

The law of gravitation as written down here (Eq. (5.5)) is for point particles. However, it also holds for spherical bodies at finite separations. This follows from the point-particle law by integration. Specifically, consider a particle P experiencing a gravitational force from a collection of point particles. The net force on P is obtained by summing over these particles. If these particles are arranged in a spherical distribution, and if the number of these particles tends to

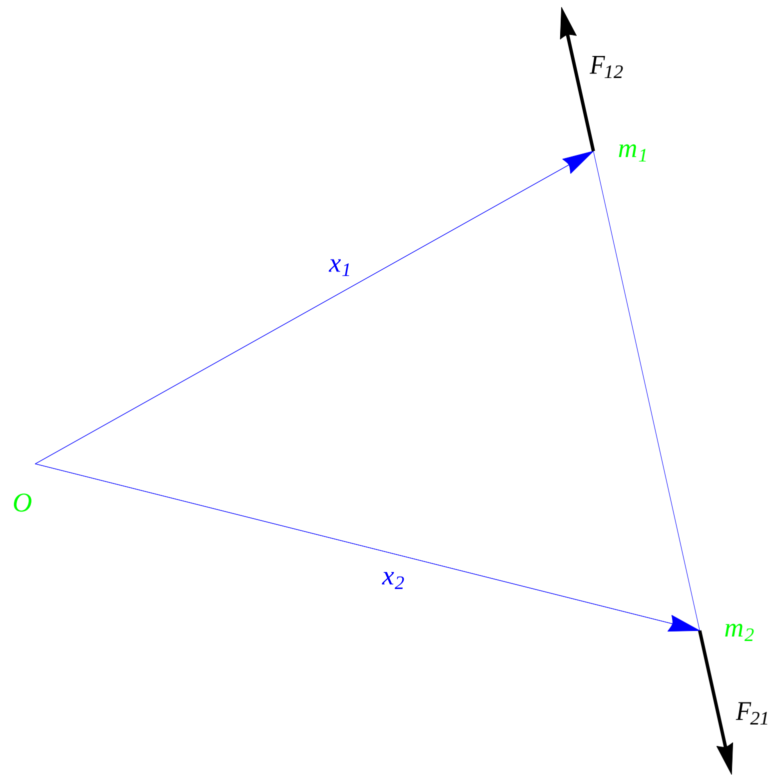


Figure 5.4: Force diagram for a repulsive Coulomb interaction.

infinity while their separation tends to zero, then we the net force on P is an integral, whose form is identical to the original law for point particles. Similarly, point P can be replaced by a spherical distribution and another integral performed, and the final result is the original force law, with the point particle masses replaced by the masses of the extended bodies.

See Note[1].

[1] This proof is not on the syllabus but can be found in University Physics, Ch. 12 (10^{th} edition), and in other places, like Wikipedia: http://en.wikipedia.org/wiki/Shell_theorem

Chapter 6

Central forces reduce to one-dimensional motion

6.1 Overview

In Chapter 5 (Angular momentum and central forces),

- We defined angular momentum: $\boldsymbol{L} = \boldsymbol{x} \times (m\boldsymbol{v})$. This depends on the coordinate origin.
- We formulated the principle of conservation of angular momentum: in the absence of net torque, the angular momentum is conserved.
- We defined central forces and showed that they were conservative.

In this chapter we demonstrate that central forces possess zero torque, when all measurements are made relative to the force centre. This enables us to reduce the problem of particle motion under central forces to a one-dimensional one.

6.2 The master theorem

Theorem: Consider a particle experiencing a central force. When measured with respect to the force centre, the angular momentum is conserved.

Proof: Consider the angular momentum $\boldsymbol{L} = \boldsymbol{x} \times (m\boldsymbol{v})$, as measured from the force centre. Recall that we showed that the rate of change of angular momentum is the torque:

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{\tau} = \boldsymbol{x} \times \boldsymbol{F} = r\hat{\boldsymbol{r}} \times \boldsymbol{F}.$$

But $\boldsymbol{F} = F(r)\,\hat{\boldsymbol{r}}$, hence

$$\boldsymbol{x} \times \boldsymbol{F} = r\hat{\boldsymbol{r}} \times \boldsymbol{F} = rF(r)\,\hat{\boldsymbol{r}} \times \hat{\boldsymbol{r}} = 0,$$

and the *vector* angular momentum is conserved. Since the direction of angular momentum is conserved, it follows that $\boldsymbol{x}$ and $\boldsymbol{v}$ lie in the same plane for all times, and that motion under a central force is in fact two dimensional. Thus, we consider central forces in two dimensions only.

6.3 Central forces in Newton's equations

We have seen that a central force can written as $\boldsymbol{F} = F(r)\,\hat{\boldsymbol{r}}$. There is therefore no force in the tangential direction. Newton's laws for such a system are therefore the following:

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = F(r), \tag{6.1}$$

$$m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0. \tag{6.2}$$

- The force in Eq. (6.1) can be re-written as

$$F = -\frac{\partial \mathcal{U}}{\partial r}, \qquad \mathcal{U}(r) = -\int^r F(s)\,\mathrm{d}s.$$

- The second equation (Eq. (6.2)) can be re-written as

$$\frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0,$$

  and immediate consequence of which is that the the quantity $h := r^2\dot{\theta}$ is conserved.

The quantity h is related to the magnitude of the angular momentum:

$$\begin{aligned}
\boldsymbol{L} &= r\hat{\boldsymbol{r}} \times (m\boldsymbol{v}), \\
&= mr\hat{\boldsymbol{r}} \times \left(\hat{\boldsymbol{r}}\,v_r + \hat{\boldsymbol{\theta}}\,v_\theta\right), \\
&= mr^2\dot{\theta}\,\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}}, \\
&= mr^2\dot{\theta}\,\hat{\boldsymbol{z}}, \\
&= mh\hat{\boldsymbol{z}}.
\end{aligned}$$

hence $h = L/m$.

Problem: Re-write Eqs. (6.1)–(6.2) as

$$\begin{aligned}
m\left(\ddot{r} - r\dot{\theta}^2\right) &= -\frac{\partial \mathcal{U}}{\partial r}, \\
m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) &= 0.
\end{aligned} \tag{6.3}$$

Identify the kinetic energy $\quad K = \frac{1}{2}m\boldsymbol{v}^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right).$

Prove that Newton's equations (6.3) satisfy the *Euler–Lagrange equations*

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{r}} - \frac{\partial K}{\partial r} = -\frac{\partial \mathcal{U}}{\partial r},$$
$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}} - \frac{\partial K}{\partial \theta} = -\frac{\partial \mathcal{U}}{\partial \theta}.$$

Next year, you will have to study these equations in more generality. For now, take them as a handy mnemonic for remembering the components of acceleration in the polar-coordinate system.

6.3.1 Reduction of the central force equations to a single equation

Take the equation (6.1)

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}r}. \tag{6.4}$$

Since $h = r^2\dot{\theta}$ is conserved, the 'weird' quantity $mr\dot{\theta}^2$ can be eliminated:

$$mr\dot{\theta}^2 = mr\left(\frac{h}{r^2}\right)^2 = \frac{mh^2}{r^3}.$$

Hence, Eq. (6.4) becomes

$$m\ddot{r} - \frac{h^2}{r^3} = -\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}r}.$$

By introducing the *effective potential*

$$\mathcal{U}_{\text{eff}} := \tfrac{1}{2}\frac{mh^2}{r^2} + \mathcal{U}, \tag{6.5}$$

the equation of motion becomes quasi one-dimensional:

$$m\ddot{r} = -\frac{\mathrm{d}\mathcal{U}_{\text{eff}}}{\mathrm{d}r}.$$

The central-force problem in three dimensions has been reduced to a problem of one-dimensional motion.

Example: In Chapter 2, we showed that conservative forces in many dimensions conserve the energy

$$E = \tfrac{1}{2}m\boldsymbol{v}^2 + \mathcal{U}(r), \qquad r := |\boldsymbol{r}|.$$

Let us study the energy for the gravitational interaction between a particle of mass m and a much more massive particle of mass M, $M \gg m$. We may approximate the particle m as experiencing a central force with a force centre at the location of M,

$$\boldsymbol{F} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}} := F(r)\,\hat{\boldsymbol{r}}.$$

Recall, from Chapter 2, the potential associated with such a central force is

$$\mathcal{U}(\boldsymbol{r}) = -\int^r F(s)\,\mathrm{d}s = GMm\int^r s^{-2}\mathrm{d}s = -\frac{GMm}{r}.$$

The energy is therefore

$$E = \tfrac{1}{2}m\boldsymbol{v}^2 - \frac{GMm}{r}.$$

We can re-write this in several ways, depending on the problem we wish to solve. It is therefore helpful to be familiar with each form:

$$\begin{aligned} E &= \tfrac{1}{2}m\boldsymbol{v}^2 - \frac{GMm}{r}, \\ &= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GMm}{r}, \\ &= \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}m\frac{h^2}{r^2} - \frac{GMm}{r}, \\ &= \tfrac{1}{2}m\dot{r}^2 + \mathcal{U}_{\text{eff}}. \end{aligned}$$

Note that this last expression is precisely the expression one would obtain by multiplying the quasi one-dimensional equation $m\ddot{r} = -\mathcal{U}'_{\text{eff}}(r)$ by $\dot{r}$ and integrating w.r.t. time.

More on the effective potential: Consider the effective potential for the basic potential $\mathcal{U} = -\lambda r^{-1}$:

$$\mathcal{U}_{\text{eff}} = \tfrac{1}{2}\frac{mh^2}{r^2} - \frac{\lambda}{r}.$$

- The initial conditions determine the magnitude of h and hence the well depth. They also control the energy E.
- If $E < 0$ the particle is constrained to lie between $r_{\min}$ and $r_{\max}$.
- The particle cannot therefore spiral in towards $r = 0$. The repulsive part of the effective potential prevents this. This is intimately related to the conservation of angular momentum.

Note further, that if the particle resides at the well minimum r_0, then $\mathcal{U}'_{\text{eff}}(r_0) = 0$, $\ddot{r} = 0$, and the particle radius remains constant. This corresponds to a stable circular orbit. There is precisely one such orbit for each h-value.

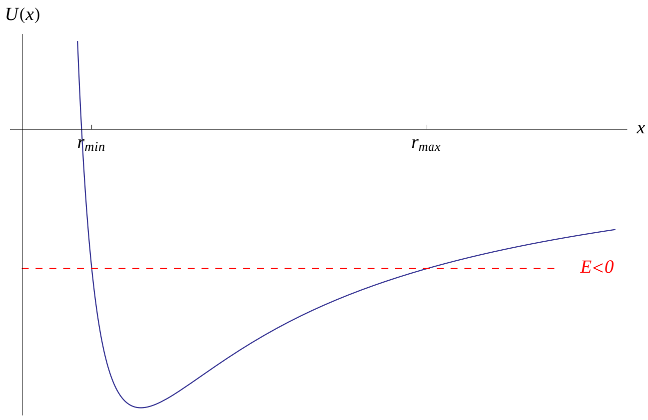


Figure 6.1: Effective potential of an attractive force (sign-negative potential).

Escape velocity of a particle in the earth's gravitational field: Suppose that a rocket of mass m is launched from the surface of the earth and that its velocity relative to the centre of the earth is $\boldsymbol{v}_0$. Find the condition that the rocket escapes the earth's pull (ignoring the effect of the atmosphere and the rotation of the earth).

The energy of the rocket-earth system is

$$E = \tfrac{1}{2}m\boldsymbol{v}_0^2 - \frac{GMm}{R}$$

where R is the initial radial location of the rocket (equal to the earth's radius). From considering the effective-potential diagram (Fig. 6.1), the condition for the rocket to escape the earth's pull is that $E = 0$, since then the motion is just barely unbounded. Hence,

$$\tfrac{1}{2}m\boldsymbol{v}_0^2 = \frac{GMm}{R},$$

and

$$|\boldsymbol{v}_0| = \sqrt{\frac{2GM}{R}}, \tag{6.6}$$

which is independent of m.

Now let us include the effects of rotation. Then, $\boldsymbol{v}_0$ contains both a radial part $\dot{r}$, and a tangential part $r_p\dot{\theta}$. The radius r_p is not the distance R to the earth's centre, but rather the distance to the earth's axis of rotation. At the equator however, $r_p = R$, and $|\boldsymbol{v}_0|$ is maximized:

$$\boldsymbol{v}_0^2 = \underbrace{\dot{r}^2}_{\text{Boost given to rocket from engines}} + \underbrace{R^2\dot{\theta}^2}_{\text{Boost given to rocket from earth's rotation, at equator}}$$

Thus, the best launch sites are near the equator.

Note finally that Eq. (6.6) is independent of mass. We may re-arrange it to give R as a function of

$v := \boldsymbol{v}_0$:

$$R = \frac{2GM}{v^2},$$

and apply it to massless particles. In particular, consider the photon, which travels at the speed of light. Then, the radius is

$$R_c = \frac{2GM}{c^2}.$$

For the earth, $R_c \approx 2\,\mathrm{cm}$. Thus, if all the earth's mass were concentrated at a point, no photon within a two $2\,\mathrm{cm}$ radius of this point could escape from that point's gravitational field, and we would have a small black hole.

Chapter 7

Kepler's Laws

7.1 Overview

In Ch. 6 (Central forces),

- We showed how the motion of a particle experiencing a central force can be reduced to a problem in one-dimensional mechanics, using the effective potential.
- We tackled some problems involving the gravitational force between two point masses in the limit where one particle was more massive than the other. Thus, we treated the lighter particle as one experiencing a central force from a fixed force centre.

In this chapter we shall show how such a reduction is possible for all pairs of particles, regardless of their mass. With this final piece of the jigsaw in place, we shall show how Newton's laws provide the theoretical explanation for the empirical laws of planetary motion observed by the astronomer Kepler.

7.2 Kepler's Laws

The following historical notes are taken from the website

`http://www.hps.cam.ac.uk/starry/`

Nicholas Copernicus (1473-1543), a Polish monk, formulated a *heliocentric view* of the solar system. In this theory, the planets move around the sun in circular orbits. The reasons for this theory were both practical, since it gave a solution to the problem of the 'retrograde motion' of the planets, and aesthetic, since a theory based on circles seemed geometrically 'pure'. Two months after publishing his book, Copernicus died.

Later, Tycho Brahe, (1546-1601)[1], a Danish astronomer, built up an enormous and very precise database over his lifetime concerning the motion of the planets then known[2]. He initially worked at Hven in Denmark but after disagreements with the king, moved to Prague. Brahe's assistant, Johannes Kepler (1571-1630), analyzed these data and formulated three empirical laws of planetary motion [3]:

1. Each planet moves in an ellipse with the sun at one focus
2. The radius vector form the sun to a planet sweeps out equal areas in equal times.
3. The period of revolution, T, is related to the semi-major axis of the ellipse, a, by $T^2 \propto a^3$.

7.3 Centre-of-mass coordinates

We begin by explaining how *all* two-body central-force problems can be reduced to a quasi one-dimensional, quasi one-particle problem. Consider two particles acting under gravity and refer to Fig. 7.1. The force on particle 1 due to particle 2 is

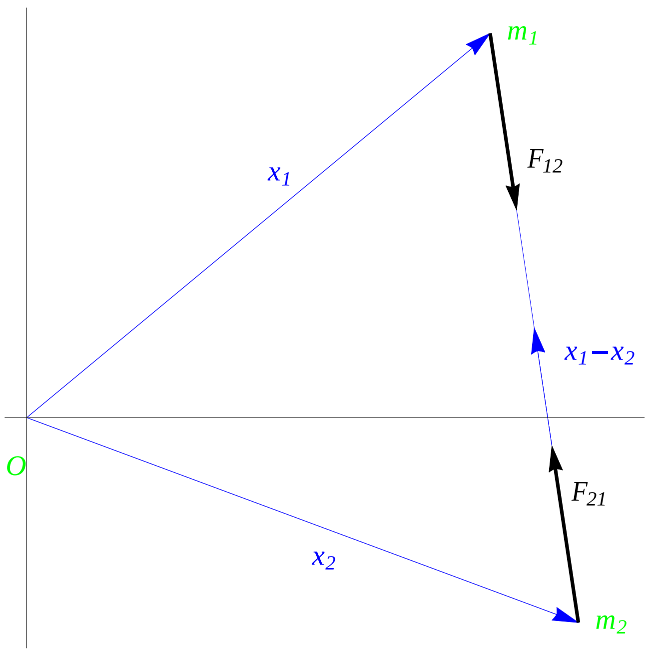


Figure 7.1: Force diagram for a two-particle gravitational interaction.

$$\boldsymbol{F}_{12} = -\frac{Gm_1m_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^2}\left(\frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\right). \tag{7.1a}$$

[1]Brahe lost his nose in a duel in Rostock in 1566. He wore a replacement nose made out of brass. He is one of several scientists to have fought an unsuccessful duel (*cf.* Evariste Galois).

[2]Mercury–Saturn

[3]The precise circumstances under which Kepler gained full access to Brahe's database are not clear.

By Newton's Third Law ,the force on particle 2 due to particle 1 is

$$\boldsymbol{F}_{21} = -\boldsymbol{F}_{12} = -\frac{Gm_1m_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^2}\left(\frac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\right). \tag{7.1b}$$

Let us introduce a new coordinate

$$\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2,$$

which points from mass 2 to mass 1. We also introduce a radial vector

$$\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}| = \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}.$$

Then, the force laws (7.1) can be rewritten as

$$\begin{aligned} \boldsymbol{F}_{12} &= -\frac{Gm_1m_2}{|\boldsymbol{r}|^3}\boldsymbol{r}, \\ \boldsymbol{F}_{21} &= +\frac{Gm_1m_2}{|\boldsymbol{r}|^3}\boldsymbol{r}, \end{aligned} \tag{7.2}$$

and the equations of motion are

$$\begin{aligned} m_1\ddot{\boldsymbol{x}}_1 &= -\frac{Gm_1m_2}{|\boldsymbol{r}|^3}\boldsymbol{r}, \\ m_2\ddot{\boldsymbol{x}}_2 &= +\frac{Gm_1m_2}{|\boldsymbol{r}|^3}\boldsymbol{r}. \end{aligned} \tag{7.3}$$

Let's add these equations (Eqs. (7.3)) together:

$$m_1\ddot{\boldsymbol{x}}_1 + m_2\ddot{\boldsymbol{x}}_2 = 0.$$

This shows that the *centre-of-mass vector*

$$\boldsymbol{R} := \frac{m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2}{m_1 + m_2}$$

does not accelerate. That is, *a frame of reference in which the centre of mass and the origin coincide is an inertial frame*. Next, let us subtract Eqs. (7.3) from each other:

$$\begin{aligned} \ddot{\boldsymbol{x}}_1 &= -\frac{Gm_2}{|\boldsymbol{r}|^3}\boldsymbol{r}, \\ \ddot{\boldsymbol{x}}_2 &= +\frac{Gm_1}{|\boldsymbol{r}|^3}\boldsymbol{r}. \end{aligned}$$

Let's SUBTRACT these equations one from the other:

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{x}}_1 - \ddot{\boldsymbol{x}}_2 = -\frac{G\left(m_1 + m_2\right)}{|\boldsymbol{r}|^3}\boldsymbol{r}. \tag{7.4}$$

Define the *reduced mass*:

$$\mu := \frac{m_1 m_2}{m_1+m_2}. \tag{7.5}$$

Multiply Eq. (7.4) by the reduced mass:

$$\mu\ddot{\boldsymbol{r}} = -\frac{Gm_1m_2}{|\boldsymbol{r}|^3}\boldsymbol{r}. \tag{7.6}$$

Therefore, if we can solve the problems $\ddot{\boldsymbol{R}}=0$ and Eq. (7.6), we can reconstruct $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ via the transformations

$$\begin{aligned}
\begin{pmatrix}\boldsymbol{r}\\ \boldsymbol{R}\end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ \dfrac{m_1}{m_1+m_2} & \dfrac{m_2}{m_1+m_2}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{pmatrix},\\
\begin{pmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{pmatrix} &= \begin{pmatrix} \dfrac{m_2}{m_1+m_2} & 1 \\ -\dfrac{m_1}{m_1+m_2} & 1\end{pmatrix}\begin{pmatrix}\boldsymbol{r}\\ \boldsymbol{R}\end{pmatrix}
\end{aligned}$$

Moreover, the angular momentum

$$\boldsymbol{L} = \mu\boldsymbol{r}\times\boldsymbol{v}, \qquad \boldsymbol{v}=\dot{\boldsymbol{r}},$$

as measured relative to the location of m_2, is conserved:

$$\begin{aligned}
\frac{d\boldsymbol{L}}{dt} &= \mu\dot{\boldsymbol{r}}\times\boldsymbol{v} + \mu\boldsymbol{r}\times\frac{d\boldsymbol{v}}{dt},\\
&= \boldsymbol{r}\times\left(\mu\ddot{\boldsymbol{r}}\right),\\
&= \boldsymbol{r}\times\left(\frac{Gm_1m_2}{|\boldsymbol{r}|^3}\boldsymbol{r}\right),\\
&= 0.
\end{aligned}$$

Similarly, the energy

$$E = \tfrac{1}{2}\mu\boldsymbol{v}^2 - \frac{Gm_1m_2}{r}, \qquad \mathcal{U} = -\frac{Gm_1m_2}{r},$$

is conserved:

$$\begin{aligned}
\frac{dE}{dt} &= \mu\boldsymbol{v}\cdot\dot{\boldsymbol{v}} + \boldsymbol{v}\cdot\nabla\mathcal{U}\left(|\boldsymbol{r}|\right),\\
&= \boldsymbol{v}\cdot\left(\mu\ddot{\boldsymbol{r}} + \mathcal{U}'\left(|\boldsymbol{r}|\right)\nabla|\boldsymbol{r}|\right),\\
&= \boldsymbol{v}\cdot\left(\mu\ddot{\boldsymbol{r}} + \mathcal{U}'\left(|\boldsymbol{r}|\right)\frac{\boldsymbol{r}}{|\boldsymbol{r}|}\right),\\
&= \boldsymbol{v}\cdot\left(\mu\ddot{\boldsymbol{r}} - \boldsymbol{F}\right),\\
&= 0.
\end{aligned}$$

Thus, the motion can be reduced to one-dimensional particle motion in the $\boldsymbol{r}$–$\boldsymbol{v}$ plane.

For the earth-sun system, the centre-of-mass vector $\boldsymbol{R}$ and the position vector of the sun $\boldsymbol{x}_2 := \boldsymbol{x}_\odot$ almost coincide:

$$
\begin{aligned}
m_2 = M_\odot &= 2 \times 10^{30}\,\mathrm{kg}, \\
m_1 = m_\mathrm{e} &= 6 \times 10^{24}\,\mathrm{kg}, \\
\frac{m_\mathrm{e}}{m_\odot} &= 3 \times 10^{-6}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{|\boldsymbol{R}|}{|\boldsymbol{x}_1|} &= \frac{|\boldsymbol{R}|}{|\boldsymbol{x}_\mathrm{e}|}, \\
&= \frac{|m_\mathrm{e}\boldsymbol{x}_\mathrm{e} + M_\odot \boldsymbol{x}_\odot|}{|\boldsymbol{x}_\mathrm{e}|\,(m_\mathrm{e} + M_\odot)}, \\
&\leq \frac{M_\odot}{M_\odot + m_\mathrm{e}}\frac{|\boldsymbol{x}_\odot|}{|\boldsymbol{x}_\mathrm{e}|} + \frac{m_\mathrm{e}}{m_\mathrm{e} + M_\odot},
\end{aligned}
$$

where the inequality follows from the Triangle Inequality. Hence,

$$
\begin{aligned}
\frac{|\boldsymbol{R}|}{|\boldsymbol{x}_1|} &\leq \frac{1}{1 + (m_\mathrm{e}/M_\odot)}\frac{|\boldsymbol{x}_\odot|}{|\boldsymbol{x}_\mathrm{e}|} + \frac{(m_\mathrm{e}/M_\odot)}{1 + (m_\mathrm{e}/M_\odot)} \\
&= \frac{1}{1 + 3 \times 10^{-6}}\frac{|\boldsymbol{x}_\odot|}{|\boldsymbol{x}_\mathrm{e}|} + \frac{3 \times 10^{-6}}{1 + 3 \times 10^{-6}}, \\
&\approx \frac{|\boldsymbol{x}_\odot|}{|\boldsymbol{x}_\mathrm{e}|}.
\end{aligned}
$$

Therefore, to simplify the solution of the equations $\ddot{\boldsymbol{R}} = 0$, $\mu\ddot{r} = -\left(GM_\odot m_\mathrm{e}/|\boldsymbol{r}|^3\right)\boldsymbol{r}$, we solve the equations in the *centre-of-mass coordinates*

$$
\boldsymbol{R} = 0,
$$

wherein the centre-of-mass location vector coincides with the origin. Hence, the origin or our coordinate system and the centre of the sun almost coincide:

$$
\boldsymbol{x}_\odot \approx \boldsymbol{R}.
$$

The same procedure works for the other planets too.

7.4 Conclusions

- Written down Kepler's Laws;
- Introduced centre of mass coordinates and the relative vector $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ for two-body

problems.

- Shown how the two-body gravitational interaction reduces to a quasi one-particle problem with a reduced mass.
- Shown how the the reduced mass gives rise to a conserved energy and angular momentum.
- Next, we will show how to solve this equation fully.

Chapter 8

Kepler's First Law

8.1 Overview

In Chapter 7 (Kepler's Laws),

- We wrote down Kepler's Laws;
- We introduced centre of mass coordinates and the relative vector for two-body problems (see Fig. 8.1).
- We showed how the two-body gravitational interaction reduces to a quasi one-particle problem with a reduced mass.
- We showed how the the reduced mass gives rise to a conserved energy and angular momentum.
- Now, we will show how to solve this equation fully.

8.2 A more detailed summary of the story so far

Recall the following quantities introduced in the last lecture:

- Relative vector $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$.
- COM vector $\boldsymbol{R} = (m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2) / (m_1 + m_2)$.
- Gravitational force on particle 1 due to particle 2: $\boldsymbol{F}_{12} = -\left(Gm_1m_2/|\boldsymbol{x}_1 - \boldsymbol{x}_2|^3\right)(\boldsymbol{x}_1 - \boldsymbol{x}_2)$.
- Newton's equation: $m_1\ddot{\boldsymbol{x}}_1 = \boldsymbol{F}_{12}$, $m_2\ddot{\boldsymbol{x}}_2 = -\boldsymbol{F}_{12}$.
- Reduced mass: $\mu = m_1m_2/(m_1 + m_2)$.
- Reduced equation:

$$\mu\ddot{\boldsymbol{r}} = -\frac{Gm_1m_2}{|\boldsymbol{r}|^3}\boldsymbol{r}. \tag{8.1}$$

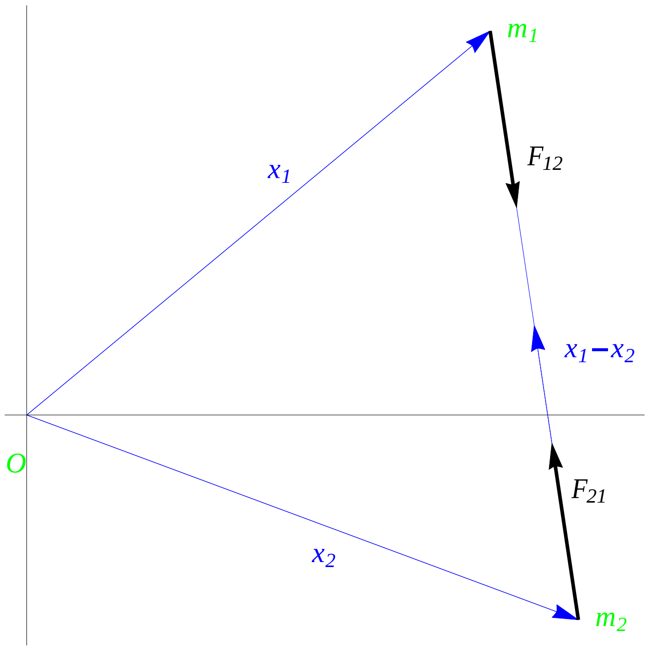


Figure 8.1: Force diagram for a two-particle gravitational interaction.

Writing Eq. (8.1) in polar coordinates, obtain

$$\mu\left(\ddot{r}-r\dot{\theta}^2\right) = -\frac{Gm_1m_2}{r^2}, \tag{8.2}$$

$$\mu\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right) = 0. \tag{8.3}$$

Writing Eq. (8.3) as

$$\frac{\mu}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)=0,$$

which shows that the angular momentum $L=\mu r^2\dot{\theta}$ is conserved. Thus, $h=r^2\dot{\theta}$ is also conserved. Next, substitute $\dot{\theta}=h/r^2$ into Eq. (8.2):

$$\mu\ddot{r}-\frac{\mu h^2}{r^3}=-\frac{Gm_1m_2}{r^2}.$$

Multiply the equation by $\dot{r}$ and recast i.t.o. perfect derivatives:

$$\frac{d}{dt}\left(\tfrac{1}{2}\mu\dot{r}^2\right)+\frac{d}{dt}\left(\frac{1}{2}\frac{\mu h^2}{r^2}\right)=\frac{d}{dt}\left(\frac{Gm_1m_2}{r}\right).$$

Identify the energy:

$$E=\tfrac{1}{2}\mu\dot{r}^2+\mathcal{U}_{\text{eff}}=\text{Const.},\qquad \mathcal{U}_{\text{eff}}=\frac{1}{2}\frac{\mu h^2}{r^2}-\frac{Gm_1m_2}{r}. \tag{8.4}$$

Of course, since $h = r^2\dot{\theta}$, the energy can be re-expressed as

$$\begin{aligned} E &= \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{Gm_1m_2}{r}, \\ &= \tfrac{1}{2}\mu\boldsymbol{v}^2 + \mathcal{U}, \qquad \boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{d}{dt}\left(\boldsymbol{x}_1 - \boldsymbol{x}_2\right) \end{aligned}$$

8.3 Solution of the orbit problem

The formal solution of the equation of motion is given as follows: Take the energy equation (8.4):

$$E = \tfrac{1}{2}\mu\dot{r}^2 + \mathcal{U}_{\mathrm{eff}}, \qquad \mathcal{U}_{\mathrm{eff}} = \tfrac{1}{2}\frac{\mu h^2}{r^2} - \frac{Gm_1m_2}{r}.$$

Re-express dr/dt as

$$\begin{aligned} \frac{dr}{dt} &= \sqrt{\frac{2}{\mu}}\sqrt{E - \mathcal{U}_{\mathrm{eff}}}, \\ \frac{dt}{dr} &= \sqrt{\frac{\mu}{2}}\frac{1}{\sqrt{E - \frac{1}{2}\frac{\mu h^2}{r^2} + \frac{Gm_1m_2}{r}}}, \\ t &= \sqrt{\frac{\mu}{2}}\int_0^r \frac{\mathrm{d}s}{\sqrt{E - \frac{1}{2}\frac{\mu h^2}{s^2} + \frac{Gm_1m_2}{s}}}. \end{aligned}$$

The solution can now be found using a table of integrals.

However, to prove Kepler's First Law, we need to find r as a function of θ. We use the chain rule:

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{dr}{dt}\frac{dt}{d\theta} \\ &= \frac{dr}{dt}\frac{1}{\dot{\theta}} \\ &= \frac{\mu r^2}{L}\frac{dr}{dt}, \qquad \text{since } L = \mu r^2\dot{\theta} = \mathrm{Const.} \end{aligned}$$

Using Eq. (8.5), this is

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{\mu r^2}{L}\sqrt{\frac{2}{\mu}}\sqrt{E - \mathcal{U}_{\mathrm{eff}}(r)}, \\ \frac{d\theta}{dr} &= \frac{L}{\sqrt{2\mu}}\frac{1}{r^2}\frac{1}{\sqrt{E - \mathcal{U}_{\mathrm{eff}}(r)}}, \end{aligned}$$

$$\theta - \theta_0 = \frac{L}{\sqrt{2\mu}} \int_{r_0}^{r} \frac{\mathrm{d}s}{s^2 \sqrt{E - \mathcal{U}_{\mathrm{eff}}(s)}}, \tag{8.5}$$

where we have chosen a coordinate system such that $\theta(r_0) = \theta_0$. Now, the solution to the orbit problem is reduced to calculating an integral.

8.4 Evaluation of the orbital integral

Consider the orbital integral (8.5). We re-express is slightly:

$$\begin{aligned} \theta - \theta_0 &= \frac{L}{\sqrt{2\mu}} \int_{r_0}^{r} \frac{\mathrm{d}s}{s^2 \sqrt{E - \frac{1}{2}\frac{L^2}{\mu s^2} + \frac{Gm_1 m_2}{s}}}, \\ &= L \int_{r_0}^{r} \frac{\mathrm{d}s}{s\sqrt{2\mu E s^2 - L^2 + 2\mu G m_1 m_2 s}}. \end{aligned}$$

This is in fact a standard integral that can be looked up in a table:

$$\mathcal{I} := \int \frac{\mathrm{d}s}{s\sqrt{\gamma s^2 + \beta s - \alpha}}. \tag{8.6}$$

Here

$$\begin{aligned} \alpha &= L^2, \\ \beta &= 2\mu G m_1 m_2 := 2\mu B, \\ \gamma &= 2\mu E; \end{aligned}$$

α and β are both positive, while γ can be positive or negative, depending on where in the effective-potential well the particle sits. Abramowitz and Stegun (AS)[1] gives

$$\begin{aligned} \mathcal{I} &= \int \frac{\mathrm{d}s}{s\sqrt{\gamma s^2 + \beta s - \alpha}} = -\int \frac{\mathrm{d}t}{\sqrt{\gamma + \beta t - \alpha t^2}}, \qquad t = 1/s, \qquad \text{Eq. 3.3.38, AS,} \\ &= \frac{1}{\sqrt{\alpha}} \sin^{-1}\left(\frac{-2\alpha t + \beta}{\sqrt{\beta^2 + 4\alpha\gamma}} \right), \qquad \text{Eq. 3.3.36, AS.} \end{aligned}$$

[1]This is a famous table of integrals and special functions and is now out of copyright: http://www.math.ucla.edu/∼cbm/aands/

Hence,

$$\begin{aligned}\theta-\theta_0 &= \sin^{-1}\left(\frac{-2L^2\left(1/r\right)+2\mu B}{\sqrt{4\mu^2B^2+8L^2\mu E}}\right),\\ &= \sin^{-1}\left(\frac{1}{r}\frac{\mu B r-L^2}{\sqrt{\mu^2B^2+2\mu E L^2}}\right),\end{aligned}$$

or

$$\mu B r - L^2 = r\sqrt{\mu^2B^2+2\mu E L^2}\sin\left(\theta-\theta_0\right).$$

Solving for r,

$$r=\frac{L^2/\mu B}{1-\sqrt{1+\left(2EL^2/\mu B^2\right)}\sin\left(\theta-\theta_0\right)}. \tag{8.7}$$

The usual convention is to take $\theta_0=-\pi/2$ and to introduce the following parameters:

$$\begin{aligned}r_0 &:= \frac{L^2}{\mu B},\\ \epsilon &:= \sqrt{1+\frac{2EL^2}{\mu B^2}}.\end{aligned} \tag{8.8}$$

Physically, r_0 is the radius of the circular orbit corresponding to the minimum of the effective potential well $\mathcal{U}_{\text{eff}}\left(r\right)$, and to the given values of L, μ, and B. The dimensionless parameter ϵ is called the *eccentricity*, and determines the shape of the orbit. To see how, let us re-write Eq. (8.7) with the replacements (8.8):

$$r=\frac{r_0}{1-\epsilon\cos\theta}.$$

In Cartesian coordinates $x=r\cos\theta$, $y=r\sin\theta$,

$$\begin{aligned}r\left(1-\epsilon\cos\theta\right) &= r_0,\\ r-\epsilon r\cos\theta &= r_0,\\ \sqrt{x^2+y^2}-\epsilon x &= r_0.\end{aligned}$$

We take the last equation and re-organize it as $\sqrt{x^2+y^2}=r_0+\epsilon x$. We square both sides to obtain:

$$x^2\left(1-\epsilon^2\right)-2r_0\epsilon x+y^2=r_0^2. \tag{8.9}$$

Equation (8.9) is the equation of a *conic section*, which can be of several forms, depending on ϵ:

1. $\epsilon = 0$ means that Eq. (8.9) becomes $x^2 + y^2 = r_0^2 = L^2/(\mu B)$, which is the equation of a circle.

2. $0 \leq \epsilon < 1$. We re-write the conic-section equation by completing the square in x:

$$(1-\epsilon^2)\left(x - \frac{\epsilon r_0}{(1-\epsilon^2)}\right)^2 + y^2 = r_0^2 + \frac{\epsilon^2 r_0^2}{1-\epsilon^2} = \frac{r_0^2}{1-\epsilon^2},$$

   which is of the form

$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

   which is the equation of an ellipse (Fig. 8.2) with *semimajor axis* a.

Note: In Figure 8.2, we have defined

$$a = \frac{r_0}{1-\epsilon^2}, \qquad b = \frac{r_0}{\sqrt{1-\epsilon^2}} = a\sqrt{1-\epsilon^2}. \tag{8.10}$$

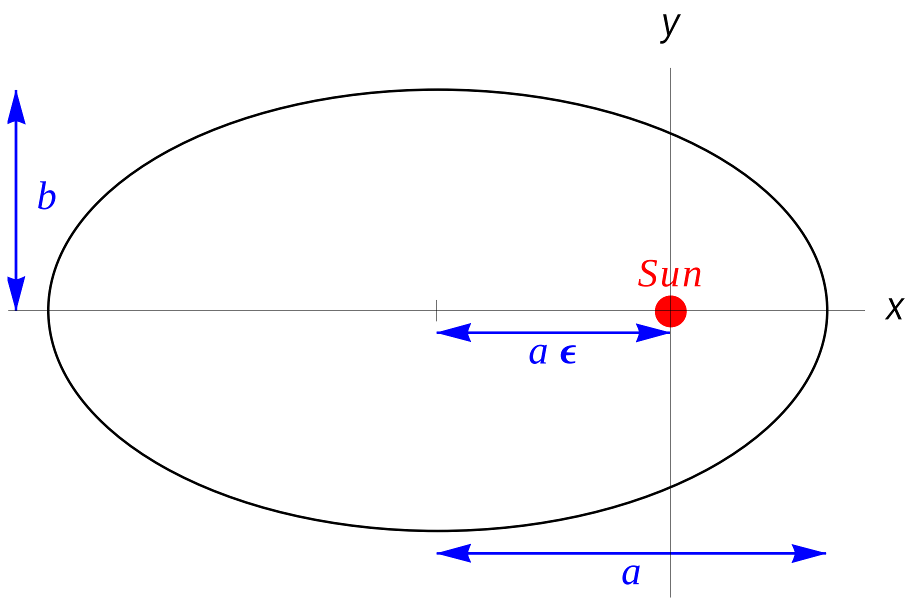


Figure 8.2: Ellipse: a is the semimajor axis, b is the semiminor axis, and ϵ is the eccentricity. The central object at $(x, y) = 0$ is at the *focus*.

3. $\epsilon = 1$, which gives

$$-2r_0 x + y^2 = r_0^2, \implies x = \tfrac{1}{2}\left(\frac{y^2}{r_0} - r_0\right).$$

   This is the equation of a *parabola* (Fig. 8.3, which is not a closed curve. This cannot describe a particle in orbit, because it is not closed; it describes a particle which comes in from infinitely far away, is deflected by the sun but never comes back.

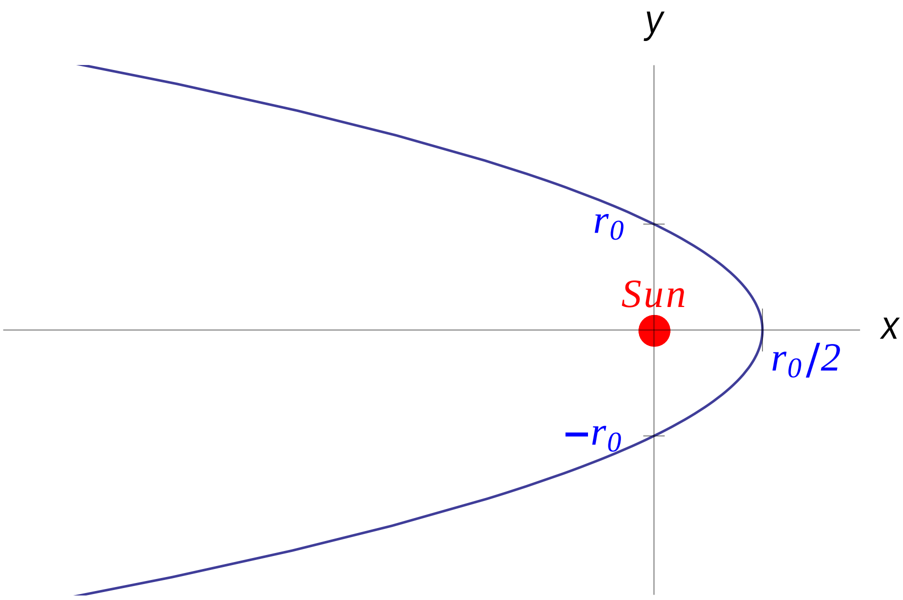


Figure 8.3: Parabola

4. $\epsilon > 1$ In this case, after some manipulation similar to the elliptic case [Exercise], we get the equation of a hyperbola,

$$\frac{(x - x_0)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and this is not a closed curve - it is describing a particle that is not bound to the central object but merely deflected by it.

8.4.1 More on the ellipse

The way in which we have defined the ellipse and introduced the semimajor axis a seems a little contrived. The more natural way of defining an ellipse is through the the specification of two *foci* O and O'. Then the ellipse is the set of all points P such that

$$|OP| + |O'P| = \text{Const.} = 2a. \tag{8.11}$$

Suppose that the focus O is at point $(0, 0)$ of a Cartesian system, and that the point O' is at point $(-2c, 0)$, where $0 < c < a$ (Fig. 8.4). Now,

$$\overrightarrow{O'P} = (2c + x, y) , \qquad \overrightarrow{OP} = (x, y) .$$

Hence

$$\sqrt{(2c + x)^2 + y^2} + \sqrt{x^2 + y^2} = 2a.$$

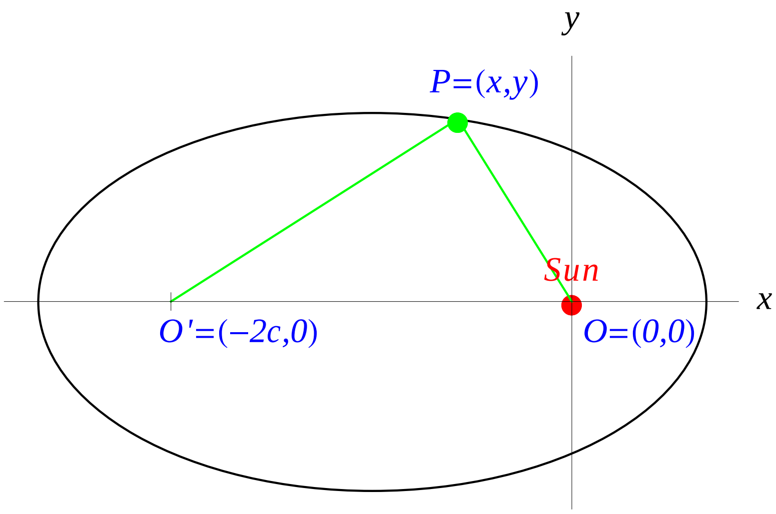


Figure 8.4: Ellipse: The sum $|OP|+|O'P|$ is constant and equal to $2a$. The constant c is necessarily less than a because when $P=(x>0,0)$, $2c+2|OP|=2c+2x=2a$, hence $c<a$.

Bring the second term to the right-hand side and square both sides:

$$\begin{aligned}\sqrt{(2c+x)^2+y^2} &= 2a-\sqrt{x^2+y^2},\\ (2c+x)^2+y^2 &= 4a^2+x^2+y^2-4a\sqrt{x^2+y^2}.\end{aligned}$$

Solve for the square root:

$$\sqrt{x^2+y^2}=\frac{1}{4a}\left\{4a^2+x^2+y^2-\left[(2c+x)^2+y^2\right]\right\}$$

Simplify this result:

$$\sqrt{x^2+y^2}=\frac{a^2-c^2-cx}{a}.$$

Eliminate the final square root:

$$x^2+y^2=\frac{\left(a^2-c^2\right)^2-2c\left(a^2-c^2\right)x+c^2x^2}{a^2},$$

or,

$$x^2\left(1-\frac{c^2}{a^2}\right)+2c\left(1-\frac{c^2}{a^2}\right)x+y^2=a^2\left(1-\frac{c^2}{a^2}\right)^2. \tag{8.12}$$

Since $c<a$ by definition, define

$$\epsilon := c/a < 1. \tag{8.13}$$

Hence, Equation (8.12) becomes

$$x^2\left(1-\epsilon^2\right)+2\epsilon a\left(1-\epsilon^2\right)x+y^2=a^2\left(1-\epsilon^2\right)^2. \tag{8.14}$$

Completing the square in x gives

$$(x + \epsilon a)^2 + \frac{y^2}{1-\epsilon^2} = a^2,$$

which is the equation of an ellipse. Shifting the x-coordinate leftwards by

$$x' = x + \epsilon a = x + c,$$

and defining $b^2 := a^2 - c^2 = a^2(1-\epsilon^2)$ gives rise to the canonical form for the equation of the ellipse:

$$\frac{x'^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Relationship between a and b: Referring back to Equation (8.10), we have

$$b = \frac{r_0}{\sqrt{1-\epsilon^2}}, \qquad a = \frac{r_0}{1-\epsilon^2}.$$

Hence,

$$\frac{b}{a} = \frac{1-\epsilon^2}{\sqrt{1-\epsilon^2}} = \sqrt{1-\epsilon^2}$$

and thus,

$$b = a\sqrt{1-\epsilon^2}. \tag{8.15}$$

This particular equation will be useful in the next chapter.

8.4.2 The Energy and ϵ

Recall formula (8.8):

$$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu B^2}}, \qquad B = Gm_1m_2.$$

Square this formula and bring the one over to the LHS. Hence, invert for E:

$$\begin{aligned}
\epsilon^2 - 1 &= \frac{2EL^2}{\mu B^2}, \\
&= \frac{2E}{\mu B^2} \times (\mu B r_0), \ \ldots \text{ from Eq. (8.8)} \\
&= \frac{2Er_0}{B}, \\
\implies E &= \frac{B}{2r_0}\left(\epsilon^2 - 1\right),
\end{aligned}$$

$$E = \frac{Gm_1m_2}{2r_0}\left(\epsilon^2 - 1\right).$$

Therefore, a particle's energy determines whether $\epsilon < 1$ or $\epsilon \geq 1$, as a result whether the particle goes into orbit

- $\epsilon \geq 1 \implies E \geq 0$: these particles have enough kinetic energy to equal, or exceed, potential energy. They are not bound to the central object and do not go into orbit but their paths are deflected into a parabolic or hyperbolic path.
- $\epsilon < 1 \implies E < 0$: these particles have a negative total energy, and their energy is dominated by the potential term. They are bound to the central object and follow an elliptic or circular path.

In the case of small eccentricities $\epsilon < 1$, we can substitute the semi-major axis:

$$E = -\frac{Gm_1m_2}{2a}. \tag{8.16}$$

This is a very useful formula because we can use it to relate a planet or satellite's speed to its position by

$$E = -\frac{GMm}{2a} = \tfrac{1}{2}m\boldsymbol{v}^2 - \frac{GMm}{r}$$

(we have set $\mu = m$ here for an earth-satellite calculation). This gives (Eq. (8.10))

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right).$$

8.4.3 Some numerical examples

- For the earth, $\epsilon \approx 1/60$, $a \approx 1.5 \times 10^8\,\mathrm{km}$, and the semiminor axis is $b = a\sqrt{1-\epsilon^2} = 0.99986a$, so the earth orbit is almost circular.
- For Halley's comet, $\epsilon = 0.9674$.

8.5 Conclusions

We have now proved Kepler's first law, namely that the trajectories of the planets are ellipses, with the sun at one focus. It remains to prove the second and the third laws. These are relatively easy compared with the first law.

Chapter 9

Kepler's Second and Third Laws

9.1 Overview

In Ch. 8 (Kepler's First Law),

- We proved Kepler's first law from first principles, namely that the orbits of planets are ellipses with the sun at one focus.
- Now we prove the second and third laws.
- First, we need to determine the area of an ellipse.

9.2 Area integrals

In this section we derive some expressions regarding area that are needed to prove Kepler's second and third laws.

9.2.1 The area of an ellipse

Recall the definition of an ellipse: Given the two *foci* O and O', the ellipse is the set of all points P such that

$$|OP| + |O'P| = \mathrm{Const.} = 2a. \tag{9.1}$$

Suppose that the focus O is at point $(0,0)$ of a Cartesian system, and that the point O' is at point $(-2c, 0)$, where $0 < c < a$ (Fig. 9.1). Now,

$$\overrightarrow{O'P} = (2c + x, y)\,, \qquad \overrightarrow{OP} = (x, y)\,.$$

Hence

$$\sqrt{(2c+x)^2 + y^2} + \sqrt{x^2 + y^2} = 2a.$$

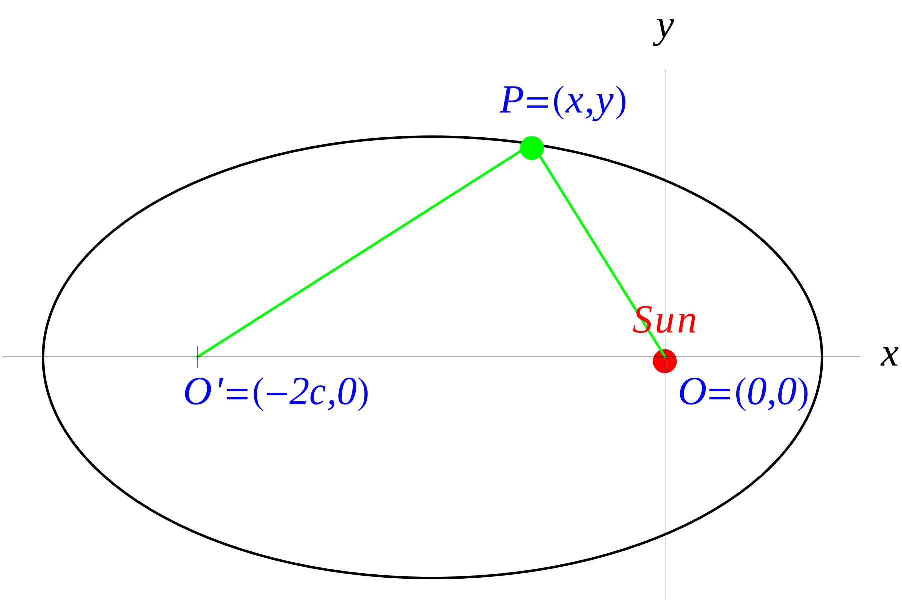


Figure 9.1: Ellipse: The sum $|OP|+|O'P|$ is constant and equal to $2a$. The constant c is necessarily less than a because when $P=(x>0,0)$, $2c+2|OP|=2c+2x=2a$, hence $c<a$.

Bring the second term to the right-hand side and square both sides:

$$\begin{aligned}\sqrt{(2c+x)^2+y^2} &= 2a-\sqrt{x^2+y^2},\\ (2c+x)^2+y^2 &= 4a^2+x^2+y^2-4a\sqrt{x^2+y^2}.\end{aligned}$$

Solve for the square root:

$$\sqrt{x^2+y^2}=\frac{1}{4a}\left\{4a^2+x^2+y^2-\left[(2c+x)^2+y^2\right]\right\}$$

Simplify this result:

$$\sqrt{x^2+y^2}=\frac{a^2-c^2-cx}{a}.$$

Eliminate the final square root:

$$x^2+y^2=\frac{\left(a^2-c^2\right)^2-2c\left(a^2-c^2\right)x+c^2x^2}{a^2},$$

or,

$$x^2\left(1-\frac{c^2}{a^2}\right)+2c\left(1-\frac{c^2}{a^2}\right)x+y^2=a^2\left(1-\frac{c^2}{a^2}\right)^2. \tag{9.2}$$

Since $c<a$ by definition, define $\epsilon := c/a<1$. Hence, Equation (9.2) becomes

$$x^2\left(1-\epsilon^2\right)+2\epsilon c\left(1-\epsilon^2\right)x+y^2=a^2\left(1-\epsilon^2\right)^2. \tag{9.3}$$

Completing the square in x gives

$$(x+\epsilon a)^2 + \frac{y^2}{1-\epsilon^2} = a^2,$$

which is the equation of an ellipse. Shifting the x-coordinate leftwards by

$$x' = x + \epsilon a = x + c,$$

and defining $b^2 := a^2 - c^2 = a^2\left(1-\epsilon^2\right)$ gives rise to the canonical form for the equation of the ellipse:

$$\frac{x'^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{9.4}$$

being the equation of an ellipse with foci at $x = \pm c = \epsilon a$, with semimajor axis a and semiminor axis b, $b^2 = a^2 - c^2$.

Theorem: The area enclosed by the curve (9.4) is πab.

Proof: Divide the ellipse into infinitesimal strips, like the one shown in the figure. The area of the strip is

$$dA = 2y\,dx = 2b\sqrt{1-\frac{x^2}{b^2}},$$

where $-a \le x \le a$. Integrate:

$$\begin{aligned} A &= \int dA = \int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x, \\ &= 2ab\int_{-1}^{1}\sqrt{1-s^2}\,\mathrm{d}s, \\ &= \pi ab. \end{aligned}$$

9.2.2 The area of a sector

A sector of a curve is the area swept out by the radius vector in moving through an angle $\mathrm{d}\theta$. It is shown in Fig. 9.3. We compute it as follows.

The area element is

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y.$$

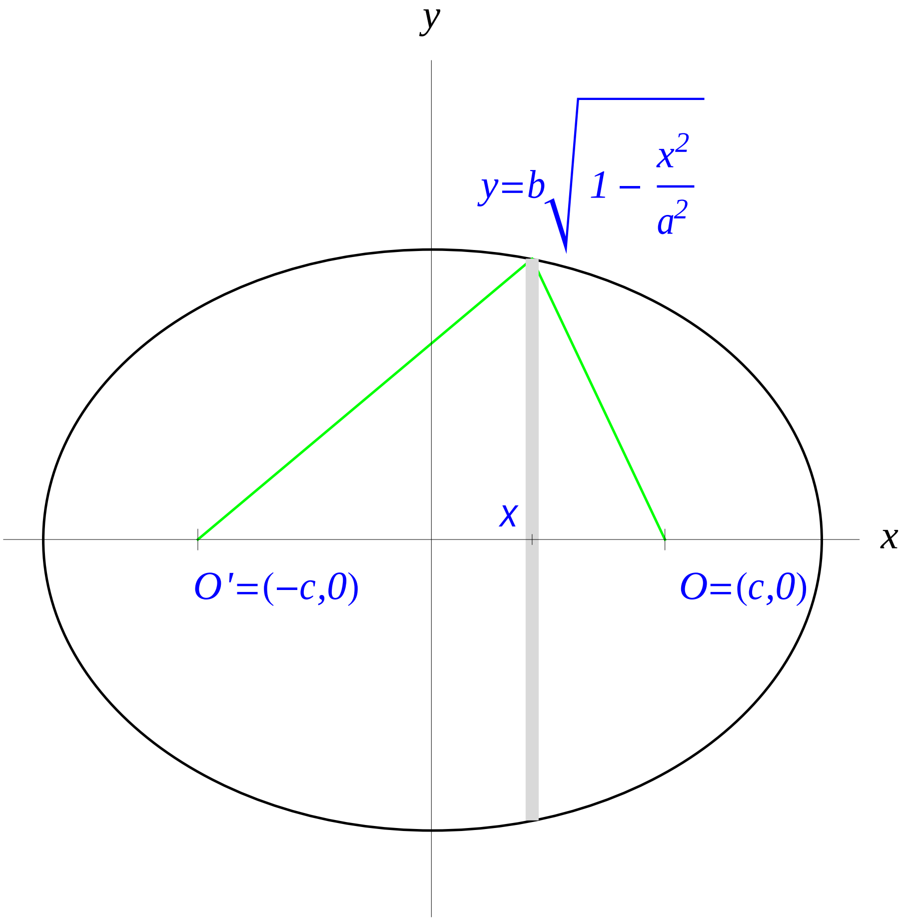


Figure 9.2: Computing the area enclosed by an ellipse.

Using the Jacobian transformation (or geometric reasoning),

$$\mathrm{d}A = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} \mathrm{d}r\mathrm{d}\theta.$$

But $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \quad , \qquad \frac{\partial y}{\partial r} = \sin\theta,$$
$$\frac{\partial x}{\partial \theta} = -r\sin\theta \quad , \qquad \frac{\partial y}{\partial r} = r\cos\theta,$$

hence

$$\mathrm{d}A = r\,\mathrm{d}r\mathrm{d}\theta.$$

A *sector* is the area between 0 and R, and between θ and $\theta + \mathrm{d}\theta$:

$$dA_{\text{sec}} = \int_0^R r\,\mathrm{d}r\mathrm{d}\theta = \tfrac{1}{2}R^2\mathrm{d}\theta.$$

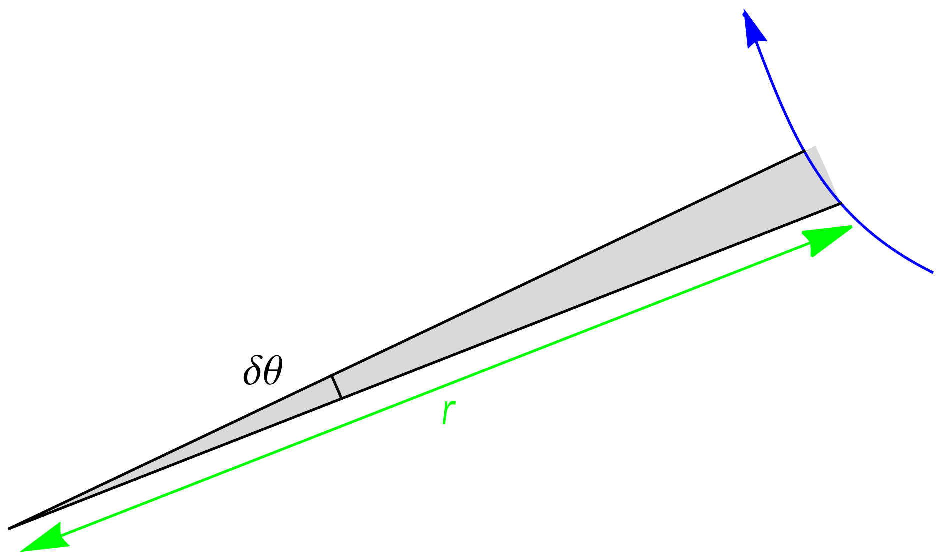


Figure 9.3: The sector is the area swept out by the radius vector in moving through an angle $\mathrm{d}\theta$.

Geometric Reasoning: The area of a circle is πR^2, or

$$\tfrac{1}{2}(2\pi)R^2.$$

The 2π here refers to the whole circle. This can be replaced by θ for a fraction of a circle. Hence, the area of a sector (a fraction of a circle) is

$$\tfrac{1}{2}\pi R^2$$

9.3 Kepler's second and third laws

2. The radius vector form the sun to a planet sweeps out equal areas in equal times.

3. The period of revolution, T, is related to the semi-major axis of the ellipse, a, by $T^2 \propto a^3$.

These laws follow directly from the conservation of angular momentum:

$$L = \mu r^2 \dot{\theta} = \mathrm{Const.},$$

valid for any central force. Indeed, Law 2 is not unique to planets, it is true for any central-force interaction.

Law 2: The area swept out by the radius vector in a time dt is

$$dA = \tfrac{1}{2} r^2 d\theta = \tfrac{1}{2} r^2 \frac{d\theta}{dt} dt = \tfrac{1}{2} r^2 \left(\frac{L}{\mu r} \right) dt,$$

hence

$$\frac{dA}{dt} = \frac{L}{2\mu} = \text{Const.}$$

Law 3: The area swept out in a finite time interval t is therefore

$$A = \frac{L}{2\mu}t.$$

In a period of revolution, the area swept out by the radius vector must equal the area of the curve traced out by the radius vector – the area of an ellipse:

$$\pi ab = \frac{L}{2\mu}T.$$

We use $b = a\sqrt{1-\epsilon^2}$ (Chapter 8, Equation (8.15)) to rewrite this as

$$\pi a^2\sqrt{1-\epsilon^2} = \frac{L}{2\mu}T. \tag{9.5}$$

But the eccentricity ϵ is (Chapter 8)

$$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu G^2 m_1^2 m_2^2}},$$

while the energy is expressible i.t.o. the semimajor axis (Chapter 8, Equation (8.16)):

$$E = -\frac{Gm_1m_2}{2a}.$$

Hence,

$$\begin{aligned} 1-\epsilon^2 &= -\frac{2EL^2}{\mu G^2 m_1^2 m_2^2}, \\ &= \frac{Gm_1m_2}{2a}\frac{2L^2}{\mu G^2 m_1^2 m_2^2}, \\ &= \frac{L^2}{\mu G m_1 m_2}a^{-1}, \\ \implies \sqrt{1-\epsilon^2} &= \frac{L}{\sqrt{\mu G m_1 m_2}}a^{-1/2}. \end{aligned}$$

Put this back into Eq. (9.5):

$$\begin{aligned} \pi a^2\sqrt{1-\epsilon^2} &= \frac{L}{2\mu}T, \\ \pi a^{3/2}\frac{L}{\sqrt{\mu G m_1 m_2}} &= \frac{J}{2\mu}T, \\ 2\pi a^{3/2}\frac{\sqrt{\mu}}{\sqrt{G m_1 m_2}} &= T. \end{aligned}$$

Now let $m_1 = M_\odot$ and $m_2 = m_\mathrm{p}$, a planetary mass. In this case, $\mu \approx m_\mathrm{p}$, the planet's mass, and

$$\sqrt{\mu/\left(m_1 m_2\right)} \approx \sqrt{m_\mathrm{p}/\left(m_\mathrm{p} M_\odot\right)} = 1/\sqrt{M_\odot}.$$

Hence,

$$T = \frac{2\pi}{\sqrt{G M_\odot}} a^{3/2}, \qquad (9.6)$$

and $T \propto a^{3/2}$, and the constant of proportionality is independent of the planet in question and only depends on the solar mass. Having now proved all of Kepler's laws, in the next Chapter, we turn to applications.

Chapter 10

Applications of Kepler's Laws in planetary orbits

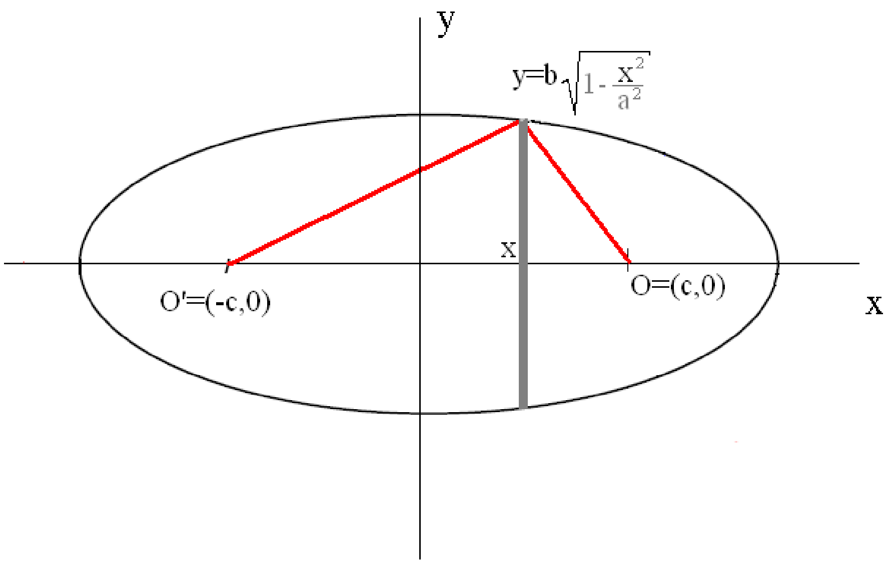


Figure 10.1: Computing the area enclosed by an ellipse.

10.1 An earth satellite

An earth satellite of mass m_0 is fired into orbit at horizontal speed $32,000\,\mathrm{km/hr}$ at an altitude of $640\,\mathrm{km}$ (Fig. 10.2). Discuss the motion. Assume that the earth radius is $R = 6,400\,\mathrm{km}$, and that $g = GM_\mathrm{e}R^{-2} = 9.8\,\mathrm{m/s^2}$.

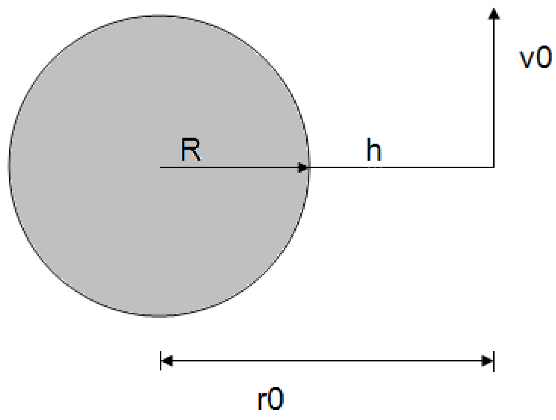


Figure 10.2: Satellite motion around the earth

Velocity:

$$\begin{aligned} v_0 &= 32,000\,\mathrm{km/hr} = \frac{3.2\times 10^7}{60\times 60}\mathrm{m/s}, \\ &= 8.88\cdots\times 10^3\,\mathrm{m/s}, \\ &\approx 8,900\,\mathrm{m/s}. \end{aligned}$$

Height above earth:

$$h = 640\mathrm{km} \;= 6.4\times 10^5\,\mathrm{m}.$$

Earth radius:

$$R = 6,400\mathrm{km} = 6.4\times 10^6\,\mathrm{m}.$$

Initial altitude of satellite orbit:

$$r_0 = R + h = 7.04\times 10^6\,\mathrm{m}.$$

Potential energy:

$$\begin{aligned} g &= GM_\mathrm{e}/R^2 \implies GM_\mathrm{e} = gR^2 = \left(9.864\,\mathrm{m/s^2}\right)\times\left(6.4\times 10^6\,\mathrm{m}\right)^2, \\ GM_\mathrm{e} &= 4.0402944\times 10^{14}\,\mathrm{m^3/s^2}, \\ \mathcal{U} &= -GM_\mathrm{e}m/r_0 = -m_0\left(GM_\mathrm{e}/r_0\right) = -m_0\left(5.739054545454546\cdots\times 10^7\right). \end{aligned}$$

Angular momentum:

$$\begin{aligned} J/m_0 \approx J/\mu = r_0 v_0 &= 6.25777\cdots\times 10^{10}\,\mathrm{m^2/\,s}, \\ &\approx 6.3\times 10^{10}\,\mathrm{m^2/\,s}. \end{aligned}$$

Total energy:

$$\begin{aligned} E &= \tfrac{1}{2}\mu v_0^2 - \frac{GM_{\mathrm{e}}m_0}{r_0}, \\ &\approx \tfrac{1}{2}m_0 v_0^2 - \frac{GM_{\mathrm{e}}m_0}{r_0}, \\ &= -m_0\left(1.788437261503929\cdots\times 10^7\,\mathrm{m}^2/\,\mathrm{s}^2\right). \end{aligned}$$

The energy is negative, so the orbit is an ellipse.

The semimajor axis: Recall the formula

$$E = -\frac{GM_{\mathrm{e}}m_0}{2a} \implies a = -\frac{GM_{\mathrm{e}}m_0}{2E}$$

for the semimajor axis of an elliptical orbit. Hence,

$$\begin{aligned} a &= \frac{GM}{2\left(E/m_0\right)} = \frac{4.0402944\times 10^{14}\,\mathrm{m}^3/\mathrm{s}^2}{2\times 1.788437261503929\cdots\times 10^7\,\mathrm{m}^2/\,\mathrm{s}^2}, \\ &= 1.129560003855669\cdots\times 10^7\,\mathrm{m}, \\ &\approx 1.1\times 10^7\,\mathrm{m}. \end{aligned}$$

The eccentricity:

$$\begin{aligned} \epsilon &= \sqrt{1+\frac{2EJ^2}{\mu G^2 m_1^2 m_2^2}}, \\ &\approx \sqrt{1+\frac{2EJ^2}{\left(GM_{\mathrm{e}}\right)^2 m_0^3}}, \\ &\approx \sqrt{1+\frac{2\left(E/m_0\right)\left(J/m_0\right)^2}{\left(GM_{\mathrm{e}}\right)^2}}, \\ &= 0.376748470557608\cdots, \\ &\approx 0.38. \end{aligned}$$

For a generic ellipse $(x/a)^2 + (y/b)^2 = 1$ (Fig. 10.1), the maximum radius occurs for along the line $y = 0$. Hence, $x = \pm a$. Applying this result to the orbital problem in hand, the maximum distance from the earth is

$$r_{\mathrm{max}} = a + c = a + \epsilon a = a\left(1+\epsilon\right), \qquad c = \epsilon a = 1.555120007711338\cdots\times 10^7\,\mathrm{m},$$

where we have used $c = \epsilon a$ (i.e. Chapter 8, Equation (8.13)). This is called the *aphelion* of the orbit – *the point furthest from the focus of the orbit* attained by the satellite. Similarly,

$$r_{\mathrm{min}} = a - c = a - \epsilon a = (1-\epsilon).$$

This is the *perihelion*[1]. [Exercise: Prove that $\dot{r} = 0$ at the perihelion and the aphelion. Hint: Work in polar coordinates.]

The maximum height of the satellite above the earth: This is

$$r_{\text{max}} - R = 9.151200077113384 \times 10^6\,\text{m} \approx 9,200\,\text{km}.$$

The period of the orbit: From Kepler's Third Law,

$$\begin{aligned} T &= \frac{2\pi}{\sqrt{GM_{\text{e}}}} a^{3/2}, \\ &= 1.186690107221579 \cdots \times 10^4\,\text{s}, \\ &= 3.3\,\text{hrs}. \end{aligned}$$

10.2 A satellite touching the orbits of Earth and Neptune

The orbits of Earth and Neptune are approximately circular and coplanar relative to a fixed force centre, the Sun. Neptune takes 164.8 years to complete an orbit around the Sun. Suppose that a satellite is given an orbit around the sun, extending from the orbit of Earth to that of Neptune, and just touching both. Find the orbital period of the satellite, assuming that the gravitational effects of the planets can be ignored. Refer to Fig. 10.3. Let a be the semimajor axis of the satellite's orbit and let ϵ be the eccentricity. Then, by construction,

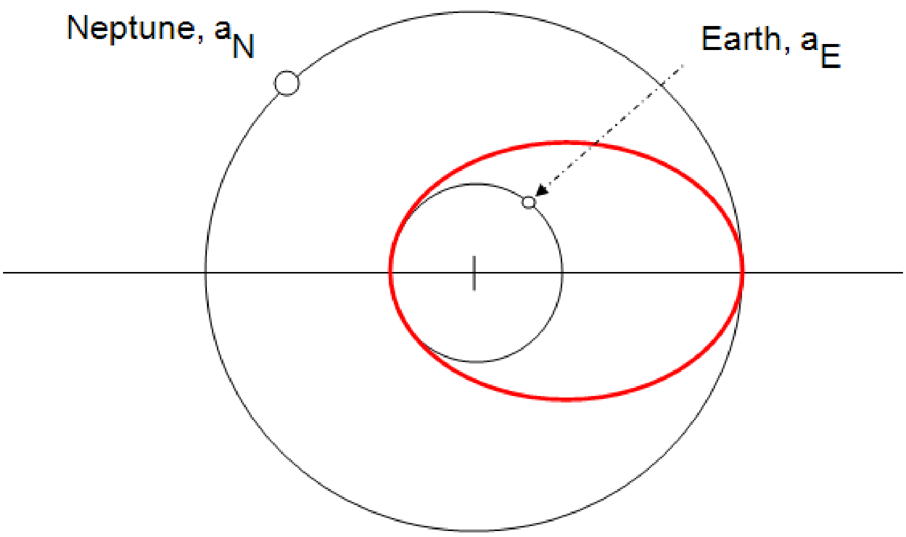


Figure 10.3: Satellite motion centred at the sun and touching the orbits of Earth and Neptune

[1] In Greek, 'apo' means 'from', 'peri' means 'around', and 'helios' means sun.

$$\begin{aligned} r_{\min} &= a\,(1-\epsilon) = a_E, \\ r_{\max} &= a\,(1+\epsilon) = a_N. \end{aligned}$$

By adding these two equations to eliminate ϵ, obtain

$$a = \tfrac{1}{2}\left(a_E + a_N\right).$$

But everything is now expressible in terms of T_E, the orbital period of the Earth:

$$\begin{aligned} T_E^2 &= \frac{4\pi^2}{GM_\odot} a_E^3, \\ T_N^2 &= \frac{4\pi^2}{GM_\odot} a_N^3 = (164.8)^2\, T_E^2 = (164.8)^2\, \frac{4\pi^2}{GM_\odot} a_E^3, \end{aligned}$$

or

$$a_N^3 = (164.8)^2\, a_E^3, \qquad a_N = (164.8)^{2/3}\, a_E.$$

Hence,

$$a = \tfrac{1}{2}\left[1 + (164.8)^{2/3}\right] a_E.$$

Finally,

$$\begin{aligned} T^2 &= \frac{4\pi^2}{GM_\odot} a^3, \\ &= \frac{4\pi^2}{GM_\odot} a_E^3 \left[\tfrac{1}{2} + \tfrac{1}{2}(164.8)^{2/3}\right]^2, \\ &= T_E^2 \left[\tfrac{1}{2} + \tfrac{1}{2}(164.8)^{2/3}\right]^3, \\ T &= T_E \left[\tfrac{1}{2} + \tfrac{1}{2}(164.8)^{2/3}\right]^{3/2}, \end{aligned}$$

or $T = 61.205\, T_E = 61.20\,\mathrm{yrs}$.

10.3 Halley's comet

Halley's comet is in an elliptic orbit around the sun. The eccentricity of the orbit is 0.967 and the period is 75.3 years.

1. Using these data, determine the distance of Halley's comet from the sun at perihelion and aphelion.

   We have $r = r_0/\left(1 - \varepsilon\cos\theta\right)$, hence $r_{\min} = r_0/\left(1+\epsilon\right)$, and $r_{\max} = r_0/\left(1-\epsilon\right)$. But from the properties of the ellipse, $r_{\min} = a - c = a - \epsilon a = a\,(1-\epsilon)$, and $r_{\max} = a + c = a\,(1+\varepsilon)$.

We know ϵ, so it suffices to compute a. From Kepler's Third Law,

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM_\odot}},$$

hence

$$a = \left(\frac{T\sqrt{GM_\odot}}{2\pi}\right)^{2/3} =$$

$$\left(\frac{76 \times 365.25 \times 24 \times 60 \times 60 \times \sqrt{6.6730 \times 10^{-11} \times 1.9889 \times 10^{30}}}{2\pi}\right)^{2/3}$$

$$= 2.667620671686515\,(e+12)\ \mathrm{m} \approx 2.67 \times 10^{12}\,\mathrm{m}$$

Hence,

$$\begin{aligned} r_{\min} &= a\,(1-0.967) = 8.80 \times 10^{10}\,\mathrm{m} = 0.588\,\mathrm{AU}, \\ r_{\max} &= a\,(1+0.967) = 5.25 \times 10^{12}\,\mathrm{m} = 35.1\,\mathrm{AU}, \end{aligned}$$

Note:

$$1\,\mathrm{AU} = 149,597,870.700 \pm 0.003\mathrm{km}.$$

2. What is the speed of Halley's comet when it is closest to the sun? We know

$$E = \tfrac{1}{2}m\boldsymbol{v}^2 - \frac{GM_\odot m}{r} = -\frac{GM_\odot m}{2a},$$

hence

$$\tfrac{1}{2}m\boldsymbol{v}^2 = \frac{GM_\odot m}{r_{\min}} - \frac{GM_\odot m}{2a}.$$

The m's cancel, and we are left with

$$\begin{aligned} v_0^2 &= 2GM_\odot\left(\frac{1}{r_{\min}} - \frac{1}{2a}\right), \\ &= 2GM_\odot\left(\frac{1}{a\,(1-\varepsilon)} - \frac{1}{2a}\right), \\ &= \frac{GM_\odot}{a}\left(\frac{1+\varepsilon}{1-\varepsilon}\right). \end{aligned}$$

Plugging in the numbers,

$$\begin{aligned}
v_0 &= \sqrt{\frac{GM_\odot}{a}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}, \\
&= \sqrt{\frac{6.6730\times10^{-11}\times1.9889\times10^{30}}{2.66762\times10^{12}}\left(\frac{1+0.967}{1-0.967}\right)}, \\
&= 5.445656070140230\times10^{4}\,\mathrm{m/s}, \\
&\approx 5.45\times10^{4}\,\mathrm{m/s}.
\end{aligned}$$

Chapter 11

Inertial frames of reference

11.1 Summary

Before looking at rotating frames in Chapter 12 we explore the very important and fundamental concept of **inertial frames**. We introduce the idea of Galilean invariance and develop the transformation laws that connect different inertial frames. We recall Newton's Laws and discover that the only hold in inertial frames. We lastly look at a frame of reference that is accelerating with respect to an inertial frame (specifically, translational acceleration), and we find that Newton's Laws have a modified form in such a frame.

11.2 Inertial frames of reference

A measurement of position, velocity or acceleration must be made with respect to a *frame of reference*. Loosely speaking, a frame of reference consists of an observer S, equipped with a metre stick and a stopwatch. *Examples:*

- Consider a person, S, standing on the platform of a train station. He measures displacements from where he is standing, and he uses the origin of a Cartesian set of axes to measure the (x, y, z) position of any object. These axes do not move relative to the platform. Any object at rest in the station has zero velocity relative to his frame of reference.
- A second person, S', is sitting on a train moving at velocity $100\,\mathrm{km/hr}$ through the station. He has his own set of axes (x', y', z') that are fixed to the train. Any object at rest in the train has zero velocity relative to the frame of reference of person S'.
- Therefore, a third person walking through the train at $5\,\mathrm{km/hr}$ in the direction of the train's motion has a velocity of $5\,\mathrm{km/hr}$ relative to person S' and a velocity of $105\,\mathrm{km/hr}$ relative to person S.

The last point is a formalized statement of something intuitive: that velocities add. However, it is WRONG. In the second part of the course, we shall find that velocities do not add in this way. This non-additivity happens in the so-called *relativistic limit*, where the velocity in question v approaches the speed of light c. When $v \ll c$, the additivity holds approximately. This is the subject of the first half of the course.

11.3 Galilean invariance

In the last section, the examples we described involved observers at rest or in uniform motion with respect to one another. This leads to a definition of *the class of inertial frames*:

> Describing frames of reference as at rest or in uniform motion relative to one another defines an equivalence relationship between frames, which in turn defines an equivalence class. A representative from this class is called an *intertial frame*.

No inertial frame can accelerate with respect to any other inertial frame: any frame that accelerates with respect to another frame is outside of this class. It is in inertial frames that we wish to formulate the laws of physics.

> The principle of Galilean invariance states that the form of all physical laws must be the same in all inertial frames of reference.

If observer S on the platform and observer S' on the train in constant motion conduct an experiment, they must obtain the same answer. Thus, if observer S' were blindfolded (but still able to do and to interpret experiments), he would have no way of knowing if his train were in motion relative to the platform in the station.

The results of an experiment performed in a frame of reference S, by Galilean invariance, must be transformable into the results of an identical experiment, performed in a frame S', moving at a velocity $\boldsymbol{V}$ relative to the first frame. This is done through a *Galilean transformation*. This is done as follows (Fig. 11.1):

- Position vector of some object relative to frame S: $\boldsymbol{x}$;
- Position vector of the same object relative to frame S': $\boldsymbol{x}'$;
- Position vector of the origin of S' relative to S: $\boldsymbol{R}$.

Vector addition:

$$\boldsymbol{x} = \boldsymbol{R} + \boldsymbol{x}'. \tag{11.1}$$

Take the derivatives. Assume

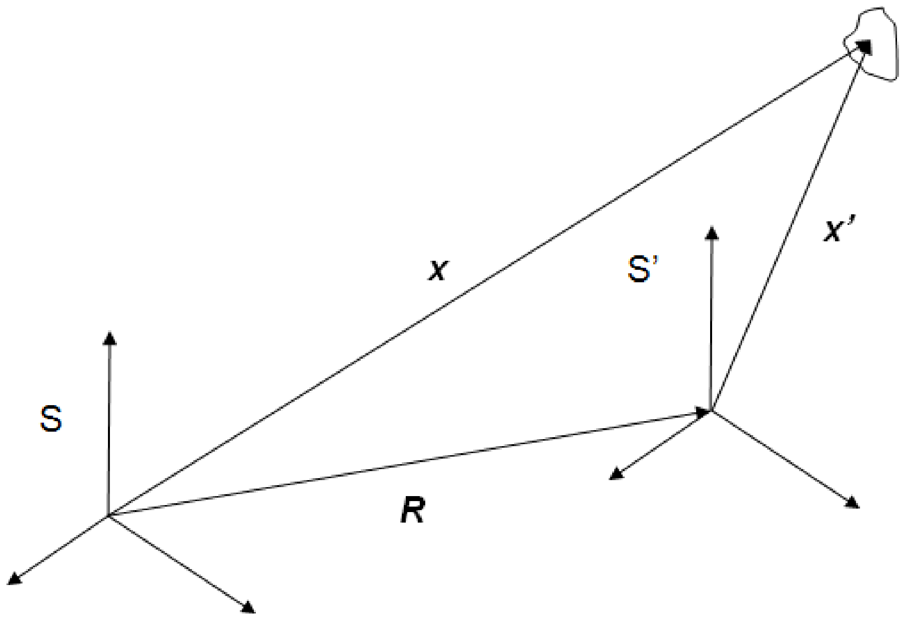


Figure 11.1: Relationship between position vectors in two distinct frames of reference.

$d/dt = d/dt'$; time is absolute.

Then,

$$\frac{d\boldsymbol{x}}{dt} = \frac{d\boldsymbol{R}}{dt} + \frac{d\boldsymbol{x}'}{dt}. \tag{11.2}$$

Introduce new notation:

$$\boldsymbol{v}_S = \boldsymbol{V}_{S'S} + \boldsymbol{v}_{S'} \tag{11.3}$$

In words,

Velocity of object relative to frame S =

Velocity of frame S' relative to frame S + Velocity of object relative to frame S'. (11.4)

Let us introduce some final notation:

$$\frac{d\boldsymbol{R}}{dt} = \boldsymbol{V}_{S'S} := \boldsymbol{V}. \tag{11.5}$$

We now specialise to inertial frames and demand that S' be in uniform motion with respect to S. Thus,

$$\frac{d\boldsymbol{V}}{dt} = 0 \text{ if and only if } \boldsymbol{V} = \text{Const}. \tag{11.6}$$

This has two consequences:

1. The acceleration is the same in both frames:

$$\frac{d\boldsymbol{v}_S}{dt} = \frac{d\boldsymbol{v}_{S'}}{dt}. \tag{11.7}$$

2. The two frames are linked via a transformation that involves $\boldsymbol{V}$. For, let us solve $\boldsymbol{V} = d\boldsymbol{R}/dt$ (Eq. (11.5)). If both frames coincide at time $t = 0$, then $\boldsymbol{R} = \boldsymbol{V}t$. Using the vector addition law $\boldsymbol{x} = \boldsymbol{R} + \boldsymbol{x}'$ (Eq. (11.1)), obtain

$$\boldsymbol{x} = \boldsymbol{V}t + \boldsymbol{x}'. \tag{11.8}$$

Let us apply this formal derivation to the train-and-platform scenario. Then, $\boldsymbol{V} = \hat{\boldsymbol{x}}V$, and

$$\begin{aligned} x &= Vt + x', \\ y &= y', \\ z &= z', \\ t &= t'. \end{aligned} \tag{11.9}$$

The velocity transformation is, as before:

$$\begin{aligned} v_x &= V + v_x', \\ v_y &= v_y', \\ v_z &= v_z'. \end{aligned} \tag{11.10}$$

The object we choose to specify with these coordinate systems is the third passenger on board the train , walking at a velocity $5\,\mathrm{km/hr}$ in the direction of motion of the train, and relative to observer S' (thus, this observer corresponds to the 'blob' in Fig. 11.1). Thus,

- $v_x' = 5\,\mathrm{km/hr}$,
- $V = 100\,\mathrm{km/hr}$,
- Using the Galilean transformation, $v_x = 105\,\mathrm{km/hr}$.

11.3.1 Absoluteness of time revisited

Einstein in his theory of special relativity also stated that all laws of physics have the same form in all inertial reference frames but that the Galilean transformations (which are based on the notion of absolute time) are wrong. They are approximately correct at speeds much slower than that of light, $c = 3 \times 10^8\,\mathrm{m/s}$ and break down completely as speeds approach c. We will examine this theory in detail during the second half of the course.

For the moment we restrict ourselves to Newtonian mechanics, where the Galilean transformation is correct.

11.4 Newton's laws of motion and Galilean invariance

Newton's laws of motion are valid only in an inertial frame of reference:

1. A body will continue in its state of rest or of uniform motion in a straight line unless an external force is applied to it.
2. The vector sum of forces acting on a body is proportional to its rate of change of momentum.

   For a body of constant mass this becomes

$$m\frac{d^2\boldsymbol{x}}{dt^2} = \boldsymbol{F}. \tag{11.11}$$

3. If a body A exerts a force on a body B then B exerts an equal and opposite force on A.

Laws (1) and (2) do not hold in all frames of reference: We can observe an object to accelerate not because of any forces acting upon it but because of the acceleration of our own reference frame. Consider a car accelerating on a straight road relative to the earth. An observer in the car who is equipped with a pendulum will see that pendulum deflected in a direction opposite to the acceleration. This is not due to any force on the pendulum itself, but rather due to a force acting on the entire frame of reference. Some more examples of non-inertial frames:

- The earth is not an inertial frame. Distant stars rotate about the earth but not because of any forces acting upon them, merely because of the earth's rotation. An observer on the equator undergoes centripetal acceleration with $\omega \approx 2\pi\,\mathrm{rad/day}$ and $r = 6,400\,\mathrm{km}$, giving

$$a = \omega^2 r = 0.03\,\mathrm{m}/s^2 = 3.45 \times 10^{-3} g.$$

  Since this is small compared to g, in practice we take the earth to be an inertial reference frame. The true angular frequency of the rotation is measured with respect to the sidereal day which is slightly shorter than the earth day – this is addressed in Chapter 12

A demonstration of the non-inertial nature of a reference frame attached to the earth's surface is the pendulum of Foucault: the direction along which a simple pendulum swings rotates with time because of Earth's daily rotation.

- Even if the earth did not turn on its own axis we would still not be in an inertial reference frame since the earth goes around the sun with $\omega = 2\pi\,\mathrm{rad/year}$ and $r = 1.5\times 10^8\,\mathrm{km}$, hence

$$a = \omega^2 r = 5.94\times 10^{-3}\,\mathrm{m/s^2} = 6.05\times 10^{-4} g$$

  We will compute the

- The sun also rotates around the centre of the Milky Way galaxy which, in turn, accelerates through the Local Group of galaxies and so on.

So, we imagine a hypothetical distant observer in deep space on whom no forces act. Such an observer will only see an object accelerating if it feels a resultant force. Any reference frame moving with constant velocity relative to this observer is also in an inertial frame.

Galilean invariance again: We have seen that the acceleration $\ddot{\boldsymbol{x}}$ is left invariant by a change of inertial frame under the transformations (11.8). Since the laws of physics are the same in all inertial frames, the force must also be left invariant by this transformation. Thus, $\boldsymbol{F}$ is not arbitrary in Eq. (11.11). We can check that central forces

$$\boldsymbol{F}_{ij} = -\lambda\left(\frac{\boldsymbol{x}_i - \boldsymbol{x}_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^p}\right), \tag{11.12}$$

are Galilean invariant, provided λ is a constant scalar. Here $\boldsymbol{F}_{ij}$ means the force on particle i due to particle j. Under a Galilean transformation,

$$\boldsymbol{x}'_i = \boldsymbol{x}_i - \boldsymbol{V}t,$$
$$\boldsymbol{x}'_j = \boldsymbol{x}_j - \boldsymbol{V}t,$$

hence,

$$\boldsymbol{x}'_i - \boldsymbol{x}'_j = \boldsymbol{x}_i - \boldsymbol{x}_j,$$

and $\boldsymbol{F}'_{ij} = \boldsymbol{F}_{ij}$.

11.5 What happens when $\boldsymbol{V}$ is no longer constant?

In this case we refer back to Figure 11.1. Suppose that S is an inertial frame and that S' is accelerating with respect to S. Then,

$$\boldsymbol{A} = \frac{\mathrm{d}V}{\mathrm{d}t} \neq 0.$$

Referring to Figure 11.1, we have

$$\ddot{\boldsymbol{x}} = \ddot{\boldsymbol{R}} + \ddot{\boldsymbol{x}}',$$

or

$$\ddot{\boldsymbol{x}} = \boldsymbol{a} + \ddot{\boldsymbol{x}}'.$$

Since $m\ddot{\boldsymbol{x}} = \boldsymbol{F}_{ext}$ by Newton's Second Law (valid in an inertial frame), we must have:

$$\boldsymbol{F}_{ext} = m\boldsymbol{A} + m\ddot{\boldsymbol{x}}',$$

hence

$$m\ddot{\boldsymbol{x}}' = \boldsymbol{F}_{ext} \underbrace{-m\boldsymbol{A}}_{\text{Fictitious force}}. \tag{11.13}$$

The 'extra term' which appears on the right-hand side of Newton's Law in S' is called the **Fictitious Force**.

It is very useful to study fictitious forces in this context because the functional form of the fictitious force in Equation (11.13) is very simple. This is a consequence of our studying frames which are accelerating frames which are in translational acceleration with respect to one another. In this case, the mathematics of fictitious forces is quite simple. In contrast, when we look at frames that are in rotational acceleration with respect to one another (in Chapter 12), the mathematics become much more involved. Therefore, looking at physics in in translationally accelerating frames is a first step in understanding non-inertial frames. To reiterate,

- Transitionally accelerating frames: the origins of S and S' coincide at only one instant but thereafter accelerate with respect to one another.
- Rotationally accelerating frames: the origins of S and S' coincide for all times but the axes in S and in S' are rotating with respect to one another – Chapter 12.

11.5.1 Example

Consider a box on a horizontal floor in the back of a van. The coefficient of static friction between the box and the floor is μ and the coefficient of sliding friction is f, with $f < \mu$.

1. What is the maximum acceleration A the van can have if the box is to remain at rest in the van?
2. Assume that the van brakes with an acceleration of a magnitude that just exceeds A so that the box slides forward. Find the speed of the box when it hits the wall of the drivers cabin assuming that the initial distance to this wall is d.

Solution:

1. The total force (net force and fictitious force) in the accelerating frame S' is

$$F = \mu mg - mA,$$

where A is the acceleration of the frame itself, relative to the ground. The total force should vanish if there is to be no motion in the accelerating frame, hence $\mu mg = mA$, hence $A = mg$. The maximum allowed acceleration is therefore $A = \mu g$.

2. In the second case, the non-inertial frame S' is decelerating, so the fictitious force is $F_{\text{fict}} = +m(A+\epsilon)$, where $\epsilon \to 0$ is a small positive constant, such that the static friction is overcome (we can now work with $\epsilon = 0$, and assume the parcel is in motion).

   Also, there is sliding friction, which opposes the motion, with $F_{\text{fric}} = -fmg$ (the minus sign is there to oppose the motion). Hence, Newton's law in the non-inertial frame is

$$ma' = -fmg + mA.$$

   Hence, $a' = A - fg$. But $A = \mu g$ from Part (a), so

$$a' = g(\mu - f).$$

   This is a positive quantity, i.e. $a' > 0$, so the parcel does indeed slide forward. It is also a constant acceleration, so we can use the kinematic equations for motion under constant acceleration, in particular,

$$v'^2 = u'^2 + 2a's$$

   As the parcel starts from rest in S', this gives $u' = 0$, hence

$$v' = \sqrt{2g(\mu - f)d}.$$

Chapter 12

Rotating Frames

12.1 Overview

When we talk of a frame we shall mean an origin together with a choice of axes, or equivalently, a right-handed set of unit vectors pointing along the respective axes. In 2-dimensions we have typically written $\boldsymbol{i}$ and $\boldsymbol{j}$ while in 3 dimensions we have used $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. So we can deal with both and also allow ourselves to prepare for Special Relativity where we will need 4 axes we will now start to use the notation $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ in 2 dimensions or $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ in 3 dimensions.

12.2 Rotating frames in 2 dimensions

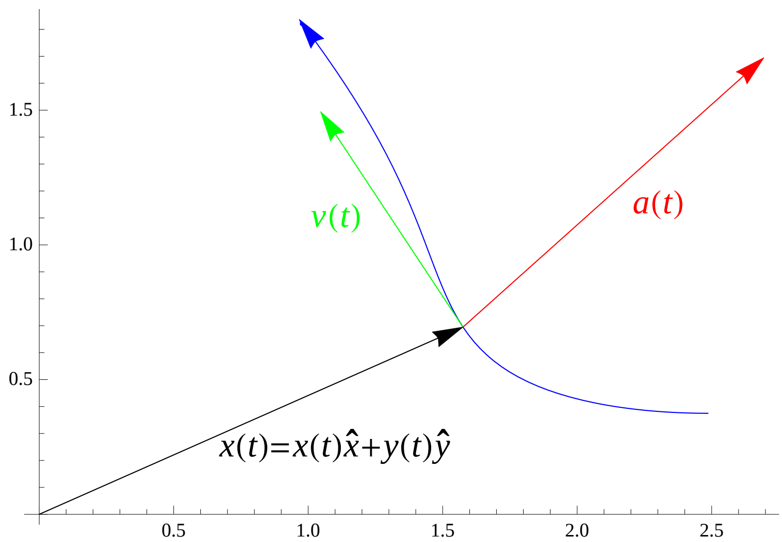


Figure 12.1: Trajectories in the plane.

In two dimensions, and in an inertial frame, two Cartesian components x and y are necessary and sufficient to specify the position of a particle:

$$\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} = x^1\boldsymbol{e}_1 + x^2\boldsymbol{e}_2 = \sum_{i=1}^{2} x^i\boldsymbol{e}_i.$$

Consider a 2-dimensional frame $S' : \big(O, (\boldsymbol{e}'_1, \boldsymbol{e}'_2)\big)$ which is rotating with respect to a frame $S : \big(O, (\boldsymbol{e}_1, \boldsymbol{e}_2)\big)$ with the same origin. For example, we could think of S as defined by the ground and S' defined by axes on a roundabout with the origin at the centre.

The angle $\theta(t)$ between $\boldsymbol{e}'_1$ and $\boldsymbol{e}_1$ is time dependent but as in Chapter 4 we can write

$$\begin{aligned} \boldsymbol{e}'_1 &= \cos\theta(t)\boldsymbol{e}_1 + \sin\theta(t)\boldsymbol{e}_2 \\ \boldsymbol{e}'_2 &= -\sin\theta(t)\boldsymbol{e}_1 + \cos\theta(t)\boldsymbol{e}_2. \end{aligned}$$

If we are considering how these change from the point of view of S we treat the basis vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ as fixed so

$$\begin{aligned} \left(\frac{\mathrm{d}\boldsymbol{e}'_1}{\mathrm{d}t}\right)_S &= -\dot{\theta}\sin\theta\boldsymbol{e}_1 + \dot{\theta}\cos\theta\boldsymbol{e}_2 = \dot{\theta}\boldsymbol{e}'_2 \\ \left(\frac{\mathrm{d}\boldsymbol{e}'_2}{\mathrm{d}t}\right)_S &= -\dot{\theta}\cos\theta\boldsymbol{e}_1 - \dot{\theta}\sin\theta\boldsymbol{e}_2 = -\dot{\theta}\boldsymbol{e}'_1. \end{aligned}$$

where we introduce the frame in which the basis taken as fixed explicitly into our notation.

If $\boldsymbol{a}$ is fixed in S', eg a particle has a fixed position in S', then

$$\boldsymbol{a} = a'^1\boldsymbol{e}'_1 + a'^2\boldsymbol{e}'_2 = \sum_{i=1}^{2} a'^i\boldsymbol{e}'_i.$$

where a'^1 and a'^2 are constant. Thus

$$\begin{aligned} \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S &= a'^1\left(\frac{\mathrm{d}\boldsymbol{e}'_1}{\mathrm{d}t}\right)_S + a'^2\left(\frac{\mathrm{d}\boldsymbol{e}'_2}{\mathrm{d}t}\right)_S \\ &= \dot{\theta}(a'^1\boldsymbol{e}'_2 - a'^2\boldsymbol{e}'_1) \\ &= \dot{\theta}\boldsymbol{k} \times (a'^1\boldsymbol{e}'_1 + a'^2\boldsymbol{e}'_2) = \dot{\theta}\boldsymbol{k} \times \boldsymbol{a}, \end{aligned}$$

where $\boldsymbol{k} = \boldsymbol{e}'_1 \times \boldsymbol{e}'_2 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$ is the unit vector normal to the plane.

The vector $\dot{\theta}\boldsymbol{k}$ is called the angular velocity of the frame S' with respect to S. It's direction defines the axis of rotation, its magnitude the angular speed.

12.3 Rotating frames in 3 dimensions

Consider a 3-dimensional frame $S' : \big(O, (\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3')\big)$ which is rotating with respect to a frame $S : \big(O, (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)\big)$ with the same origin.

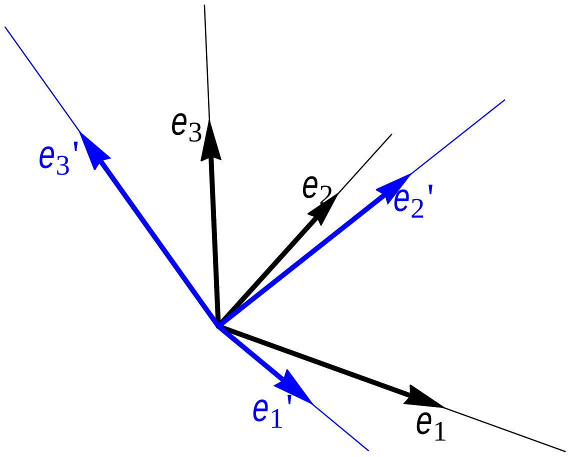


Figure 12.2: 3D of the frame S' rotating with respect to the frame S.

Does there exist an angular velocity vector $\boldsymbol{\omega}$ for motion in 3 dimensions? Yes:

Theorem: There exists a unique angular vector $\boldsymbol{\omega}$ such that if $\boldsymbol{a}$ is a vector fixed in S' i.e. $\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{S'} = \boldsymbol{0}$ then

$$\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S = \boldsymbol{\omega} \times \boldsymbol{a},$$

where $\boldsymbol{\omega}$ is independent of $\boldsymbol{a}$.

Definition: $\boldsymbol{\omega}$ is called the angular velocity of S' relative to S.

Proof: (1) Existence

We can write

$$\boldsymbol{a} = \sum_{i=1}^{3} (\boldsymbol{a}\cdot\boldsymbol{e}_i')\boldsymbol{e}_i',$$

where as $\boldsymbol{a}$ is fixed in S'

$$(\boldsymbol{a}\cdot\boldsymbol{e}_i') = \text{constant}$$

so

$$\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S = \sum_{i=1}^{3} (\boldsymbol{a}\cdot\boldsymbol{e}_i') \left(\frac{\mathrm{d}\boldsymbol{e}_i'}{\mathrm{d}t}\right)_S. \tag{12.1}$$

Also as $\boldsymbol{a}\cdot\boldsymbol{e}'_i$) is a scalar it does not depend on a choice of basis and so we get the same result whatever frame we use. Hence, as it is constant

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}\cdot\boldsymbol{e}'_i = \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S \cdot\boldsymbol{e}'_i + \boldsymbol{a}\cdot\left(\frac{\mathrm{d}\boldsymbol{e}'_i}{\mathrm{d}t}\right)_S,$$

hence

$$\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S \cdot\boldsymbol{e}'_i = -\boldsymbol{a}\cdot\left(\frac{\mathrm{d}\boldsymbol{e}'_i}{\mathrm{d}t}\right)_S.$$

Now,

$$\boldsymbol{v} = \sum_{i=1}^{3} (\boldsymbol{v}\cdot\boldsymbol{e}'_i)\,\boldsymbol{e}'_i$$

for any vector $\boldsymbol{v}$. So let $\boldsymbol{v} = \mathrm{d}\boldsymbol{a}/\mathrm{d}t$ to get

$$\begin{aligned}\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S &= \sum_{i=1}^{3}\left(\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S\cdot\boldsymbol{e}'_i\right)\boldsymbol{e}'_i \\ &= -\sum_{i=1}^{3}\boldsymbol{a}\cdot\left(\frac{\mathrm{d}\boldsymbol{e}'_i}{\mathrm{d}t}\right)_S\boldsymbol{e}'_i \qquad (12.2)\end{aligned}$$

Adding (12.1) and (12.2)

$$2\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S = \sum_{i=1}^{3}\left[(\boldsymbol{a}\cdot\boldsymbol{e}'_i)\left(\frac{\mathrm{d}\boldsymbol{e}'_i}{\mathrm{d}t}\right)_S - \boldsymbol{a}\cdot\left(\frac{\mathrm{d}\boldsymbol{e}'_i}{\mathrm{d}t}\right)_S\boldsymbol{e}'_i\right]$$

so, recalling $\boldsymbol{x}\times(\boldsymbol{y}\times\boldsymbol{z}) = (\boldsymbol{x}\cdot\boldsymbol{z})\boldsymbol{y} - (\boldsymbol{x}\cdot\boldsymbol{y})\boldsymbol{z}$,

$$\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S = \frac{1}{2}\sum_{i=1}^{3}\boldsymbol{a}\times\left[\left(\frac{\mathrm{d}\boldsymbol{e}'_i}{\mathrm{d}t}\right)_S\times\boldsymbol{e}'_i\right]$$

or

$$\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S = \left[\frac{1}{2}\sum_{i=1}^{3}\boldsymbol{e}'_i\times\left(\frac{\mathrm{d}\boldsymbol{e}'_i}{\mathrm{d}t}\right)_S\right]\times\boldsymbol{a}.$$

Defining

$$\boldsymbol{\omega} = \frac{1}{2}\sum_{i=1}^{3}\boldsymbol{e}'_i\times\left(\frac{\mathrm{d}\boldsymbol{e}'_i}{\mathrm{d}t}\right)_S$$

this is the result we want (note that our definition of $\boldsymbol{\omega}$ is independent of $\boldsymbol{a}$).

(2) Uniqueness:

Suppose there exists another vector $\boldsymbol{\omega}'$ such that

$$\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S = \boldsymbol{\omega}' \times \boldsymbol{a} \qquad \forall \boldsymbol{a},$$

then

$$0 = \boldsymbol{\omega} \times \boldsymbol{a} - \boldsymbol{\omega}' \times \boldsymbol{a} = (\boldsymbol{\omega} - \boldsymbol{\omega}') \times \boldsymbol{a} \qquad \forall \boldsymbol{a}.$$

So, $(\boldsymbol{\omega} - \boldsymbol{\omega}') \times \boldsymbol{a} = 0$ for all vectors $\boldsymbol{a}$. So, $\boldsymbol{\omega} - \boldsymbol{\omega}'$ is parallel to every vector. This can only be true if $\boldsymbol{\omega} - \boldsymbol{\omega}' = 0$. Hence, $\boldsymbol{\omega} = \boldsymbol{\omega}'$, i.e. $\boldsymbol{\omega}$ is unique.

12.4 The meaning of $\boldsymbol{\omega}$

To clarify the physical meaning of $\boldsymbol{\omega}$ consider the case when $\boldsymbol{\omega}$ is constant in time. [Note that this can mean in either frame as if it's true in S' then $\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_S = \boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$.]

If $\boldsymbol{a}$ is fixed in S' then

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}\cdot\boldsymbol{a} = 2\boldsymbol{a}\cdot\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S = 2\boldsymbol{a}\cdot(\boldsymbol{\omega} \times \boldsymbol{a}) = 0$$

so $|\boldsymbol{a}| =$ constant. Also

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}\cdot\boldsymbol{a} = \boldsymbol{\omega}\cdot\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S = \boldsymbol{\omega}\cdot(\boldsymbol{\omega} \times \boldsymbol{a}) = 0$$

and as

$$\boldsymbol{\omega}\cdot\boldsymbol{a} = |\boldsymbol{\omega}|\,|\boldsymbol{a}|\cos\alpha$$

where α is the angle between $\boldsymbol{\omega}$ and $\boldsymbol{a}$ we can conclude, as $|\boldsymbol{\omega}|$ and $|\boldsymbol{a}|$ are constant, that α is constant. [Note: we could also derive these from the fact that both $\boldsymbol{\omega}$ and $\boldsymbol{a}$ are fixed in S' so the results are obvious there - but it's good to see consistency with our new formula.]

In pictures, the point described by the tip of the vector $\boldsymbol{a}$ describes a circle lying on a cone with axis defined by $\boldsymbol{\omega}$ and semi-vertical angle α.

Also

$$\left|\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S\right|^2 = |\boldsymbol{\omega} \times \boldsymbol{a}|^2 = |\boldsymbol{\omega}|^2\,|\boldsymbol{a}|^2\sin^2\alpha$$

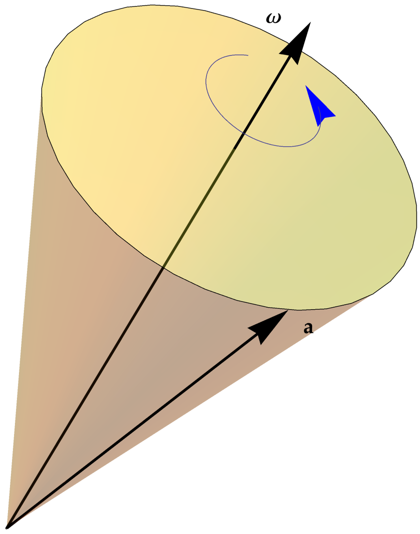


Figure 12.3: 3D of the frame S' rotating with respect to the frame S.

so the point moves with constant speed $|\boldsymbol{\omega}|\,|\boldsymbol{a}|\sin\alpha$ and the time to complete one full revolution is

$$T = \frac{\text{circumference of circle}}{\text{speed}} = \frac{2\pi|\boldsymbol{a}|\sin\alpha}{|\boldsymbol{\omega}|\,|\boldsymbol{a}|\sin\alpha} = \frac{2\pi}{|\boldsymbol{\omega}|}.$$

independent of $\boldsymbol{a}$. So, the direction of $\boldsymbol{\omega}$ defines the axis of rotation and $|\boldsymbol{\omega}|$ is the angular speed with which any point fixed in S' rotates about it.

Example:

Let S is the 'frame of the fixed stars' and let S' have axes centred at the centre of the Earth and fixed in it with the z-axis running pole to pole with the North pole on the positive z-axis. Then the direction of $\boldsymbol{\omega}$ is that of the z-axis, and as the Earth turns on its axis relative to the fixed stars once in a Sidereal day (24 hours $\times$ $365.25/(366.25)$) which is approximately $86,164\,\text{s}$ the magnitude of $\boldsymbol{\omega}$ is given by

$$|\boldsymbol{\omega}| = \frac{2\pi}{86164\,\text{s}} = 7.29 \times 10^{-5}\,\text{radians}\,\text{s}^{-1}.$$

More on the sidereal day:

An earth day (also called a solar day) is the time from one noon on earth to the next. This is equal to 24 hours, or a well-defined number of seconds:

$$1 \text{ Earth day} = 24 \text{ hours } = 24 \times 60 \times 60 = 86,400 \text{ seconds}.$$

Think about a sundial. At noon, the sundial casts the shortest shadow. At noon the next day, the sundial again casts the shortest shadow of the day. On both these occasions, we can draw a line that that joins the centre of the earth to the centre of the sun, and goes through the sundial – this is called the meridian line – see Figure 12.4. Therefore, the sundial resumes its orientation

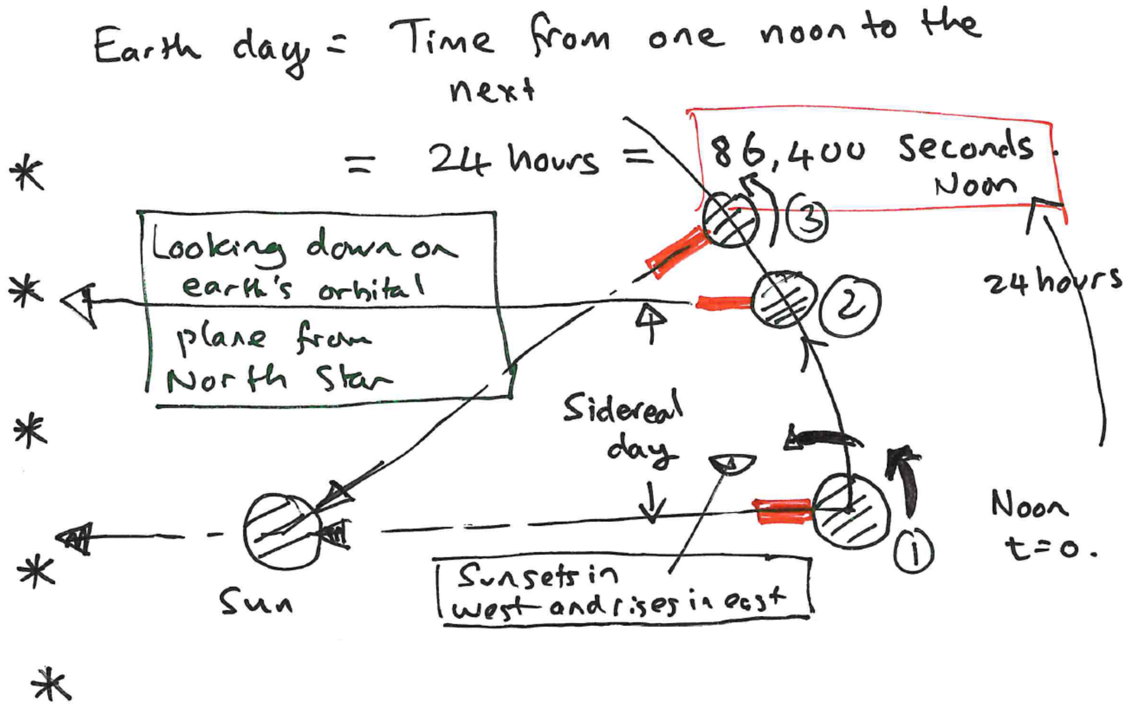


Figure 12.4: Schematic diagram showing the length of the sidereal day versus the length of the solar day. As a rough approximation, the orbit of the earth around the sun can be assumed to be a circle for the purpose of this explanation.

with respect to the sun every 24 hours. However, from the sketch, it can be seen that the sundial resumes its orientation with respect to the fixed stars after slightly less than 24 hours. This period is called the **sidereal day**.

The question then is, how much shorter is the sidereal day compared to the earth day. To answer this, let us take yet another reference point, namely the centre of the sun. Relative to the centre of the sun, the meridian has a position vector that is made up of two parts:

$$\boldsymbol{r}_{\text{meridian}} = \text{Position vector from centre of sun to centre of earth} \\ + \text{Position vector of meridian relative to centre of earth.}$$

Both of these position vectors are rotating with time – but they are doing so in a very well-defined way. As such, we can write:

$$\boldsymbol{r}_{\text{meridian}} = r_0 \left[\cos(\omega_{\text{Yrt}})\hat{\boldsymbol{x}} + \sin(\omega_{\text{Yr}} t)\hat{\boldsymbol{y}}\right] + R_e \left[\cos(\omega_R t)\hat{\boldsymbol{x}} + \sin(\omega_R t)\hat{\boldsymbol{y}}\right],$$

where

- r_0 is the mean distance between the earth centre and the sun centre. Here, for simplicity, we approximate the earth's orbit around the sun as a circle.
- ω_{Yr} corresponds to the earth's orbital period around the sun, specifically,

$$\omega_{\mathsf{Yr}} = 2\pi/T, \qquad T = 365.25 \text{ days} = 365.25 \times 86,400 \text{ seconds}.$$

- R_e is the earth's radius.
- ω_{Yr} is to be specified in what follows.

After making one orbital period (in time T), the meridian ends up back where it starts. We therefore require $\omega_{\mathsf{Yr}}T = 2\pi$ – this is already the case; we also require

$$\omega_R T = 2\pi n,$$

where n is an integer. We now specify ω_R precisely. We have:

$$\omega_{R,\mathsf{Earth}} T = 2\pi n;$$

also,

$$\omega_{R,\mathsf{Sidereal}} T = 2\pi(n+1);$$

In this way, the frotational frequency of the sidereal day is just slightly longer than the rotational frequency of the earth day, and hence, the period (length) of the sidereal day is just slightly **shorter** than the length of the earth day. We combine these two equations together as:

$$\omega_{R,\mathsf{Sidereal}} T = \omega_{R,\mathsf{Earth}} T + 2\pi.$$

Hence,

$$\omega_{R,\mathsf{Sidereal}} = \omega_{R,\mathsf{Earth}} + \frac{2\pi}{T}.$$

We now express things in terms of periods:

$$\frac{2\pi}{T_{R,\mathsf{Sidereal}}} = \frac{2\pi}{T_{R,\mathsf{Earth}}} + \frac{2\pi}{T}.$$

We measure time now in units of earth days, with $T_{R,\mathsf{Earth}} = 86,400$ seconds. As such,

$$\frac{1}{T_{R,\mathsf{Sidereal}}} = \frac{1}{1 \text{ Earth Day}} + \frac{1}{365.251 \text{ Earth Days}}$$

Re-arranging gives:

$$T_{R,\mathsf{Sidereal}} = \frac{365.25}{366.25} \text{ Earth Days} = 86,164 \text{ seconds}.$$

12.4.1 If ω is not constant

Then, we can still say that at any given instant the points on the line $\boldsymbol{r} = \lambda\boldsymbol{\omega}$ $(\lambda \in \mathbb{R})$ will have

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_S = \boldsymbol{\omega} \times \boldsymbol{r} = \lambda\boldsymbol{\omega} \times \boldsymbol{\omega} = 0,$$

that is they are *instantaneously* at rest. Accordingly the line is called the instantaneous axis of rotation.

12.5 The Rotating Axes Theorem

What happens if the vector $\boldsymbol{a}$ depends on time in S'? We then have two time derivatives depending on the choice of frame:

$$\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S = \sum_{i=1}^{3} \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{e}_i)\right) \boldsymbol{e}_i,$$
$$\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{S'} = \sum_{i=1}^{3} \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{e}'_i)\right) \boldsymbol{e}'_i.$$

How are they related?

The Rotating Axes Theorem:

$$\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \boldsymbol{a},$$

where $\boldsymbol{\omega}$ is the angular velocity of S' with respect to S.

Proof:

$$\boldsymbol{a} = \sum_{i=1}^{3} (\boldsymbol{a}\cdot\boldsymbol{e}'_i)\boldsymbol{e}'_i$$

so

$$\begin{aligned}
\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S &= \sum_{i=1}^{3} \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{e}'_i)\right) \boldsymbol{e}'_i + \sum_{i=1}^{3} (\boldsymbol{a}\cdot\boldsymbol{e}_i) \left(\frac{\mathrm{d}\boldsymbol{e}'_i}{\mathrm{d}t}\right)_S \\
&= \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{S'} + \sum_{i=1}^{3} (\boldsymbol{a}\cdot\boldsymbol{e}_i)\, \boldsymbol{\omega} \times \boldsymbol{e}'_i \\
&= \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \left(\sum_{i=1}^{3} (\boldsymbol{a}\cdot\boldsymbol{e}_i)\boldsymbol{e}'_i\right) = \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \boldsymbol{a}
\end{aligned}$$

There are a number of immediate consequences:

1.
$$\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{S'} .$$

2. If S'' rotates with angular velocity $\boldsymbol{\omega}'$ with respect to S' while S' rotates with angular velocity $\boldsymbol{\omega}$ with respect to S, then S'' rotates with angular velocity $\boldsymbol{\omega} + \boldsymbol{\omega}'$ with respect to S.

   Proof: For any vector $\boldsymbol{a}$

$$\begin{aligned}
\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{S'} &= \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{S''} + \boldsymbol{\omega}' \times \boldsymbol{a} \\
\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_S &= \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \boldsymbol{a} \\
&= \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{S''} + \boldsymbol{\omega}' \times \boldsymbol{a} + \boldsymbol{\omega} \times \boldsymbol{a} = \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{S''} + (\boldsymbol{\omega}' + \boldsymbol{\omega}) \times \boldsymbol{a}
\end{aligned}$$

3. In particular, taking $S'' = S$ we must have $\boldsymbol{\omega} + \boldsymbol{\omega}' = \mathbf{0}$, that is, if $\boldsymbol{\omega}$ is the angular velocity of S' with respect to S then the angular velocity of S with respect to S' is $-\boldsymbol{\omega}$.

Chapter 13

Particle Moving in a Rotating Frame of Reference

13.1 Overview

We use the results of the previous Chapter to right the write the equations of motion for a particle written as a description with respect to a frame S' which is rotating with respect to an inertial frame (equivalent to 'the frame of the fixed stars').

13.2 Equations of motion

Let $\boldsymbol{r}(t)$ denote the position of a particle relative to the common origin O of S and S'. Then if S' rotates with angular velocity $\boldsymbol{\omega}$ with respect to S then

$$\begin{aligned}
\text{Velocity of particle according to } S &= \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_S \\
\text{Velocity of particle according to } S' &= \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'} \\
\text{Acceleration of particle according to } S &= \left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_S \\
\text{Acceleration of particle according to } S' &= \left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{S'}.
\end{aligned}$$

Now by the rotating axes theorem

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \boldsymbol{r},$$

and so

$$\begin{aligned}\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_S &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_S\right)_S = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_S\right)_{S'} + \boldsymbol{\omega}\times\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_S \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\boldsymbol{r}\right)_{S'} + \boldsymbol{\omega}\times\left(\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\boldsymbol{r}\right) \\ &= \left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{S'} + \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{S'}\times\boldsymbol{r} + 2\boldsymbol{\omega}\times\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\end{aligned}$$

Thus if we consider a particle moving in a frame S' rotating with angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame S and acted on by a total force $\boldsymbol{F}_{\text{tot}}$ then on using Newton's second law in the inertial frame and switching to the rotating frame we have

$$\boldsymbol{F}_{\text{tot}} = m\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_S = m\left[\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{S'} + \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{S'}\times\boldsymbol{r} + 2\boldsymbol{\omega}\times\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\right]. \tag{13.1}$$

One way to interpret this is to write it as

$$\boldsymbol{F}_{\text{tot}} \underbrace{-m\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{S'}\times\boldsymbol{r}}_{\text{Coriolis}} \underbrace{-2m\boldsymbol{\omega}\times\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'}}_{\text{Centripetal}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) = m\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{S'}$$

and to interpret the extra three terms on the left hand side as 'fictitious forces' required to ensure that Newton's second law takes the standard (inertial) form in the rotating (non-inertial) frame. In this context the third term is called the 'Coriolis force' and the fourth term the 'centrifugal force' However, it is really easier and more correct to think of these as part of the acceleration.

13.3 Equations of motion

Example – The Gravitron: In a fairground attraction a student is pinned to the inside of a rotating drum, given the coefficient of friction is μ and radius a find the minimum constant angular speed required for the student to stick.

Choose axes S' fixed in the drum with $\boldsymbol{e}_3'$ vertical through the centre and such that the position of the student is $a\boldsymbol{e}_1'$ (assumed in equilibrium).

Then as the student doesn't move relative to the drum

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'} = 0 \quad \text{and} \quad \left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{S'} = 0,$$

while

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) = \omega\boldsymbol{e}_3'\times(\omega\boldsymbol{e}_3'\times a\boldsymbol{e}_1') = -\omega^2 a\boldsymbol{e}_1'$$

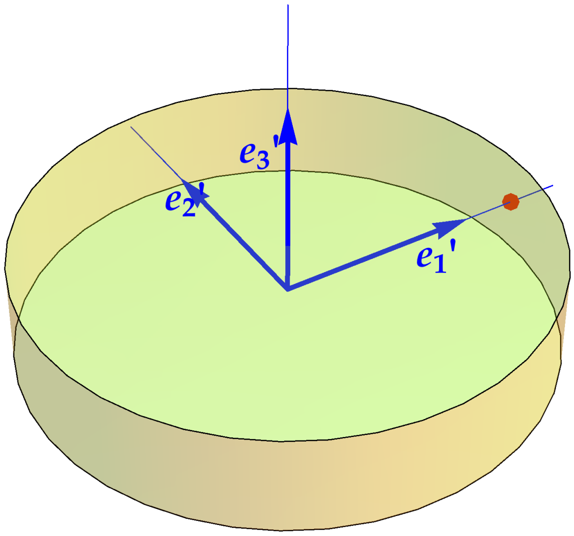


Figure 13.1: The *'Rotor'* from the UK National Fairground Archive http://www.nfa.dept.shef.ac.uk/history/rides/rotor.html

The forces acting are:

1. Gravity $-mg\boldsymbol{e}_3'$;
2. The Normal Reaction – the floor in the Gravitron drops away, meaning that the student is pinned to the wall by friction, hence a normal force $-R\boldsymbol{e}_1'$;
3. Friction – The floor in the Gravitron drops away; the tendency for the student to slide down under gravity is opposed by friction, hence a force $F\boldsymbol{e}_3'$.

The forces balance because the student is stationary in the frame S'. Hence, by Equation in equilibrium we must have

$$-mg\boldsymbol{e}_3' - R\boldsymbol{e}_1' + F\boldsymbol{e}_3' = -m\omega^2 a\boldsymbol{e}_1'$$

or, equating coefficients,

$$R = m\omega^2 a \quad \text{and} \quad F = mg.$$

Finally, since $F \leq \mu R$ we need $mg \leq \mu m\omega^2 a$ or

$$\omega \geq \sqrt{\frac{g}{\mu a}}.$$

Example: A smooth wire shaped as an ellipse with semi-major axis a and semi-minor axis b rotates in a horizontal plane about a vertical axis through its centre with constant angular speed ω. A bead is threaded onto the wire; show that it is in equilibrium and either end of the semi-major or semi-minor axis. Which, if any, of these equilibria are stable?

The angular velocity $\boldsymbol{\omega} = \omega \boldsymbol{e}_3 = \omega \boldsymbol{e}_3'$ where ω is constant.

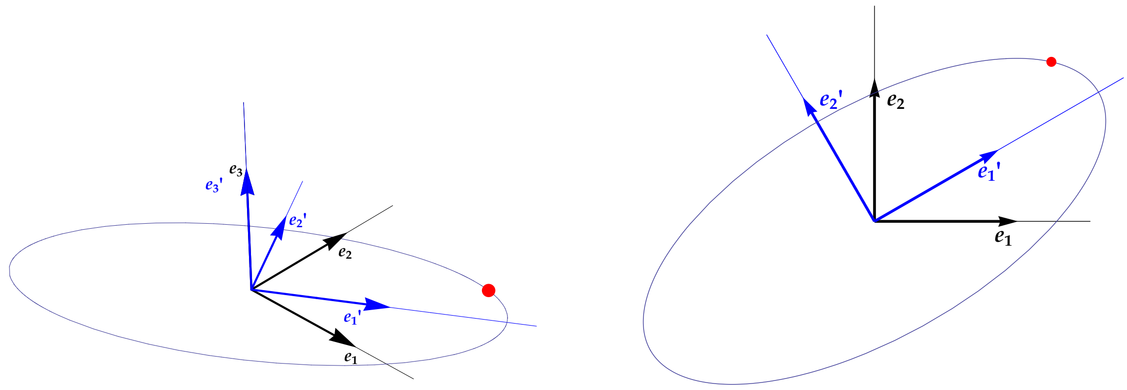


Figure 13.2: 3D and 2D view of the rotating elliptical wire which is fixed with respect to the frame S'.

The wire can be parameterised as

$$\boldsymbol{r} = a\cos\psi\,\boldsymbol{e}_1' + b\sin\psi\,\boldsymbol{e}_2' \tag{13.2}$$

where we have chosen axes in S' along the semi-major and semi-minor axis and without loss of generality we may assume that $a > b$. [Note that ψ is not the plane polar angle but a value of ψ uniquely determines the position.]

We start our calculations with the parametrization (13.2), where ψ depends on time. By the chain rule, we have:

$$\left(\frac{\partial \boldsymbol{r}}{\partial t}\right)_{S'} = \frac{\partial \boldsymbol{r}}{\partial \psi}\dot{\psi}, \tag{13.3a}$$

$$\left(\frac{\partial^2 \boldsymbol{r}}{\partial t^2}\right)_{S'} = \frac{\partial \boldsymbol{r}}{\partial \psi}\ddot{\psi} + \frac{\partial^2 \boldsymbol{r}}{\partial \psi^2}\dot{\psi}^2. \tag{13.3b}$$

We substitute into Newton's Equation in the rotating frame, recalled here as

$$m\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{S'} = \boldsymbol{F}_{\text{tot}} - m\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{S'} \times \boldsymbol{r} - 2m\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{S'} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\,.$$

Re-arranging terms, setting $\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t = 0$, and using the expressions (13.3) gives:

$$\left(\frac{\partial \boldsymbol{r}}{\partial \psi}\ddot{\psi} + \frac{\partial^2 \boldsymbol{r}}{\partial \psi^2}\dot{\psi}^2\right) + 2\left(\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial \psi}\right)\dot{\psi} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = \boldsymbol{F}_{\text{tot}}/m. \tag{13.4}$$

The total external force $\boldsymbol{F}_{\text{tot}}$ is identified with the reaction force keeping the particle on the wire. Another word for the reaction force is the **normal force** because it acts in the normal direction.

On the other hand, $\partial\boldsymbol{r}/\partial\psi$ is a vector in the direction **tangent** to the wire – this is a standard geometric fact due to the fact that the wire is a curve parametrized by ψ. Hence,

$$\boldsymbol{F}_{\mathrm{tot}}\cdot\left(\frac{\partial\boldsymbol{r}}{\partial\psi}\right)=0.$$

We therefore take the dot product of Equation (13.4) with $\partial\boldsymbol{r}/\partial\psi$ to project out the normal force. We obtain:

$$\left(\frac{\partial\boldsymbol{r}}{\partial\psi}\right)^2\ddot{\psi}+\frac{\partial^2\boldsymbol{r}}{\partial\psi^2}\cdot\frac{\partial\boldsymbol{r}}{\partial\psi}\dot{\psi}^2+\underbrace{2\left(\boldsymbol{\omega}\times\frac{\partial\boldsymbol{r}}{\partial\psi}\right)\cdot\frac{\partial\boldsymbol{r}}{\partial\psi}}_{\text{Zero}}+[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})]\cdot\frac{\partial\boldsymbol{r}}{\partial\psi}=0.$$

We use the following expressions, valid for the parametrization $\boldsymbol{r}=a\cos\psi\boldsymbol{e}_1'+b\sin\psi\boldsymbol{e}_2'$:

$$\begin{aligned}\frac{\partial^2\boldsymbol{r}}{\partial\psi^2}\cdot\frac{\partial\boldsymbol{r}}{\partial\psi} &= \tfrac{1}{2}\frac{\partial}{\partial\psi}\left(\frac{\partial\boldsymbol{r}}{\partial\psi}\right)^2,\\ \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) &= -\omega^2\boldsymbol{r}.\end{aligned}$$

Hence, Newton's equation becomes:

$$\left(\frac{\partial\boldsymbol{r}}{\partial\psi}\right)^2\ddot{\psi}+\tfrac{1}{2}\frac{\partial}{\partial\psi}\left(\frac{\partial\boldsymbol{r}}{\partial\psi}\right)^2\dot{\psi}^2-\omega^2\boldsymbol{r}\cdot\frac{\partial\boldsymbol{r}}{\partial\psi}=0.$$

We also use $\boldsymbol{r}\cdot(\partial\boldsymbol{r}/\partial\psi)=(1/2)(\partial/\partial\psi)\boldsymbol{r}^2$ to write the Newton Equation in one last form:

$$\left(\frac{\partial\boldsymbol{r}}{\partial\psi}\right)^2\ddot{\psi}+\tfrac{1}{2}\frac{\partial}{\partial\psi}\left(\frac{\partial\boldsymbol{r}}{\partial\psi}\right)^2\dot{\psi}^2-\tfrac{1}{2}\omega^2\frac{\partial}{\partial\psi}\boldsymbol{r}^2=0. \tag{13.5}$$

Only now do we evaluate the different partial derivatives explicitly:

$$\begin{aligned}\left(\frac{\partial\boldsymbol{r}}{\partial\psi}\right)^2 &= a^2\sin^2\psi+b^2\cos^2\psi,\\ \tfrac{1}{2}\frac{\partial}{\partial\psi}\left(\frac{\partial\boldsymbol{r}}{\partial\psi}\right)^2 &= (a^2-b^2)\sin\psi\cos\psi,\\ |\boldsymbol{r}|^2 &= a^2\cos^2\psi+b^2\sin^2\psi,\\ \tfrac{1}{2}\frac{\partial}{\partial\psi}|\boldsymbol{r}|^2 &= (b^2-a^2)\sin\psi\cos\psi.\end{aligned}$$

Hence, Equation (13.5) becomes:

$$\left(a^2\sin^2\psi+b^2\cos^2\psi\right)\ddot{\psi}+(a^2-b^2)\sin\psi\cos\psi\dot{\psi}^2+\underbrace{\omega^2(a^2-b^2)\sin\psi\cos\psi}_{\text{Note the sign!}}=0. \tag{13.6}$$

Given an initial value of ψ and $\dot{\psi}$ we can solve this equation numerically to give the solution for all time.

Let us now look for equilibrium solutions, that is $\psi(t) = \psi_0$, constant. We insert this proposed constant solution into Equation (13.6):

$$0 = \omega^2(a^2 - b^2)\sin\psi_0\cos\psi_0.$$

So

$$\sin\psi_0 \Rightarrow \psi_0 = 0 \text{ or } \pi \qquad \text{or} \qquad \cos\psi_0 \Rightarrow \psi_0 = \pi/2 \text{ or } 3\pi/2,$$

i.e., the particle is at the end of either the semi-major or semi-minor axis.

To examine stability we write $\psi(t) = \psi_0 + \epsilon(t)$ and assume $|\epsilon(t)| \ll 1$ so that we may approximate $\cos\epsilon \approx 1$ and $\sin\epsilon \approx \epsilon$. Correspondingly

$$\cos(\psi_0 + \epsilon) \approx \cos\psi_0 - \epsilon\sin\psi_0 \qquad \text{and} \qquad \sin(\psi_0 + \epsilon) \approx \sin\psi_0 + \epsilon\cos\psi_0.$$

Then taking $\psi_0 = 0$, Equation (13.6) gives

$$b^2\ddot{\epsilon} + \omega^2(a^2 - b^2)\epsilon = 0, \qquad a > b.$$

Recalling that we assumed $a > b$ we have stable equilibrium with oscillatory solutions $\begin{matrix}\cos\\ \sin\end{matrix}\left(\omega\sqrt{\frac{a^2 - b^2}{b^2}}t\right)$ with period

$$\frac{2\pi}{\omega}\sqrt{\frac{b^2}{a^2 - b^2}}.$$

On the other hand, taking $\psi_0 = \pi/2$, Equation eq:EqnofMotion gives

$$a^2\ddot{\epsilon} - \omega^2(a^2 - b^2)\epsilon = 0,$$

so we have unstable equilibrium with exponential solutions $\exp\left(\pm\omega\sqrt{\frac{a^2 - b^2}{a^2}}t\right)$ with the solution with the positive sign growing exponentially with time.

Chapter 14

Introduction to Einstein's theory of Special Relativity

14.1 Introduction

- Relativity is not a new concept to us: we have formulated classical mechanics on the basis of Galilean relativity, which says that *the laws of physics are the same in all inertial frames*.
- One further postulate of Galilean relativity is that time is absolute: given a pair of inertial frames in uniform motion relative to one another, clocks in both frames run at the same rate.
- This assumption breaks down at velocities close to the speed of light, and this requires the introduction of a new postulate of relativity, namely that *the speed of light is the same in all inertial frames of reference*.
- These two postulates form the basis of Special Relativity (SR). It is called special because it is a special case wherein gravity is absent (cf. General Relativity).
- It is not a purely kinematic theory however, because the interaction of charged particles with electromagnetic fields can be described using SR.

14.2 Electromagnetic radiation

In 1861, James Clark Maxwell formulated the equations for electromagnetic radiation (light, radiowaves, microwaves &c). A solution to Maxwell's equations in free space is the wave equation:

$$\frac{1}{c^2}\frac{\partial\psi}{\partial t^2} = \nabla^2\psi,$$

where ψ is the $\boldsymbol{E}$ or $\boldsymbol{B}$ field, and c is the velocity of light, which is a universal constant. But this is inconsistent with the idea that velocities depend on the frame of reference with respect to which they are measured. Contemporary physicists explained this by postulating the existence of the *aether*, an elastic medium fixed in space. Vibrations in this medium would correspond to EM radiation propagating at a speed c w.r.t. the fixed aether. An experiment by Michelson and Morley in 1882 failed to detect any relative motion of the earth with respect to the aether. This made the entire concept of the aether problematic. Some solutions to the problem were proposed by Poincaré, Lorentz, and Fitzgerald[1], but the most complete and ultimately, the successful one, was given by Einstein in 1905.

Einstein proposed to dispense with the unobservable aether altogether, and introduce a second postulate of relativity:

1. The laws of physics have the same form in all inertial frames of reference.
2. The speed of light in a vacuum is a universal constant, independent of the velocity of the light source.

Note that the second postulate is INCONSISTENT with the Galilean law of velocity additions and so produces an entirely new physics. The first consequence of this new physics is that *time is not absolute*.

14.3 Time is not absolute

Consider the familiar scenario of a train in motion relative to the platform of a station. The train is equipped with a clock. During one 'tick' of the clock, a beam of light is emitted from a source located on the floor the carriage, travels upwards, hits a mirror on the carriage ceiling, and arrives back at the floor. Let the height of the carriage be d. The time for one tick, as measured in the carriage, is

$$t' = \frac{2d}{c}.$$

(See Fig. 14.1). Now in the FOR of the platform, the light beam travels a distance

$$s = 2\sqrt{d^2 + \tfrac{1}{4}v^2t^2},$$

where v is the velocity of the train w.r.t. the platform and t is the tick time measured by the observer on the platform. But $s = ct$, hence

$$t = \frac{s}{c} = \frac{2}{c}\sqrt{d^2 + \tfrac{1}{4}v^2t^2}.$$

Re-arranging gives

$$t^2\left(1 - \frac{v^2}{c^2}\right) = \frac{4d^2}{c^2} = t'^2,$$

[1]Bio: see http://www.ucc.ie/academic/undersci/pages/sci_georgefrancisfitzgerald.htm

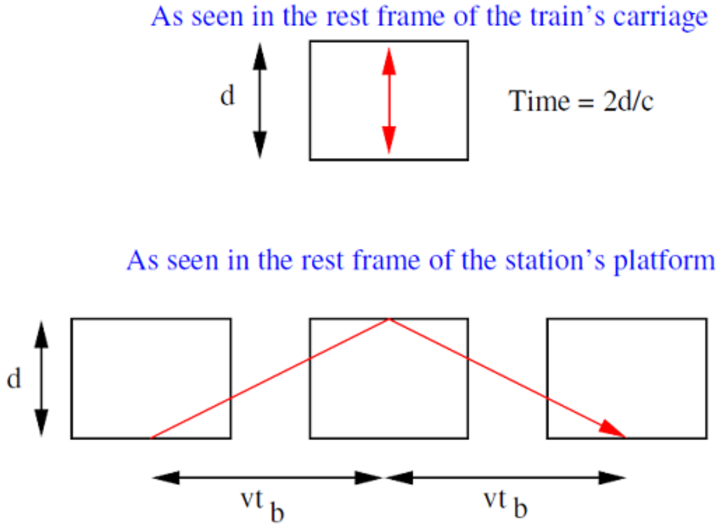


Figure 14.1: Schematic description of the light clock, $t_b = t/2$.

hence

$$t = \frac{t'}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{14.1}$$

or

$$t' = t\sqrt{1 - \frac{v^2}{c^2}}.$$

We shall see that $v < c$ always, so that

$$t' < t,$$

a result encapsulated in the phrase 'moving clocks run slow'.

Could we ever measure this for a real train ? If we take $d = 3\,\mathrm{m}$, then

$$t' = 6/c = 2 \times 10^{-8}\,\mathrm{s}.$$

If we have a very fast train we could take $v = 200\,\mathrm{kmhr}^{-1}$, but this gives $v/c = 1.85 \times 10^{-7}$, and, using the binomial theorem to estimate the denominator in Eq. (14.1),

$$t \approx 2 \times 10^{-8} \left(1 + 1.7 \times 10^{-14}\right).$$

Hence, the station master measures a duration equal to that measured by someone on the train plus 3.4×10^{-22} seconds – which is why we don't notice disagreements about time in our everyday experience.

Einstein's second postulate implies that we must revise the Galilean transformations and these are known as the Lorentz transformations.

14.4 Discussion question

Consider identical twin astronauts named Eartha and Astrid. Eartha remains on earth while her twin Astrid takes off on a high-speed trip through the galaxy. Because of time dilation, Eartha sees Astrid's heartbeat and all other life processes proceeding more slowly than her own. Thus, to Eartha, Astrid ages more slowly; when Astrid returns to earth she is younger (has aged less) than Eartha.

Now here is the paradox: All inertial frames are equivalent. Can't Astrid make exactly the same arguments to conclude that Eartha is in fact the younger? Then each twin measures the other to be younger when they're back together, and that is a paradox. What is the resolution to this apparent contradiction?

The answer is that the frames are inequivalent: Astra's frame is non-inertial becuase it undergoes accelearation at the point when Astra turns back towards earth. On the other hand, Earth's frame remains approximately inertial at all times. It turns out that Eartha is correct: Astra ages more slowly than Eartha, so that when she comes back, she is in fact younger than Eartha.

Chapter 15

The Lorentz Transformations

15.1 Overview

We want to derive the relation between an event occuring at position (x, y, z) and time t wrt observer S to the same event as seen by observer S' at (x', y', z') and time t'. Here frame S' moves at constant velocity V w.r.t. frame S. At speeds that are small compared to light we must recover the Galilean transformations but our new Lorentz transformations must be consistent with Einstein's second postulate which in turn implies that $t \neq t'$. Previously, we used the following diagram to derive the Galilean transformations: Thus, the relative motion is only in the x-direction,

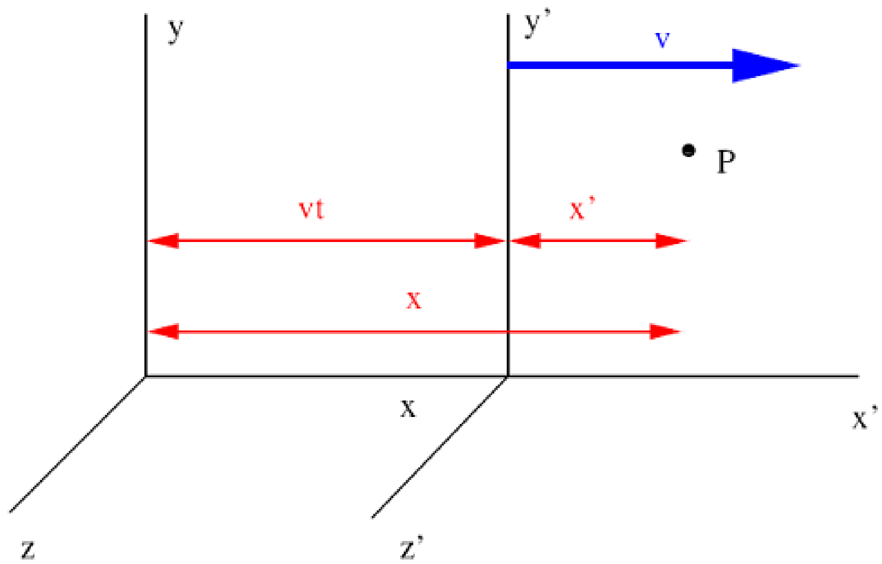


Figure 15.1: Relationship between two inertial frames of reference.

hence $y = y'$ and $z = z'$. As before, we arrange for the clocks to be synchronised so that when $x = x' = 0$, $t = t' = 0$. The transformations between frames S and S' must be linear, so we have the general form

$$\begin{aligned} x' &= Ax + Bt, \\ t' &= Cx + Dt. \end{aligned}$$

15.2 Fixing the coefficients in the linear transformation

We can determine the four constants, and how they depend on v , from the following four constraints.

1. Observer S sees the origin of S' moving along the x-axis with velocity V . We write this path for the origin of S' in each frame as

   $$x = Vt \text{ and } x' = 0.$$

   Therefore,

   $$x' = Ax + Bt \implies 0 = AVt + Bt \implies B = -AV,$$

   and now we have only three unknowns,

   $$\begin{aligned} x' &= A\left(x - Vt\right), \\ t' &= Cx + Dt. \end{aligned}$$

2. Observer S' sees the origin of S moving along the x' axis with velocity $-V$. The path for the origin of S is then

   $$x = 0 \text{ and } x' = -Vt',$$

   to give

   $$A\left(0 - Vt\right) = -V\left(0 + Dt\right) \implies D = A,$$

   which leaves two constants,

   $$\begin{aligned} x' &= A\left(x - Vt\right), \\ t' &= Cx + At. \end{aligned}$$

3. A pulse of light emitted from the origin in the x-direction at moves at speed c relative to both frames. Therefore, the path of this pulse of light in each frame is given by

   $$x = ct \text{ and } x' = ct'$$

   to give

   $$A\left(ct - Vt\right) = c\left(Cct + At\right) \implies C = -\frac{AV}{c^2},$$

and we are left with one remaining constant

$$\begin{aligned} x' &= A\left(x - Vt\right), \\ t' &= A\left(t - \frac{Vx}{c^2}\right). \end{aligned}$$

4. A light pulse is sent along the y-axis at $t = 0$. In the S frame we have

$$x = 0, \text{ and } y = ct,$$

while in the S' frame the pulse also moves at speed c but now in the (x', y') plane:

$$x'^2 + y'^2 = c^2 t'^2,$$

which becomes

$$A^2\left(0 - Vt\right)^2 + \left(ct\right)^2 = c^2 A^2 \left(0 + t\right)^2,$$

which tidies up to give

$$A = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

The quantity A is called the *Lorentz factor* and usually given the symbol γ:

$$\gamma := \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

We now have the equations which relate the coordinates and time for an event as measured by two inertial observers, the *Lorentz Transformations*:

$$\begin{aligned} x' &= \gamma\left(x - Vt\right), \\ t' &= \gamma\left(t - \frac{Vx}{c^2}\right), \\ y' &= y, \\ z' &= z. \end{aligned}$$

Once we have the coordinates for an event in one frame we can work out the coordinates in the other inertial frame. Note that, as required by the principle of relativity, neither frame is more important than the other and the inverse transformations are obtained by replacing $-V$ by $+V$:

$$x = \gamma(x' + Vt'),$$
$$t = \gamma\left(t' + \frac{Vx'}{c^2}\right).$$

15.3 Spacetime diagrams

Note that the Lorentz transformations for x' and t' are more alike that they look at first sight. If we were to change our units, from SI, to a system where $c = 1$ which is more 'natural', then we would have

$$x' = \gamma(x - Vt),$$
$$t' = \gamma(t - Vx),$$

and we can represent both sets of axes on one diagram. Note that the t'-axis, for example, is defined by $x' = 0$, which becomes $t = x/V$. This is an example of a *Minkowski or Space-Time diagram*. Note that we will always work in SI so that $c = 3 \times 10^8$, and we can still draw these diagrams, but

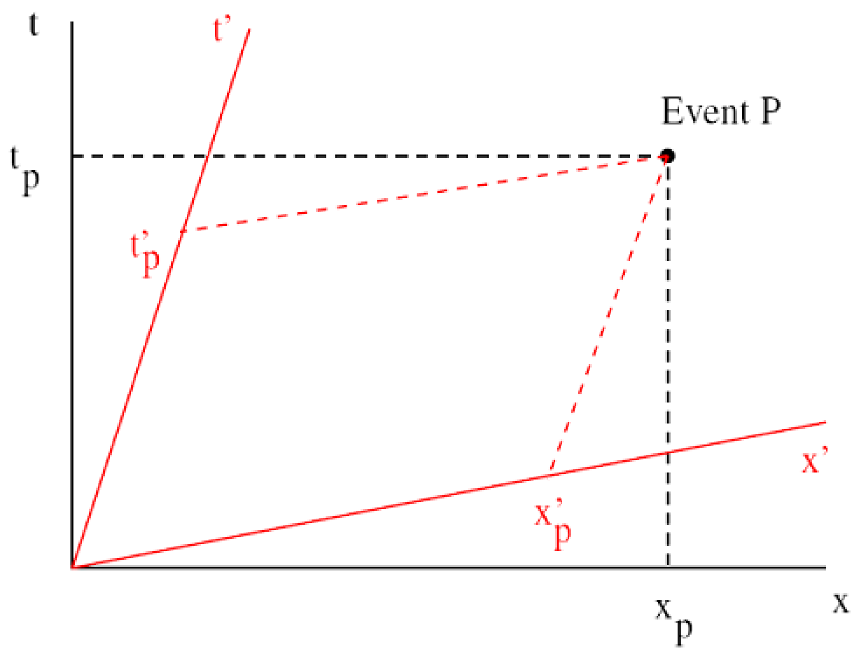


Figure 15.2: Minkowski diagram

the t- and t'-axes would be very close to one another. In more advanced work, we typically work in units where $c = 1$. It becomes immediately apparent from this diagram that if two events are simultaneous in one frame then they will not be simultaneous in the other frame.

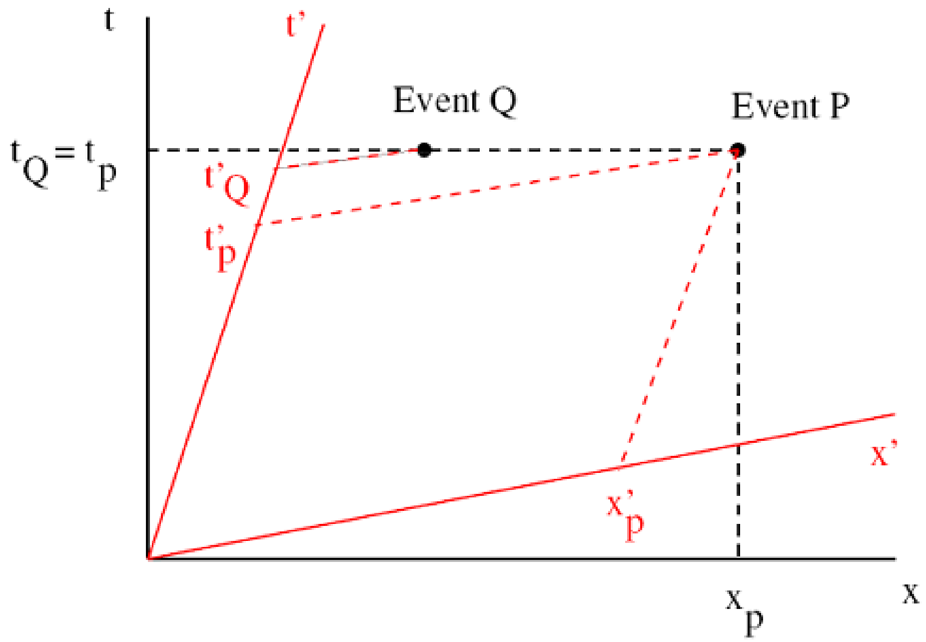


Figure 15.3: Minkowski diagram and the relativity of simulaneity: simultaneity is relative in special relativity because the transformed time axis is no longer parallel to the old time axis.

15.4 The Relativity of Simultaneity

As a thought experiment we imagine a person standing in the middle of a train carriage, of length $2L$ (in S') with a lamp that emits a pulse of light. This light arrives at each end of the carriage, A and B , after a time interval of L/c. The two events of light arriving at A and light arriving at B are therefore simultaneous relative to an observe on the train. However, an observer standing on the station's platform, S , observes the train to be moving at speed V but the forward and reverse pulse of light each move at speed c because of Einstein's second postulate. Therefore, observer S sees the backwards pulse hitting the back of the carriage before the forward pulse hits the front of the carriage. The events are not simultaneous in S. The Lorentz transformations make quantitative statements about this system. We take the origin of S' to be the centre of the carriage and the flashes of light are emitted when $t' = 0$. Therefore, the flashes are emitted at $x' = 0$, $t' = 0$, which corresponds to $x = 0$, $t = 0$. We now look at each pulse arriving at the front and back.

Event 1: Pulse arrives at the back of the train. As seen by S', this event is

$$x'_1 = -L, \qquad t'_1 = \frac{L}{c} := T.$$

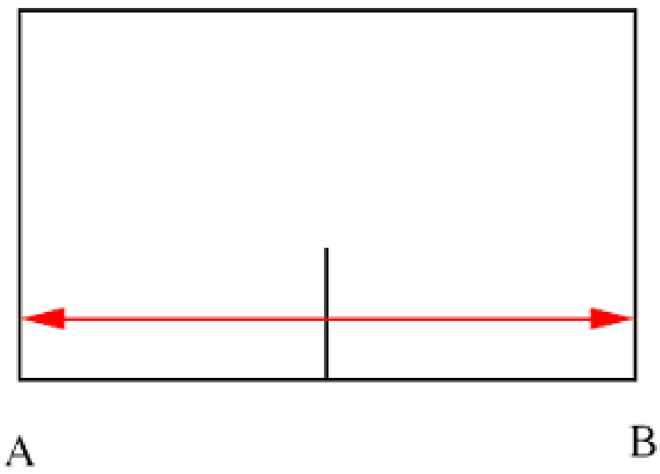


Figure 15.4: In frame S' light hits A and B simultaneously.

The time at which this event takes place as seen by S is

$$\begin{aligned} t_1 &= \gamma\left(t_1' + \frac{Vx_1'}{c^2}\right), \\ &= \gamma\left(T - \frac{VL}{c^2}\right), \\ &= \sqrt{1-\frac{V^2}{c^2}}\left(T - \frac{V}{c}T\right), \\ &= T\sqrt{\frac{1-\frac{V}{c}}{1+\frac{V}{c}}}, \end{aligned}$$

which is less than T .

Event 2: Pulse arrives at the front of the train. As seen by S' this event is

$$x_2' = L, \qquad t_2' = \frac{L}{c} = T.$$

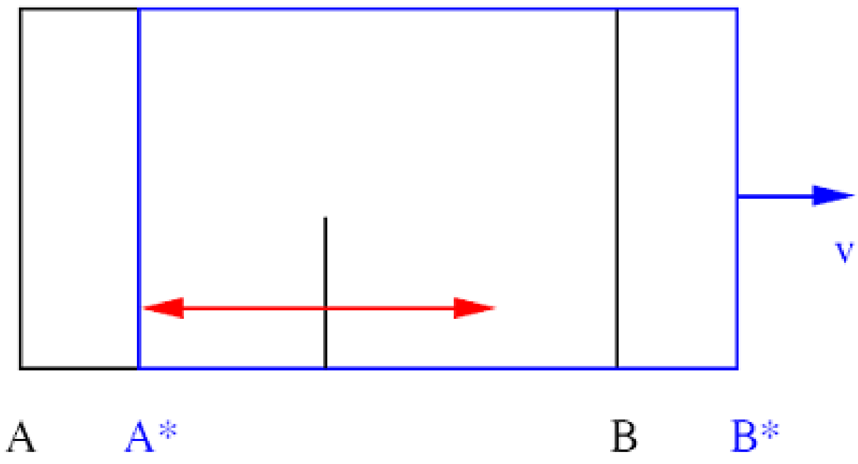


Figure 15.5: In frame S light hits back of train first.

The time at which this event takes place as seen by S is

$$\begin{aligned} t_2 &= \gamma\left(t_2' + \frac{Vx_2'}{c^2}\right), \\ &= \gamma\left(T + \frac{VL}{c^2}\right), \\ &= \sqrt{1-\frac{V^2}{c^2}}\left(T + \frac{V}{c}T\right), \\ &= T\sqrt{\frac{1+\frac{V}{c}}{1-\frac{V}{c}}}, \end{aligned}$$

which is greater than T.

Note: If we consider a very fast train then

$$V = 200\,\mathrm{km, hr}^{-1}, \qquad \frac{t_{1,2}}{T} \approx 1 \pm 1.85 \times 10^{-7},$$

which is a very small difference ! However, for $V = 0.5c$ (a very fast and very unrealistic rocket), then

$$t_1 = \frac{T}{\sqrt{3}}, \qquad t_2 = T\sqrt{3}.$$

15.5 A spacetime invariant

Although our two observers will disagree on the *when* and the *where* of events, there is a quantity that has the same value for each of them. Starting from the transformation law for an event,

$$\begin{aligned} x' &= \gamma\,(x - Vt)\,, \\ t' &= \gamma\left(t - \frac{xV}{c^2}\right), \end{aligned}$$

we calculate $c^2t'^2 - x'^2$:

$$\begin{aligned} c^2t'^2 - x'^2 &= \gamma^2\left[c^2\left(t - \frac{xV}{c^2}\right)^2 - (x - Vt)^2\right], \\ &= \gamma^2\left[\left(ct - \frac{xV}{c}\right)^2 - (x - Vt)^2\right], \\ &= \gamma^2\left[c^2t^2 - 2\frac{xV}{c}ct + \left(\frac{V}{c}\right)^2 x^2 - x^2 + 2xVct - V^2t^2\right], \\ &= \gamma^2\left[\left(c^2 - V^2\right)t^2 - x^2 + \left(\frac{V}{c}\right)^2 x^2\right], \\ &= \gamma^2\left[\left(c^2 - V^2\right)t^2 - \left(1 - \frac{V^2}{c^2}\right)x^2\right], \\ &= c^2t^2 - x^2. \end{aligned}$$

Hence, we have a quantity that is the same for all inertial observers,

$$s^2 = c^2t^2 - x^2.$$

The quantity s is a Lorentz Invariant.

Chapter 16

Length contraction and time dilation

16.1 Overview

In Chapter 15 (The Lorentz Transformations) we saw how events with given coordinates in one frame can be re-expressed in terms of other coordinates in another frame. In this chapter we look at the transformation of intervals in space and time. We then use the Lorentz Transformations to derive the transformation laws relating velocities in different frames.

16.2 Length contraction

Consider a stick at rest in frame S' lying along the x-axis. One end of the stick is at x'_A while the other is at x'_B . The length of the stick is given by

$$L_0 = x'_B - x'_A$$

and this is called the *proper length*; **i.e. the length of the stick in the frame where the stick is at rest.**

What is the length of the stick as measured by observer S? The length of the stick is the distance between the end points measured at the same time!. In other words the positions of the end points must be determined at the same time in S. If observer S determines the front of the stick to be at position x_B at time t and the back of the stick to be at x_A at the same time t then the length of the stick as measured by S is

$$L = x_B - x_A.$$

Moreover, by the Lorentz Transformations we have

$$x'_B = \gamma\left(x_B - vt\right),$$
$$x'_A = \gamma\left(x_A - vt\right).$$

Subtracting we have

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1 - \frac{v^2}{c^2}}.$$

The longest length a stick can have is when viewed in its own rest frame – in every other frame it is shorter. A stick has a shorter length the faster it travels.

Note that everything is relative – imagine observer S' measuring his carriage to be of length L_0; he is at rest relative to the carriage so this is the proper length. Observer S measures this carriage to be of length L_0/γ. Now place observer S in his own carriage which is at rest in the train station and also has proper length L_0 . Observer S' will now measure that carriage to be contracted to the length L_0/γ.

16.3 Time dilation

We have already seen two chapters ago that time intervals are not absolute. Let us make that more exact here. Consider two events occurring on the train *at the same position*:

- Event A at position x'_0 and time t'_A;
- Event B at position x'_0 and time t'_B.

The time interval between these events is known as the **proper time** since the events occur at the same position and is

$$T_0 = t'_B - t'_A.$$

In frame S we have a time interval of $T = t_B - t_A$ between these events, where

$$\begin{aligned} t_A &= \gamma\left(t'_A + \frac{vx'_0}{c^2}\right), \\ t_B &= \gamma\left(t'_B + \frac{vx'_0}{c^2}\right), \end{aligned}$$

so that

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

with the result that $T > T_0$. The shortest interval of time between two events is when they are observed in a frame where they occur at the same position, their proper frame, in all other frames the time interval is longer.

Note that is not always possible to find a proper frame for two events. This occurs when one would have to travel faster than light for the two events to occur at the same position – such events are known as space-like.

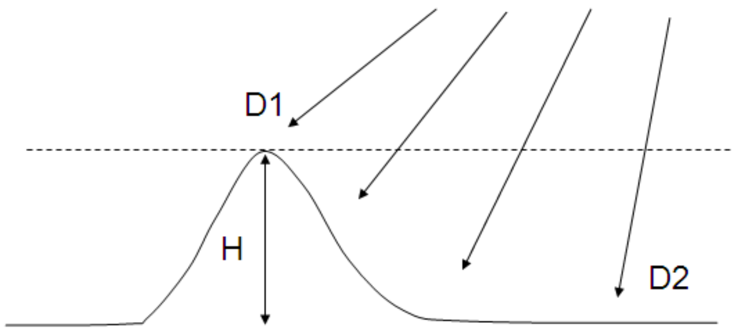


Figure 16.1: Schematic description of muon decay at two different locations.

16.3.1 Time dilation and meson decay

The muon is an unstable subatomic particle that decays into a more stable state. Muons constantly bombard us on earth through cosmic rays. The lifetime of the particles depends on the frame in which the decay events are viewed, so providing an experimental verification of time dilation.

The muon decays into an electron and a neutrino-antineutrino pair. Symbolically,

$$\mu^{\pm} \to e^{\pm} + \nu + \overline{\nu}.$$

The decay of the muons follows the law of radioactive decay: if there are N_0 particles present at time $t = 0$ then at time t, there are $N(t)$ particles, where $N(t) = N_0 e^{-t/\tau_0}$, and τ_0 is the *lifetime* of the particles. The decay time can be measured in the laboratory, and it is found that $\tau_0 = 2.15 \times 10^{-6}$ s. Now, suppose that we have two muon detectors, one on top of a mountain, and one at sea level, as in Figure 16.1. Let N_1 be the number of particles detected at the mountaintop per unit area of detector per unit time. Similarly, let N_2 be the number of particles detected at sea-level per unit area of detector per unit time. Now the muons are travelling at a velocity $v \lesssim c$. Because the particles decay in going through a distance H, and the time for this travel is $T \approx H/c$, we might expect that

$$\frac{N_2}{N_1} \stackrel{?}{=} \mathrm{e}^{-T/\tau_0} \approx e^{-H/(c\tau_0)}.$$

However, because we make the observation in a frame that is different from the muon's rest frame, the observed lifetime is dilated, and equal to

$$\tau = \gamma \tau_0$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Thus, the correct value for the flux ratio is

$$\frac{N_2}{N_1} = \mathrm{e}^{-T/\gamma\tau_0} \approx e^{-H/(\gamma c \tau_0)}.$$

Putting in a value $H = 2,000\,\mathrm{m}$ and $\gamma = 10$, we obtain

$$\frac{N_2}{N_1} = 0.7,$$

while $\mathrm{e}^{-H/(c\tau_0)} = 0.045$. Therefore, time dilation means that fewer decays happen in the frame of the earth, and we see more particles at sea level than might otherwise be expected.

16.3.2 Measurement of the Neutrino lifetime

The lifetime τ_0 of a muon at rest can be measured in a laboratory. This is done by passing the muons through a scintillator, which slows them down, see e.g.

`http://web.mit.edu/8.13/www/JLExperiments/JLExp14.pdf`

16.4 The Lorentz transformation law for velocities

Consider a particle moving with velocity u along the x-axis relative to observer S. Relative to observer S' this particle has a velocity u'. According to Galilean relativity we should have

$$u' = u - v$$

but, in the case of a photon with $u = c$ this would violate Einstein's second postulate. This law for transforming velocities must be incorrect.

16.4.1 The derivation

Recall that the definitions for velocity are

$$u = \frac{\mathrm{d}x}{\mathrm{d}t}, \qquad u' = \frac{\mathrm{d}x'}{\mathrm{d}t'}$$

and the Lorentz transformations allow us to relate space-time intervals from one frame to another. Firstly,

$$x' = \gamma\,(x - vt) \implies \mathrm{d}x' = \gamma\,(\mathrm{d}x - v\,\mathrm{d}t)\,.$$

and for time,

$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \qquad \mathrm{d}t' = \gamma\left(\mathrm{d}t - \frac{v}{c^2}\mathrm{d}x\right).$$

Formally dividing these equations now gives

$$\frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x - v\,\mathrm{d}t}{\mathrm{d}t - \frac{v}{c^2}\mathrm{d}x}.$$

Divide the RHS above and below by dt:

$$u' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{u - v}{1 - \frac{uv}{c^2}}.$$

Now, in the case that our particle is a photon so that $u = c$ we get

$$u' = \frac{c - v}{1 - \frac{v}{c}} = c,$$

consistent with Einstein's second postulate. Moreover, as usual in relativity, at normal everyday terrestrial speeds where $u, v \ll c$ the above result reduces to the Galilean limit that $u' \approx u - v$.

Example: Consider an observer that sees rocket A moving along the positive x-axis with speed $0.9c$ and rocket B moving along the negative x-axis at speed $0.9c$. What is the speed of one rocket relative to the other?

According to Galileo it would be $1.8c$ but using the correct law for adding velocities we get

$$u' = \frac{0.9c + 0.9c}{1 + 0.81} = 0.994475c,$$

which is less than the speed of light.

Notice the following important fact:

> The law for adding velocities always gives $u' < c$ when $u < c$, hence no material particle can move faster than the speed of light.

This is left as an exercise for the reader to prove.

16.5 Velocity transformations in three spatial dimensions

What if our particle were not moving purely along the x-axis, i.e. the velocity in frame S had three components

$$\boldsymbol{u} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right) = \left(u_x, u_y, u_z\right),$$

with the corresponding velocity as measured in S'

$$\boldsymbol{u}' = \left(\frac{\mathrm{d}x'}{\mathrm{d}t'}, \frac{\mathrm{d}y'}{\mathrm{d}t'}, \frac{\mathrm{d}z'}{\mathrm{d}t'}\right) = \left(u'_x, u'_y, u'_z\right),$$

We can now generalize the argument we presented above by noting that

$$\begin{aligned} \mathrm{d}x' &= \gamma\left(\mathrm{d}x - v\,\mathrm{d}t\right), \\ \mathrm{d}t' &= \gamma\left(\mathrm{d}t - \frac{v}{c^2}\mathrm{d}x\right), \\ \mathrm{d}y' &= \mathrm{d}y, \\ \mathrm{d}z' &= \mathrm{d}z. \end{aligned}$$

Straightforward division gives the transformation laws

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}, \tag{16.1a}$$

$$u'_x = \frac{u_y/\gamma}{1 - \frac{u_x v}{c^2}}, \tag{16.1b}$$

$$u'_x = \frac{u_z/\gamma}{1 - \frac{u_x v}{c^2}}. \tag{16.1c}$$

The laws for u'_y and u'_z are now substantially different from their simple Galilean counterparts. It is straightforward to show that if

$$u = \sqrt{u_x^2 + u_y^2 + u_z^2} = c,$$

then

$$u' = c$$

too, so that, as expected, a beam of light will move with speed c as measured in either frame but the direction of that beam will appear different to each observer.

Chapter 17

The Geometry of Space-Time

17.1 Minkowski Space

We begin by re-writing the Lorentz Transformations (Chapter 15) in matrix form:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} \\ -\gamma\frac{v}{c} & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \tag{17.1}$$

From now on, we place space and time on an equal footing by taking $ct = X^0$ and $x = x^1 = X^1$; if there are other spatial dimensions under consideration, then we write them as $y = x^2 = X^2$ and $z = x^3 = X^3$. The convention here therefore is to view X as a 4-tuple:

$$X^a = (X^0, X^1, X^2, X^3), \qquad a = 0, 1, 2, 3.$$

There are two more conventions that we have to familiarize ourselves with in Special Relativity:

- The index a is allowed to range from $a = 0$ to $a = 3$, whereas indices i, j, etc. are typically taken to range from $i = 1$ to $i = 3$, etc.
- The variables X^i and x^i are taken to mean the same thing, for $i = 1, 2, 3$.

Thus, it is common to write X^a in many different, but equivalent forms:

$$\begin{aligned} X^a &= (ct, x, y, z), \\ &= (ct, x^1, x^2, x^3), \\ &= (X^0, X^1, X^2, X^3). \end{aligned}$$

In any case, now Equation (17.1) can be viewed as a linear transformation in a four-dimensional space:

$$\begin{pmatrix} (X^0)' \\ (X^1)' \\ (X^2)' \\ (X^3)' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma(v/c) & 0 & \\ -\gamma(v/c) & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 01 & \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix}. \tag{17.2}$$

Definition 17.1 *The four-dimensional space in which different frames of reference are connected via transformations such as Equation* (17.2) *is called* **Space-Time** *or* **Minkowski Space**.

Clearly, Minkowski space allows for Lorentz formations with velocities pointing in other directions other than the x-direction – this is handled by modifying the matrix in Equation (17.2) in an obvious fashion. This motivates the following definition of four-vectors:

Definition 17.2 *A* **four-vector** *is correspondingly defined to be any four-tuple* $(z^0, z^1, z^2, z^3) \in \mathbb{R}^4$ *that transforms under a linear transformation such as Equation* (17.2) *under a change of inertial frame.*

For now however it suffices to focus on one space and one time dimension, as the geometry of Minkowski space becomes clear in that context.

As such, in Equation (17.1) we introduce $\gamma = 1/\sqrt{1-\beta^2}$, with $\beta = v/c$. Since $1 < \gamma < \infty$, we define γ to be equal to $\cosh \alpha$, where α is a real number. We have $\cosh^2 \alpha - \sinh^2 \alpha = 1$, hence

$$\sinh \alpha = \sqrt{\cosh^2 \alpha - 1} = \sqrt{\frac{1}{1-\beta^2} - 1} = \beta\gamma = (v/c)\gamma.$$

Notice also, $\tanh \alpha = v/c$, with $\tanh \alpha \in (-1, 1)$, hence

$$\tanh \alpha = v/c \iff \alpha = \tanh^{-1}(v/c).$$

Hence, Equation (17.1) can be re-written as:

$$\begin{pmatrix} (X^0)' \\ (X^1)' \end{pmatrix} = \begin{pmatrix} \cosh \alpha & -\sinh \alpha \\ -\sinh \alpha & \cosh \alpha \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \end{pmatrix} \tag{17.3}$$

The inverse transformation is also readily worked out:

$$\begin{pmatrix} X^0 \\ X^1 \end{pmatrix} = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} \begin{pmatrix} (X^0)' \\ (X^1)' \end{pmatrix} \tag{17.4}$$

17.2 The Lorentz Group

We call the set

$$\mathcal{L} = \{\mathbf{L} \in \mathbb{R} \in \mathbb{R}^{2\times 2} \,\big|\, \mathbf{L} = \begin{pmatrix} \cosh a & -\sinh a \\ -\sinh a & \cosh a \end{pmatrix}, \, a \in \mathbb{R}\} \tag{17.5}$$

the set of Lorentz matrices. Using trigonometric identities, it can be shown that

$$\mathbf{L}_1, \mathbf{L}_2 \in \mathcal{L} \implies \mathbf{L}_1\mathbf{L}_2 \in \mathcal{L}.$$

Specifically,

$$\begin{aligned} \mathbf{L}_1, \mathbf{L}_2 = &\begin{pmatrix} \cosh\alpha' & -\sinh\alpha' \\ -\sinh\alpha' & \cosh\alpha' \end{pmatrix}\begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cosh\alpha'\cosh\alpha + \sinh\alpha'\sinh\alpha & -\cosh\alpha'\sinh\alpha - \sinh\alpha'\cosh\alpha \\ -\cosh\alpha'\sinh\alpha - \sinh\alpha'\cosh\alpha & \cosh\alpha'\cosh\alpha + \sinh\alpha'\sinh\alpha \end{pmatrix}, \end{aligned}$$

and this new matrix is also in $\mathcal{L}$. This means that the set $\mathcal{L}$ is **closed** under matrix multiplication. We also have:

- $\mathbb{I} \in \mathcal{L}$ (take $a = 0$ in Equation (17.5)).
- Let $\mathbf{L} \in \mathcal{L}$, such that
$$\mathbf{L} = \begin{pmatrix} \cosh a & -\sinh a \\ -\sinh a & \cosh a \end{pmatrix}.$$
  Then (Section 17.1),
$$\mathbf{L}^{-1} = \begin{pmatrix} \cosh a & \sinh a \\ \sinh a & \cosh a \end{pmatrix}$$
  is also in $\mathcal{L}$.

These properties (closure, identity element, existence of inverses), together with the associativity of matrix multiplication make the set $\mathcal{L}$ into a group under matrix multiplication. This is called the **Lorentz Group**.

Clarification

Properly speaking, $\mathcal{L}$ is a subgroup of the Lorentz group of 4×4 matrices which preserve the indefinite length $c^2t^2 - x^2 - y^2 - z^2$ under transformations. Two-by-two matrices in $\mathcal{L}$ can be embedded in an obvious way into this group of 4×4 matrices. For now, it suffices to focus on the 2×2 case to get a good understanding of the geometry of spacetime. Also, manipulations of 2×2

matrices are easy compared with manipulations of 4×4 matrices! We do however take tentative first steps into the theory of 4×4 Lorentz matrices in Section 17.6, below.

17.3 Proper Time Invariant

We have already seen in Chapter 15 that the quantity $s^2 = c^2t^2 - x^2$ is invariant, i.e. the same in all inertial frames of reference. Since Lorentz transformations are linear, it follows that differences between coordinates are invariant under such transformations, hence:

$$c^2\Delta t^2 - \Delta x^2 = c^2\Delta t'^2 - \Delta x'^2,$$

for space- and time-intervals in different frames.

We take $\Delta x' = 0$ (object at rest in S') and take the intervals Δt, Δx, and $\Delta t'$ to be very small, such that $\Delta t' \to \mathrm{d}\tau$. This gives:

$$\mathrm{d}\tau = \sqrt{\mathrm{d}t^2 - \frac{1}{c^2}\mathrm{d}x^2}. \tag{17.6}$$

This is the proper time interval, and it is the infinitesimal time interval as measured in a frame in which the object in question is at rest.

17.4 4-Velocity

From Chapter 16, we know that the ordinary velocity in frame S is

$$\boldsymbol{u} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}x^1}{\mathrm{d}t}, \frac{\mathrm{d}x^2}{\mathrm{d}t}, \frac{\mathrm{d}x^3}{\mathrm{d}t}\right) = \left(u_x, u_y, u_z\right),$$

It is tempting to make this into a four-vector in Minkowski space by taking

$$\left(c, \frac{\mathrm{d}x^1}{\mathrm{d}t}, \frac{\mathrm{d}x^2}{\mathrm{d}t}, \frac{\mathrm{d}x^3}{\mathrm{d}t}\right)$$

to be the four-velocity. However, the components x^i, with $i = 1, 2, 3$ transform according to a Lorents transformation, as does $\mathrm{d}t$. Therefore, the quantity $\mathrm{d}x^i/\mathrm{d}t$ transforms according to some mixture of Lorentz transformations for the numerator and denominator. The overall transformation can not however be a Lorentz transformation. Therefore, in order to make a four-velocity that transforms according to Equation (17.2), we must take it to be:

$$V^a = \left(\frac{\mathrm{d}X^0}{\mathrm{d}\tau}, \frac{\mathrm{d}X^1}{\mathrm{d}\tau}, \frac{\mathrm{d}X^2}{\mathrm{d}\tau}, \frac{\mathrm{d}X^3}{\mathrm{d}\tau}\right), \qquad a = 0, 1, 2, 3, \tag{17.7}$$

where we do the time derivative with respect to the proper time. Note also that we use $X^i = x^i$ interchangeably, for $i = 1, 2, 3$. Equation (17.7) is the (relativistic) four-velocity. Correspondingly, the relativistic four-momentum is:

$$P^a = mV^a.$$

We have $P^a = m\mathrm{d}X^a/\mathrm{d}\tau$, hence

$$P^a = m\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}X^a}{\mathrm{d}t} = m\frac{\mathrm{d}t}{\mathrm{d}\tau}\left(c, \frac{\mathrm{d}x^i}{\mathrm{d}t}\right). \tag{17.8}$$

We now evaluate $\mathrm{d}t/\mathrm{d}\tau$. We start with the case of $1+1$ dimensions, where we have (from Equation (**??**)):

$$\begin{aligned} \mathrm{d}\tau &= \sqrt{\mathrm{d}t^2 - (1/c^2)\mathrm{d}x^2}, \\ &= \mathrm{d}t\sqrt{1 - \frac{1}{c^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}, \\ &= \mathrm{d}t\sqrt{1-(v^2/c^2)}, \end{aligned}$$

hence

$$\mathrm{d}t = \mathrm{d}\tau/\sqrt{1-(v^2/c^2)} = \gamma\,\mathrm{d}\tau. \tag{17.9}$$

The extension to $1+3$ diemnsions is straightforward – there, $\mathrm{d}\tau$ has the same form as in Equation (17.9) but v^2 now means the same things as $\boldsymbol{v}^2$, where $\boldsymbol{v}$ is the three spatial components of the velocity. In any case, combining Equations (17.8) and (17.9) now gives:

$$P^a = \gamma m\left(c, \frac{\mathrm{d}x^i}{\mathrm{d}t}\right) \tag{17.10}$$

Depending on the application / problem at hand, we often also write P^a in various other forms:

$$P^a = \gamma\left(c, \frac{\mathrm{d}x^i}{\mathrm{d}t}\right) = \gamma m(c, \boldsymbol{v}) = (\gamma mc, \boldsymbol{p}).$$

Here, we have identified $\boldsymbol{p} = \gamma m\boldsymbol{v}$ with the spatial components of the momentum 4-vector.

17.5 Euclidean versus Minkowski space

Given the form of the transformation law (17.2), it is very tempting (and right) to make an analogy between rotations in Euclidean space and Lorentz transformations ('boosts') in Minkowski space. This is done in Table 17.1. Although the mathematical structure of Euclidean and Minkowski space is the same, there is one key difference: the metric tensor in Minkowski space is not positive definite. Thus, while $\boldsymbol{x}\cdot\boldsymbol{x} = \sum_{i=1}^{2}\delta_{ij}x^ix^j = x^2+y^2 \geq 0$ for all $\boldsymbol{x}$ in Euclidean space, it is not the case that $\eta(X,X) = (X^0)^2 - \sum_i \delta_{ij}x^ix^j = (X_0)^2 - \boldsymbol{x}^2$ is positive definite – it can have positive or negative sign, or be zero.

We may now explain why we put the index on X^a upstairs. An vector with the index upstairs is called

	Euclidean	Minkowski
Vector	$\boldsymbol{x} = (x, y)$	$X^a = (X^0, X^1)$
Transformation	Rotation	Boost / Lorentz Transformation
Transformation Law:	$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	$\begin{pmatrix} (X^0)' \\ (X^1)' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \end{pmatrix}$
Invariant Quantity	The Euclidean Length (squared) $x^2 + y^2 = \sum_{i,j=1}^{2} \delta_{ij} x^i x^j$ $x^1 = x,\ x^2 = y$ $\delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	The indefinite length $(ct)^2 - x^2 = \sum_{a,b=0}^{1} \eta_{ab} X^a X^b$ $X^0 = ct,\ X^1 = x$ $\eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Metric tensor:	δ_{ij}	η_{ab}
Scalar Product:	$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i,j=1}^{2} \delta_{ij} x^i y^j$	$\eta(X, Y) = \sum_{a,b=0}^{1} \eta_{ab} X^a Y^b$

Table 17.1: Analogy between Euclidean space and Minkowski space

a **contravariant vector**. The corresponding covariant vector (or vector with the index downstairs) is obtained by **lowering the index** by **contraction** with the metric tensor. In other words,

$$X_a = \sum_b \eta_{ab} X^b.$$

For a four-vector $X^a = (X^0, x, y, z)$, this gives

$$X_a = (X^0, -x, -y, -z).$$

Notice that in Euclidean space, there is not much difference between co-variant and contra-variant vectors: if $\boldsymbol{x} = (x, y, z)$ is a contra-variant Euclidean vector with components x^i, then the corresponding co-variant vector can be obtained by contraction with the metric tensor δ_{ij}:

$$x_i = \sum_j \delta_{ij} x^j,$$

hence $x_i = x^i$.

Finally, notice that we are using the convention

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This is the 'Particle Physics convention'. The convention in General Relativity is to have the signs on the diagonal of the metric tensor the other way around – see e.g. Figure 17.1.

The choice of metric signature

In general, but with several exceptions, mathematicians and general relativists prefer spacelike vectors to yield a positive sign, $(-+++)$, while particle physicists tend to prefer timelike vectors to yield a positive sign, $(+---)$. Authors covering several areas of physics, e.g. Steven Weinberg and Landau and Lifshitz ($(-+++)$ and $(+---)$ respectively) stick to one choice regardless of topic. Arguments for the former convention include "continuity" from the Euclidean case corresponding to the non-relativistic limit $c \to \infty$. Arguments for the latter include that minus signs, otherwise ubiquitous in particle physics, go away. Yet other authors, especially of introductory texts, e.g. Kleppner & Kolenkow (1978), do *not* choose a signature at all, but instead opt to coordinatize spacetime such that the time *coordinate* (but not time itself!) is imaginary. This removes the need of the *explicit* introduction of a metric tensor (which may seem as an extra burden in an introductory course), and one needs *not* be concerned with covariant vectors and contravariant vectors (or raising and lowering indices) to be described below. The inner product is instead effected by a straightforward extension of the dot product in $\mathbb{R}^3$ to $\mathbb{R}^3 \times \mathbb{C}$. This works in the flat spacetime of special relativity, but not in the curved spacetime of general relativity, see Misner, Thorne & Wheeler (1973, Box 2.1, Farewell to *ict*) (who, by the way use $(-+++)$). MTW also argues that it hides the true *indefinite* nature of the metric and the true nature of Lorentz boosts, which aren't rotations. It also needlessly complicates the use of tools of differential geometry that are otherwise immediately available and useful for geometrical description and calculation – even in the flat spacetime of special relativity, e.g. of the electromagnetic field.

Figure 17.1: The different conventions concerning the signature of the metric tensor – from `https://en.wikipedia.org/wiki/Minkowski_space`.

17.6 Algebriac treatment of Lorentz invariance

We embed the one-space-dimension-and-one-time-dimension Lorentz transformations of the form (17.3) into $\mathbb{R}^{4\times 4}$ by looking at matrices of the form:

$$\Lambda = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{17.11}$$

such that the upper left 2×2 block corresponds to a matrix in the set $\mathcal{L}$. As such, Λ in Equation (17.11) is a linear map

$$\begin{aligned} \Lambda : \mathbb{R}^4 &\to \mathbb{R}^4, \\ X &\mapsto \Lambda X, \end{aligned}$$

where we use the index conventions that

$$X' = \Lambda X, \qquad (X')^a = \sum_b \Lambda^a_b X^b,$$

such that the components of Λ are denoted by Λ^a_b.

We have the following definition:

Definition 17.3 *A Lorentz matrix is a matrix $\Lambda \in \mathbb{R}^{4\times 4}$ such that*

$$\Lambda^T \boldsymbol{\eta} \Lambda = \boldsymbol{\eta}, \tag{17.12}$$

where $\boldsymbol{\eta}$ is the metric tensor with entries $(+---)$ along the diagonal and zeros everywhere else.

We have the following theorem:

Theorem 17.1 *Matrices of the form* (17.11) *are Lorentz matrices.*

We prove this theorem by direct computation:

$$
\begin{aligned}
\Lambda^T\boldsymbol{\eta}\Lambda &= \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
&= \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ \sinh\alpha & -\cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
&= \begin{pmatrix} \cosh^2\alpha-\sinh^2\alpha & 0 & 0 & 0 \\ 0 & -\sinh^2\alpha-\cosh^2\alpha & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
&= \boldsymbol{\eta}.
\end{aligned}
$$

We now state and prove the theory of Lorentz invariance in a proper mathematical way:

Theorem 17.2 *The inner product of any two vectors in space-time is the same in all inertial frames:*

$$\eta(X, Y) = \eta(X', Y'),$$

where $X' = \Lambda X$ and $Y' = \Lambda Y$, and where Λ is a Lorentz matrix.

Proof:

$$
\begin{aligned}
\eta(X', Y') &= \eta(\Lambda X, \Lambda Y), \\
&= \sum_{ab} (\Lambda X)^a \, (\Lambda Y)^b \, \eta_{ab}.
\end{aligned}
$$

Continue thus:

$$\begin{aligned}\eta(X',Y') &= \sum_{ab}\left(\sum_c \Lambda^a_c X^c\right)\left(\sum_d \Lambda^b_d Y^d\right)\eta_{ab},\\ &= \sum_{cd} X^c Y^d\left(\sum_{ab}\Lambda^a_c\Lambda^b_d\eta_{ab}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\eta(X',Y') &= \sum_{cd} X^c Y^d\left(\sum_{ab}(\Lambda^T)^c_a\eta_{ab}\Lambda^b_d\right),\\ &= \sum_{cd} X^c Y^d\left(\Lambda^T\boldsymbol{\eta}\Lambda\right)_{cd},\\ &= \sum_{cd} X^c Y^d\eta_{cd},\\ &= \eta(X,Y).\end{aligned}$$

This concludes the proof.

17.6.1 Clarification

The set of all Lorentz matrices $\Lambda \in \mathbb{R}^{4\times 4}$ includes boosts in different directions. In this module we focus only on boosts in the x-direction. As such, we have looked at Lorentz matrices that consist of a boost in only the x-direction. It should be noted for future reference (e.g. ACM 40010) that there are other more general matrices for which Equation (17.12) also holds.

17.6.2 Analogue in Euclidean space

Notice that the analogue of $\eta(X,Y)=\eta(X',Y')$ in Euclidean space is the invariance of the dot product under rotations: $\boldsymbol{x}\cdot\boldsymbol{y}$ is the same before and after a rotation. Indeed, since the metric tensor in Euclidean space is just δ_{ij}, the analogue of Equation (17.12) in Euclidean space is just

$$\mathbf{R}^T\mathbf{R}=\mathbb{I},\qquad \mathbf{R}\in\mathbb{R}^{3\times 3}$$

which is exactly the definition of a rotation matrix.

17.7 The Light Cone

We look at $\eta(X,X)=(X^0)^2-|\boldsymbol{x}|^2$, where $|\boldsymbol{x}|^2=\sum_{i=1}^3 X^iX^i$.

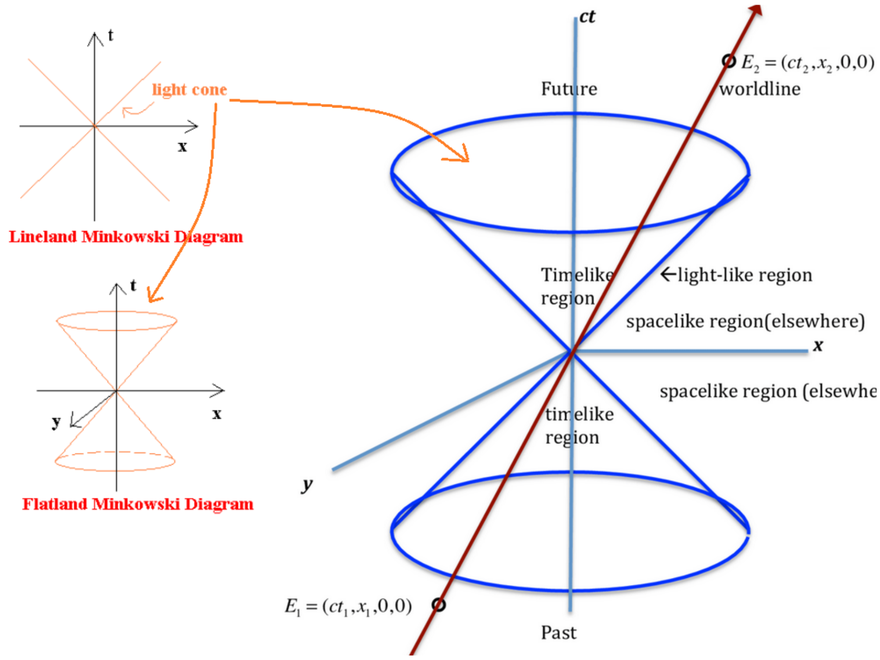


Figure 17.2: The light cone in Minkowski space

- If $\eta(X, X) > 0$, then $c^2t^2 - |\boldsymbol{x}|^2 > 0$, hence $|\boldsymbol{x}|/t < c$ – then, the vector X is time-like and there is a causal connection between an event at the origin in space-time and an event at X – **time-like**.
- If $\eta(X, X) = 0$, then $ct = |\boldsymbol{x}|$ and the vector X corresponds to a pulse of light emanating from the origin in space-time – **light-like**
- If $\eta(X, X) < 0$, then $c^2t^2 - |\boldsymbol{x}|^2 < 0$, hence $|\boldsymbol{x}|/t >$ – then, there is no causal connection between an event at the origin in space-time and an event at X – **space-like**.

Thus, $\eta(X, X) = 0$ defines the **light cone** in Minkowski space – see Figure 17.2.

Chapter 18

Relativistic momentum and energy

18.1 Overview

In this chapter we generalize the notions of momentum and kinetic energy in such a way that these concepts become relativistically correct. These quantities can then be used to solve a variety of *kinematic* problems, involving collisions between particles.

Recall the contrasting notions of kinematics and dynamics. Kinematics is the study of motion; dynamics is the study of changes to motion. In kinematics we simply want to describe motion (energy, velocity, momentum), without being interested in changes in this motion.

18.2 Relativistic momentum

It turns out that momentum is still conserved under special relativity. That is, when two bodies interact, the total momentum is conserved, provided that the net external force acting on the bodies in an inertial reference frame is zero.

However, in Special Relativity, the momentum is **not** $\boldsymbol{p} = m\boldsymbol{v}$. We have already seen in Chapter 17 that the momentum P^a is a four-vector, with $P^a = (\gamma mc, \boldsymbol{p})$, and

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1-(v/c)^2}} := \gamma m\boldsymbol{v}.$$

Example: An electron (rest mass $9.11 \times 10^{-31}\,\mathrm{kg}$ is moving at velocity v in a linear accelerator. Find the momentum of the electron if (a) $v = 0.01c$; (b) $v = 0.99c$.

Now $p_{\text{Case } a} = \gamma_a m v_1 = (1-0.01^2)^{-1/2}\,(9.11 \times 10^{-31})\,(0.01 \times 3.00 \times 10^8) = 2.7 \times 10^{-24}\,\mathrm{kgm/s}$.

Similarly, $p_{\text{Case } b} = \gamma_b m v_b = (1-0.99^2)^{-1/2}\,(9.11 \times 10^{-31})\,(0.99 \times 3.00 \times 10^8) = 1.9{\times}10^{-21}\,\mathrm{kgm/s}$.

18.3 Relativistic kinetic energy

Recall the energy framework we built up in ordinary Newtonian mechanics. The kinetic energy was identified with a change in momentum through the work-energy relation:

$$K_2 - K_1 = \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{x}.$$

We therefore introduce Kinetic Energy in an analogous fashion in the Relativistic case:

$$\begin{aligned}
K_2 - K_1 &= \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{x}. \\
&= \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{m\boldsymbol{v}}{\sqrt{1-(v/c)^2}} \right] \cdot \mathrm{d}\boldsymbol{x}, \\
&= \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{m\boldsymbol{v}}{\sqrt{1-(v/c)^2}} \right] \cdot \boldsymbol{v}\,\mathrm{d}t, \\
&= \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \boldsymbol{v} \cdot \mathrm{d} \left[\frac{m\boldsymbol{v}}{\sqrt{1-(v/c)^2}} \right].
\end{aligned}$$

But the integrand is $\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{p} = \mathrm{d}\left(\boldsymbol{v} \cdot \boldsymbol{p}\right) - \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{v}$ (integration by parts). Hence,

$$\begin{aligned}
K_2 - K_1 &= \left(\boldsymbol{v} \cdot \boldsymbol{p}\right)\Big|_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} - \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{v}, \\
&= \left.\frac{mv^2}{\sqrt{1-v^2/c^2}}\right|_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} - \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \frac{mvdv}{\sqrt{1-v^2/c^2}},
\end{aligned}$$

where we have used $\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{v} = v\mathrm{d}v$. The integral can be looked up in a table, and we find

$$K_2 - K_1 = \left.\frac{mv^2}{\sqrt{1-v^2/c^2}}\right|_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} + mc^2\sqrt{1-v^2/c^2}\Big|_{\boldsymbol{x}_1}^{\boldsymbol{x}_2}$$

(note the sign change in the second term!). If we take the point $\boldsymbol{x}_2$ to be arbitrary, and let the particle be at rest at $\boldsymbol{x}_1$, $\boldsymbol{v}\left(\boldsymbol{x}_1\right) = 0$, then

$$\begin{aligned}
K &= \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} - mc^2, \\
&= \frac{m\left[v^2 + c^2\left(1 - v^2/c^2\right)\right]}{\sqrt{1-v^2/c^2}} - mc^2, \\
&= \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2,
\end{aligned}$$

hence

$$K = (\gamma - 1)\, mc^2. \tag{18.1}$$

This result bears little resemblance to its Newtonian analogue, $K = mv^2/2$. However, if we expand γ in small v/c, we obtain

$$\gamma = 1 + \tfrac{1}{2}\frac{v^2}{c^2} + \tfrac{3}{8}\frac{v^4}{c^4} + \cdots .$$

Keeping only terms to second order on v/c, we obtain

$$K = (\gamma - 1)\, mc^2 = \tfrac{1}{2}mv^2 + v^2 O\left(v^2/c^2\right).$$

Equation (18.1) for the kinetic energy – the energy due to the motion of the particle – contains an energy term $mc^2/\sqrt{1 - v^2/c^2}$ that depends on the motion and a second energy term mc^2 that is independent of the motion. It seems that the kinetic energy of a particle is the difference between some **total energy** E and an energy mc^2 that the particle has when it is at rest. Thus, we re-write Eq. (18.1) as

$$E = K + mc^2 = \frac{mc^2}{1 - v^2/c^2} = \gamma mc^2. \tag{18.2}$$

For a particle at rest ($K = 0$), we see that $E = mc^2$. The energy mc^2 associated with rest mass m rather than motion is called the **rest energy** of the particle. The total energy E is conserved in time-independent processes (conservation of mass-energy). Thus, in some processes, neither the sum of the rest masses of the particles nor the total energy (other than the rest energy) is separately conserved, but rather the sum implied in Equation (18.2) is conserved.

The energy and the momentum can be related together as follows. We re-write the formula for E as

$$\left(\frac{E}{mc^2}\right)^2 = \frac{1}{1 - v^2/c^2},$$

and the formula for p as

$$\left(\frac{p}{mc}\right)^2 = \frac{v^2/c^2}{1 - v^2/c^2}.$$

Subtracting these equations gives

$$E^2 - (pc)^2 = \left(mc^2\right)^2. \tag{18.3}$$

This might be the most important equation of all in classical physics. Just as $x^2 - c^2t^2$ is an *invariant* under the Lorentz transformations, so too is $E^2 - p^2c^2$, since this difference is a constant in all frames.

We can also now derive Equation (18.3) using techniques from the theory of four-vectors. Since $P^0 = \gamma mc$ (Chapter 17) and $E = \gamma mc^2$ (Equation (18.2)), we can write $P^0 = E/c$, hence

$$P^a = \left(\frac{E}{c}, \boldsymbol{p}\right).$$

Since P^a is a four-vector, $\sum_{a,b} \eta_{ab} P^a P^b$ is invariant under Lorentz transformations:

$$(P^0)^2 - \boldsymbol{p}^2 = \text{Invariant}.$$

In particular, the value of $(P^0)^2 - \boldsymbol{p}^2$ must be equal to its value in the rest frame ($\boldsymbol{p} = 0$), where it is equal to $(mc)^2$, hence

$$(P^0)^2 - \boldsymbol{p}^2 = (mc)^2.$$

But $P^0 = E/c$, hence

$$(E/c)^2 - \boldsymbol{p}^2 = (mc^2),$$

which is equivalent to Equation (18.3).

Example: Two protons (each with mass $M = 1.67 \times 10^{-27}\,\mathrm{kg}$ are initally moving with equal speeds in opposite directions. They continue to exist after a head-on collision that also produces a neutral pion of mass $m = 2.40 \times 10^{-28}\,\mathrm{kg}$. If the protons and the pion are at rest after the collision, find the initial speed of the protons. Energy is conserved in this collision.

Solution: Try conservation of momenta: $P_{\text{init}} = \gamma M\,(v - v) = 0$, and $P_{\text{fin}} = 0$, since the particles emerge from the collision at rest. This gives no information. So we turn to conservation of energy. That is, the total energy $E = \sum_i \gamma_i m_i c^2$ is conserved. Initially, $E = 2\gamma M c^2$. After the collision, the energy is pure rest energy and is equal to $E = 2Mc^2 + mc^2$. Equating gives

$$\gamma = 1 + \frac{m}{2M},$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the gamma-factor of the initial proton pair. Inverting for v/c gives

$$\frac{v}{c} = \left[1 - \left(1 + \frac{m}{2M}\right)^{-2}\right]^{1/2}.$$

Plugging in the numerical values gives

$$v = 0.360c.$$

18.4 Photons

There are two complementary ways of viewing light (electromagnetic radiation). In one picture, light is a wave, which satisfies a wave equation (Maxwell's equations). In the other picture, light behaves like mass-zero particles called *photons*. Both pictures are needed for a full description of light-related phenomena. Although these pictures are complementary, they are unified on the level of quantum field theory (Quantum Electrodynamics). Whenever a particle picture of light is needed, SR provides the framework for this description.

Here are some relations that relate the particle nature of light to its wave nature: In this table,

Mass of a photon	$m = 0$
SR formula for photon energy	$E = pc$
Quantum mechanics formula for photon energy	$E = h\nu$, $\nu =$light frequency
Equating the energy formulas	$E = h\nu = pc \implies p = h\,(\nu/c) = h/\lambda.$

Table 18.1: Properties of the photon

$$h = 6.62607015 \times 10^{-34}\,\mathrm{m^2 kg/s} = 6.62607015 \times 10^{-34}\mathrm{J \cdot s}$$

is Planck's constant, a fundamental constant of quantum mechanics, and $\lambda = \nu/c$ is the wavelength of the light.

Notice that the equation $E = pc$ for a photon comes from the four-vector equation $\eta_{ab}P^aP^b = (mc)^2$. Since $m = 0$ for a photon, we must have $\eta_{ab}P^aP^b = 0$, hence $(E/c)^2 = \boldsymbol{p}^2$, hence $E = pc$, where $p = |\boldsymbol{p}|$.

18.5 Compton scattering

Consider very energetic photons of energy $E = h\nu$ incident on a target. Each photon strikes an electron and sets the electron into motion. During the collision, the photon's momentum changes. If $\lambda = c/\nu$ is the wavelength of a photon, compute the wavelength of the scattered photons. See Figure 18.1 for the schematic diagram.

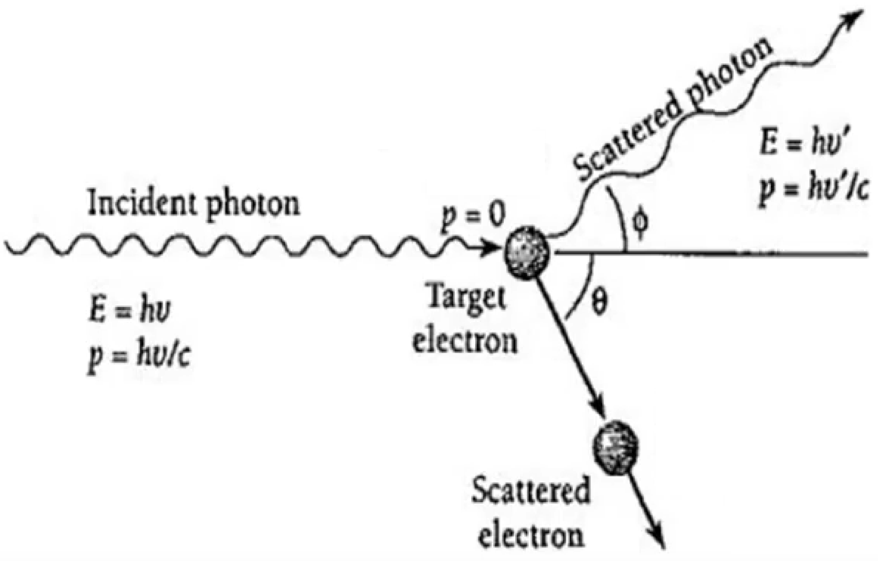


Figure 18.1: Schematic diagram of Compton Scattering

Solution: Schematically, the process we are considering is this:

$$[\text{Photon}] + [\text{Electron at rest}] \to [\text{Scattered photon}] + [\text{Electron in motion}]\,.$$

We use the principles of conservation and energy here. This can be viewed on the level of four-vectors:

$$\sum P^a_{\text{initial}} = \sum P^a_{\text{final}}, \tag{18.4}$$

where the sum is over all the particles (photon plus electron). The $a = 0$ component of this principle corresponds to conservation of energy. As such, we focus first on the initial energy:

$$pc + m_\text{e}c^2 = h\nu + m_\text{e}c^2,$$

and on the final energy:

$$p'c + \gamma mc^2 = h\nu' + \gamma m_\text{e}c^2.$$

We are going to call the electron energy $\gamma m_\text{e}c^2$ E. Thus,

$$h\left(\nu - \nu'\right) + m_\text{e}c^2 = E.$$

We can square this if we want to:

$$\left[h\left(\nu - \nu'\right) + m_\text{e}c^2\right]^2 = E^2.$$

We now look at $a = i = 1, 2, 3$ in Equation (18.4). The initial momentum is $(h\nu/c)\,\hat{\boldsymbol{x}}$. The final momentum is $\boldsymbol{p} + (h\nu'/c)\,\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is just some unit vector. By conservation of momentum,

$$(h\nu/c)\,\hat{\boldsymbol{x}} = \boldsymbol{p} + (h\nu'/c)\,\hat{\boldsymbol{n}}.$$

Re-arrange this:

$$h\nu\hat{\boldsymbol{x}} - h\nu'\hat{\boldsymbol{n}} = \boldsymbol{p}c.$$

Squaring gives

$$p^2c^2 = (h\nu)^2 + (h\nu')^2 - 2\,(h\nu)\,(h\nu')\cos\phi,$$

where $\cos\phi = \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{n}}$ is an angle. Now $E^2 = \left(mc^2\right)^2 + (pc)^2$, and we have

$$\begin{aligned} E^2 &= \left[h\left(\nu - \nu'\right) + m_\text{e}c^2\right]^2, \\ (pc)^2 &= (h\nu)^2 + (h\nu')^2 - 2\,(h\nu)\,(h\nu')\cos\phi. \end{aligned}$$

Hence,

$$\left[h\left(\nu - \nu'\right) + m_\text{e}c^2\right]^2 = \underline{\left(mc^2\right)^2} + (h\nu)^2 + (h\nu')^2 - 2\,(h\nu)\,(h\nu')\cos\phi$$

Expand the LHS:

$$\begin{aligned} &(h\nu)^2 + (h\nu')^2 + \left(m_\text{e}c^2\right)^2 + 2\,(h\nu)\,m_\text{e}c^2 - 2\,(h\nu')\,m_\text{e}c^2 - 2\,(h\nu)\,(h\nu') = \\ &\qquad\qquad \left(mc^2\right)^2 + (h\nu)^2 + (h\nu')^2 - 2\,(h\nu)\,(h\nu')\cos\phi \end{aligned}$$

Effecting cancellations and tidying up gives

$$\frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{mc}\left(1 - \cos\phi\right).$$

But $c = \lambda\nu$ and $\lambda = c/\nu$ for waves, so

$$\lambda' - \lambda = \frac{h}{mc}\left(1 - \cos\phi\right). \tag{18.5}$$

Example: X-rays emanate from a source have a wavelength $\lambda = 0.124\,\mathrm{nm}$ and are Compton-scattered off a target. At what scattering angle ϕ is the wavelength of the scattered X-rays 1.0% longer than that of the incident X-rays?

Solution: In Equation 18.5 we want $\Delta\lambda := \lambda' - \lambda$ to be 1.0% of $0.124\,\mathrm{nm}$. That is, $\Delta\lambda = 0.00124\,\mathrm{nm} = 1.24 \times 10^{-12}\mathrm{m}$. Using the value $h/mc = 2.426 \times 10^{-12}\,\mathrm{m}$, we find

$$\Delta\lambda = \frac{h}{mc}\left(1 - \cos\phi\right),$$

or

$$\cos\phi = 1 - \frac{\Delta\lambda}{(h/mc)} = 1 - \frac{1.24 \times 10^{-12}}{2.426 \times 10^{-12}} = 0.4889.$$

Hence, $\phi = 60.7°$.

Chapter 19

Relativisitic Billiard Balls

19.1 Overview

In this chapter we look at the collision problem in Figure 19.1 in the relativistic setting. We aim to compute the scattering angles θ_1 and θ_2 from the principle of conservation of mass-energy. The relationship between θ_1 and θ_2 should depend on the incident velocity u alone.

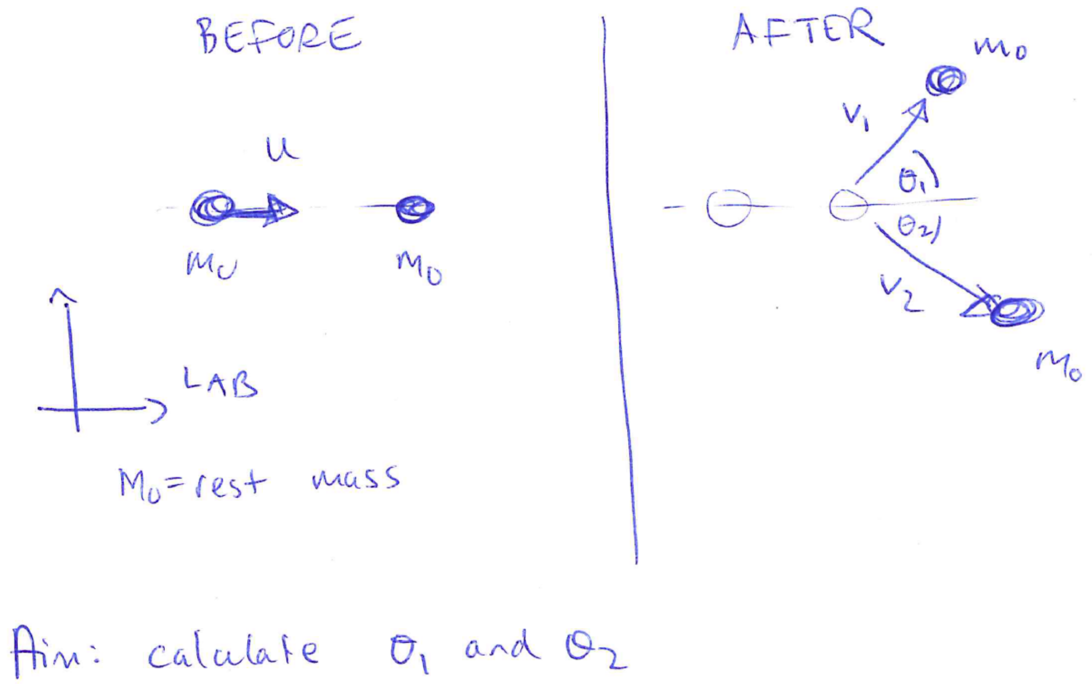


Figure 19.1: Schematic diagram for the collision problem of the 'relativistic billiard balls'

19.2 Background

In Newtonian mechanics, we use

$$E_{\text{initial}} = \tfrac{1}{2}m_0 u^2 = E_{\text{fin}} = \tfrac{1}{2}m_0 v_1^2 + \tfrac{1}{2}m_0 v_2^2$$

and $\boldsymbol{p}_{\text{initial}} = \boldsymbol{p}_{\text{final}}$ to show that $\theta_1 + \theta_2 = \pi/2$. In this chapter, we look at the relativisitic setting, and we use the principle of conservation of four-momentum,

$$\sum P^a_{\text{initial}} = \sum P^a_{\text{final}}, \qquad a = 0, 1, 2, 3, \tag{19.1}$$

to compute θ_1 and θ_2. Here, the sum in Equation (19.1) is over the two particles involved in the collision.

19.3 Conservation of Energy

We take $a = 0$ in Equation (19.1):

$$\sum P^0_{\text{initial}} = \sum P^0_{\text{final}}. \tag{19.2}$$

Initial: The are two particles:

$$\begin{aligned} P^0_{\text{initial},1} &= \gamma(u) m_0 c, \qquad \text{(moving particle)}, \\ P^0_{\text{initial},2} &= \gamma(0) m_0 c = m_0 c. \end{aligned}$$

Hence,

$$\sum P^0_{\text{initial}} = [\gamma(u) + 1]\, m_0 c := (\gamma_u + 1) m_0 c.$$

Final: Two particles:

$$\begin{aligned} P^0_{\text{final},1} &= \gamma(v_1) m_0 c := \gamma_1 m_0 c, \\ P^0_{\text{final},2} &= \gamma(v_2) m_0 c := \gamma_2 m_0 c. \end{aligned}$$

Equate (initial = final):

$$(\gamma_u + 1)\, m_0 c = (\gamma_1 + \gamma_2)\, m_0 c,$$

hence

$$\gamma_u + 1 = \gamma_1 + \gamma_2.$$

19.4 Conservation of Momentum

We now look at the components $a = i = 1, 2, 3$ in Equation (19.1). Since the velocities are in a plane, we can take $P^3 = 0$ both before and after the collision. As such, we look at $i = 1, 2$ only.

Initial: The are two particles:

$$\begin{aligned} P^1_{\text{initial},1} &= \gamma(u) m_0 u, \qquad \text{(moving particle)}, \\ P^1_{\text{initial},2} &= \gamma(0) m_0 0, \end{aligned}$$

Hence,

$$\sum P^1_{\text{initial}} = \gamma_u m_0 u.$$

Final: We draw in the vector momenta on a diagram (Figure 19.2). We have:

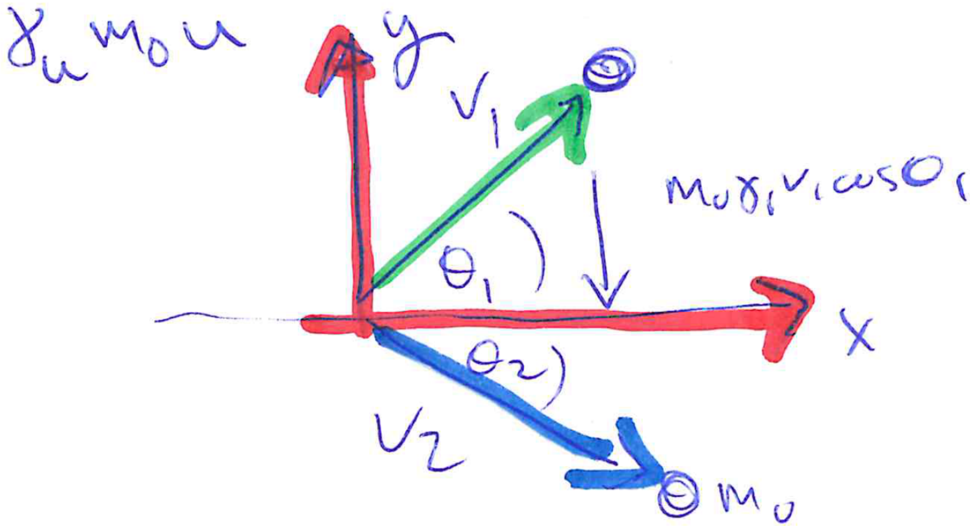


Figure 19.2: Momentum balance for the relativistic billiard balls

$$\begin{aligned} P^1_{\textsf{final,total}} &= m_0\gamma_1 v_1 \cos\theta_1 + m_0\gamma_2 v_2 \cos\theta_2, \\ P^2_{\textsf{final,total}} &= m_0\gamma_1 v_1 \sin\theta_1 - m_0\gamma_2 v_2 \sin\theta_2, \end{aligned}$$

Use:

$$P^i_{\textsf{initial,total}} = P^i_{\textsf{final,total}}, \qquad i = 1, 2.$$

Hence, in the x-direction ($i = 1$), we obtain:

$$\gamma_u m_0 u = \gamma_1 m_0 v_1 \cos\theta_1 + \gamma_2 m_0 v_2 \cos\theta_2.$$

In the the y-direction ($i = 2$), we obtain:

$$0 = \gamma_1 m_0 v_1 \sin\theta_1 - \gamma_2 m_0 v_2 \sin\theta_2.$$

We put all our results to date together in one place:

$$\begin{aligned} \gamma_u + 1 &= \gamma_1 + \gamma_2, && \text{(19.3a)} \\ \gamma_u u &= \gamma_1 v_1 \cos\theta_1 + \gamma_2 v_2 \cos\theta_2, && \text{(19.3b)} \\ 0 &= \gamma_1 v_1 \sin\theta_1 - \gamma_2 v_2 \sin\theta_2. && \text{(19.3c)} \end{aligned}$$

19.5 The strategy behind the calculation

The aim now is to study Equation (19.3), and hence to find θ_1 and θ_2 in terms of u. As a first step, we introduce the following useful identity:

Theorem 19.1 *Let* $\gamma_w = 1/\sqrt{1-(w/c)^2}$*. Then*

$$\gamma_w^2 w^2 = c^2(\gamma_w^2 - 1). \qquad (19.4)$$

Proof: We check the RHS:

$$\begin{aligned} \text{RHS} &= c^2\left(\frac{1}{1-(w/c)^2} - 1\right), \\ &= c^2\left(\frac{1-[1-(w/c)^2]}{1-(w/c)^2}\right), \\ &= \frac{w^2}{1-(w/c)^2}, \\ &= w^2\gamma_w^2, \\ &= \text{LHS}. \end{aligned}$$

Hence, the result us shown.

We now use Equation (19.4) to eliminate the **velocities** from Equation (19.3) – this makes the resulting calculations easier. We use: $\gamma_w w = c\sqrt{\gamma_w^2 - 1}$. Hence, Equation (19.3) becomes:

$$\gamma_u + 1 = \gamma_1 + \gamma_2, \tag{19.5a}$$
$$\sqrt{\gamma_u^2 - 1} = \sqrt{\gamma_1^2 - 1}\cos\theta_1 + \sqrt{\gamma_2^2 - 1}\cos\theta_2, \tag{19.5b}$$
$$0 = \sqrt{\gamma_1^2 - 1}\sin\theta_1 - \sqrt{\gamma_2^2 - 1}\sin\theta_2. \tag{19.5c}$$

The **strategy** of the calculation comes now in several parts:

- First part: We will show (not straightforwardly) that

$$\cos\theta_1 = \sqrt{\frac{\gamma_1 - 1}{\gamma_1 + 1}}\sqrt{\frac{\gamma_u + 1}{\gamma_u - 1}},$$

  and

$$\cos\theta_2 = \sqrt{\frac{\gamma_2 - 1}{\gamma_2 + 1}}\sqrt{\frac{\gamma_u + 1}{\gamma_u - 1}}.$$

- Second part: Develop expressions for $\sin\theta_1$ and $\sin\theta_2$ to show that

$$\tan\theta_1\tan\theta_2 = \pm\frac{\sin^2\theta_2}{\cos^2\theta_2}\left|\frac{\gamma_2 - 1}{\gamma_1 - 1}\right|$$

- Put Parts 1 and 2 together to obtain

$$\tan\theta_1\tan\theta_2 = -\frac{2}{1 + \gamma_u}.$$

The interpretation here is that as $u/c \to 0$ (i.e. the non-relativistic limit), that $\gamma_u \to 1$, and hence, $\tan\theta_1\tan\theta_2 \to -1$, thus, the two particles emerge from the collision at right angles to one another, in the limit as $u/c \to 0$ (e.g. Figure 19.3).

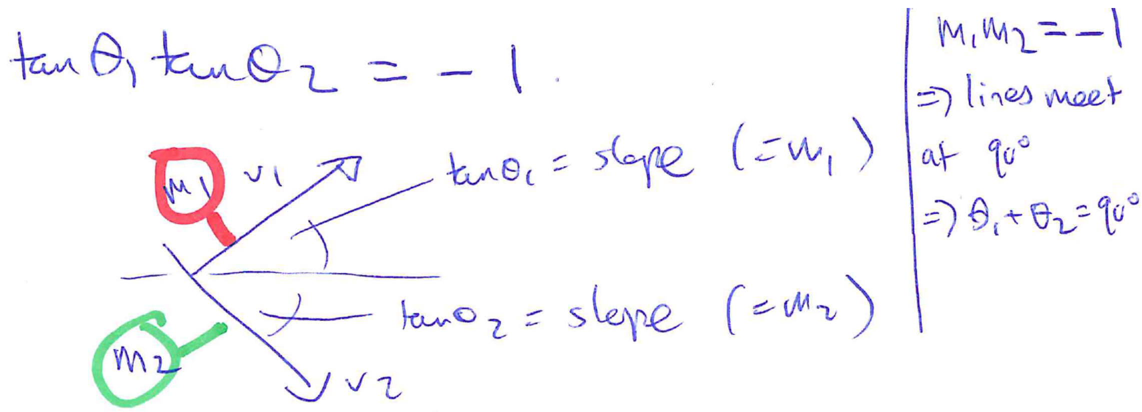


Figure 19.3: The billiard-ball collision in the non-relativistic limit as $u/c \to 0$

19.6 First Part of Strategy

We now re-write Equation (19.5)(b)–(c) as follows:

$$\begin{aligned} X &= a\cos\theta_1 + b\cos\theta_2, &\quad (19.6a)\\ 0 &= a\sin\theta_1 - b\sin\theta_2, &\quad (19.6b)\end{aligned}$$

with the obvious identifications

$$X = \sqrt{\gamma_u^2 - 1}, \qquad a = \sqrt{\gamma_1^2 - 1}, \qquad b = \sqrt{\gamma_2^2 - 1}.$$

Tactics:

- Eliminate θ_1 and find a formula for θ_2 (or rather, $\cos\theta_2$).
- Use $\sin^2\alpha + \cos^2\alpha = 1$, for any real number α.

As such, we re-write Equation (19.6)(b) as:

$$\begin{aligned} a\sin\theta_1 &= b\sin\theta_2,\\ a\sqrt{1-\cos^2\theta_1} &= b\sqrt{1-\cos^2\theta_2},\\ a^2 - a^2\cos^2\theta_1 &= b^2 - b^2\cos^2\theta_2,\\ -a^2\cos^2\theta_1 &= -a^2 + b^2 - b^2\cos^2\theta_2,\\ a^2\cos^2\theta_1 &= a^2 - b^2 + b^2\cos^2\theta_2.\end{aligned}$$

We also take Equation (19.6)(a) and re-write it as

$$X - b\cos\theta_2 = a\cos\theta_1.$$

Square up:

$$\begin{aligned} X^2 - 2bX\cos\theta_2 + b^2\cos^2\theta_2 &= a^2\cos^2\theta_1,\\ &= a^2\cos^2\theta_1,\\ &= a^2 - b^2 + b^2\cos^2\theta_2.\end{aligned}$$

Effect cancellations and obtain:

$$\frac{X^2 - a^2 + b^2}{2bX} = \cos\theta_2.$$

But $X = \sqrt{\gamma_u^2 - 1}$, hence:

$$\cos\theta_2 = \frac{(\gamma_u^2 - 1) - (\gamma_1^2 - 1) + (\gamma_2^2 - 1)}{2\sqrt{\gamma_2^2 - 1}\sqrt{\gamma_u^2 - 1}} \qquad (19.7)$$

This can be re-written more tidily as:

$$\cos\theta_2 = \frac{\gamma_u^2 - \gamma_1^2 + \gamma_2^2 - 1}{2\sqrt{\gamma_2^2 - 1}\sqrt{\gamma_u^2 - 1}} \tag{19.8}$$

We now use **something new** – we have not yet used conservation of energy, i.e. $1 + \gamma_u = \gamma_1 + \gamma_2$, hence

$$\gamma_1 = 1 + \gamma_u - \gamma_2.$$

Hence, starting with Equation (19.8) we have:

$$\begin{aligned}
\cos\theta_2 &= \frac{\gamma_u^2 - \gamma_1^2 + \gamma_2^2 - 1}{2\sqrt{\gamma_2^2 - 1}\sqrt{\gamma_u^2 - 1}}, \\
&= \frac{\gamma_u^2 - (1 + \gamma_u - \gamma_2)^2 + \gamma_2^2 - 1}{2\sqrt{\gamma_2^2 - 1}\sqrt{\gamma_u^2 - 1}}, \\
&= \frac{\gamma_u^2 - (1 + \gamma_u^2 + \gamma_2^2 + 2\gamma_u - 2\gamma_2 - \gamma_u\gamma_1) + \gamma_2^2 - 1}{2\sqrt{\gamma_2^2 - 1}\sqrt{\gamma_u^2 - 1}}, \\
&= \frac{\gamma_2 + \gamma_u\gamma_2 - \gamma_u - 1}{\sqrt{\gamma_2^2 - 1}\sqrt{\gamma_u^2 - 1}}, \\
&= \frac{(\gamma_2 - 1)(\gamma_u + 1)}{\sqrt{\gamma_2^2 - 1}\sqrt{\gamma_u^2 - 1}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\cos\theta_2 &= \frac{\sqrt{(\gamma_2 - 1)(\gamma_2 - 1)}}{\sqrt{(\gamma_2 - 1)(\gamma_2 + 1)}} \times \frac{\sqrt{(\gamma_u + 1)(\gamma_u + 1)}}{\sqrt{(\gamma_u - 1)(\gamma_2 + 1)}}, \\
&= \sqrt{\frac{\gamma_2 - 1}{\gamma_2 + 1}}\sqrt{\frac{\gamma_u + 1}{\gamma_u - 1}}.
\end{aligned}$$

Because the two particles are identical (up to labels), we can let $1 \leftrightarrow 2$ in this formula to obtain also that

$$\cos\theta_1 = \sqrt{\frac{\gamma_1 - 1}{\gamma_1 + 1}}\sqrt{\frac{\gamma_u + 1}{\gamma_u - 1}}.$$

We put these two equations together here on one place:

$$\cos\theta_1 = \sqrt{\frac{\gamma_1 - 1}{\gamma_1 + 1}}\sqrt{\frac{\gamma_u + 1}{\gamma_u - 1}}, \tag{19.9a}$$

$$\cos\theta_2 = \sqrt{\frac{\gamma_2 - 1}{\gamma_2 + 1}}\sqrt{\frac{\gamma_u + 1}{\gamma_u - 1}}. \tag{19.9b}$$

19.7 Second Part of Strategy

We go back to Equation (19.5)(c):

$$\begin{aligned} a\sin\theta_1 &= b\sin\theta_2, \\ \sqrt{\gamma_1^2-1}\sin\theta_1 &= \sqrt{\gamma_2^2-1}\sin\theta_2. \end{aligned}$$

Tactic: We will eliminate θ_1 and leave everything in terms of θ_2:

$$\sin\theta_1 = \frac{\sqrt{\gamma_2^2-1}}{\sqrt{\gamma_1^2-1}}\sin\theta_2. \tag{19.10}$$

We use Equation (19.9) and do something similar for $\cos\theta_1$. From Equation (19.9), we have:

$$\begin{aligned} \frac{\cos\theta_1}{\cos\theta_2} &= \frac{\sqrt{\frac{\gamma_1-1}{\gamma_1+1}}\sqrt{\frac{\gamma_u+1}{\gamma_u-1}}}{\sqrt{\frac{\gamma_2-1}{\gamma_2+1}}\sqrt{\frac{\gamma_u+1}{\gamma_u-1}}}, \\ &= \sqrt{\frac{\gamma_1-1}{\gamma_1+1}}\sqrt{\frac{\gamma_2+1}{\gamma_2-1}}, \end{aligned}$$

hence

$$\cos\theta_1 = \sqrt{\frac{\gamma_2-1}{\gamma_2+1}}\sqrt{\frac{\gamma_u+1}{\gamma_u-1}}\cos\theta_2. \tag{19.11}$$

Combine Equations (19.10)–(19.11):

$$\begin{aligned} \frac{\sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2} &= \frac{\left(\frac{\sqrt{\gamma_2^2-1}}{\sqrt{\gamma_1^2-1}}\sin\theta_2\right)\sin\theta_2}{\left(\sqrt{\frac{\gamma_2-1}{\gamma_2+1}}\sqrt{\frac{\gamma_u+1}{\gamma_u-1}}\cos\theta_2\right)\cos\theta_2}, \\ &= \frac{\sin^2\theta_2}{\cos^2\theta_2}\frac{\sqrt{(\gamma_2-1)^2}}{\sqrt{(\gamma_1-1)^2}}. \end{aligned}$$

Hence,

$$\tan\theta_1\tan\theta_2 = \pm\frac{\sin^2\theta_2}{\cos^2\theta_2}\left|\frac{\gamma_2-1}{\gamma_1-1}\right|.$$

19.8 Third Part of Strategy

We choose the minus sign in the above equation to emphasize that θ_1 and θ_2 are in different quadrants; they can't be in the same quadrants that would violate conservation of linear momentum.

As such, we have:

$$\begin{aligned}
\tan\theta_1 \tan\theta_2 &= -\frac{\sin^2\theta_2}{\cos^2\theta_2}\left(\frac{\gamma_2-1}{\gamma_1-1}\right),\\
&= -\frac{1-\cos^2\theta_2}{\cos^2\theta_2}\left(\frac{\gamma_2-1}{\gamma_1-1}\right),\\
&= -\left(\frac{1}{\cos^2\theta_2}-1\right)\left(\frac{\gamma_2-1}{\gamma_1-1}\right),\\
&= -\left[\frac{\gamma_2+1}{\gamma_2-1}\frac{\gamma_u-1}{\gamma_u+1}-1\right]\left(\frac{\gamma_2-1}{\gamma_1-1}\right),\\
&= -\frac{1}{\gamma_1-1}\left[(\gamma_2+1)\frac{\gamma_u-1}{\gamma_u+1}-(\gamma_2-1)\right],\\
&= -\frac{1}{\gamma_1-1}\left[\frac{(\gamma_2+1)(\gamma_u-1)-(\gamma_2-1)(\gamma_u+1)}{\gamma_u+1}\right],\\
&= -\frac{1}{\gamma_1-1}\left[\frac{\gamma_2\gamma_u-\gamma_2+\gamma_u-1-\gamma_2\gamma_u-\gamma_2+\gamma_u+1}{\gamma_u+1}\right],\\
&= -\frac{2}{\gamma_1-1}\frac{\gamma_u-\gamma_2}{\gamma_u+1}.
\end{aligned}$$

We use conservation of energy, i.e. $1+\gamma_u=\gamma_1+\gamma_2$, hence $\gamma_1-1=\gamma_u-\gamma_2$, hence:

$$\begin{aligned}
\tan\theta_1 \tan\theta_2 &= -\frac{2}{\gamma_u-\gamma_2}\left(\frac{\gamma_u-\gamma_2}{\gamma_u+1}\right),\\
&= -\frac{2}{\gamma_u+1}.
\end{aligned}$$

This is the required final result.

For consistency, we check the non-relativistic limit: $\gamma_u\to 1$ as $u/c\to 0$, hence $\tan\theta_1\tan\theta_)2\to -1$ in the same limit. In this limit, we have the situation described in Figure 19.4. From the figure, we have $m_1=\tan\theta_1$ and $m_2=\tan\theta_2$ (slopes of lines made by outgoing momenta), hence $m_1\times m_2 = -1$, hence, the two line segments meet at right angles, hence $\theta_1+\theta_2=\pi/2$.

19.9 Explicit Non-relativistic calculation

It is instructive to do the same 'billiard balls' problem directly in the Newtonian limit, i.e. without any Special Relativity, meaning that the following calculations are valid in the limit as $u/c\to 0$. As such, we refer to Figure 19.5 We look at elastic collisions, i.e. where the kinetic energy is the same before an dafter the collision. We also have the more general principle that the total linear momentum is conserved. As such we have:

$\frac{1}{2}m_0^2=\frac{1}{2}m_0v_1^2+\frac{1}{2}m_0v_2^2,\qquad$ Conservation of Kinetic Energy,

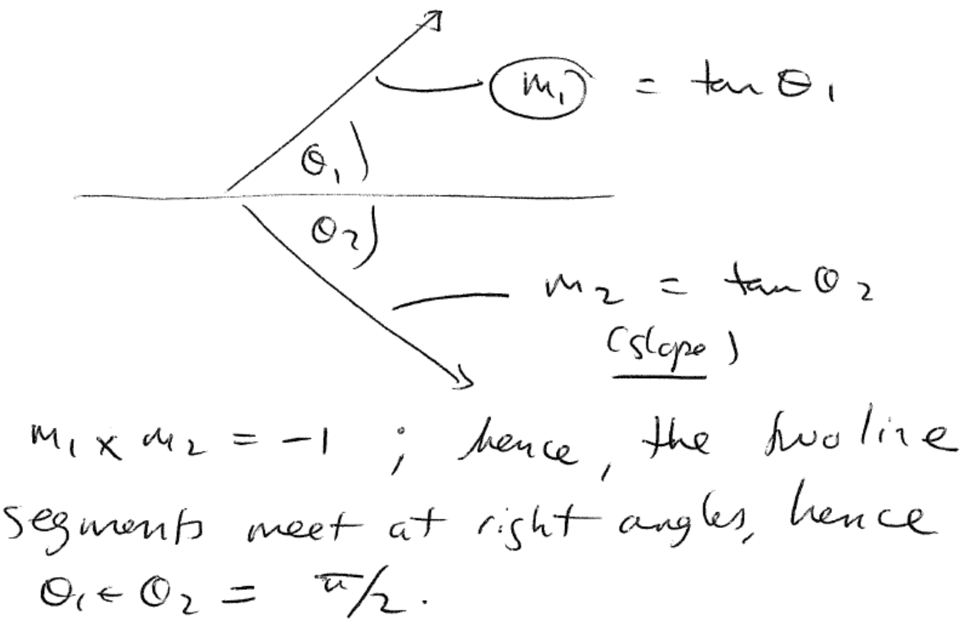


Figure 19.4: Scattering angles for non-relativistic billiard balls

hence

$$u^2 = v_1^2 + v_2^2.$$

We also have conservation of momentum in both directions in the $x - y$ plane:

$$\begin{aligned} m_0 u &= m_0 v_1 \cos\theta_1 + m_0 v_2 \cos\theta_2, \\ 0 &= m_0 v_1 \sin\theta_1 - m_0 v_2 \sin\theta_2, \end{aligned}$$

hence

$$u = v_1 \cos\theta_1 + v_2 \cos\theta_2, \tag{19.12a}$$

$$0 = v_1 \sin\theta_1 - v_2 \sin\theta_2. \tag{19.12b}$$

The aim of the calculation in the limiting case $u/c \to 0$ is the same as that in the relativistic case – to compute $\tan\theta_1 \tan\theta_2$ in terms of u. The **strategy** is very similar:

- Compute $\cos\theta_1$ and $\cos\theta_2$.
- Show that $\tan\theta_1 \tan\theta_2 = -1$.

We start with conservation of momentum in the y-direction, which gives $v_1 \sin\theta_1 - v_2 \sin\theta_2$. Use trigonometric identities:

$$\begin{aligned} v_1\sqrt{1-\cos^2\theta_1} &= v_2\sqrt{1-\cos^2\theta_2}, \\ v_1^2 - v_1^2\cos^2\theta_1 &= v_2^2 - v_2^2\cos^2\theta_2, \end{aligned}$$

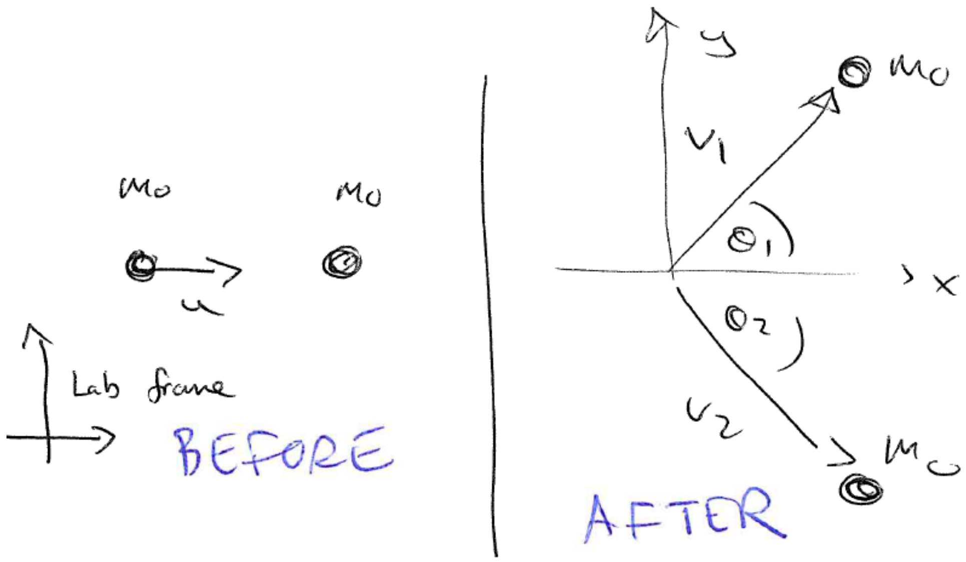


Figure 19.5: Scattering angles for non-relativistic billiard balls – setup for conservation of Energy and Momentum

hence

$$v_1^2 \cos^2\theta_1 = v_1^2 - v_2^2 + v_2^2 \cos^2\theta_2. \tag{19.13}$$

Back to conservation of momentum in the x-direction, i.e. $u = v_1 \cos\theta_1 + v_2 \cos\theta_2$, hence

$$v_1 \cos\theta_1 = u - v_2 \cos\theta_2.$$

Square up:

$$v_1^2 \cos^2\theta_1 = u^2 + v_2^2 \cos^2\theta_2 - 2uv_2 \cos\theta_2. \tag{19.14}$$

Equate Equations (19.13) and (19.15):

$$u^2 + v_2^2 \cos^2\theta_2 - 2uv_2 \cos\theta_2 = v_1^2 - v_2^2 + v_2^2 \cos^2\theta_2.$$

Make cancellations:

$$u^2 - v_1^2 + v_2^2 = 2uv_2 \cos\theta_2. \tag{19.15}$$

Use conservation of kinetic energy: $u^2 = v_1^2 + u_2^2$, hence $u^2 - v_2^2 = v_1^2$, hence Equation (**??**) becomes:

$$u^2 - \left(u^2 - v_2^2\right) + v_2^2 = 2uv_2 \cos\theta_2,$$

hence

$$v_2 = u \cos\theta_2,$$

or $\cos\theta_2 = v_2/u$. Since the two particles are identical up to relabelling, we also have $\cos\theta_1 = v_1/u$.

Use trigonometry to show that $\tan\theta_2 = \sqrt{u^2 - v_2^2}/v_2$, and similarly for $\tan\theta_1$, hence

$$\tan\theta_1 \tan\theta_2 = \pm\frac{\sqrt{u^2 - v_2^2}}{v_2}\frac{\sqrt{u^2 - v_1^2}}{v_1}.$$

Take the minus sign because the particles must fly off into different quadrants to conserve momentum:

$$\tan\theta_1 \tan\theta_2 = -\frac{\sqrt{u^2 - v_2^2}}{v_2}\frac{\sqrt{u^2 - v_1^2}}{v_1}.$$

Use conservation of energy again: $u^2 = v_1^2 + v_2^2$, hence

$$u^2 - v_2^2 = v_1^2, \qquad u^2 - v_1^2 = v_2^2,$$

hence

$$\tan\theta_1 \tan\theta_2 = -\frac{v_1}{v_2}\frac{v_2}{v_1} = -1,$$

as required.

Chapter 20

Special topics in Special Relativity

20.1 Overview

We look at various problems of kinematics in Special Relativity. We focus especially on problems involving collisions of particles. Throughout this section, we use the following conversion factor relating the electron-volt to Joules: $1\,\mathrm{eV} = 1.60217646 \times 10^{-19}\,\mathrm{Joules}$, where the Joule is the SI unit of energy, $\mathrm{J} = \mathrm{kg}\,\mathrm{m}^2/\mathrm{s}^2$.

20.2 Example – Time Dilation

After being produced in a collision between elementary particles, a positive pion must travel down a $1.20\,\mathrm{km}$-long tube to reach a detector. A positive pion has an average lifetime (measured in its rest frame) of $\tau_0 = 2.60 \times 10^{-8}\,\mathrm{s}$; the pions we consider have this lifetime. (a) How fast must the pion travel if is not to decay before it reaches the end of the tube? (Since the speed v of the pion is close to c, write $v = (1-\Delta)\,c$ and give your answer in terms of Δ); (b) The pion has a rest energy of $139.6 \times 10^6\,\mathrm{eV} = 139.6\,\mathrm{MeV}$. What is the total energy of the pion at the speed calculated in part (a)?

Solution: Time of travel: $T = H/v$, $v = (1-\Delta)\,c$. We require $T < \tau = \gamma\tau_0$. Hence,

$$H = \tau_0 c \frac{1-\Delta}{\sqrt{1-(1-\Delta)^2}}.$$

Using the binomial expansion $\sqrt{1-(1-\Delta)^2} \approx \sqrt{2\Delta}$. To lowest order then

$$H = \frac{\tau_0 c}{\sqrt{2\Delta}},$$

and

$$\Delta = \tfrac{1}{2}\left(\frac{\tau_0 c}{H}\right)^2.$$

Plug in the numbers: $\Delta = 2.11 \times 10^{-5}$.

In the second part, $E = \gamma mc^2$, and $\gamma \approx 1/\sqrt{2\Delta}$. Hence,

$$E \approx \frac{mc^2}{\sqrt{2\Delta}}.$$

Plugging in the numbers,

$$E = \frac{139.6 \times 10^6\,\mathrm{eV}}{0.0065} = 2.147692307692308e + 10\,\mathrm{eV} = 2.15 \times 10^4\,\mathrm{MeV}.$$

20.3 Example – Length Contraction

A rod of length ℓ_0 lies in the $x'y'$ plane of its rest system and makes an angle θ_0 with the x' axis. What is the length and orientation of the rod in the lab system xy in which the rod moves to the right with velocity v?

Call the ends of the rod A and B. In the rest system these points have coordinates

$$\begin{array}{lll} A: & x'_A = 0, & y'_A = 0 \\ B: & x'_B = \ell_0 \cos\theta_0, & y'_B = \ell_0 \sin\theta_0. \end{array}$$

We Lorentz-transform the lengths using $x' = \gamma\,(x - vt)$, $y' = y$ and obtain

$$\begin{array}{lll} A: & x'_A = 0 = \gamma\,(x_A - vt)\,, & y'_A = 0 = y_A \\ B: & x'_B = \ell_0 \cos\theta_0 = \gamma\,(x_B - vt)\,, & y'_B = \ell_0 \sin\theta_0 = y_B. \end{array}$$

Inverting for the unprimed quantities gives

$$\begin{aligned} x_B - x_A &= \frac{\ell_0 \cos\theta_0}{\gamma}, \\ y_B - y_A &= \ell \sin\theta_0. \end{aligned}$$

The length is

$$\ell = \sqrt{(x_B - x_A)^2 + (y_A - y_B)^2} = \ell_0\sqrt{(1 - v^2/c^2)\cos^2\theta_0 + \sin^2\theta_0}$$

$$= \ell_0\sqrt{1 - \frac{v^2}{c^2}\cos^2\theta_0}.$$

The angle that the rod makes with the x-axis in the lab frame is

$$\theta = \arctan\frac{y_B - y_A}{x_B - x_A} = \arctan\left(\gamma\frac{\sin\theta_0}{\cos\theta_0}\right) = \arctan\left(\gamma\tan\theta_0\right).$$

20.4 Example: Pair production

Under certain conditions, a photon can spontaneously decay into an electron-positron pair. This cannot happen in empty space however. To see why, let us perform the energy-momentum budget for the reaction $\gamma \to e^+ + e^-$. Conservation of energy gives

$$h\nu = \left(\gamma_+ + \gamma_-\right) mc^2.$$

Conservation of momentum gives

$$h\nu/c = |\gamma_+\boldsymbol{v}_+ + \gamma_- + \boldsymbol{v}_-|m$$

Equating the two expressions for $h\nu/c$ gives

$$\left(\gamma_+ + \gamma_-\right) c = |\gamma_+\boldsymbol{v}_+ + \gamma_- + \boldsymbol{v}_-|.$$

But this violates the following string of inequalities:

$$\begin{aligned}
\left(\gamma_+ + \gamma_-\right)^2 c^2 &= |\gamma_+\boldsymbol{v}_+ + \gamma_- + \boldsymbol{v}_-|^2, \\
&= \gamma_+|\boldsymbol{v}_+|^2 + \gamma_-|\boldsymbol{v}_-|^2 + \gamma_+\gamma_-|\boldsymbol{v}_+||\boldsymbol{v}_-|\cos\phi, \\
&\leq \gamma_+|\boldsymbol{v}_+|^2 + \gamma_-|\boldsymbol{v}_-|^2 + \gamma_+\gamma_-|\boldsymbol{v}_+||\boldsymbol{v}_-|, \\
&= \left(\gamma_+|\boldsymbol{v}_+| + \gamma_-|\boldsymbol{v}_-|\right)^2, \\
&< \left(\gamma_+ + \gamma_-\right)^2 c^2.
\end{aligned}$$

Therefore, pair production is impossible in empty space. However, suppose that a photon spontaneously decays in the vicinity of a massive nucleus at rest (mass M). Schematically, the decay process is $\gamma + M \to e^+ + e^- + M$. Now suppose that the decaying photon gives the nucleus M a small kinetic energy, such that the post-collision nucleus travels in the same direction as the incident photon with a velocity $V \ll c$. Moreover, suppose that the pair produced is at rest in the lab frame. Then, the energy balance gives

$$h\nu = 2mc^2 + \tfrac{1}{2}MV^2, \tag{20.1}$$

while the momentum balance (for $V \ll c$) gives

$$\frac{h\nu}{c} = MV. \tag{20.2}$$

Substituting Equation (20.2) into Eq. (20.1), we obtain

$$h\nu = 2m_0c^2 + \tfrac{1}{2}\frac{(h\nu)^2}{Mc^2}.$$

However, the assumption that the nucleus behaves in a Newtonian way requires that the incident photon does not affect the nucleus that much. Therefore, the energy of the photon must be much less than the nucleus rest energy, $h\nu \ll Mc^2$. Thus, to lowest order, the photon energy required for pair production in the vicinity of a very massive nucleus is

$$h\nu = 2m_0c^2.$$

Plugging in the electron mass, we require

$$h\nu_{\text{threshold}} = 1.02\,\text{MeV}.$$

For comparison, the rest energy of hydrogen is $940\,\text{MeV}$, and at threshold, the photon energy is therefore much less than the nucleus rest energy.

20.5 Cockroft and Walton's Experiment; Mass-energy equivalence

In 1932, Cockroft and Walton successfully operated the first high-energy proton accelerator and succeeded in causing a nuclear disintegration. Their experiment provided one of the earliest confirmations of the relativistic mass-energy relation. They studied a reaction wherein protons hit a Lithium target at rest. This precipitated a nuclear reaction in which the proton-lithium pair were converted into two Helium atoms (α-particles). Schematically, they observed the reaction

$$^1\text{H} +^7\text{Li} \rightarrow^4\text{He} +^4\text{He}.$$

The mass-energy balance for this reaction is

$$K\left(^1\text{H}\right) + M\left(^1\text{H}\right)c^2 + M\left(^7\text{Li}\right)c^2 = 2K\left(^4\text{He}\right) + 2M\left(^4\text{He}\right).$$

Calling $K = 2K\left(^4\text{He}\right) - K\left(^1\text{H}\right)$ and $\Delta M = M\left(^1\text{H}\right) + M\left(^7\text{Li}\right) - 2M\left(^4\text{He}\right)$, we have

$$K = \Delta Mc^2.$$

Cockroft and Walton knew the mass of each of the elements from mass spectrometry. They had

$$\begin{aligned} M\left({}^{1}\mathrm{H}\right) &= 1.0072, \\ M\left({}^{7}\mathrm{Li}\right) &= 7.0104 \pm 0.0030, \\ M\left({}^{4}\mathrm{He}\right) &= 4.0011, \end{aligned} \tag{20.3}$$

where the units here are atomic mass units, and $1\,\mathrm{amu} = 931\,\mathrm{MeV}$. These numbers yield

$$\Delta M = (1.0072 + 7.0104) - 2\,(4.0011) = (0.0154 \pm 0.0030)\ \mathrm{amu},$$

and $\Delta Mc^2 = (14.3 \pm 2.7)\,\mathrm{MeV}$. From particle detectors, C&W measured $\Delta K = 17.2\,\mathrm{MeV}$, which implies a ΔMc^2 of $17.2\,\mathrm{MeV}$ also. The experimental value of ΔMc^2 is thus $17.2\,\mathrm{MeV}$, while the upper-limit of the theoretical value is $17.0\,\mathrm{MeV}$. With more modern techniques of mass spectrometry, the rest masses in Eq. (20.3) are better known, and this gap has been closed.

Historical Note

E.T.S. Walton (1903 –1995) is celebrated as Ireland's only nobel Laureate in science. He studied at Trinity College Dublin and then worked at Cambridge, under Rutherford. He later taught back in Trinity. Arguably his real *alma mater* is the Methodist College Belfast. Other famous Irish / British alumni of the Methodist College include W. M. F. Orr (1866–1934), J. E. Campbell (1862-1924), and John Herivel (1918-2011). Orr was a professor at UCD and co-discovered the Orr–Sommerfeld equation, an equation vital to the understanding of turbulence. Campbell was one of the discovers of the Baker–Campbell–Hausdorff identity in the theory of Lie Algebras. Herivel helped to break the Enigma Code in World War II.

20.6 Example – Kinematics

Two particles of rest mass m_0 approach each other with equal and opposite velocity, v, in the laboratory frame. What is the total energy of one particle as measured in the rest frame of the other?

Vector addition: this is the analogue of the problem in Test 1. Let $V = \hat{\boldsymbol{x}}v$ be the velocity of frame A w.r.t. the lab. Let $v' = -\hat{\boldsymbol{x}}v$ be the velocity of particle B w.r.t. the lab frame. Then,

$$v' = \frac{(-v) - V}{1 - (-v)\,V/c^2} = \frac{-2v}{1 + v^2/c^2}.$$

Use

$$E = \gamma\,(v')\,mc^2.$$

$$E = \frac{1}{\sqrt{1 - \frac{4v^2/c^2}{(1+v^2/c^2)^2}}} mc^2 = \left(\frac{1 + v^2/c^2}{1 - v^2/c^2} \right) mc^2.$$

Alternative derivation: Let E_t and P_t denote the total energy in either frame. Now $E_t^2 - (P_t c)^2$ is invariant, so it is the frame in both the lab and the moving frames. In the lab frame, $E_t = 2\gamma_0 mc^2$, where $\gamma_0 = \gamma(v)$, and $P_t = 0$. In the moving frame, $E_t = mc^2 + \gamma_1 mc^2$ and $P_t = \gamma_1 mv'$, where $\gamma_1 = \gamma(v')$. Writing down the invariant, we have

$$\left(2\gamma_0 mc^2\right)^2 = \left(mc^2 + \gamma_1 mc^2\right)^2 - (\gamma_1 v' c)^2.$$

Expanding the squares, using $\gamma_1 (1 - v'^2/c^2) = 1$ and eliminating terms, we obtain

$$\gamma_1 = 2\gamma_0^2 - 1,$$

hence

$$E = \gamma_1 mc^2 = \left(2\gamma_0^2 - 1\right) mc^2 = \left(\frac{1 + v^2/c^2}{1 - v^2/c^2} \right) mc^2,$$

as before.

20.7 Example – More Kinematics

A particle of rest mass m and speed v_0 collides and sticks to a stationary particle of mass M. What is the final speed of the composite particle?

Initial state:

Energy:	$E = \gamma_0 mc^2 + Mc^2,$
Momentum:	$\gamma m v_0 \hat{x}$

Final state:

Energy:	$E = \gamma_1 \mu c^2,$
Momentum:	$\gamma \mu v_1 \hat{\boldsymbol{x}},$

where $\mu \neq m + M$ is the final mass. NOTE THE NONEQUALITY! Equating energies gives $\gamma_0 mc^2 + Mc^2 = \gamma_1 \mu c^2$; equating momenta gives $\gamma_1 \mu v_1 = \gamma_0 m v_0$. The energy equation implies that $\gamma_1 \mu = \gamma_0 m + M$. Plug this into the momentum equation:

$$v_1 \left(\gamma_0 m + M\right) = \gamma_0 m v_0 \implies v_1 = \frac{\gamma_0 m v_0}{\gamma_0 m + M}.$$

Note that the final mass is

$$\mu = \frac{\gamma_0 m + M}{\gamma_1} = \frac{\gamma_0}{\gamma_1} m + \frac{1}{\gamma_1} M.$$

Let us compute the first-order correction to the mass assuming that $v/c \ll 1$. We have $\mu = \mu(x)$, where $x = v^2/c^2$. To first order,

$$\mu = \mu(0) + \frac{d\mu}{dx}(0)\, x.$$

Now $\mu(0) = m + M$.

$$\frac{d\mu}{dx} = \frac{m}{\gamma_1}\frac{d\gamma_0}{dx} - \frac{m{\gamma_0}^2}{\gamma_1}\frac{d\gamma_1}{dx} - \frac{M}{\gamma_1^2}\frac{d\gamma_1}{dx}.$$

Here $\gamma_0(x) = (1-x)^{-1/2}$, hence $\gamma_0'(0) = 1/2$, and $\gamma_0(0) = 1$. Moreover,

$$\gamma_1(x) = \frac{1}{\sqrt{1 - \frac{\gamma_0^2 m^2 x}{(\gamma_0 m + M)^2}}}.$$

Hence,

$$\begin{aligned}\frac{d\gamma_1}{dx} &= \tfrac{1}{2}\frac{1}{\left[1 - \frac{\gamma_0^2 m^2 x}{(\gamma_0 m + M)^2}\right]^{3/2}}\frac{d}{dx}\left(\frac{\gamma_0^2 m^2 x}{(\gamma_0 m + M)^2}\right)\\ &= \tfrac{1}{2}\frac{1}{\left[1 - \frac{\gamma_0^2 m^2 x}{(\gamma_0 m + M)^2}\right]^{3/2}}\left[\frac{\gamma_0^2 m^2}{(\gamma_0 m + M)^2} + x\frac{d}{dx}\left(\frac{\gamma_0^2 m^2}{(\gamma_0 m + M)^2}\right)\right].\end{aligned}$$

Evaluating this at zero, it is

$$\gamma_1'(0) = \tfrac{1}{2}\frac{m^2}{(m+M)^2};$$

similarly, $\gamma_1(0) = 1$. Thus,

$$\mu'(0) = \tfrac{1}{2}m - \tfrac{1}{2}m\frac{m^2}{(m+M)^2} - \tfrac{1}{2}M\frac{m^2}{(m+M)^2}.$$

Thus,

$$\mu = (m+M) + \tfrac{1}{2}x\left[m - m\frac{m^2}{(m+M)^2} - M\frac{m^2}{(m+M)^2}\right] + O\left(x^2\right).$$

Tidying this up by introducing $\delta = m/M$ and restoring $x = v^2/c^2$, the correction to the mass is

$$\mu = m + M + \tfrac{1}{2}\frac{v^2}{c^2}\left[m - (m+M)\frac{\delta^2}{(1+\delta)^2}\right] + O\left(v^4/c^4\right).$$

Appendix A

Physical units

A.1 Overview

We review the basic physical units and especially the three (or possibly four) that are relevant for mechanics. We look at various quantities whose physical units are derived from these basic units. We also look at **dimensional analysis** – using intuition about physical units to derive approximate formulae (and very good approximations that) for various processes in mechanics even without having full access to the underlying mathematical theory.

A.2 Physical Units

In science, there are only seven basic physical units, and all other physical units can be constructed from these seven by multiplication. For mechanics, we are only interested in the first three (or possibly four).

Quantity	SI unit	Symbol
Time	Second	s
Length	Metre	m
Mass	Kilogram	kg
Charge	Coulomb	C
Temperature	Kelvin	K
Amount of substance	Mol	mol
Luminous intensity	Candela	cd

One second is defined as follows:

> One second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom,

and

> One metre is the distance travelled by light in free space in $1/299,792,458$ of a second.

Since 2019, the kilogram is defined as follows:

> $1\,\mathrm{kg}$ is defined by taking the fixed numerical value of the Planck constant h to be $6.62607015 \times 10^{-34}$ when expressed in the unit $\mathrm{kg\,m^2\,s^{-1}}$.

(previously, the kilogram was defined with respect to a 'standard' kilogram held in a vault in Paris).

A.3 Examples

Energy

The units of energy are not fundamental. For example, the kinetic energy of a body is $mv^2/2$, hence the units of energy are

$$[E] = \frac{\text{Mass} \times \text{Length}^2}{\text{Time}^2},$$

where the square brackets mean 'dimensions of'. Thus, the SI units of energy are $\mathrm{kg\,m^2\,s^{-2}}$, also called the *Joule*.

Force

The units of force are not fundamental. Since $F = ma$,

$$[F] = \frac{\text{Mass} \times \text{Length}}{\text{Time}^2},$$

and the SI units of force are $\mathrm{kg\,m\,s^{-2}}$. This combination of basic units is referred to as the *Newton*.

Hooke's constant

The constant in Hooke's law is not fundamental. Since $F = -kx$, and $[F] = \text{Mass}\times\text{Length}/\text{Time}^2$, we have

$$\frac{\text{Mass} \times \text{Length}}{\text{Time}^2} = [k]\text{Length},$$

hence

$$[k] = \frac{\text{Mass}}{\text{Time}^2}.$$

Note also,

$$[(k/m)^{1/2}] = \frac{1}{\text{Time}},$$

and thus, $\sqrt{k/m}$ is a frequency.

Newton's constant, G

The gravitational constant G is a derived quantity, because $F = Gm_1m_2/r^2$, hence

$$\frac{\mathrm{Mass} \times \mathrm{Length}}{\mathrm{Time}^2} = [G]\frac{\mathrm{Mass}^2}{\mathrm{Length}^2},$$

and

$$[G] = \frac{\mathrm{Length}^3}{\mathrm{Time}^2 \times \mathrm{Mass}};$$

the SI units are thus $\mathrm{m}^3\,\mathrm{s}^{-2}\,\mathrm{kg}^{-1}$.

A.4 Dimensional analysis

Sometimes it is possible to solve a problem in mechanics without solving any differential equations. For example, to estimate the energy of a system from the system parameters, we combine those parameters in such a way as to give an energy. This estimate usually gives the correct answer, up to a nondimensional prefactor.

The ground-state energy of hydrogen:

In a hydrogen atom, an electron and a proton interact via the Coulomb force

$$F = -\frac{Ke^2}{r^2},$$

where the minus sign indicates attraction, and e is the unit charge. Hence,

$$[Ke^2] = [F] \times \mathrm{Length}^2 = \frac{\mathrm{Mass} \times \mathrm{Length}^3}{\mathrm{Time}^2}$$

In every quantum mechanics problem, Planck's constant h appears, and

$$[h] = \mathrm{Momentum} \times \mathrm{Length} = \frac{\mathrm{Mass} \times \mathrm{Length}^2}{\mathrm{Time}}$$

We expect the mass of the electron to be important too. Thus, we write

$$E = m_e^a h^b \left(Ke^2\right)^c,$$

and solve for a, b, and c. The dimensions must match in this equation:

$$\frac{ML^2}{T^2} = M^a \left(\frac{ML^2}{T}\right)^b \left(\frac{ML^3}{T^2}\right)^c.$$

Or,

$$ML^2T^{-2} = M^{a+b+c}L^{2b+3c}T^{-b-2c}.$$

This gives a system of equations to solve:

$$\begin{aligned} 1 &= a+b+c, \\ 2 &= 2b+3c, \\ -2 &= -b-2c. \end{aligned}$$

Solving the last two equations together gives $c = 2$, $b = -2$. Backsubstitution into the first equation gives $a = 1$. Hence, the binding energy of the hydrogen atom is

$$E_{\text{da}} = \frac{m_e \left(Ke^2\right)^2}{h^2}$$

The true energy from solving the Schrödinger equation is

$$E = -2\pi^2 \frac{m_e \left(Ke^2\right)^2}{h^2},$$

which is not far off our guesstimate!

The gravitational self-energy of the earth:

We know that there is a gravitational potential energy between two particles:

$$U = -Gm_1m_2r^{-1},$$

where r is the particles' relative separation and m_1 and m_2 denote their masses. However, an extended body (such as the earth) can be treated as a collection of infinitesimal particles. What if we integrate over this collection of particles? We will get an energy that is called the *self-energy* of the extended body. This is the binding energy associated with all the attractive forces interacting within the system.

If we were to estimate the self-energy of the earth, there are only three parameters to play with: G, M_e, the earth mass, and R_e, the earth radius. Based on dimensional analysis, we would guess that

$$U_{\text{self}} = -\frac{GM_e^2}{R_e}.$$

The true value, obtained from integration, is

$$U_{\text{self}} = -\tfrac{3}{5}\frac{GM_e^2}{R_e}.$$

The energy required for a solar system to form from interstellar dust:

Imagine a collection of interstellar dust in deep space. Suppose that it has uniform density ρ and that this collection of dust is of extent ℓ. Then its mass is approximately

$$M \approx \rho\left(\tfrac{4}{3}\pi\ell^3\right),$$

and its gravitational self-energy is approximately

$$E_{\text{self}} \approx \frac{GM^2}{\ell}.$$

The particles will have some random motion associated with the finite temperature of the dust. This gives rise to an energy

$$E_{\text{thermal}} \approx Nk_BT,$$

where T is the temperature, k_B is Boltzmann's constant, and N is the number of dust particles. Now

$$\rho = \frac{M}{V} = \frac{Nm_0}{V} \implies N = \frac{\rho}{m_0}\left(\tfrac{4}{3}\pi\ell^3\right),$$

where m_0 is the mass of each dust particle. Thus we have two energies:

$$\begin{aligned} E_{\text{self}} &= \rho^2\left(4\pi/3\right)^3\ell^5, \\ E_{\text{thermal}} &= \frac{\rho\left(\frac{4}{3}\pi\ell^3\right)k_BT}{m_0}. \end{aligned}$$

Gravitational collapse occurs if the binding energy can overcome the random thermal motion:

$$E_{\text{self}} > E_{\text{thermal}},$$

and collapse just barely occrus if

$$E_{\text{self}} = E_{\text{thermal}},$$

or

$$\rho\left(\tfrac{4}{3}\pi\ell^3\right)\left(\rho G\tfrac{4}{3}\pi\ell^2\right) = \frac{\rho\left(\frac{4}{3}\pi\ell^3\right)k_BT}{m_0}.$$

Effecting the cancellations implied gives

$$\ell = \sqrt{\frac{3k_BT}{4\pi m_0\rho G}}.$$

Thus, if the extent of the dust cloud exceeds this critical length (called the Jeans length), gravitational collapse will occur, possibly giving rise to a primordial solar system.