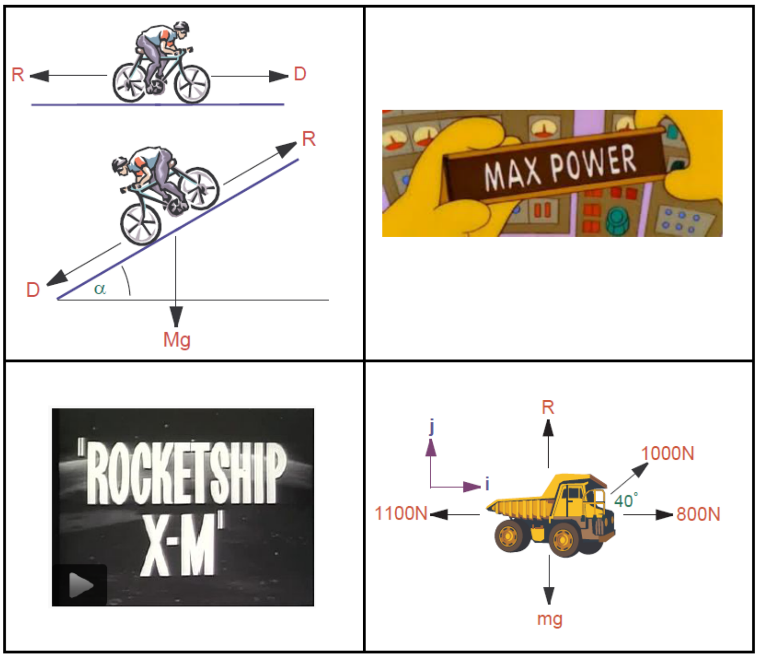
University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

School of Mathematics and Statistics
Scoil na Matamaitice agus Staitisticí
Applied Mathematics: Mechanics and Methods
(ACM 10080)



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Lecture notes in Applied Mathematics: Mechanics and Methods September 2022

Applied Mathematics: Mechanics and Methods (ACM10080)

- Subject: Applied and Computational Mathematics
- School: Mathematics and Statistics
- Module coordinator: Dr Lennon Ó Náraigh
- Credits: 5
- Level: 1
- Semester: First

This module introduces students to the basic principles of Newtonian Mechanics. The module starts with a description of motion (kinematics) including notions of displacement, velocity, and acceleration. The module then moves on to an understanding of the causes of motion and changes in motion (dynamics) via an elementary introduction to Newton's laws. Key topics here include force and momentum. Concurrently with this discussion, there will be an introduction to elementary mathematical methods that can be used to frame all of the above in mathematical language, including the mathematical treatment of kinematics and dynamics via Calculus and vector algebra.

What will I learn?

On completion of this module students should be able to

1. Perform mathematical calculations using vectors, including vector addition, scalar multiplication, and dot products.
2. Use the Cartesian notation for vectors with proficiency.
3. Explain precisely the ideas of vector, velocity, acceleration, momentum, kinetic energy, potential energy.
4. Use Newton's laws to express and solve problems in mathematical terms.

Editions

First edition: September 2015

Second edition: September 2019

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Acknowledgements

These lecture notes are based almost entirely on previous generations of lecture notes for this module or for a similar one, written and updated by Professor Joe Pulé, Professor Peter Duffy, Professor Adrian Ottewill, and Professor Peter Lynch.

If I have seen further, it is by standing on the shoulders of giants.

Contents

Chapter 1

Vectors

Overview

We define vectors and scalars and give examples. We introduce vector addition using the Parallelogram Law.

1.1 Introduction

When dealing with motion on a straight line ('one-dimensional motion') we will measure things like distances and speeds with respect to a fixed point of interest, which we call the origin. A particle can either be to the left or right of the origin. By convention,

- Distances to the right of the origin are **positive**.
- Distances to the left of the origin are **negative**.

Displacement is then the signed distance i.e. distance taking direction into account, and Figure 1.1 is just a glorified number line. The question then is how to generalize this to motion in higher

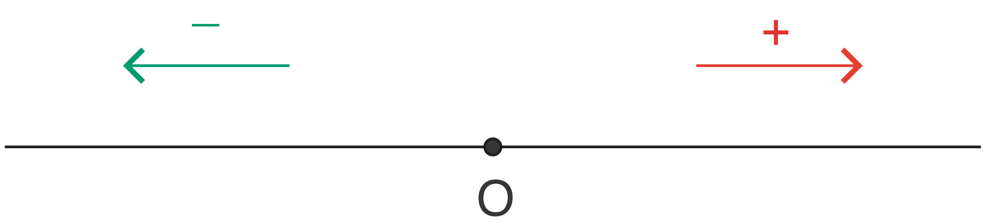


Figure 1.1: Schematic description of one-dimensional displacements with respect to the origin O .

dimensions (e.g. motion in the plane or in full three-dimensional space). In this context, we need to be able to measure things like distances and speeds again with respect to a fixed point of interest. We then need to generalize the number line in Figure 1.1 to planes and full three-dimensional space. This is where the concept of **vectors** is needed:

Definition 1.1 *Quantities with magnitude and direction are called* **vectors***; quantities with magnitude only are called* **scalars**.

For example, **displacement** is distance in a particular direction:

$$5\ km \leftrightarrow \text{distance} \leftrightarrow \text{SCALAR}$$

$$5\ km\ NW \quad \leftrightarrow \text{VECTOR}$$

Velocity is speed in a specified direction:

$$15\ \text{km/hr} \leftrightarrow \text{speed} \leftrightarrow \text{SCALAR}$$

$$15\ \text{km/hr}\ east \leftrightarrow \text{velocity} \leftrightarrow \text{VECTOR}$$

1.1.1 Quiz

What about the following quantities – vector or scalar?

- Temperature
- Pressure
- Wind
- Energy
- Momentum

1.2 Mathematical description of vectors

A vector can be represented by a **directed line segment** (Figure 1.2):

- direction of line → gives direction of vector.
- length of line → magnitude of vector.

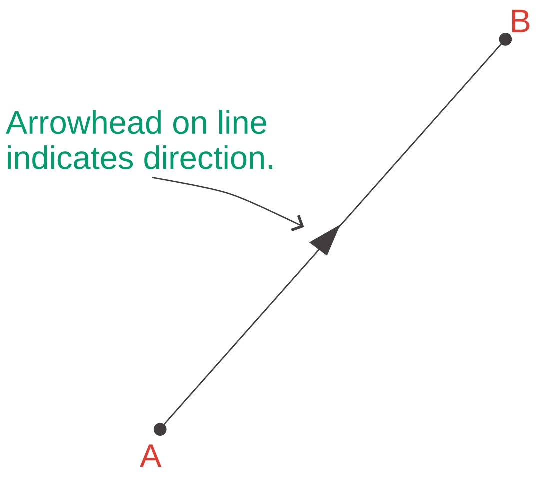
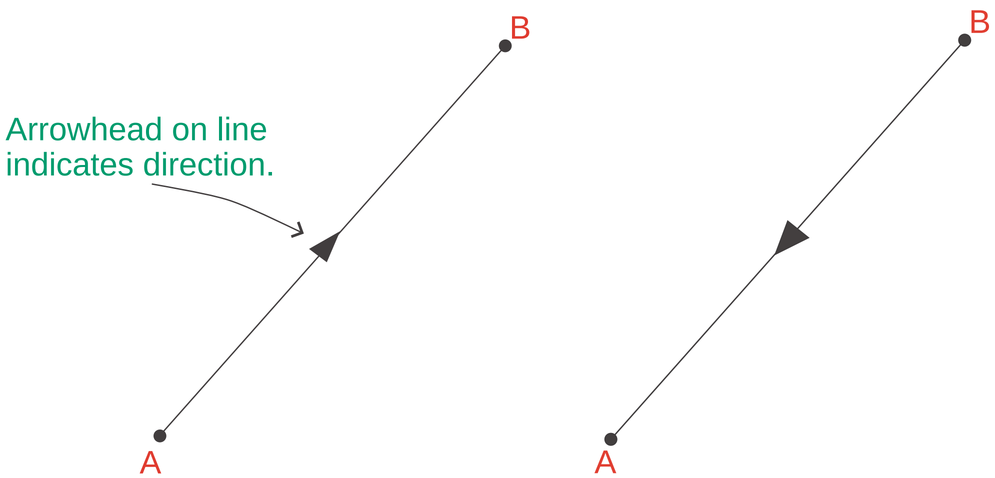


Figure 1.2: The vector $\overrightarrow{AB}$ represented by a directed line segment. The direction of the vector is from A to B and the magnitude of the vector corresponds to the length of the line segment from A to B.

The line segment AB represents a vector denoted by $\overrightarrow{AB}$. The direction of the vector is from A to B and the magnitude of the vector corresponds to the length of the line segment from A to B. Thus, vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ represent two vectors having the same magnigude but opposite directions (Figure 1.3).



(a) (b)

Figure 1.3: Vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same magnitude but opposite directions.

1.2.1 Other notation for vectors:

Vectors are often denoted by boldface type (Figure 1.4).

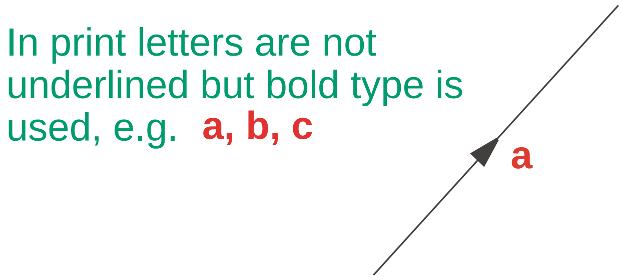


Figure 1.4: A vector quantity denoted by a boldface letter ‘a’

There are several other ways to write vectors:

- Arrows: $\overrightarrow{AB}$
- Underbars: $\underline{\mathbf{a}}$
- Tildes: $\tilde{\mathbf{a}}$
- Others . . .

1.2.2 Relations between vectors

Definition 1.2 *Vectors which have the same magnitude and direction are* **equal**.

Definition 1.3 *If* $\mathbf{a}$ *has the same magnitude but the opposite direction to* $\mathbf{b}$, *then* $\mathbf{a} = -\mathbf{b}$.

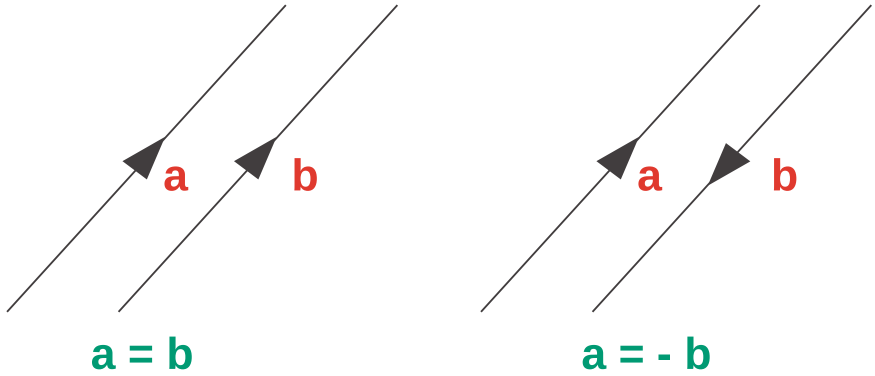


Figure 1.5: Equality of vectors

Definition 1.4 *Let* $\overrightarrow{AB}$ *be a vector. Let the point* A *be moved through a distance* D *to produce a new point* A', *and a line segment* AA'. *Also, let the point* B *be moved through the same distance* D *and in parallel to the line segment* AA' *to produce a new point* B'. *Then the vector* $\overrightarrow{A'B'}$ *is the vector* $\overrightarrow{AB}$ *parallel-transported through distance* D *– see Figure 1.6.*

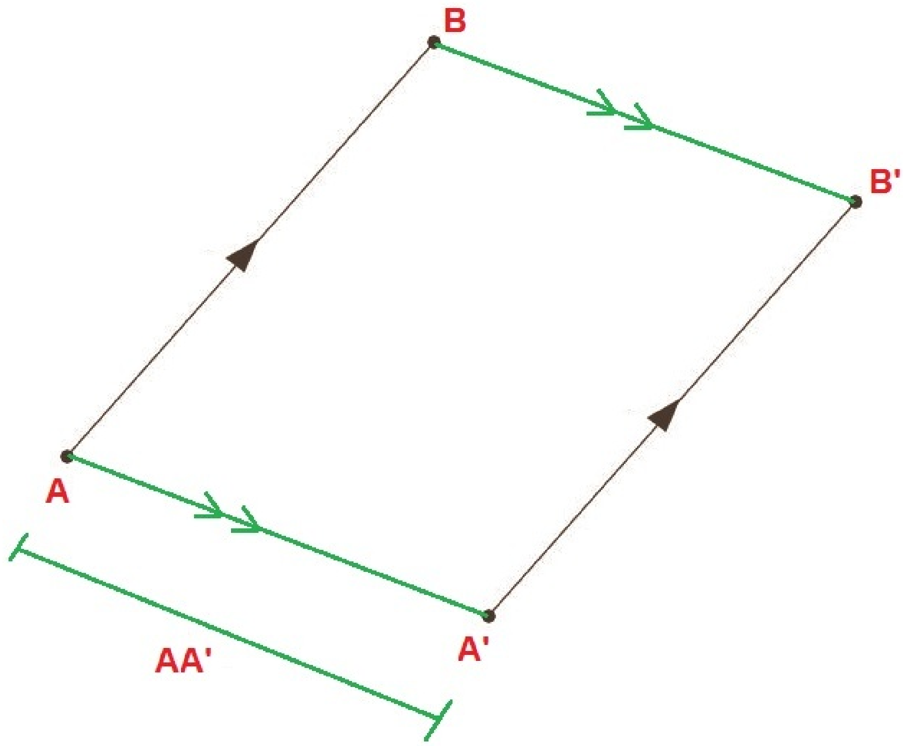


Figure 1.6: Parallel transport of the vector $\overrightarrow{AB}$.

By definition 1.2 the vector $\overrightarrow{AB}$ and its parallel-transported version $\overrightarrow{A'B'}$ are equal. Parallel-transport of a vector leaves the vector unchanged. This is very much like the idea in geometry of congruent triangles. In particular, two triangles that can be parallel-transported so that they coincide are congruent. Thus, equality of vectors can be thought of as being like congruency – it is a coincidence of two geometric objects that is brought about by a geometric transformation.

Definition 1.5 *Let λ be a real number and let $\overrightarrow{AB}$ be a vector. Then $\lambda(\overrightarrow{AB})$ is also a vector, with the following properties:*

- *The magnitude of the new vector is $|\lambda||AB|$.*
- *Direction:*
  - *If $\lambda > 0$, then the direction of the new vector is the same is the direction of the old one, i.e. from A to B.*
  - *If $\lambda < 0$, then the direction of the new vector is the opposite as that of the old one, i.e. from B to A.*

Then $\lambda(\overrightarrow{AB})$ is called a **scalar multiple** *of $\overrightarrow{AB}$.*

Geometrically, $\lambda\overrightarrow{AB}$ can be parallel-transported so that it is collinear with $\overrightarrow{AB}$ and points either in the same direction as $\overrightarrow{AB}$ or in the opposite direction. Thus, scalar multiplication of a vector results in a new vector, parallel or anti-parallel to the starting vector. For examples, see Figure 1.7

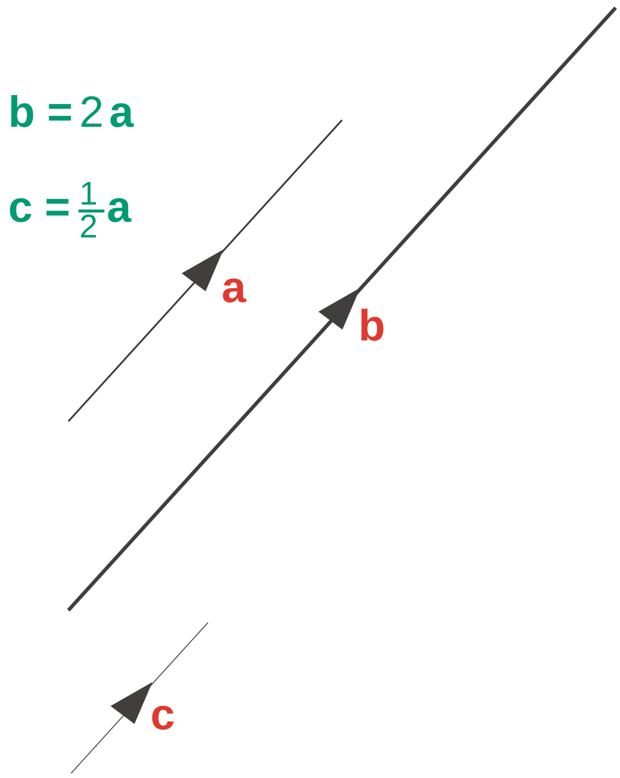


Figure 1.7: Scalar multiples of the vector $\boldsymbol{a}$

1.3 Addition of vectors – parallelogram law

In what follows, it will be very helpful to be able to add two vectors together. There is a recipe for doing this called the **parallelogram law** of vector addition, which starts with two vectors $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{XY}$ and generates the vector sum $\boldsymbol{a} + \boldsymbol{b}$:

1. Parallel-transport the two vectors so that their bases coincide, and such that

$$\boldsymbol{a} = \overrightarrow{OA}, \qquad \boldsymbol{b} = \overrightarrow{OB}.$$

   See Figure 1.8(a).

2. Form a parallelogram $OACB$ by further parallel-transport operations (Figure 1.8(b)):

   - Parallel-transport the vector $\boldsymbol{a}$ through the vector $\boldsymbol{b}$ to produce a new vector $\overrightarrow{BC}$;
   - Parallel-transport the vector $\boldsymbol{b}$ through the vector $\boldsymbol{a}$ to produce a new vector $\overrightarrow{AC}$.

3. The vector $\overrightarrow{OC}$ is the sum $\boldsymbol{a}+\boldsymbol{b}$ (Figure 1.8).

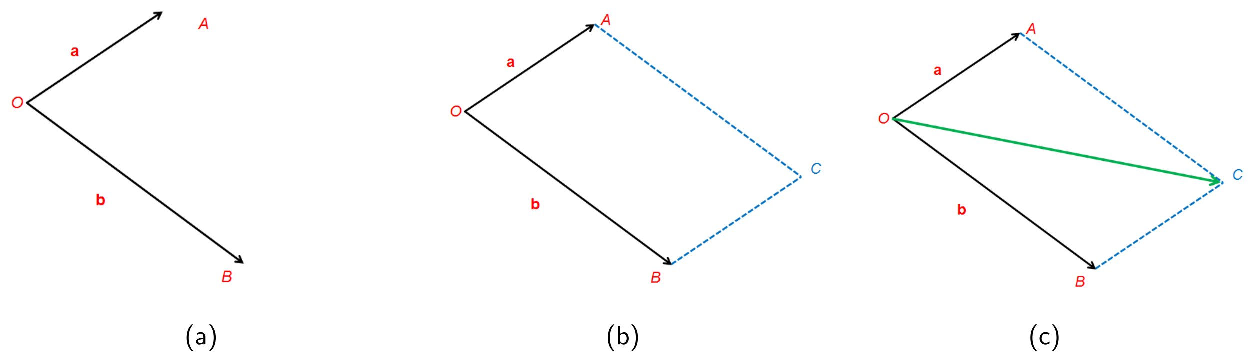


Figure 1.8: Parallelogram law of vector addition

A nice feature of this law is that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are used interchangeably – neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ are singled out for special treatment in the definition. Therefore, it follows that $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$, and a familiar property of (ordinary) addition is retained here, namely that the vector addition is **commutative**.

Furthermore, because the vectors $\boldsymbol{b}$ and $\boldsymbol{b}' := \overrightarrow{AC}$ are the same vector (by parallel transport), we can formulate a totally equivalent (but sometimes more convenient) recipe for adding vectors called the **triangle law** of vector addition, which can be summarized in the following sequence of steps:

1. Parallel-transport the two vectors so that the base of $\boldsymbol{b}$ coincides with the tip of $\boldsymbol{a}$, such that, under parallel transport,

$$\boldsymbol{a}=\overrightarrow{OA}, \qquad \boldsymbol{b}=\boldsymbol{b}'=\overrightarrow{AC}.$$

2. Consider the resulting triangle OAC. The vector $\overrightarrow{OC}$ is the vector sum $\boldsymbol{a}+\boldsymbol{b}$.

The triangle law gives a good way of thinking addition of two arbitrary vectors is in terms of the particular vector displacement. Suppose I walk from O to A and from A to C. The net effect of my journey is the same as if I would travel directly from O to C. This is my displacement. So my net displacement can be got from vector addition of the two vectors $\overrightarrow{OA}$ and $\overrightarrow{AC}$. Mathematically,

$$\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC}.$$

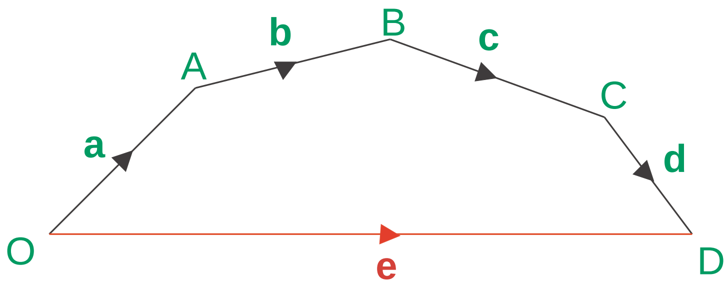


Figure 1.9: Any number of vectors can be added by successive application of the parallelogram law.

Any number of vectors can be added by successive application of the parallelogram law. For example, in Figure 1.9 we have

$$\mathbf{e} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$$

or

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$$

The order in which these applications are done does not matter, i.e.

$$\begin{aligned} \mathbf{e} &= (\mathbf{a} + \mathbf{b}) + (\mathbf{c} + \mathbf{d}), \\ &= [(\mathbf{a} + \mathbf{b}) + \mathbf{c}] + \mathbf{d}, \qquad \text{etc.} \end{aligned}$$

This is another familiar property of addition that is reproduced by the vector addition operation.

At this point, it is helpful to introduce the notion of the **zero vector**: this is a vector with zero magnitude and arbitrary direction, and is denoted simply by 0, as with ordinary real numbers. Consider for a moment the vector sum $\boldsymbol{a} + \boldsymbol{b}$. One can imagine shrinking down the magnitude of $\boldsymbol{b}$ until it becomes the zero vector. From the triangle law, one can see that as $\boldsymbol{b}$ is shrunk to the zero vector, the sum $\boldsymbol{a} + \boldsymbol{b}$ reduces simply to $\boldsymbol{a}$. Hence,

$$\boldsymbol{a} + 0 = \boldsymbol{a}.$$

Finally, vectors can also be subtracted: $\boldsymbol{a} - \boldsymbol{b}$ means the same thing as the vector addition of $\boldsymbol{a} + (-\boldsymbol{b})$, where $-\boldsymbol{b}$ is the vector $\boldsymbol{b}$ with its direction reversed. From the triangle law, it follows that $\boldsymbol{a} - \boldsymbol{a} = 0$.

Interestingly, we may summarize the properties of the vector addition, valid for all vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$:

1. Closure: The sum of any two vectors is also a vector: by the parallelogram law in Figure 1.8, the sum $\boldsymbol{a} + \boldsymbol{b}$ is a quantity with both magnigude and direction, and is therefore yet another vector.

2. Associative:

$$(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c},$$

3. Existence of zero vector 0, such that

$$\boldsymbol{a}+0=\boldsymbol{a}$$

4. Existence of inverses: for each vector $\boldsymbol{a}$, we have

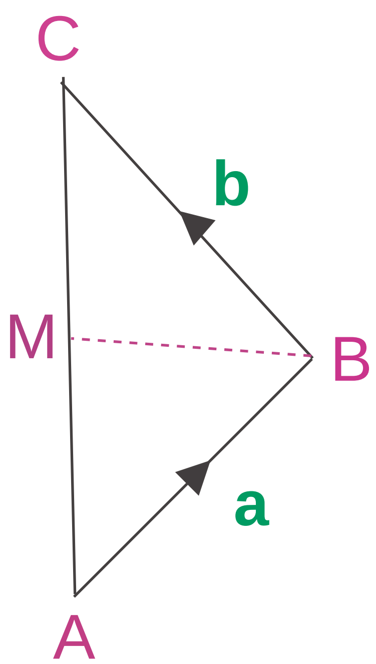
$$\boldsymbol{a}+(-\boldsymbol{a})=0.$$

5. Commutative:

$$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}.$$

Later on you will see that these properties make the set of all vectors into a mathematical structure called an **Abelian group**.

1.4 Worked examples



Example: Refer to the figure. The point M is at the mid-point of AC Find, in terms of **a** and **b**

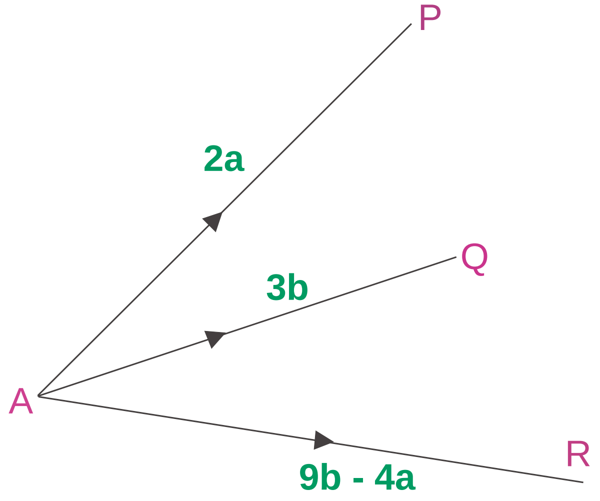
$$\overrightarrow{AC}, \quad \overrightarrow{CA}, \quad \overrightarrow{AM}, \quad \overrightarrow{MB}.$$

Solution:

$$\overrightarrow{AB} = \mathbf{a} \qquad \overrightarrow{BC} = \mathbf{b}$$

$$\begin{aligned}
\text{(a)} \quad \overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{a} + \mathbf{b} \\
\text{(b)} \quad \overrightarrow{CA} &= -\overrightarrow{AC} = -(\mathbf{a} + \mathbf{b}) \\
\text{(c)} \quad \overrightarrow{AM} &= \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\mathbf{a} + \mathbf{b})
\end{aligned}$$

$$\begin{aligned}
\text{(d)} \quad \overrightarrow{MB} &= \overrightarrow{MA} + \overrightarrow{AB} \\
&= -\overrightarrow{AM} + \overrightarrow{AB} \\
&= -\frac{1}{2}(\mathbf{a} + \mathbf{b}) + \mathbf{a} \\
&= -\frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} + \mathbf{a} \\
&= \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a} - \mathbf{b}) \; .
\end{aligned}$$



Example: Refer to the figure. A, P, Q and R are four points:

$$\begin{aligned}
\overrightarrow{AP} &= 2\mathbf{a} \\
\overrightarrow{AQ} &= 3\mathbf{b} \\
\overrightarrow{AR} &= 9\mathbf{b} - 4\mathbf{a}
\end{aligned}$$

Show that P, Q and R are co-linear.

Solution:

$$\begin{aligned}
\overrightarrow{PQ} &= \overrightarrow{PA} + \overrightarrow{AQ} \\
&= -\overrightarrow{AP} + \overrightarrow{AQ} \\
&= -2\mathbf{a} + 3\mathbf{b} \\
\overrightarrow{PR} &= \overrightarrow{PA} + \overrightarrow{AR} \\
&= -\overrightarrow{AP} + \overrightarrow{AR} \\
&= -2\mathbf{a} + (9\mathbf{b} - 4\mathbf{a}) \\
&= -6\mathbf{a} + 9\mathbf{b} \\
&= 3(-2\mathbf{a} + 3\mathbf{b}) \\
&= 3\overrightarrow{PQ} \\
\overrightarrow{PR} &= 3\overrightarrow{PQ}
\end{aligned}$$

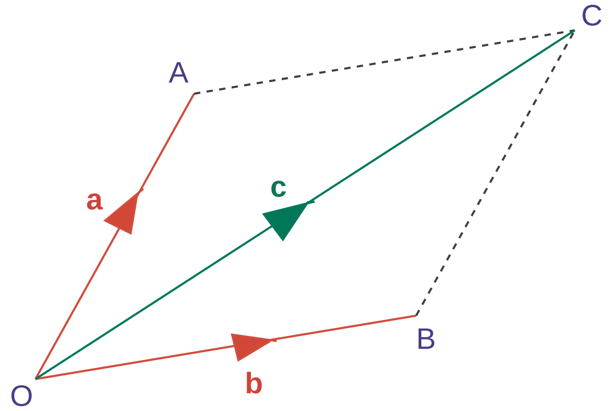
Chapter 2

Vectors – representation in Cartesian coordinate systems

Overview

In the last section we introduced vectors and vector addition. The discussion was a little bit on the qualitative side. A convenient way to make this discussion more formal and to enable efficient computations is to represent vectors in a Cartesian coordinate system. This is the subject of this chapter.

2.1 Components of a vector

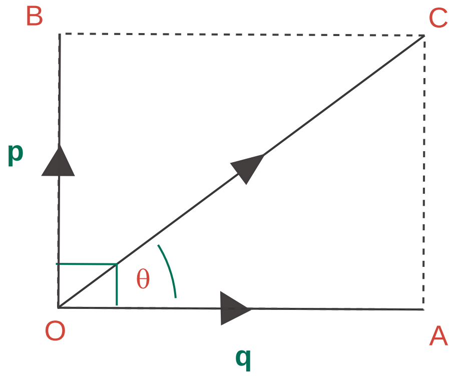


It is useful to split up a vector into its **components**. This is called **resolving** the vector into components. To see what these terms mean, consider the figure on the left, where we have $\mathbf{c} = \mathbf{a} + \mathbf{b}$. Then

$\mathbf{c}$ is the resultant of the vectors $\mathbf{a}$ and $\mathbf{b}$.

$\mathbf{a}$ and $\mathbf{b}$ are the components of $\mathbf{c}$.

Usually we resolve a vector in two orthogonal (perpendicular) components – as in Figure 2.1.



If

- the magnitude of $\overrightarrow{OB}$ is p (we write $|\overrightarrow{OB}| = p$) and
- the magnitude of $\overrightarrow{OA}$ is q (we write $|\overrightarrow{OA}| = q$)

then

$$|\overrightarrow{OC}| = \sqrt{p^2 + q^2}. \tag{2.1}$$

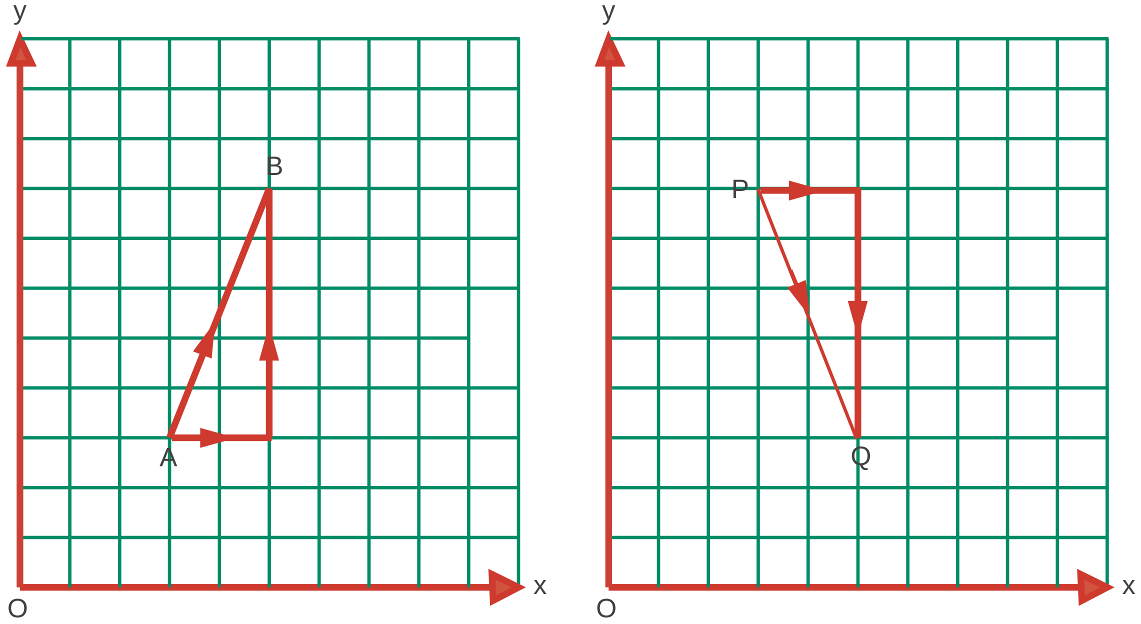
We will discuss Equation (2.1) in more detail below. However, we first of all introduce distinguished vectors to help us resolve any vector into orthogonal components. To do this, we need the notion of a unit vector:

Definition 2.1 *A vector of magnitude one is called a* **unit vector**.

More specifically, we have the following two distinguished unit vectors in a specific Cartesian coordinate system:

Definition 2.2 *In the x-y plane, unit vectors in the Ox and Oy directions are denoted by* $\mathbf{i}$ *and* $\mathbf{j}$.

Any vector in the x-y-plane can be resolved in terms of $\mathbf{i}$ and $\mathbf{j}$ – e.g. Figure 2.1.



(a) $\overrightarrow{AB} = 2\mathbf{i} + 5\mathbf{j}$ (b) $\overrightarrow{PQ} = 2\mathbf{i} - 5\mathbf{j}$

In general, a vector $\boldsymbol{a}$ is represented in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ as

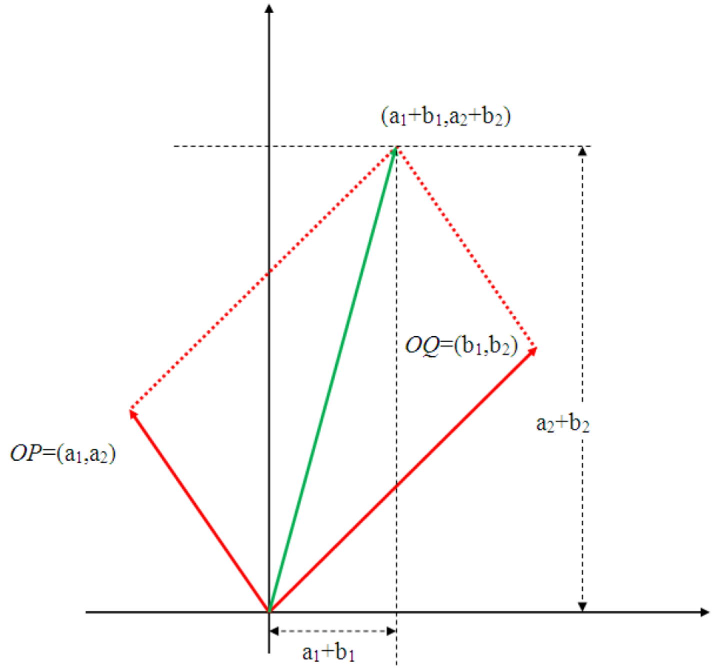
$$\boldsymbol{a} = a_1\mathbf{i} + a_2\mathbf{j},$$

where a_1 and a_2 are real numbers. Addition and subtraction of vectors then become very easy – and the machinery of the parallelogram law can be placed firmly in the background. For example,

Theorem 2.1 *Let $\boldsymbol{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\boldsymbol{b} = b_1\mathbf{i} + b_2\mathbf{j}$ be two co-planar vectors, and let $\boldsymbol{a} + \boldsymbol{b}$ be given by the paralllelogram law of vector addition. Then*

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{k}.$$

Proof: Refer to Figure 2.1. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are already positioned such that their bases coincide. A parallelogram is completed by parallel transport:



- The vector $\boldsymbol{a}$ is parallel-transported through the vector $\boldsymbol{b}$. To every point with coordinates (x, y) along the line segment OP the amount (b_1, b_2) is added. Thus, the tip of the parallel-transported vector is at $(a_1 + b_1, a_2, b_2)$.

- Similarly, the vector $\boldsymbol{b}$ is parallel-transported through the vector $\boldsymbol{b}$. To every point on the line segment OQ an amount (a_1, a_2) is added. Thus, the tip of the parallel-transported vector is again at $(a_1 + b_1, a_2 + b_2)$, and the parallelogram is completed.

It follows that $\boldsymbol{a} + \boldsymbol{b} = \overrightarrow{OR}$, whose tip has coordinates $(a_1 + b_1, a_2 + b_2)$; hence, this is the vector

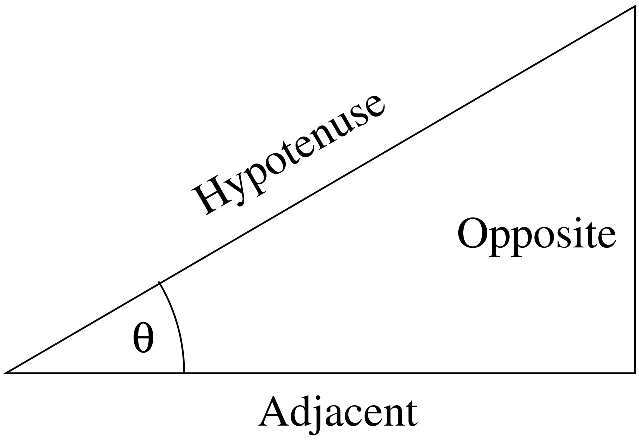
$$\boldsymbol{a} + \boldsymbol{b} = \overrightarrow{OR} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}. \quad \blacksquare$$

2.1.1 Examples

$$\begin{aligned} (3\mathbf{i} + 2\mathbf{j}) + (4\mathbf{i} + 5\mathbf{j}) &= 7\mathbf{i} + 7\mathbf{j}, \\ (3\mathbf{i} + 2\mathbf{j}) - (4\mathbf{i} + 5\mathbf{j}) &= -\mathbf{i} - 3\mathbf{j}. \end{aligned}$$

2.2 The magnitude of a vector

We revert to Equation (2.1) for the magnitude of a vector expressed in terms of Cartesian coordinates. To do this properly, we first of all need to review some trigonometry:



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

How do you remember this?

> Silly Old Harry Caught A Herring Trawling Off America
> Two Old Angels Sitting On High Chatting About Heaven
> sock a toe-a (SOHCAHTOA)
> Tom's Old Aunt Sat On Her Coat And Hat
> sin, cos, tan: Orace Had A Heap Of Apples

I went to a Gaelscoil, so we had

$$\sin(\theta) = \frac{\text{úrcóireach}}{\text{taobhagán}}, \qquad \cos(\theta) = \frac{\text{cóngrach}}{\text{taobhagán}}, \qquad \tan(\theta) = \frac{\text{úrcóireach}}{\text{cóngrach}},$$

and

> sin, cos, tan: Under The Crappy Teacher's Useless Car

which was certainly no comment on my mathematics teacher, who was outstanding. In any event, for an arbitrary vector $\boldsymbol{a} = \overrightarrow{OA}$ expressed in Cartesian coordinates as

$$\boldsymbol{a} = a_1\mathbf{i} + a_2\mathbf{j},$$

the magnitude of the vector is the length of the line segment OA, which by Pythagoras's theorem is

$$|OA| = \sqrt{a_1^2 + a_2^2},$$

hence

$$\text{magnitude of } \boldsymbol{a} \equiv |\boldsymbol{a}| \equiv |\overrightarrow{OA}| = \sqrt{a_1^2 + a_2^2},$$

which is simply Equation (2.1) re-expressed in slightly different notation. An example of a magnitude calculation is shown in Figure 2.1.

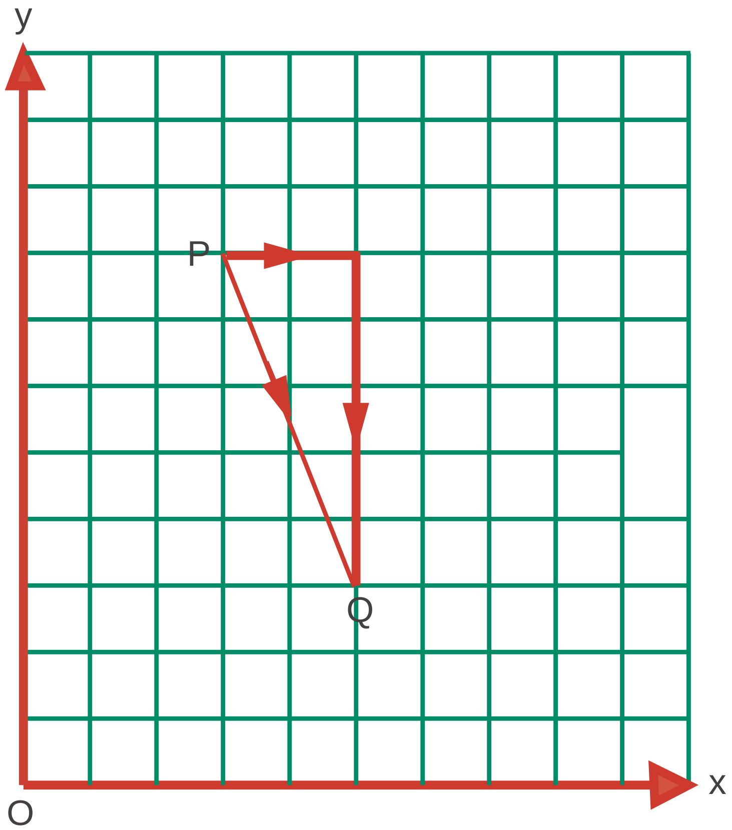
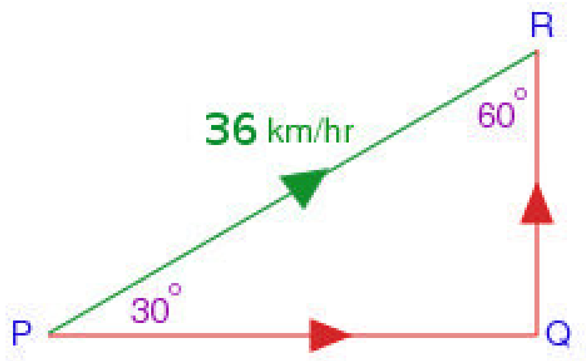


Figure 2.1: $|\overrightarrow{PQ}| = |2\mathbf{i} - 5\mathbf{j}| = \sqrt{4+25} = \sqrt{29}$

2.2.1 Worked examples



A plane taking off at an angle of $30°$ to the runway at 36 km/hr. Find the velocity components.

Solution: The vectors in the figure and their associated magnitudes are converted into an abstract trigonometric problem in Figure 2.2 below.

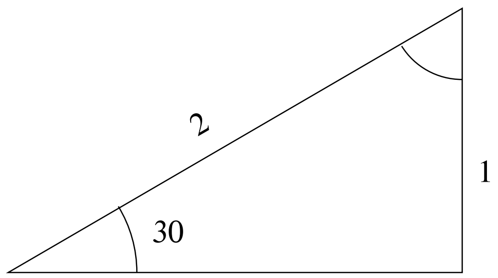


Figure 2.2:

We have,

$$36 \text{ km/hr} = \frac{36\times1000}{60\times60} = \frac{36000}{3600} = 10 \text{ m/s}.$$

$$\text{Horizontal component: } PQ = 10\cos 30^\circ = 10\frac{\sqrt{3}}{2} = 8.7 \text{ m/s}.$$

$$\text{Vertical component: } QR = 10\sin 30^\circ = 10\frac{\sqrt{1}}{2} = 5 \text{ m/s}.$$

Example: $\boldsymbol{v}$ is in the direction of the vector $12\mathbf{i} - 5\mathbf{j}$ and has magnitude 39. Find $\boldsymbol{v}$.

Solution: $\boldsymbol{v}$ must be a multiple of $12\mathbf{i} - 5\mathbf{j}$, so we have

$$\begin{aligned}
\boldsymbol{v} &= k(12\mathbf{i} - 5\mathbf{j}) \qquad k > 0 \\
&= 12k\mathbf{i} - 5k\mathbf{j} \\
|\mathbf{v}|^2 &= 144k^2 + 25k^2 \\
&= 169k^2 \\
|\mathbf{v}| &= 13k = 39 \quad k = 3 \\
\mathbf{v} &= 36\mathbf{i} - 15\mathbf{j}
\end{aligned}$$

Example: Let $\boldsymbol{p} = 4\mathbf{i} - 3\mathbf{j}$ and $\boldsymbol{q} = -12\mathbf{i} + 5\mathbf{j}$. The vector $\boldsymbol{v}$ has the same direction as $\boldsymbol{p} - \boldsymbol{q}$ and has half the size of $\boldsymbol{p} + \boldsymbol{q}$. Find $\boldsymbol{v}$.

Solution: We have

$$\boldsymbol{p} - \boldsymbol{q} = (4 + 12)\mathbf{i} + (-3 - 5)\mathbf{j} = 16\mathbf{i} - 8\mathbf{j}.$$

Also,

$$\boldsymbol{p} + \boldsymbol{q} = (4 - 12)\mathbf{i} + (-3 + 5)\mathbf{j} = -8\mathbf{i} + 2\mathbf{j},$$

with

$$|\boldsymbol{p}+\boldsymbol{q}| = \sqrt{8^2+2^2} = \sqrt{68}.$$

In the question, we are given that

$$\boldsymbol{v} = k(\boldsymbol{p}-\boldsymbol{q}) = k(16\mathbf{i}-8\mathbf{j}), \qquad k>0,$$

hence

$$|\boldsymbol{v}| = k\sqrt{16^2+64^2} = k\sqrt{320}.$$

But $|\boldsymbol{v}| = (1/2)|\boldsymbol{p}+\boldsymbol{q}|$ from the question, hence

$$|\boldsymbol{v}| = (1/2)\sqrt{68} = k\sqrt{320},$$

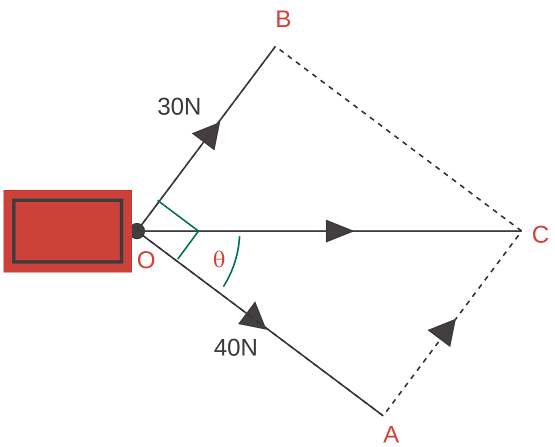
and

$$k = \tfrac{1}{2}\sqrt{\frac{68}{320}} = \sqrt{\frac{68}{4\times 320}} = \sqrt{\frac{17}{320}}.$$

Thus

$$\boldsymbol{v} = k(16\mathbf{i}-8\mathbf{j}) = \sqrt{\frac{17}{320}}(16\mathbf{i}-8\mathbf{j}),$$

which is the final answer.



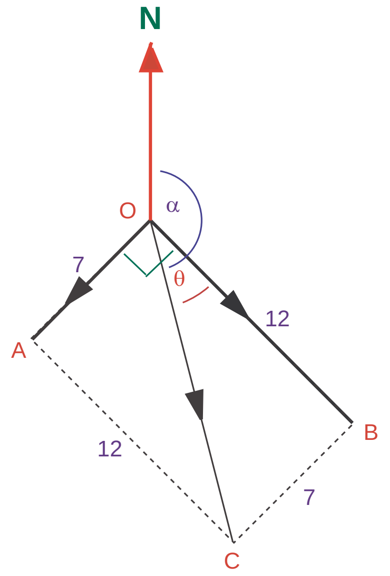
Example: Refer to the figure. Find:

- Magnitude of $\overrightarrow{OC}$
- θ

Solution: Magnitude of $\overrightarrow{OC} = \sqrt{(40)^2+(30)^2} = \sqrt{1600+900} = \sqrt{2500} = 50$

$$\tan\theta = \frac{30}{40} = \frac{3}{4} = 0.75$$
$$\theta = \tan^{-1}(\tfrac{3}{4}).$$

Correct to two significant figures, we have (using a calculator) $\theta \approx 37°$.



Example: Refer to the figure. Find the resultant (magnitude and α) of two velocities 7 km/hr SW and 12 km/hr SE.

Solution: We have

$$\begin{aligned} |\overrightarrow{OC}|^2 &= (12)^2 + 7^2 \\ &= 144 + 49 = 193 \\ |\overrightarrow{OC}| &= \sqrt{193} \end{aligned}$$

Correct to three significant figures this is 13.9: Resultant has magnitude 13.9 km/hr (correct to three significant figures). Also,

$$\begin{aligned} \tan\theta &= 7/12 \\ \theta &= \tan^{-1}(\tfrac{7}{12}) \\ \alpha &= \tfrac{\pi}{2} + \tfrac{\pi}{4} + \theta = \tfrac{3\pi}{4} + \theta. \end{aligned}$$

Thus,

$$\alpha = \tfrac{3\pi}{4} + \tan^{-1}(\tfrac{7}{12}).$$

Correct to three significant figures, this is

$$\alpha = 165.3^\circ, \text{ correct to three significant figures.}$$

Chapter 3

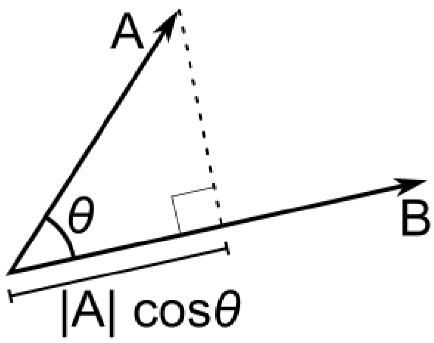
The dot product

Overview

We introduce a kind of multiplication between two vectors that takes in two vectors and returns a scalar quantity. This is called the **dot product**. The dot product provides a very useful way of computing the magnitude of a vector.

3.1 The definition

Definition 3.1 *The* **dot product** *of two vectors is the product of the magnitudes of the two vectors and the cosine of the angle between them. Equivalently, it is the projection of the first vector onto the second vector. Referring to the figure, the dot product between the vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is*



$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta.$$

For technical reasons, and by convention, the angle θ between the two vectors is taken to be the smaller of the two angles that can be produced between the vectors. Also,

- The quantity $\boldsymbol{a} \cdot \boldsymbol{b}$ is a scalar, hence the dot product is equivalently referred to as the "scalar product".
- If $\theta = 0$, then $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$.
- Hence, the dot product of a vector with itself is $|\boldsymbol{a}|^2$:

$$\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2.$$

Thus, another way to look at the magnitude of a vector is as follows:

$$\text{magnitude of } \boldsymbol{a} \equiv |\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}.$$

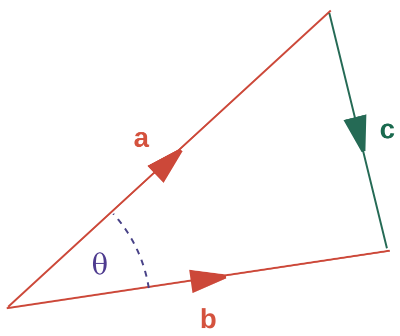
- Just as importantly, if $\theta = \pi/2$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$: two vectors are are right angles (orthogonal) if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

The dot product is also very intimately connected to the cosine rule, as the following theorem shows:

Theorem 3.1 *Let* $\boldsymbol{c} = \boldsymbol{b} - \boldsymbol{a}$*. Then*

$$\boldsymbol{a} \cdot \boldsymbol{b} = \tfrac{1}{2}\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - |\boldsymbol{c}|^2\right).$$

Proof: Refer to the figure, noting in particular the vector $\boldsymbol{c} = \boldsymbol{b} - \boldsymbol{a}$. By the cosine rule,



$$\begin{aligned} |\boldsymbol{c}|^2 &= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta \\ &= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\boldsymbol{a} \cdot \boldsymbol{b}. \end{aligned}$$

Re-arranging gives

$$\boldsymbol{a} \cdot \boldsymbol{b} = \tfrac{1}{2}\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - |\boldsymbol{c}|^2\right). \quad \blacksquare$$

3.2 The dot product in a Cartesian coordinate system

Now, the neatest thing of all about the dot product is that it is very easily computed in Cartesian coordinates:

Theorem 3.2 *Let* $\boldsymbol{a} = a_1\mathbf{i} + a_2\mathbf{j}$ *and let* $\boldsymbol{b} = b_1\mathbf{i} + b_2\mathbf{j}$ *be two co-planar vectors. Then*

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2.$$

Proof: Refer to Figure 3.1.

We have

$$\boldsymbol{c} = \boldsymbol{b} - \boldsymbol{a} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j}.$$

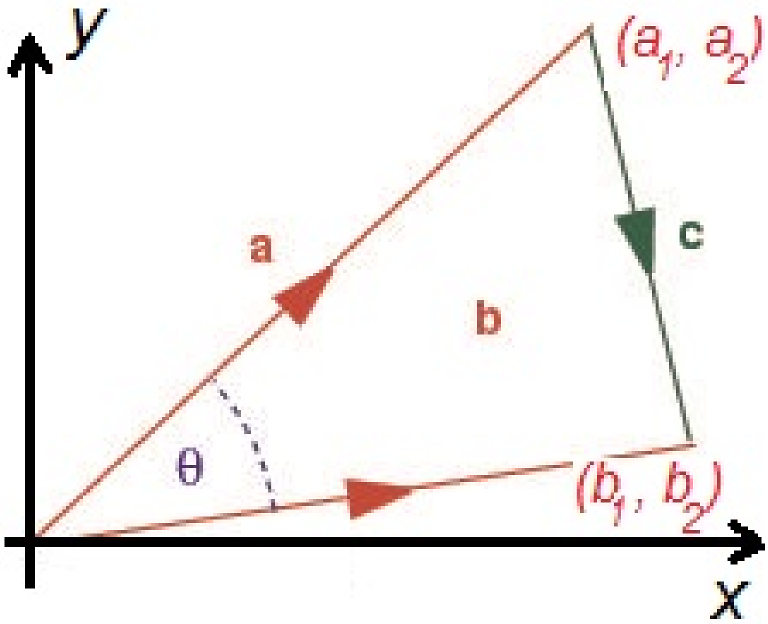


Figure 3.1: Sketch for the proof of Theorem 3.2

Using Theorem 3.1 we have

$$\begin{aligned}
\boldsymbol{a}\cdot\boldsymbol{b} &= \tfrac{1}{2}\left(|\boldsymbol{a}|^2+|\boldsymbol{b}|^2-|\boldsymbol{c}|^2\right),\\
&= \tfrac{1}{2}\left\{\left(a_1^2+a_2^2\right)+(b_1^2+b_2^2)-\left[(b_1-a_1)^2+(b_2-a_2)^2\right]\right\},\\
&= \tfrac{1}{2}\left[a_1^2+a_2^2+b_1^2+b_2^2-b_1^2-a_1^2+2a_1b_1-b_2^2-a_2^2+2b_2a_2\right].
\end{aligned}$$

Tidying up, this is

$$\boldsymbol{a}\cdot\boldsymbol{b}=a_1b_1+a_2b_2. \qquad \blacksquare$$

3.3 Worked examples

Example: If $\boldsymbol{a}=2\mathbf{i}+3\mathbf{j}$ and $\boldsymbol{b}=4\mathbf{i}+15\mathbf{j}$, find $\boldsymbol{a}\cdot\boldsymbol{b}$.

Solution:

$$\begin{aligned}
\boldsymbol{a}\cdot\boldsymbol{b} &= (2\mathbf{i}+3\mathbf{j})\cdot(4\mathbf{i}+15\mathbf{j}),\\
&= 2\times 4+3\times 15,\\
&= 53.
\end{aligned}$$

Example: Let $\boldsymbol{a} = 3\mathbf{i} + \mathbf{j}$ and $\boldsymbol{b} = \mathbf{i} - \mathbf{j}$. Find the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

Solution: we have $|\boldsymbol{a}| = \sqrt{3^2 + 1} = \sqrt{10}$ and $|\boldsymbol{b}| = \sqrt{2}$. Also, $\boldsymbol{a} \cdot \boldsymbol{b} = 3 - 1 = 2$. We now use

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta \implies \cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

to write

$$\cos\theta = \tfrac{2}{\sqrt{10}\sqrt{2}} = \sqrt{\tfrac{4}{20}} = \tfrac{1}{\sqrt{5}},$$

hence

$$\theta = \cos^{-1}\left(\tfrac{1}{5}\right).$$

Example: Find a unit vector perpendicular to $\boldsymbol{a} = \mathbf{i} + 3\mathbf{j}$.

Solution: let $\boldsymbol{b} = x\mathbf{i} + y\mathbf{j}$ be the required vector. We have $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, hence

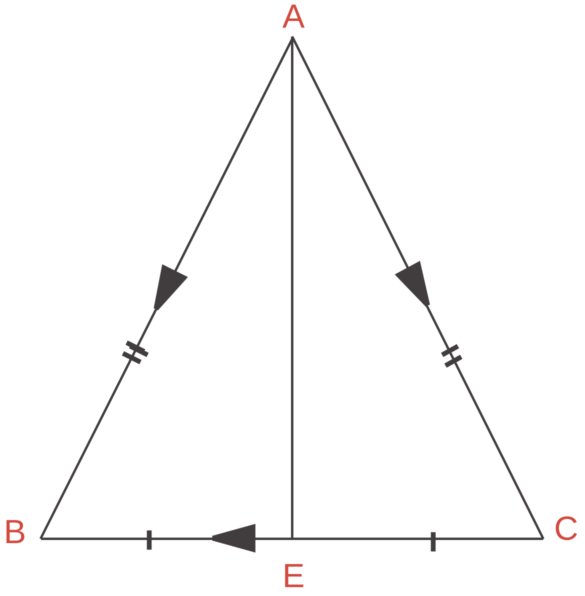
$$x + 3y = 0,$$

hence $x = -3y$. Thus, the vector can be rewritetn as $\boldsymbol{b} = y(-3\mathbf{i} + \mathbf{j})$. The second condition is that this is a unit vector, such that $\boldsymbol{b} \cdot \boldsymbol{b} = 1$, or

$$y^2(3 \cdot 3 + 1 \cdot 1) = 1,$$

hence $y^2 = 1/10$, hence $y = \pm 1/\sqrt{10}$, and the positive sign is chosen (this is an arbitary choice). Thus, the required vector is

$$\boldsymbol{b} = \tfrac{1}{\sqrt{10}}(-3\mathbf{i} + \mathbf{j}).$$



Example: Refer to the figure. Show that $AE \perp BC$.

Solution: We have

$$\overrightarrow{AE} = \overrightarrow{AC} + \tfrac{1}{2}\overrightarrow{CB}, \qquad \overrightarrow{CB} = \overrightarrow{AB} - \overrightarrow{AC}.$$

Hence,

$$\overrightarrow{AE} = \overrightarrow{AC} + \tfrac{1}{2}\left(\overrightarrow{AB} - \overrightarrow{AC}\right) = \tfrac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right).$$

Compute $\overrightarrow{CB} \cdot \overrightarrow{AE}$:

$$\begin{aligned} \overrightarrow{CB} \cdot \overrightarrow{AE} &= \tfrac{1}{2}\left(\overrightarrow{AB} - \overrightarrow{AC}\right) \cdot \left(\overrightarrow{AB} + \overrightarrow{AC}\right), \\ &= \tfrac{1}{2}\left(|\overrightarrow{AB}|^2 + \overrightarrow{AB} \cdot \overrightarrow{AC} - \overrightarrow{AC} \cdot \overrightarrow{AB} - |\overrightarrow{AC}|^2\right), \\ &= \tfrac{1}{2}\left(|\overrightarrow{AB}|^2 - |\overrightarrow{AC}|^2\right), \end{aligned}$$

which is zero, because $|AB| = |AC|$, hence $\overrightarrow{CB} \cdot \overrightarrow{AE} = 0$, hence

$$\overrightarrow{CB} \perp \overrightarrow{AE},$$

as required.

Chapter 4

The cross product

Overview

In the previous chapter, we saw how two vectors can be combined together to produce a scalar. Now, we look at how two vectors can be combined together to produce another vector. To do this, we must make an excursion into the third dimension.

4.1 The third dimension

In order to construct the cross product, we must look at vectors in three-dimensional space. Therefore, we introduce a Cartesian coordinate frame with coordinates (x, y, z) – the z-coordinate labels the third dimension. We have:

- The unit vector in the x-direction is $\mathbf{i}$.
- The unit vector in the y-direction is $\mathbf{j}$.
- The unit vector in the z-direction is $\mathbf{k}$.

As these are mutually orthogonal directions, we have:

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0.$$

A general vector $\boldsymbol{a}$ in three dimensions can therefore be written as:

$$\boldsymbol{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

where a_1, a_2, and a_3 are the components of $\boldsymbol{a}$ in the different Cartesian dimensions.

4.2 The definition

The cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be defined with respect to Cartesian coordinates. We start with the expressions

$$\boldsymbol{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \qquad \boldsymbol{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}.$$

Then, the cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted by $\boldsymbol{a}\times\boldsymbol{b}$, is constructed via the following **determinant**:

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{4.1}$$

This is probably a novel expression for students. What it means will be explained in later modules. For now, we will just learn how to evaluate the determinant. This is done as follows:

$$\begin{aligned} \boldsymbol{a}\times\boldsymbol{b} = \mathbf{i}\,&(\text{Strike out the first row and the first column, and evaluate the resulting sub-determinant}) \\ -\mathbf{j}\,&(\text{Strike out the first row and the second column, and evaluate the resulting sub-determinant}) \\ +\mathbf{k}\,&(\text{Strike out the first row and the third column, and evaluate the resulting sub-determinant}) \end{aligned} \tag{4.2}$$

This works out as:

$$\boldsymbol{a}\times\boldsymbol{b} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

There is now a very simple way to work out these sub-determinants via a kind of 'cross-multiplication':

$$\boldsymbol{a}\times\boldsymbol{b} = \mathbf{i}\left(a_2b_3 - a_3b_2\right) - \mathbf{j}\left(a_1b_3 - a_3b_1\right) + \mathbf{k}\left(a_1b_2 - a_2b_1\right).$$

4.2.1 Properties of the vector or cross product

1. Skew-symmetry: $\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$,
2. Linearity: $(\lambda\boldsymbol{a})\times\boldsymbol{b} = \boldsymbol{a}\times(\lambda\boldsymbol{b}) = \lambda\left(\boldsymbol{a}\times\boldsymbol{b}\right)$, for $\lambda\in\mathbb{R}$.

These results readily follow from the determinant definition. Result (1) is particularly weird. Note:

$$\begin{aligned} \boldsymbol{a}\times\boldsymbol{a} &= -\boldsymbol{a}\times\boldsymbol{a}, \qquad \text{Result (1)}, \\ 2\boldsymbol{a}\times\boldsymbol{a} &= 0, \\ \boldsymbol{a}\times\boldsymbol{a} &= 0. \end{aligned}$$

Example: Let

$$\boldsymbol{a} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}, \qquad \boldsymbol{b} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}.$$

Compute $\boldsymbol{a} \times \boldsymbol{b}$.

Solution:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\mathbf{i} - 7\mathbf{k}.$$

Show that the so-called *orthonormal triad* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, where

$$\mathbf{i} = 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = (1, 0, 0), \qquad \text{\&c.}$$

satisfies the relations

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k}, \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i}, \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}. \end{aligned} \tag{4.3}$$

4.3 Geometrical treatment of cross product

So far, our treatment of the cross product has been in terms of a particular choice of Cartesian axes. However, the definition of the cross product is in fact independent of any choice of such axes. To demonstrate this, we re-construct the cross product.

Step 1: Finding the length of $\boldsymbol{a} \times \boldsymbol{b}$ Note that

$$\begin{aligned} |\boldsymbol{a} \times \boldsymbol{b}|^2 + (\boldsymbol{a} \cdot \boldsymbol{b})^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\ &+ (a_1 b_1 + a_2 b_2 + a_3 b_3)^2, \\ &= \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right), \\ &= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2. \end{aligned}$$

Hence,

$$\begin{aligned} |\boldsymbol{a}\times\boldsymbol{b}|^2 &= |\boldsymbol{a}|^2|\boldsymbol{b}|^2-(\boldsymbol{a}\cdot\boldsymbol{b})^2\,, \\ &= |\boldsymbol{a}|^2|\boldsymbol{b}|^2\left(1-\cos^2\theta\right), \\ &= |\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta \end{aligned}$$

and

$$|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta,$$

where $0\le\theta\le\pi$, such that the relation $|\boldsymbol{a}\times\boldsymbol{b}|\ge 0$ is satisfied.

Step 2: Finding the direction of $\boldsymbol{a}\times\boldsymbol{b}$ Note that

$$\begin{aligned} \boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{b}) &= a_1\left(a_2b_3-a_3b_2\right)+a_2\left(a_3b_1-a_1b_3\right)+a_3\left(a_1b_2-a_2b_1\right), \\ &= 0. \end{aligned}$$

Similarly, $\boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{b})=0$. Hence, $\boldsymbol{a}\times\boldsymbol{b}$ is a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. It remains to find the sense of $\boldsymbol{a}\times\boldsymbol{b}$. Indeed, this is arbitrary and must be fixed. We fix it such that we have a right-handed system, and such that the following rule-of-thumb is satisfied (Fig. 4.1).

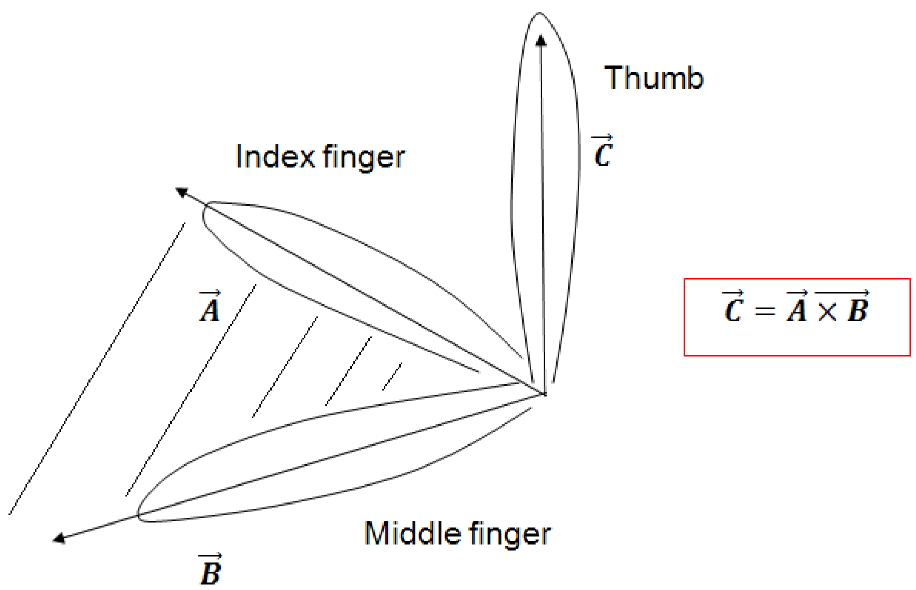


Figure 4.1: The right-hand rule.

Choosing a right-hand rule means that relations (4.3) are satisfied ($\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ form a 'right-handed' system). This also corresponds to putting a plus sign in front of the determinant in the original definition of the cross product.

In summary, $\boldsymbol{a} \times \boldsymbol{b}$ is a vector of magnitude $|\boldsymbol{a}||\boldsymbol{b}| \sin\theta$, that is normal to both $\boldsymbol{a}$ and $\boldsymbol{b}$, and whose sense is determined by the right-hand rule.

Chapter 5

Kinematics of a particle

Overview

In kinematics we are concerned with *describing a particle's motion* without analysing what causes or changes that motion (forces). In this chapter we look at particles moving in a straight line. The objectives here are as follows:

- To define concepts such as distance, displacement, speed, velocity, and acceleration.
- To study motion in a straight line with constant acceleration mathematically.

We will use **metric units:** metres for distance and seconds for time.

Before defining the fairly basic concepts mentioned above, it is helpful to define what we mean by a particle. A particle is a mathematical abstraction: it is a physical object concentrated at a single point - that is, an object with no extension in space. The kinematic laws written down in this chapter hold for such abstract entities; the laws apply equally to extended objects, as we shall demonstrate in later chapters. This generalization requires extra concepts and mathematical tools. To avoid this extra layer of complication, we shall for the present deal only with point particles.

5.1 Distance, Displacement, Speed, Velocity, Acceleration

Our first definition is that **speed**. This is easy for the case of an object that is moving **steadily**:

Definition 5.1 *The speed of an object moving steadily is*

$$\textit{speed} = \frac{\textit{distance travelled}}{\textit{time taken}}.$$

If the object is not moving steadily, that is, if the speed in the definition is not constant, we can still look at the average speed, which is defined in all cases as

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}.$$

Speed is therefore measured in metres per second (m/s).

> Example: A cyclist travels on a straight road. The first mile takes one hour. The second mile takes two hours (bad cyclist!). Draw the cyclist's speed on a distance-time graph. Hence, compute the cyclist's average speed.

Solution:

$$1\,\text{hour} = 3600\,\text{s} \qquad\qquad 1\,\text{mile} = \frac{8}{5}\,\text{km} = 1600\,\text{m}$$

Distance-time graph:

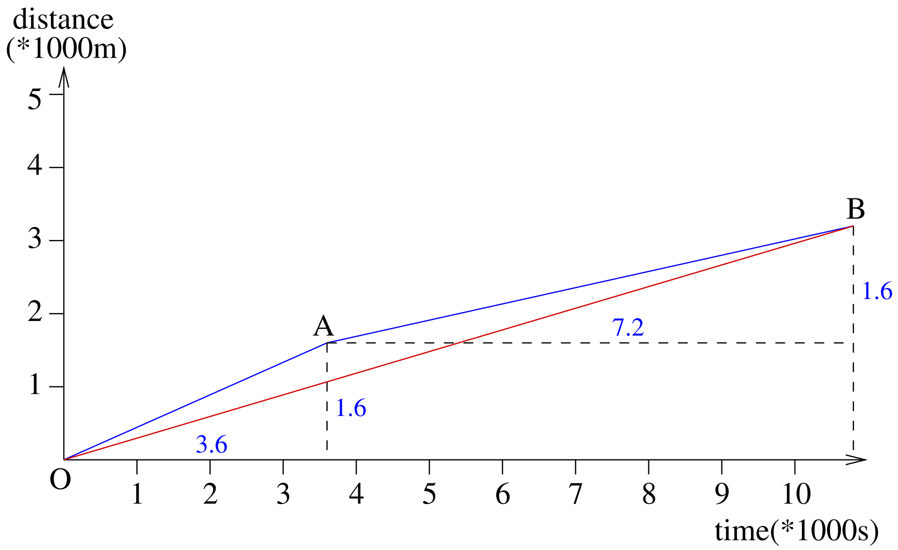


Figure 5.1:

- **Cyclist's speed** from O to A $= \frac{1600}{3600} = 0.44\,\text{m/s}$

  Slope of OA $= \frac{1600}{3600} = 0.44$

- **Cyclist's speed** from A to B $= \frac{1600}{7200} = 0.22\,\text{m/s}$

  Slope of AB $= \frac{1600}{7200} = 0.22$

- **Cyclist's average speed** from O to B

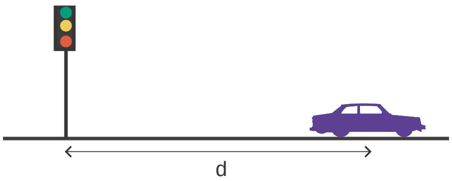
$$= \frac{1600 + 1600}{3600 + 7200} = \frac{32}{108} = 0.30\,\text{m/s} = \textbf{Slope of OB}$$

In the previous example, it can be seen that the **slope** of the distance-time graph is the cyclist's speed. The gradient changes is piecewise-constant and hence, the cyclist's speed is too – at each point in time the cyclist's speed is constant and the speed changes abruptly at one point. In contrast, objects whose speed varies smoothly will have a curves distance-time graph, as in the following example.

Example: A car starts from rest at a traffic light and moves away a distance d after time t, according to the following table:

t	0	1	2	3	4	5
d	0	2	8	18	32	50

What is the average speed of the car, between $2s$ and $4s$? Here, time is expressed in seconds and distance in metre.



Solution: A distance-time graph is shown in Figure 5.2.

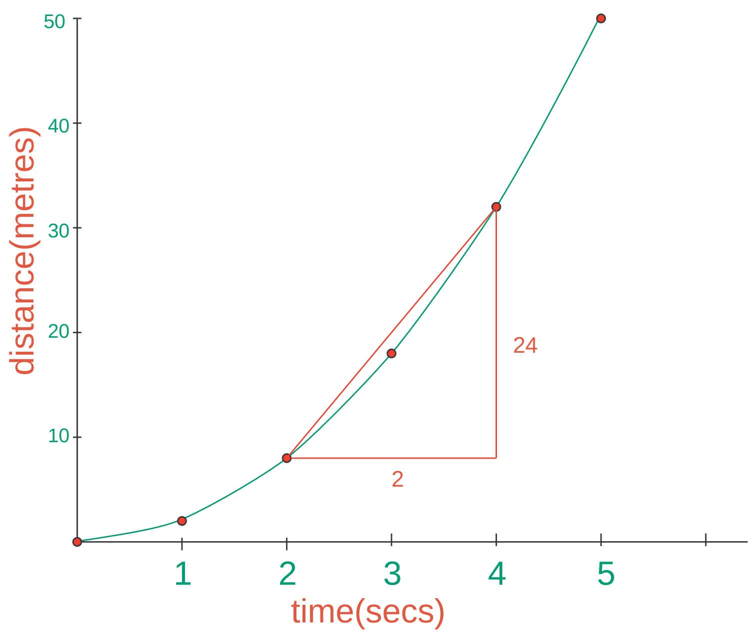
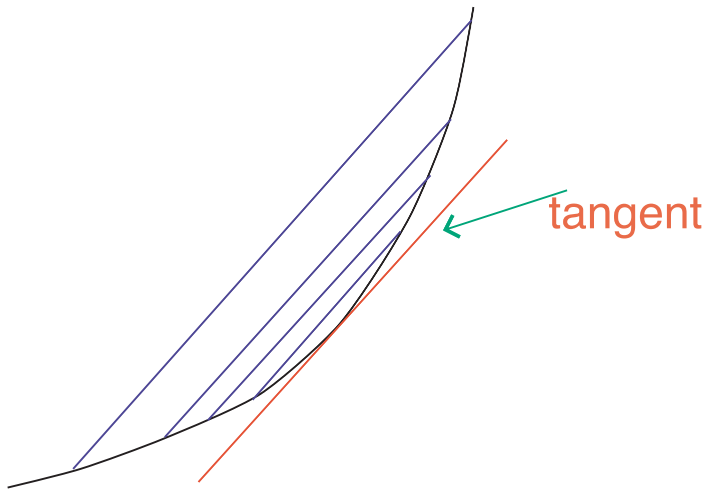


Figure 5.2: Distance-time graph

Reading off from the graph (or from the table), the verage speed from 2 sec to 4 sec is $\frac{24}{2} = 12$ m/s.

5.1.1 The instantaneous speed

If we want to find the speed at a particular time we have to take the **average over smaller and smaller intervals**. The slope of these chords approaches the slope of the tangent (Figure 5.1.1):



We then have the following very important result / definition:

Definition 5.2 *The* **instantaneous speed** *at a particular value of t is given by the slope of the tangent of the distance-time graph at the point where t has that value.*

Equivalently, using Calculus, we can write

$$v = \frac{\mathrm{d}s}{\mathrm{d}t},$$

where

- v is the instantaneous speed (function of time)
- s is the distance travelled at time t.

5.1.2 Velocity

Recall the notion of **displacement** as the **signed distance** from the origin. This gives rise to related notion of **instantaneous velocity**

Definition 5.3 *The instantaneous velocity is the rate of change of the displacement with respect to time.*

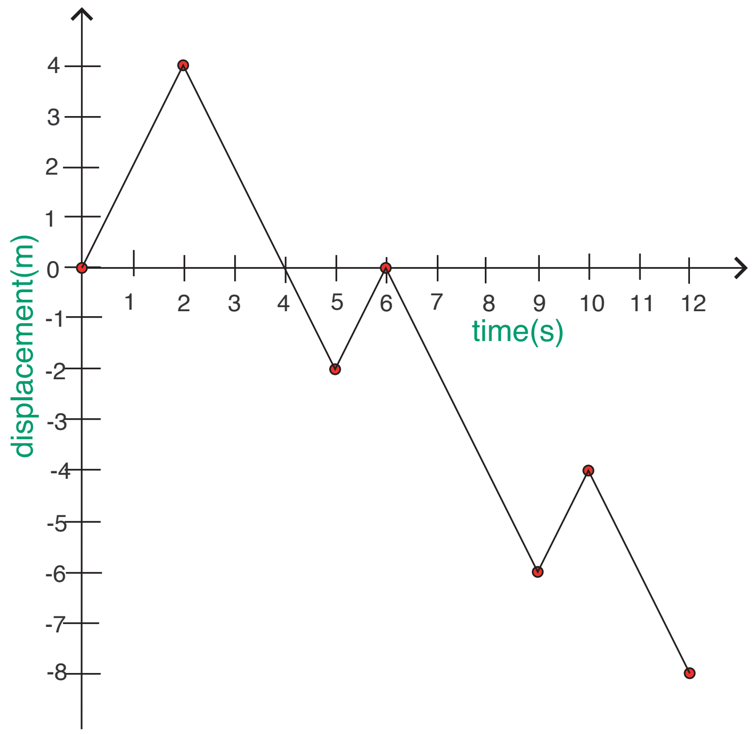
Equivalently, it is the speed with directional information.

Example: A particle moves at **constant speed** but its displacement-time graph has the following characteristics:

Time (s)	0	2	5	6	9	10	12
Displacement from Origin	0	+4	-2	0	-6	-4	-8

Draw the displacement-time graph.

Solution – Figure 5.1.2



- Particle travels $4\,\mathrm{m}$ in 2 seconds.
- Speed is $2\,\mathrm{m/s}$, and is constant!
- But Slope of graph is ± 2 – **Velocity** alternates between $+2$ and -2.
- A change in the velocity is called **acceleration**.

Definition 5.4 *The instantaneous acceleration is the rate of change of velocity with respect to time:*

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}, \tag{5.1}$$

where v is the velocity.

- If $a > 0$ then the velocity is increasing – 'speeding up'.
- If $a < 0$ then the velocity is decreasing – 'slowing down'. Sometimes a negative acceleration is called **deceleration**.

Also,

- Units of acceleration are therefore units of velocity divided by units of time.
- We write this as $\mathbf{m/s}^2$ and we say this is 'metres per second per second' or 'metres per second squared'.
- The slope of a velocity-time plot tells us the acceleration.

Generally, the acceleration will depend on time. However, there is a very nice mathematical theory of kinematics for the case of constant acceleration, which we study below. We first of all have a look at some examples.

> Example: A racing car accelerates rapidly from the grid. Starting from a stationary state, it reaches 100 kilometres per hour in just 4 seconds. What is the average acceleration over this period?

Solution: The instantaneous acceleration is given by Equation (5.1). The **average acceleration** is got by replacing the infinitesimal amounts $\mathrm{d}v$ and $\mathrm{d}t$ with finite increments:

$$\text{Average Acceleration over a period } \Delta t = \frac{\Delta v}{\Delta t},$$

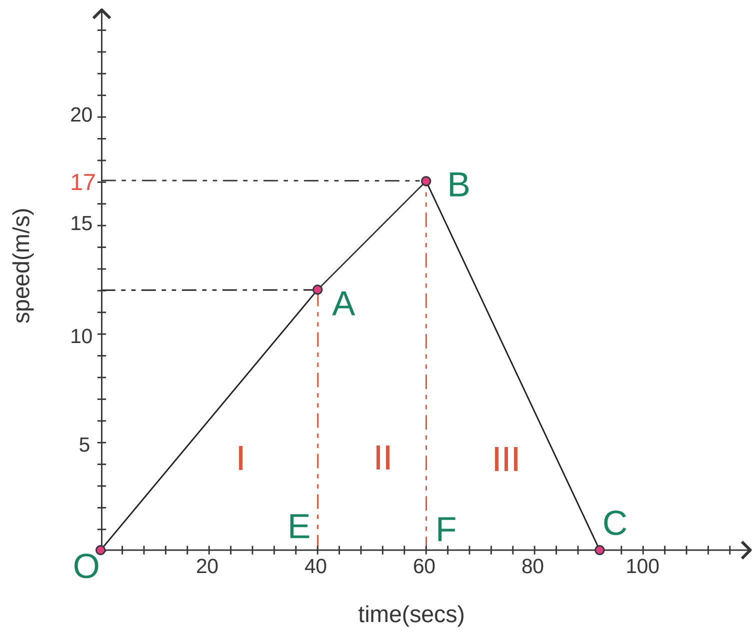
and in the present example, $\Delta v = 100\,\text{kph}$ and $\Delta t = 4\,\text{sec}$. Hence,

$$\text{Average acceleration} = \frac{\text{change in velocity}}{\text{time taken}} = \frac{100\ \text{kph}}{4\ \text{sec}}$$

$$100\ \text{kph} = \frac{100\,000}{3\,600}\ \text{m/s} \approx 28\ \text{m/s}$$

$$\text{Average acceleration} = \frac{28\ \text{m/s}}{4\ \text{sec}} = 7\ \text{m/s}^2$$

Compare this to the acceleration of gravity, $9.81\ \text{m/s}^2$!



Example: A particle moves in the positive direction and has the velocity-time curve shown in Figure 5.1.2. Compute the (constant) acceleration in each of the phases I, II, and III.

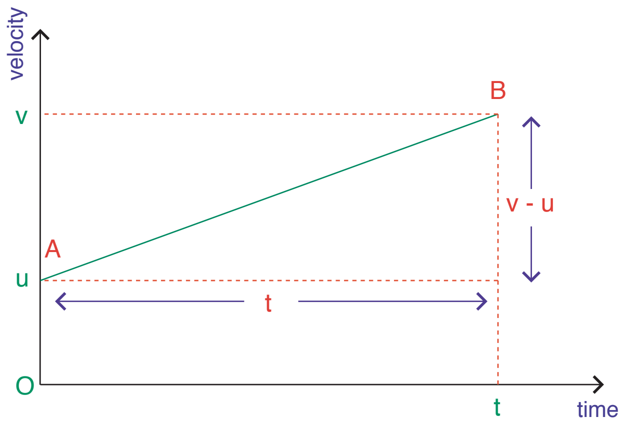
Solution:

- **Phase I**: The speed increases from 0 to 12 m/s in 40s. The acceleration $= \frac{12}{40} = 0.3$ m/s^2. The slope of OA $= \frac{12}{40} = 0.3$.

- **Phase II**: The speed increases form 12m/s to 17m/s. The acceleration $= \frac{5}{20} = 0.25$ m/s^2. The slope of AB $= \frac{5}{20} = 0.25$.

- **Phase III**: The speed **decreases** from 17 m/s to 0 m/s in 32 s. The deceleration $= \frac{17}{32} = 0.53$ m/s^2.

5.2 Mathematical description of kinematics with constant acceleration



Consider a particle launched from the origin (zero initial displacement) with initial velocity u. At a final time t, the particle has a velocity v (Figure 5.2). The particle experiences a **constant acceleration** a. We now work out a mathematical description between the final displacement s, and the quantities a, u, and v.

Our starting-point is the definition of the instantaneous acceleration in calculus notation:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}.$$

We integrate this expression:

$$\begin{aligned} \int_0^t a\,\mathrm{d}t &= \int_0^t \frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t, \\ &= v(t) - v(0), \\ &= v - u. \end{aligned}$$

However, the acceleration is constant and therefore comes outside the integral, to give $\int_0^t a\,\mathrm{d}t = at$, hence

$$\boxed{v = u + at} \tag{5.2}$$

We can also make use of the expression

$$v = \frac{\mathrm{d}s}{dt}$$

and integrate as before to obtain

$$s = \int_0^t vt.$$

Use the expression (5.2) in the integrand here:

$$s = \int_0^t (u + at)\,\mathrm{d}t.$$

Hence,

$$\boxed{s = ut + \tfrac{1}{2}at^2.} \tag{5.3}$$

It is sometimes useful to work with expressions that do not involve t. We can eliminate t between Equations (5.2) and (5.3) by taking Equation (5.2) and writing

$$t = \frac{v - u}{a}.$$

Substitute into Equation (5.3):

$$\begin{aligned} s &= ut + \tfrac{1}{2}at^2, \\ &= u\left(\frac{v-u}{a}\right) + \tfrac{1}{2}\left(\frac{v-u}{a}\right)^2, \\ &= \frac{uv - u^2 + \tfrac{1}{2}v^2 - uv + \tfrac{1}{2}v^2}{a}. \end{aligned}$$

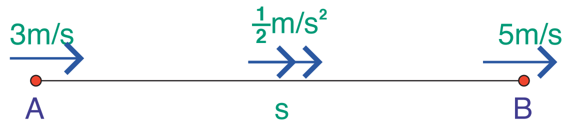
Hence,

$$as = \tfrac{1}{2}\left(v^2 - u^2\right).$$

Tidy up to obtain

$$\boxed{v^2 = u^2 + 2as.} \tag{5.4}$$

> Example: A particle starts from a point A with velocity 3 m/s and moves with a constant acceleration of $\frac{1}{2}$m/s^2 along a straight line AB. It reaches B with a velocity of 5 m/s.
> Find:
> (a) the displacement from A to B.
> (b) the time taken from A to B.



Solution:

- For motion in a straight line with constant acceleration we have five quantities: u, v, s, a, t.
- Each of the equations contains four of the five.
- Which of the equations should we use?
- It depends on the problem. In the present context we anticipate that different formulas should be for (a) and (b) above. In particular, we know: $u = 3$ $v = 5$ $a = \frac{1}{2}$, so we should focus either on eliminating s or t.

(a) $s = ?$ – Avoid formula with t in it:

$$\begin{aligned} v^2 - u^2 &= 2as \\ 25 - 9 &= 2 \cdot \frac{1}{2} \cdot s \\ s &= 16 \text{ metres} \end{aligned}$$

(b) $t = ?$ – Avoid formula with s in it:

$$\begin{aligned} v &= u + at \\ 5 &= 3 + \frac{1}{2}t \\ t &= 4 \text{ seconds} \end{aligned}$$

> Example: The driver of a train begins the approach to the station by applying the brakes to produce a steady deceleration of 0.2 m/s^2 and bring the train to rest at the platform in 1 min 30 secs.

Find:
(a) the speed when the brakes were applied,
(b) the distance travelled.

Solution: We know: $t = 90$, $v = 0$, and $a = -0.2$. (a) Initial speed

$$\begin{aligned} u &= ? \\ v &= u + at. \\ 0 &= u - 0.2 \times 90. \\ u &= \frac{2}{10} \times 90 = 18 \text{ m/s}. \end{aligned}$$

(b) Distance travelled

$$\begin{aligned} s &= ? \\ v^2 - u^2 &= 2as. \\ 0 - (18)^2 &= 2(-0.2)s. \\ s &= 810 \text{ m}. \end{aligned}$$

5.3 Sprinting

Example: A world-class sprinter accelerates with constant acceleration to his maximum speed in $4.0\,\mathrm{s}$. he then maintains this speed for the remainder of a 100-m race, finishing with a total time of $9.1\,\mathrm{s}$.

(a) What is the runner's average acceleration during the first 4.0s?
(b) What is his average acceleration during the last 5.1s?
(c) What is his average acceleration for the entire race?
(d) Explain why your answer to part (c) is not the average of the answers to parts (a) and (b).

Solution: It is helpful to view a velocity-versus-time plot, as in the sketch in Figure 5.3.

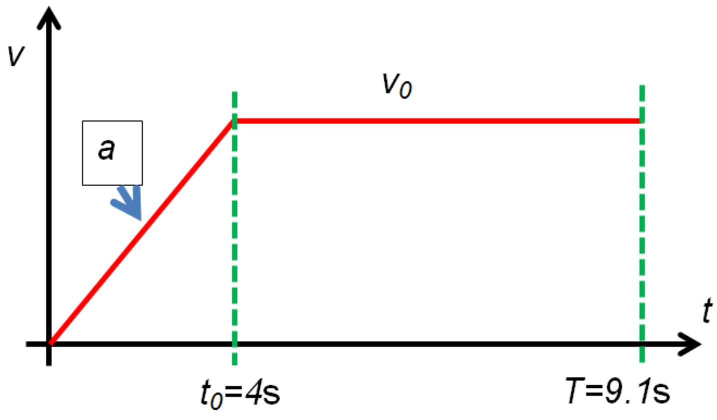


Figure 5.3:

The strategy for Part (a) is to find the displacements s_0 and s_1 in Phases I and II in terms of the acceleration a. In Phase I we have

$$s_0 = \frac{1}{2}at_0^2,$$

where $t_0 = 4$s is the duration of Phase I. In Phase II we have

$$s_1 = v_0(T - t_0),$$

where v_0 is the unknown final velocity, $T = 9.1$s and $T - t_0 = 5.1$ is the duration of Phase II. The velocity v_0 and the acceleration a are connected via

$$a = v_0/t \implies v_0 = at_0.$$

Combine these results to get

$$s_0 = \frac{1}{2}at_0^2, \qquad s_1 = at_0\left(T - t_0\right).$$

But

$$s_0 + s_1 = 100\text{m},$$

hence

$$\frac{1}{2}at_0^2 + at_0\left(T - t_0\right) = 100\text{m},$$

hence

$$at_0\left(T - \frac{1}{2}t_0\right) = 100\text{m}.$$

Fill in the numbers:

$$a\left(4\text{s}\right)\left(7.1\text{s}\right) = 100\text{m},$$

hence

$$a = (100/28.4) = 3.5\,\mathrm{m/s^2},$$

correct to two significant figures.

For part (b) the velocity is constant and hence the acceleration is zero.

For part (c) the average acceleration over the whole race is

$$a_{\rm av} = \frac{v_{\rm final} - v_{\rm initial}}{T} = \frac{v_0}{T} = a(t_0/T).$$

Filling in the numbers, this is

$$a_{\rm av} = (3.52) \times (4/9.1) = 1.5\mathrm{m/s^2},$$

correct to two significant figures.

For part (d), a simple average of the two accelerations gives $(3.5+0)/2 = 1.75\mathrm{m/s^2}$, which is the wrong answer. The reason this is the wrong answer is because the average should be weighted by how much time is spent in each phase. The corrected weighted average would be

$$a_{\rm av} = a\left(\frac{t_0}{T}\right) + 0\left(\frac{T - t_0}{T}\right) = a(t_0/T),$$

in agreement with the average computed in part (c).

5.3.1 Realism

Video analysis of a sprinter's behaviour during a race shows that (to a first approximation) breaking up the race into a constant-acceleration phase and a constant-velocity phase is a good approximation. However, the numbers in this example are not realistic. The world records are listed below.

- Women: Florence Griffin Joyner $10.49\,\mathrm{s}$ (US Olympics Trials, Indianapolis 1988). It is suggested that there was a faulty anemometer at these trials, and that 'Flo Jo' may have benefited from a greater-than-regulation tailwind. In contrast, Flo Jo's Olympic Record was $10.62\,\mathrm{s}$ at the Seoul Olympic Games in 1988. This was improved in the Tokyo Olympic Games (2020, held in 2021) by Elaine Thompson–Herah ($10.61\,\mathrm{s}$).
- Men: Usain Bolt $9.58\,\mathrm{s}$ (World Championships, Berlin, 2009).

Interestingly, Florence Griffin Joyner's record means she is ranked 6804 in the list of fastest sprinters in history. The fact that the other 6803 sprinters ahead of Flo Jo in the list are male is because on average, men are bigger, stronger, and faster than women. That is why the IAAF maintains a protected female classification, to ensure fair competition for women athletes[1].

[1] https://worldathletics.org/news/press-release/questions-answers-iaaf-female-eligibility-reg

Chapter 6

Free-fall motion under gravity

Overview

Galileo showed that all falling objects accelerate towards the Earth at a rate g that is **independent of their mass**, with

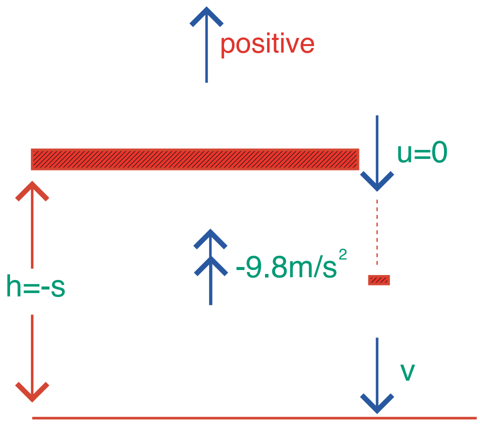
$$\boxed{g \approx 9.81\,\mathrm{m/s^2}.} \tag{6.1}$$

This means that we can solve simple problems involving falling objects using the formulae from the last chapter.

6.1 Calculations

In a later chapter we will give a theoretical underpinning to Equation (6.1). However, in the present chapter we simply use the equation as-is.

Example: A brick is dropped from a scaffold board and hits the ground 3 secs later. Find the height of the scaffold.



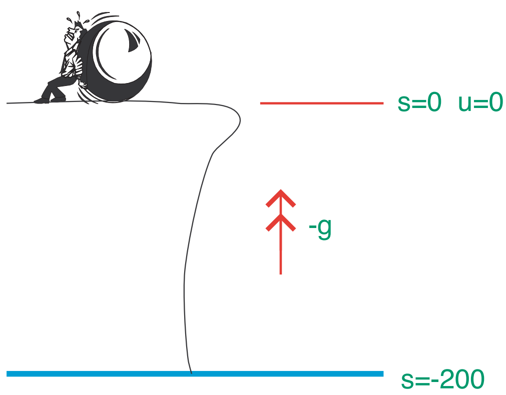
Solution: We know: $t = 3$, $u = 0$, and $a = g$. Hence,

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ &= 0 \times 3 - \frac{1}{2} \times 9.8 \times 9 \\ &= -44.1 \end{aligned}$$

Thus,

$$h = 44.1 \quad \text{metres.}$$

Example: A boulder slips from the top of a precipice and falls vertically downwards to a plain 200m below.
Find the speed when the boulder hits the plain:
(a) In a polar region where g=9.830 m/s^2.
(b) In an equatorial region where g=9.781 m/s^2.



Solution: We have

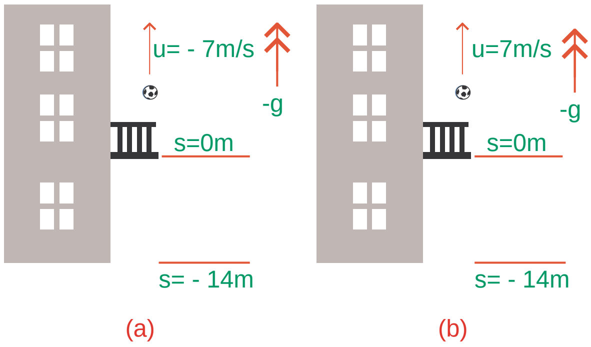
$$\begin{aligned} v^2 &= u^2 + 2as \\ &= 0 + 2(-g)(-200) \\ &= 400g, \end{aligned}$$

thus $v = 20\sqrt{g}$. We now fill in the specific values for g at the different locations on the earth's surface:

(a) $g = 9.830\,\text{m/s}^2$ and $v = 62.7$ m/s.

(b) $g = 9.781\,\text{m/s}^2$ and $v = 62.5$ m/s.

Example: A ball is thrown vertically, with a speed of 7 m/s from a balcony 14 m above the ground.
How long does it take to reach the ground if thrown:
(a) downwards,
(b) upwards.
Find the speed with which it reaches the ground.



Solution: Use $s = ut + (1/2)at^2$, with $u = \pm 7$m/s and $a = -g = 9.8\text{m}/s^2$. Choose the positive sign $u = +7$m/s for the ball being thrown vertically upwards and the negative sign for the ball being thrown vertically downwards. Also, use $s = -14$m. For the two cases, we have

Negative case – vertically downwards	Positive case – vertically upwards
$-14 = -7t - \frac{1}{2}(9.8)t^2$	$-14 = 7t - \frac{1}{2}(9.8)t^2$
$9.8t^2 + 14t - 28 = 0$	$9.8t^2 - 14t - 28 = 0$
$0.7t^2 + t - 2 = 0$	$0.7t^2 - t - 2 = 0$
$7t^2 + 10t - 20 = 0$	$7t^2 - 10t - 20 = 0$
$t = \frac{-10 \pm \sqrt{100 + 560}}{14}$	$t = \frac{10 \pm \sqrt{100 + 560}}{14}$
$t = 1.1\,s$	$t = 2.6\,s$

For the second part (speed on impact), this is

$$\begin{aligned} v^2 &= u^2 + 2as \\ v^2 &= 49 + 2(-9.8) \times (-14) \\ &= 324 \\ v &= 18 \text{ m/s} \end{aligned}$$

Note that the answer here is independent of whether the ball is initially thrown vertically upwards or downwards.

6.2 Vector Analysis

Suppose now that the projectile is projected from ground level at an initial velocity

$$\boldsymbol{v}_0 = v_{0x}\mathbf{i} + v_{0y}\mathbf{j}.$$

The angle that the velocity makes with the x-axis is therefore

$$\tan\alpha = \frac{v_{0y}}{v_{0x}}.$$

Then, gravity should also be written in vector form:

$$\boldsymbol{g} = -g\mathbf{j},$$

where $g = 9.81\,\mathrm{m/s}^2$ is intrinsically a positive quantity.

The acceleration is still constant, only now we need to write it explicitly as a vector:

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \text{Const.} = \boldsymbol{g} = -g\mathbf{j}.$$

We integrate once to find the velocity:

$$\boldsymbol{v}(t) - \boldsymbol{v}(0) = \int_0^t (-g\mathbf{j})\mathrm{d}t = -(gt)\mathbf{j}.$$

But $\boldsymbol{v}(0) = \boldsymbol{v}_0$, hence

$$\boxed{\boldsymbol{v}(t) = \boldsymbol{v}_0 - (gt)\mathbf{j}.} \tag{6.2}$$

We recall that velocity is the rate of change of displacement with respect to time:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \boldsymbol{v}_0 - (gt)\mathbf{j}.$$

We integrate once to find the displacement:

$$\boldsymbol{s}(t) - \boldsymbol{s}(0) = \int_0^t [\boldsymbol{v}_0 - gt(\mathbf{j})]\mathrm{d}t = \boldsymbol{v}_0 t - \left(\tfrac{1}{2}gt^2\right)\mathbf{j}.$$

By choosing the coordinate system so that the point of projection is also the origin, we have $\boldsymbol{s}(0) = 0$. Also,

$$\boldsymbol{s}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

Hence,

$$\boxed{x(t) = v_{0x}t, \qquad y(t) = v_{0y}t - \tfrac{1}{2}gt^2.} \tag{6.3}$$

Let $v_0 = |\boldsymbol{v}_0| = \sqrt{v_{0x}^2 + v_{0y}^2}$. Then,

$$v_{0x} = v_0 \cos\alpha, \qquad v_{0y} = v_0 \sin\alpha.$$

Hence, we can also write the **trajectory** $(x(t), y(t))$ as:

$$x(t) = v_0 \cos\alpha\, t, \qquad y(t) = v_0 \sin\alpha t - \tfrac{1}{2} g t^2.$$

6.2.1 Parabolas

Drop the explicit t-dependence from the trajectory, such that

$$x = v_0 \cos\alpha t, \qquad y = v_0 \sin\alpha t - \tfrac{1}{2} g t^2.$$

Notice that

$$t = \frac{x}{v_0 \cos\alpha}.$$

Substitute into the expression for y:

$$\begin{aligned} y &= v_0 \sin\alpha \frac{x}{v_0 \cos\alpha} - \tfrac{1}{2} g \frac{x^2}{v_0^2 \cos^2\alpha}, \\ &= (\tan\alpha)x - \tfrac{1}{2} \frac{g}{v_0^2 \cos^2\alpha} x^2. \end{aligned}$$

Thus, $y = (\text{Const.})x + (\text{Another Const.})x^2$, hence, the trajectory is a **parabola**.

6.2.2 Worked Example

A motorcycle stunt rider rides off the edge of a cliff. Just t the edge his velocity is horizontal, with magnitude $9.0\,\mathrm{m/s}$. Find the motorcycles position, distance from the edge of the cliff, and velocity $0.50\,\mathrm{s}$ after it leaves the edge of the cliff. Take $g = 9.8\,\mathrm{m/s^2}$.

Solution: Refer to Figure 6.1. The motorcyclist is experiencing only the acceleration due to gravity ('projectile motion') as soon as he leaves the edge of the cliff, which we take to be the origin. His initial velocity $\boldsymbol{v}_0$ at the edge of the cliff is horizontal, so $\boldsymbol{v}_0 = (9.0\,\mathrm{s})\mathbf{i} + 0\mathbf{j}$, hence, $v_{0x} = 9.0\mathrm{m/s}$ and $v_{0y} = 0$.

- Position: We use Equation (6.3), to compute

$$x(t) = v_{0x} t = (9.0\,\mathrm{m/s})(0.50\,\mathrm{s}) = 4.5\,\mathrm{m}.$$

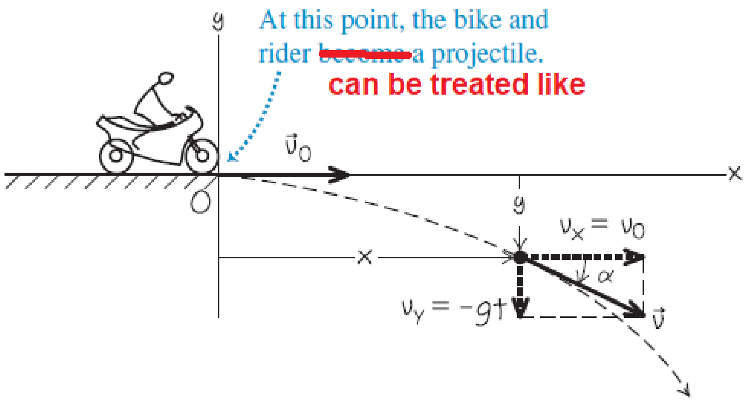


Figure 6.1: Definition sketch for the problem

Also,

$$y(t) = 0 - \tfrac{1}{2}gt^2 = -\tfrac{1}{2}(9.8\,\mathrm{m/s^2})(0.50\,\mathrm{s})^2 = -1.2\,\mathrm{m}.$$

The negative value of y shows that the motorcycle is below its starting point.

- Distance: From the figure, the motorcycles distance from the origin at $t = 0.50\,\mathrm{s}$ is:

$$r = \sqrt{x^2 + y^2} = \sqrt{(4.5\,\mathrm{m})^2 + (-1.2\,\mathrm{m})^2} = 4.7\,\mathrm{m}.$$

- Velocity: We use Equation (6.2) to compute:

$$\begin{aligned} v_x &= v_{0x} = 9.0\,\mathrm{m/s}, \\ v_y &= -gt = (-9.8\,\mathrm{m/s^2})(0.50\,\mathrm{s}) = -4.9\,\mathrm{m/s}. \end{aligned}$$

Thus, the motorcycle has the same horizontal velocity v_x as when it left the cliff at $t = 0$, but in addition, there is a downward (negative) vertical velocity v_y. The velocity vector at $t = 0.50\,\mathrm{s}$ is:

$$\boldsymbol{v} = v_x\mathbf{i} + v_y\mathbf{j} = (9.0\,\mathrm{m/s})\mathbf{i} + (-4.9\,\mathrm{m/s})\mathbf{j}.$$

Chapter 7

Forces and Newton's first two laws of motion

Overview

We have already described motion in kinematic terms, discussing ideas about velocity and acceleration. Understanding what causes such motion requires a deeper theory, called dynamics. Basically, **forces** cause motion. In this section we introduce the notion of force - first in a qualitative way and then in a rigorous mathematical framework, which requires the introduction of **Newton's laws of motion**. We discuss Newton's first two laws of motion, postponing discussion of the third law until the next chapter.

7.1 Forces

Definition 7.1 *Force is that which changes or tends to change the state of rest or uniform motion of a body in a straight line.*

The SI unit of force is the **Newton** – we will define this unit precisely in what follows. Also, force is a vector – it has magnitude and direction.

On a fundamental level in physics there are only four forces and they are

- gravitational
- electromagnetic
- weak nuclear force
- strong nuclear force

However, such brutally elegant simplicity has no place in the kind of macroscopic world that we deal with here, wherein we are concerned with the dynamics and statics of "large" objects. If I fall off a ladder unharmed the first thing I do is check if the base of the ladder was secure. I do not worry about the weak nuclear force in the constituent atoms and molecules of the ladder. Therefore, for applications in the macroscopic world around us, we classify forces differently (but on a fundamental level, perfectly equivalently), in the following way:

- Gravitational attraction and Weight
- Electric and Magnetic Forces
- Contact forces
- Attachment

7.2 Gravitational Attraction and Weight

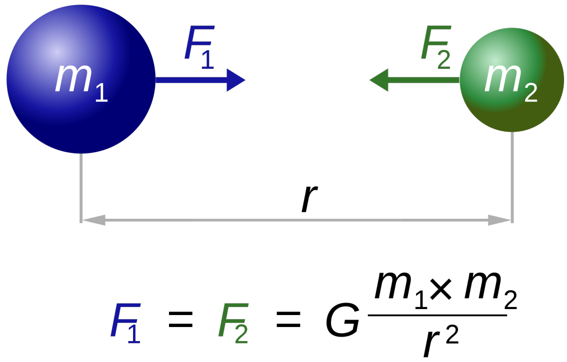
Definition 7.2 *The* **mass** *of a body is the quantity of matter that it contains.*

Newton's law of gravitation states the following:

> Any two bodies in the universe attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

The law can be written down mathematically for point particles:

$$F = -\frac{Gm_1m_2}{r^2}, \tag{7.1}$$



where

- F is the gravitational force between the two particles,
- G is the universal gravitational constant,
- m_1 and m_2 are the masses of the two particles,
- r is the distance separating the two particles,
- The minus sign indicates attraction.

This would not be a very good law if it only applied to point particles. However, the mathematical law (7.1) applies exactly to spherical bodies also, replacing the word 'particle' with 'body'; r is then the distance between the centre of mass of the two bodies.

7.2.1 Weight

Definition 7.3 *The gravitational attraction on an object on the earth's surface is called the* **weight** *of the object.*

For a point particle of mass m, by Equation (7.1), we have

$$F = \frac{GM_em}{r^2}, \tag{7.2}$$

where M_e is the mass of the earth, and r is the distance between the point particle and the earth's centre (we omit the minus sign, on the understanding that we know the force is attractive). For any extended body on the earth's surface, the body will 'look like' a point particle when viewed in comparison to the earth itself, and thus Equation (7.3) holds equally well for extended bodies. Furthermore, letting

$$r = R_e + h,$$

where

- R_e is the radius of the earth,

- h is the elevation of the body from the earth's surface,

we see that $h \ll R_\mathrm{e}$ and thus, $r = R_\mathrm{e}$ is a good approximation and finally, Equation (7.3) can be well approximated by

$$F = \left(\frac{GM_\mathrm{e}}{R_\mathrm{e}^2}\right) m. \tag{7.3}$$

We identify $GM_\mathrm{e}/R_\mathrm{e}^2$ as a constant independent of the body in question. Indeed, we shall later be able to demonstrate that g, the gravitational acceleration of Chapter 6 is expressed in fundamental terms as

$$g = \frac{GM_\mathrm{e}}{R_\mathrm{e}^2}.$$

Thus, the force F of the earth's gravity on the body can be written as

$$F = mg.$$

This is the object's weight, sometimes denoted by W:

$$W = mg.$$

Caution! Mass should not be confused with weight.

7.3 Electric and Magnetic Forces

Often, particles come with an electric charge (measured in SI units of Coulomb). The charge can be positive or negative. If two particles have a charge of q_1 and q_2 (either sign), then there is a force between them. Analogous to the gravitational force for point particles, the force for **point charges** is:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}, \tag{7.4}$$

where

- F is the **electrostatic force** between the two particles,
- $1/4\pi\epsilon_0$ is a universal constant,
- q_1 and q_2 are the charges of the two point particles,
- r is the distance separating the two particles.

As q_1 and q_2 can have either sign, F in Equation (7.4) can be positive or negative:

- If $q_1q_2 > 0$, $F > 0$, and the electrostatic force is **repulsive** – same-charged particles repel.
- If $q_1q_2 < 0$, then $F < 0$, and the electrostatic force is **attractive** – opposite-charged particles attract.

7.3.1 Electric Field

Just as the gravititational force between the Earth and a point particle can be approximated as $W = mg$, often, there are situations where the electrical (Coulomb) force between a point particle and another object can be approximated by $F = Eq$, or in vector terms, as

$$\boldsymbol{F} = \boldsymbol{E}q,$$

where q is the charge of the point particle (positive or negative) and $\boldsymbol{E}$ is a vector quantity which we call the **Electric Field** (an field is a vector that is defined at every point in space). The case of a constant electric field is particularly interesting, and there are analogies here between $d\boldsymbol{v}/dt = -g\mathbf{j}$ for gravity and the electric field, which we will explore later in Chapter 15.

7.3.2 Lorentz Force

We will not describe in detail the meaning of a magnetic field in this module. Therefore, for the time being, you can think of a constant magnetic field as simply a constant vector quantity that is present at every point in space. If a charged particle (charge q) is placed in such a constant magnetic field $\boldsymbol{B}$, the particle experiences a force called the **Lorentz force**,

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B},$$

where $\boldsymbol{v}$ is the velocity of the particle. Here, we are using the vector cross product introduced in Chapter 4.

The Lorentz force for a constant magnetic field causes a particle to move in a helix – we will look at this case in Chapter 15. The Lorentz force can be thought of as an **electrodynamic** force because it applies to moving particles.

We will not go into the physics of the magnetic field – other than to note that its SI units are the **tesla** (T). The Tesla is a derived unit, meaning it can be made up of more basic physical units:

$$\mathrm{T} = \frac{\mathrm{kg}}{\mathrm{A} \cdot \mathrm{s}^2},$$

where A is the Ampere, which is the SI unit of current, where

$$1\mathrm{A} = 1\mathrm{C/s}.$$

7.4 Contact forces

These are best understood in the context of an example – see Figure 7.1.

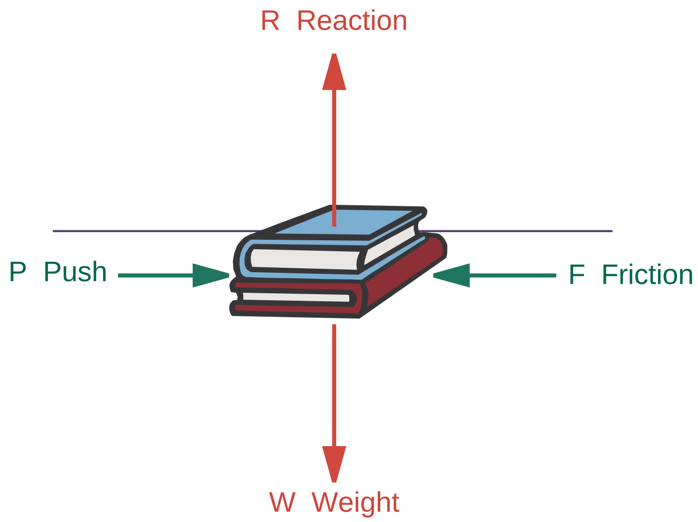
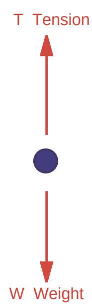


Figure 7.1: Contact forces and other forces exerted on a book

When a book is resting on a table, the table exerts a **normal reaction** force on the book, which is equal to the book's own weight. When the table is rough a **frictional force** acts on the book when we slide it across the table. This acts along the surface of contact and in a direction opposing the (potential) motion of the body.

7.4.1 Forces of attachment

Ropes, strings, etc.



7.5 The resultant force and force diagrams

Definition 7.4 *The* **resultant force** *or* **net force** *acting on a body is the vector sum of all forces acting on the body.*

In symbols, the resultant force is $\boldsymbol{R}$, and

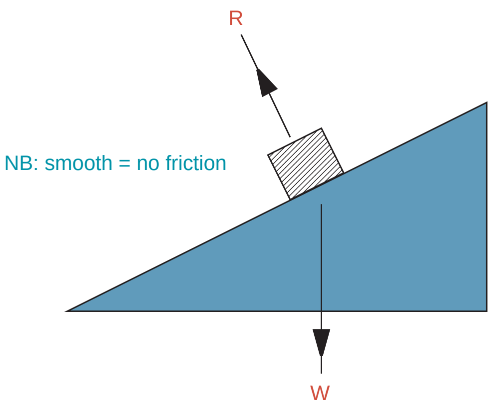
$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots = \sum \boldsymbol{F}.$$

Recall, force is a vector quantity. In what follows, it will be very useful to add up all of the forces acting on a body using vector addition. Of course, the parallelogram law is the basis of vector addition. But it can sometimes be helpful to represent the process of adding up all the forces by using what is called a **force diagram**. The vector sum of all the forces acting on a body is called the **net force**, or the **resultant**. The **rules** of force diagrams are:

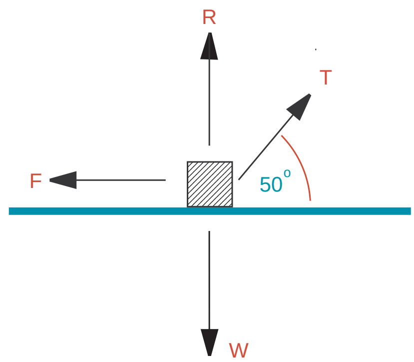
1. Weight acts vertically downwards
2. Reaction is normal to surface (smooth surface)
3. If surface is not smooth there is a frictional force.
4. Attachment forces act at the point of attachment.

Two very qualitative examples of force diagrams are shown below.

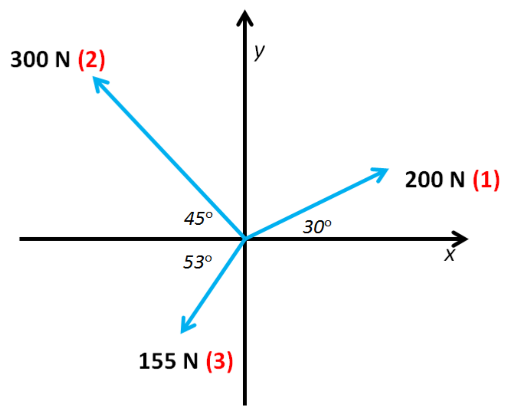
Example: A block sliding down a smooth inclined plane:



Example: A block pulled along a **rough** horizontal surface by a rope at 50° to the surface:



Example: Three customers are fighting over the same coat in the Christmas sales. They apply the three horizontal forces to the coat that are shown in the figure, where the coat is located at the origin. Find the net force on the coat in terms of its x- and y-components. Hence, find the magntiude and direction of the net force.
All units are SI. Hence, the units of force are the Newton – to be elaborated upon below.



Solution: Call the vector forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$. We resolve the forces along the Cartesian axes:

$$\begin{aligned}\boldsymbol{F}_1 &= (200\text{N})\left(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}\right),\\ \boldsymbol{F}_3 &= (300\text{N})\left(-\cos 45°\mathbf{i} + \sin 45°\mathbf{j}\right),\\ \boldsymbol{F}_3 &= (155\text{N})\left(-\cos 53°\mathbf{i} - \sin 53°\mathbf{j}\right).\end{aligned}$$

These work out as follows:

$$\begin{aligned}\boldsymbol{F}_1 &= (173\mathbf{i} + 100\mathbf{j})\text{N},\\ \boldsymbol{F}_3 &= (-212\mathbf{i} + 212\mathbf{j})\text{N},\\ \boldsymbol{F}_3 &= (-93\mathbf{i} - 124\mathbf{j})\text{N}.\end{aligned}$$

The results here are correct to three significant figures, which is the precision of the given forces in the question.

Do the vector sum:

$$\begin{aligned}\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 &= \quad (173\mathbf{i} + 100\mathbf{j})\text{N}\\ &\quad +(-212\mathbf{i} + 212\mathbf{j})\text{N},\\ &\quad +(-93\mathbf{i} - 124\mathbf{j})\text{N},\\ &= \left[(173 - 212 - 93)\mathbf{i} + (100 + 212 - 124)\mathbf{j}\right]\text{N},\\ &= (-132\mathbf{i} + 188\mathbf{j})\text{N}.\end{aligned}$$

This is the resultant force, $\boldsymbol{R}$. We have

$$|\boldsymbol{R}| = \sqrt{132^2 + 188^2} = 230\text{N}.$$

The angle that the resultant force makes with the x-axis is θ, where

$$\tan\theta = \frac{R_y}{R_x} = -\frac{188}{132} \implies \theta = -\tan^{-1}\tfrac{188}{132} = -55^\circ + \cdots,$$

where the inverse tangent is defined only up to a shift of an integer multiple of 180°, and where the angle is given to two significant figures, which is the precision given for the angles in the question. Clearly, since $R_x < 0$ and $R_y > 0$ the resultant is in the second quadrant, hence

$$\theta = -55^\circ + 180^\circ = 125^\circ.$$

7.6 Newton's first law

Equipped with the notion of net force, we can write down Newton's first law:

> A body acted on by no net force moves with constant velocity (which may be zero) and zero acceleration.

Equivalently, we say that a body is in **equilibrium** if the vector sum of forces on the body is zero:

$$\sum \boldsymbol{F} = 0 \qquad \text{(body in equilibrium)}$$

Therefore, by Newton's third law, a body in equilibrium must either be at rest or be moving in a straight line with constant velocity. We also have the following further consequences of the first law:

- If body is at rest or moving with constant velocity in a straight line then there is no net force on it.
- If the **speed** is changing, there must be a net force on the body.

The following third point is sometimes not picked up on by students on a first glance at this theory:

- If the **direction** of motion is changing then there must be a **resultant** force on the body.

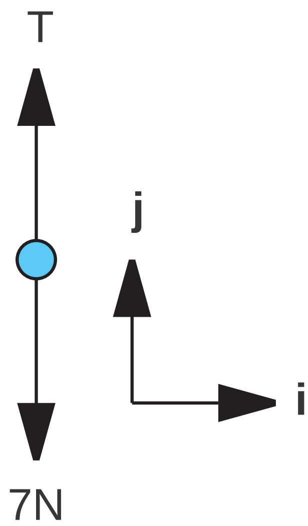
In particular, this means that a particle moving in a circle experiences a net force (circular motion).

Caution! In the above presentation, we are assuming that the body can be represented adequately as a point particle. When the body has finite size, we also have to consider *where* on the body the forces are applied. This will require us to generalize the notion of equilibrium – an extended body is in equilibrium if (i) the net force on the body is zero and (ii) the net moment of the forces is zero. We will return to this topic later in the module.

Example: In the classic 1950 science fiction movie *Rocketship X-M*, a spaceship is moving in the vacuum of outer space, far from any planet, when its engine dies. As a result, the spaceship slows down and stops. What does Newton's first law say about this event?

Solution: In this situation, there are no forces acting on the spaceship, so according to Newton's first law, it will not stop. It continues to move in a straight line with constant speed. This example of science fiction contains more fiction than science!

Example: A particle of weight 7N is hanging at the end of a light vertical string. Find the tension in the string. What is the tension in the string if the particle is moving upwards with constant velocity 9.81 m/s?

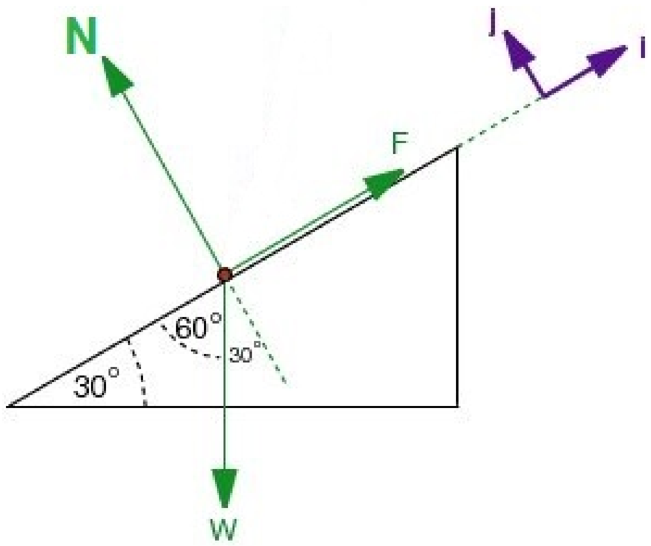


Solution:

$$\begin{aligned} \boldsymbol{R} &= (T-7)\mathbf{j}0, \\ T &= 7\,\mathsf{N}. \end{aligned}$$

For the second part, the particle moves with a constant **velocity**. This does not affect the force balance, so the tension is still 7 N.

Example: Refer to the figure. A particle rests on a rough plane inclined at an angle $30°$ to the horizontal. Compute the friction and reaction force experienced by the particle. Leave your answer in terms of the particle's weight W.

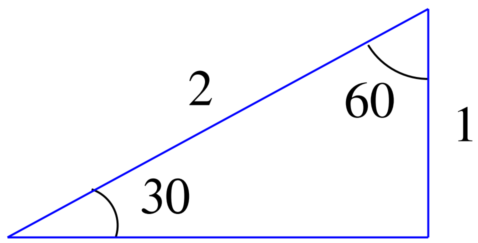


Solution: the particle is at rest, so the resultant force is zero. It remains therefore to compute the resultant force. A coordinate system is set up as shown, such that

- Reaction force $N\mathbf{j}$
- Friction force $F\mathbf{i}$. The friction force can carry any sign here, and its definite value will be fixed in the course of the calculation. So it doesn't matter if you can't figure out which direction this should point in (so long as it is parallel to the plane)!

It remains to write down the gravitational force on the particle in terms of the chosen coordinate system:

$$\begin{aligned}\mathbf{W} &= -W\sin 30°\mathbf{i} - W\cos 30°\mathbf{j} \\ &= -\frac{1}{2}W\mathbf{i} - \frac{\sqrt{3}}{2}W\mathbf{j}.\end{aligned}$$



So adding the three forces together gives

$$Resultant \quad \boldsymbol{R} = (F - \frac{1}{2}W)\mathbf{i} + (N - \frac{\sqrt{3}}{2}W)\mathbf{j}.$$

But $\boldsymbol{R} = 0$, hence

$$F = \tfrac{1}{2}W, \qquad N = \tfrac{\sqrt{3}}{2}W.$$

7.7 Newton's second law

The second law makes a quantitative statement which relates the force acting on a particle to the change in **momentum**. Therefore, we first of all define the momentum of a particle:

Definition 7.5 *The momentum* $\boldsymbol{p}$ *of a particle of mass* m *moving with velocity* $\boldsymbol{v}$ *is*

$$\boldsymbol{p} = m\boldsymbol{v}. \tag{7.5}$$

Note that momentum is a vector quantity. Loosely speaking, momentum = Mass × Velocity, and hence the SI units of momentum are kg m s^{-1}.

Equipped with this definition, Newton's second law is simply expressed as:

The resultant force on a body is proportional to its rate of change of momentum

For a body of constant mass m, and using $\boldsymbol{F}$ to denote the net force, this becomes

$$\boldsymbol{F} \propto \frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \tag{7.6}$$

We are now in a position to definite the SI unit of force (the Newton) precisely:

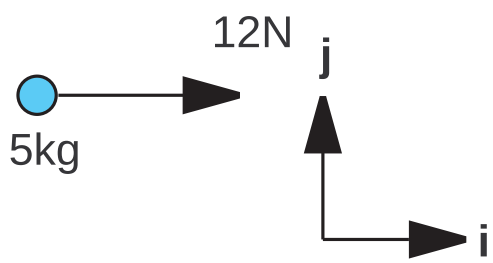
Definition 7.6 *We define the unit of force (the Newton) to be the force required to give a mass of* $1\,\text{kg}$ *an acceleration of* $1\,\text{m/s}^2$.

In this way, the constant of proportionality in Equation (7.6) becomes unity, and then Newton's second law for a body of constant mass becomes

$$\boxed{\boldsymbol{F} = m\boldsymbol{a}} \tag{7.7}$$

7.7.1 Worked examples

Example: Refer to the figure. Find the acceleration.



Solution:

$$5\boldsymbol{a} = 12\mathbf{i}$$
$$\boldsymbol{a} = \frac{12}{5}\mathbf{i} = 2.4\mathbf{i}$$

Acceleration $= 2.4\,\text{m/s}^2$ in the $\mathbf{i}$ direction.

Example: Forces $4\mathbf{i} - 7\mathbf{j}$ N and $-\mathbf{i} + 3\mathbf{j}$ N act on a particle of mass 2kg.

Find the acceleration and the angle that it makes with $\mathbf{i}$.

Solution: The net force is

$$\boldsymbol{F} = (4\mathbf{i} - 7\mathbf{j}) + (-\mathbf{i} + 3\mathbf{j}) = 3\mathbf{i} - 4\mathbf{j}.$$

By Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$. The mass is $2\,\text{kg}$. Hence,

$$\boldsymbol{F} = 3\mathbf{i} - 4\mathbf{j} = 2\boldsymbol{a},$$

hence

$$\boldsymbol{a} = \frac{1}{2}(3\mathbf{i} - 4\mathbf{j}) = 1.5\mathbf{i} - 2\mathbf{j}.$$

The angle with $\mathbf{i}$ is θ, where

$$\mathbf{i} \cdot \boldsymbol{a} = |\boldsymbol{a}| \cos\theta.$$

We have

$$\mathbf{i} \cdot \boldsymbol{a} = \tfrac{3}{2}, \qquad |\boldsymbol{a}| = \sqrt{\tfrac{9}{4} + 4} = \sqrt{\tfrac{25}{4}} = \tfrac{5}{2},$$

hence

$$\cos\theta = \frac{\mathbf{i} \cdot \boldsymbol{a}}{|\boldsymbol{a}|} = \frac{3/2}{5/2} = \tfrac{3}{5},$$

hence

$$\theta = \cos^{-1} \tfrac{3}{5}.$$

Example: A rope with a bucket attached to the end is used to raise water from a well. The mass of the empty bucket is 1.2 kg and it can raise 10 kg of water when full.

Find the tension in the rope when
(a) the empty bucket is lowered with an acceleration of $2\,\text{m/s}^2$,
(b) the full bucket is raised with an acceleration of $0.3\,\text{m/s}^2$.

Solution: For part (a), we have

- Mass: $1.2\,\mathrm{kg}$
- Acceleration: $\boldsymbol{a} = -2\mathbf{j}\,\mathrm{m/s^2}$
- Weight: $\boldsymbol{W} = -(1.2 \times 9.8\mathbf{j})\,\mathrm{N} = -11.76\mathbf{j}\,\mathrm{N}$
- Tension: $\boldsymbol{T} = +T\mathbf{j}$, with T to be determined.

Assemble Newton's second law for the problem:

$$\begin{aligned} m\boldsymbol{a} &= \sigma\boldsymbol{F} = \boldsymbol{W} + \boldsymbol{T}, \\ (1.2\,\mathrm{kg}) \times (-2\mathbf{j}\,\mathrm{m/s^2}) &= -11.76\mathbf{j}\,\mathrm{N} + T\mathbf{j}. \end{aligned}$$

Dropping the dimensional notation for now, this is

$$-2.4\mathbf{j} = (-11.76 + T)\mathbf{j},$$

or

$$(-2.4 + 11.76 - T)\mathbf{j} = 0.$$

Since $\mathbf{j}$ is not the zero vector, the only way for this expression to hold is if

$$(-2.4 + 11.76 - T) = 0,$$

i.e. $T = 9.36$. Restoring the dimensions, this is $T = 9.4\,\mathrm{N}$, keeping two significant figures.

For (b) we have we have

- Mass: $11.2\,\mathrm{kg}$
- Acceleration: $\boldsymbol{a} = +0.3\mathbf{j}\,\mathrm{m/s^2}$
- Weight: $\boldsymbol{W} = -(11.2 \times 9.8\mathbf{j})\,\mathrm{N} = -108.76\mathbf{j}\,\mathrm{N}$
- Tension: $\boldsymbol{T} = +T\mathbf{j}$, with T to be determined.

Assemble Newton's second law for the problem:

$$\begin{aligned} m\boldsymbol{a} &= \boldsymbol{F} = \boldsymbol{W} + \boldsymbol{T}, \\ (11.2\,\mathrm{kg}) \times (0.3\mathbf{j}\,\mathrm{m/s^2}) &= -108.76\mathbf{j}\,\mathrm{N} + T\mathbf{j}. \end{aligned}$$

Dropping the dimensional notation for now, this is

$$3.36\mathbf{j} = (-108.76 + T)\mathbf{j},$$

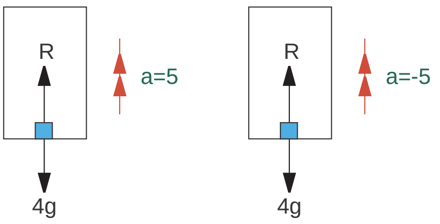
We skip straight to the solution

$$3.36 + 108.76 = T,$$

hence

$$T = 112.12\,\mathrm{N}.$$

> Example: A block of mass $4\,\mathrm{kg}$ is lying on the floor of a lift that is accelerating at $5\,\mathrm{m/s^2}$. Find the normal reaction exerted on the block by the lift floor if the lift is
> (a) going up,
> (b) going down.



Solution: For (a) we have

- Mass: $4\,\mathrm{kg}$, acceleration $+5\mathbf{j}\,\mathrm{m/s^2}$, $m\boldsymbol{a} = 20\mathbf{j}\,\mathrm{N}$.
- Weight: $\boldsymbol{W} = -4 \times 9.8 = -39.2\mathbf{j}\,\mathrm{N}$
- Reaction: $R\mathbf{j}$

Newton's second law:

$$m\boldsymbol{a} = \boldsymbol{W} + R\mathbf{j},$$

hence

$$20 = -39.2 + R,$$

hence

$$R = 59.2\,\mathrm{N}.$$

For (b) we have

- Mass: $4\,\mathrm{kg}$, acceleration $-5\mathbf{j}\,\mathrm{m/s^2}$, $m\boldsymbol{a} = -20\mathbf{j}\,\mathrm{N}$.
- Weight: $\boldsymbol{W} = -4 \times 9.8 = -39.2\mathbf{j}\,\mathrm{N}$
- Reaction: $R\mathbf{j}$

Newton's second law:

$$m\boldsymbol{a} = \boldsymbol{W} + R\mathbf{j},$$

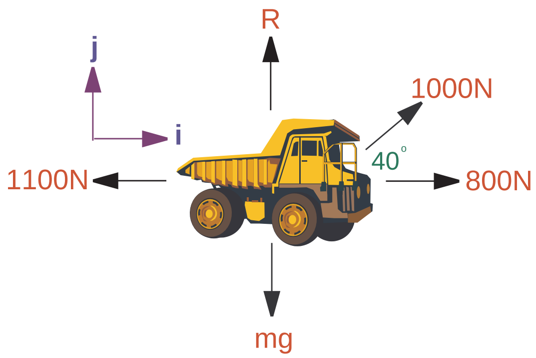
hence

$$-20 = -39.2 + R,$$

hence

$$R = 19.2\,\text{N}.$$

> Example: A truck is being pulled along a horizontal track by two cables against resistances totalling 1100N with an acceleration of $0.8\,\text{m/s}^2$. One cable is horizontal and the other is inclined at 40° to the track. The tensions in the cable are as shown. Find
> (a) the mass of the truck,
> (b) the vertical force of the track on the truck.



Solution: The acceleration is in the x-direction only, so we compute the net force in the x direction, which is

$$F_x = -1100 + 800 + 1000\cos 40^\circ = 466\,\text{N}.$$

Use Newton's law for the x-direction:

$$ma_x = F_x,$$

hence

$$ma_x = 466\,\text{N}.$$

But $a_x = 0.8\,\text{m/s}^2$, hence

$$m = \frac{466\,\text{kg m/s}^2}{0.8\,\text{m/s}^2} = \frac{466}{0.8}\,\text{kg} = 582.5\,\text{kg}.$$

Newton's law in the y-direction reads

$$ma_y = F_y,$$

where the resultant force in the y-direction is $R + 1000\sin 40^\circ - mg$. But $a_y = 0$, hence $R = mg - 1000\sin 40^\circ$, hence

$$R = (583\,\text{kg}) \times (9.8\,\text{m/s}^2) - 642.78\cdots\text{N} = 5070.61\cdots\text{N} \approx 5071\,\text{N}.$$

7.8 Weight revisited

Recall Equation (7.3), where we found that a body of mass m falling to the ground under the earth's gravitational attraction experiences a force

$$F = \left(\frac{GM_{\mathrm{e}}}{R_{\mathrm{e}}^2}\right) m.$$

Neglecting air resistance, this is the only force on the body. Therefore, this is the net force. Hence, under Newton's second law, the body experiences an acceleration a, where

$$ma = F = \left(\frac{GM_{\mathrm{e}}}{R_{\mathrm{e}}^2}\right) m.$$

Crucially, m **cancels on both sides of this equation now**:

$$a = \frac{GM_{\mathrm{e}}}{R_{\mathrm{e}}^2}.$$

Hence, we have rigorously shown that a particle in the earth's gravitational field experiences a uniform constant acceleration of magnitude $GM_{\mathrm{e}}/R_{\mathrm{e}}^2$, which we identify as the already-studied constant g:

$$g = \frac{GM_{\mathrm{e}}}{R_{\mathrm{e}}^2}.$$

Therefore, the weight of the body can be written as

$$W = mg = \frac{GM_{\mathrm{e}}m}{R_{\mathrm{e}}^2}.$$

Chapter 8

Newton's third law of motion

Overview

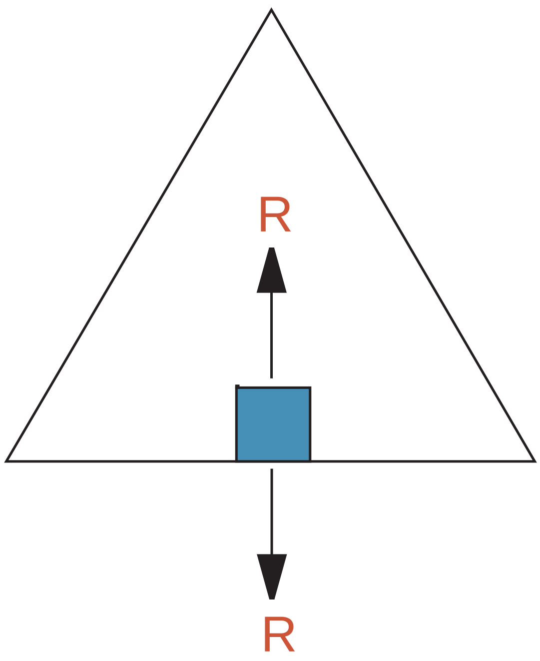
We complete our discussion of Newton's laws of motion by introducing the Third Law. We show how this can be linked to the principle of conservation of momentum.

8.1 Newton's third law – motivation and definition

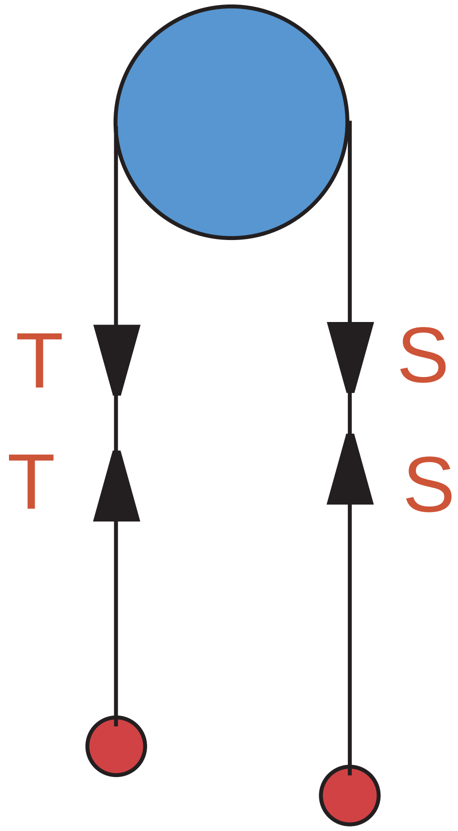
So far we have discussed the resultant force and acceleration on one particular body in any given problem. However, particles interact with each other (by exerting forces!) and we need a principle which describes this interaction. This is provided by Newton's Third Law.

> Action and Reaction are equal and opposite. If a body A exerts a force on a body B then B exerts an equal and opposite force on A.

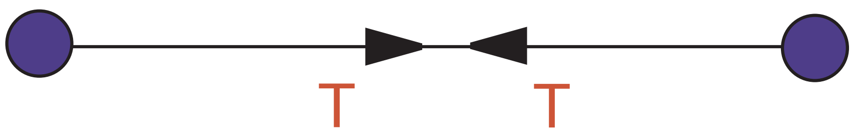
Some examples are shown in Figure 8.1.



(a) Scales pan: the pan exerts a downward force (weight) on the scales; the scales exert an equal and opposite upward force on the pan (reaction)



(b) Pulley: if the pulley is smooth then $T = S$



(c) Two bodies connected by a string

Figure 8.1:

8.1.1 Discussion

The Earth's gravity exerts a force mg on a football (pulling it "down"). The football exerts exactly the same force on the Earth (pulling it "up"). But we have Newton's law for the earth,

$$M_e a = F = mg$$

which mean's that the earth accelerates towards the football! Why don't we observe the earth's acceleration in practice?

The reason of course is that the acceleration is tiny:

$$a = \frac{m}{M_e} g.$$

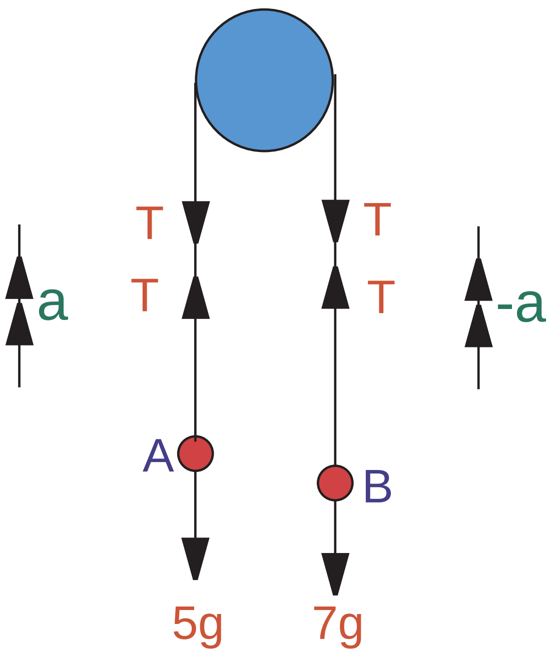
The ball's mass should be approximately $1\,\mathrm{kg}$, the earth's mass is approximately $10^{24}\,\mathrm{kg}$, and acceleration due to gravity is approximately $10\,\mathrm{m/s^2}$, so the acceleration will be

$$a = 10^{-23}\,\mathrm{m/s^2},$$

which is tiny! In this case the earth's large mass appears as **inertia** – the ability of the body in question to resist changes in its motion.

8.2 Worked examples

Example: A light inextensible string passes over a smooth pulley and carries particles of masses 5 kg and 7 kg. If the system is moving freely find:
(a) the acceleration of each particle,
(b) the tension in the string,
(c) the force exerted by the string on the pulley.



Solution: Every word in the problem matters. So, 'light' means the string's inertia can be neglected, 'inextensibile' means the string is totally rigid, so the force of one of the particles is transmitted perfectly to the other. Finally, 'smooth' means that the tensions on both sides of the pulley are the same – as shown in the figure. Therefore, the force balance for the two particles is rather straightforward. Finally, the acceleration of the two particles is equal and opposite (otherwise there would be acceleration in the string).

We first of all guess that the heavier particle B is moving downwards, and hence has an acceleration $-a$. This is not necessary, as the maths will pick the sign of the acceleration – all that matters is that the accelerations should be equal and opposite. But we have to start somewhere. Therefore, the force balance for particle B,

$$-7a = T - 7g.$$

Similarly, for the lighter particle (particle A):

$$5a = T - 5g$$

We eliminate T between these simultaneous equations by subtracting one from the other:

$$\begin{aligned} -7a &= T - 7g, \\ 5a &= T - 5g, \end{aligned}$$

hence

$$12a = 2g,$$

hence $a = g/6$. The tension can be found by backsubstitution into (say) the second equation:

$$\begin{aligned} 5a &= T - 5g, \\ \tfrac{5}{6}g &= T - 5g, \\ \tfrac{5}{6}g + 5g &= T, \end{aligned}$$

hence

$$T = \tfrac{35}{6}g.$$

For (c), the total force on the pulley is the sum of

1. The tension T exerted by mass A on the pulley (downward direction)

2. The tension T exerted by mass B on the pulley (downward direction),

hence the force is $2T$, or $(35/6)g$, in a downwards direction.

Example: A small block of mass $6\,\mathrm{kg}$ rests ona tabletop and is connected by a light inextensible string that passes over a smooth pulley, fixed on the edge of the table, to another block of $5\,\mathrm{kg}$ which is hanging freely.

Find the acceleration of the system and the tension in the string
(a) if the table is smooth,
(b) if the table is rough and the frictional force is $2g\,\mathrm{N}$.

Solution: A force diagram is shown in Figure 8.2.

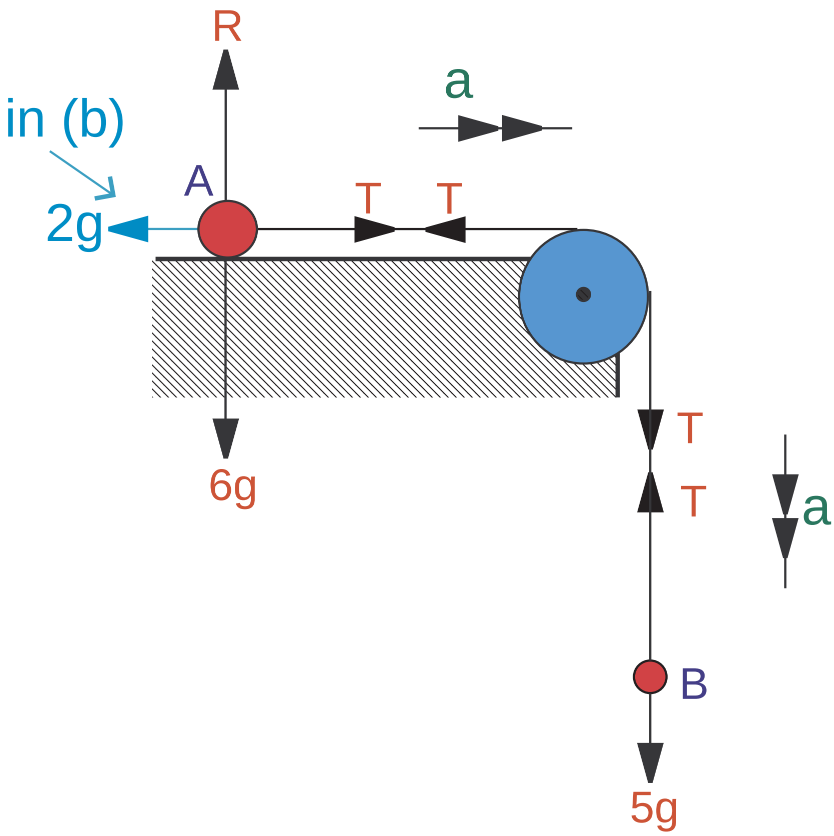


Figure 8.2:

Solution – Part (a): As before, the acceleration a for particle A is written down. The force balance for A is simply

$$6a = T.$$

The acceleration of particle B must also be a – but in the downwards direction. Thus, the force balance for B is

$$-5a = T - 5g.$$

Eliminating T beteween these equations gives

$$a = \tfrac{5}{11}g$$

Back-substitution into $6a = T$ gives $T = (30/11)g$.

Part (a): The acceleration a for particle A is written down. The force balance for A is modified:

$$6a = T - F = T - 2g,$$

where the sign of the frictional force is known – it opposes the motion. The second equation (for B) is unchanged. We therefore have simulataneous equations:

$$\begin{aligned} 6a &= T - 2g, \\ -5a &= T - 5g. \end{aligned}$$

Eliminate T by subtraction to get $11a = 3g$, hence $a = (3/11)g$. Back-substitution into the first equation gives

$$\begin{aligned} 6a &= T - 2g, \\ 6\left(\tfrac{3}{11}g\right) &= T - 2g, \\ \tfrac{18}{11}g + 2g &= T, \end{aligned}$$

hence $T = (40/11)g$.

8.3 Conservation of momentum

In 'mechanics' a 'system' is one of those terms which is kept delibeately vague. Yet it can be thought of as a collection of particles or in slightly more generality, a collection of objects. Consider a system of n particles each with momentum $\boldsymbol{p}_i$. The **total momentum** of the system $\boldsymbol{P}$ is simply the vector sum of the individual momenta:

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \cdots \boldsymbol{p}_n = \sum_{i=1}^{n} \boldsymbol{p}_i.$$

It is of interest to study how $\boldsymbol{P}$ changes over time. We therefore differentiate it:

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t}. \tag{8.1}$$

We use Newton's second law to write

$$\frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \boldsymbol{F}_{i,\mathrm{int}} + \boldsymbol{F}_{i,\mathrm{ext}}, \tag{8.2}$$

which is the total force on the i^{th} particle, made up of internal and external contributions:

- $\boldsymbol{F}_{i,\mathrm{int}}$ is the force on the i^{th} particle due to all the other particles in the system. Let $\boldsymbol{F}_{ij}$ denote the force exerted on the i^{th} particle by the j^{th} particle, with $i \neq j$. Then

$$\boldsymbol{F}_{i,\mathrm{int}} = \sum_{\substack{j \\ j\neq i}} \boldsymbol{F}_{ij}.$$

- $\boldsymbol{F}_{i,\mathrm{ext}}$ is the force on the i^{th} particle due to other influences, i.e. not due to the other particles, i.e. external forces.

Thus, Equation (8.2) becomes

$$\frac{\mathrm{d}\boldsymbol{p}_k}{\mathrm{d}t} = \sum_{\substack{j \\ j\neq k}} \boldsymbol{F}_{kj} + \boldsymbol{F}_{k,\mathrm{ext}}, \tag{8.3}$$

Insert Equation (8.3) into Equation (8.1):

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} &= \sum_{k=1}^{n} \frac{\mathrm{d}\boldsymbol{p}_k}{\mathrm{d}t}, \\
&\overset{Eq.\ (8.3)}{=} \sum_{k=1}^{n} \left(\sum_{\substack{j \\ j\neq k}} \boldsymbol{F}_{kj} + \boldsymbol{F}_{k,\mathrm{ext}} \right), \\
&= \underbrace{\sum_{k=1}^{n} \sum_{\substack{j \\ j\neq k}} \boldsymbol{F}_{kj}}_{(*)} + \sum_{k=1}^{n} \boldsymbol{F}_{k,\mathrm{ext}}.
\end{aligned}$$

Here's the thing: the first complicated-looking sum here (call it $(*)$) is the sum over all forces $\boldsymbol{F}_{kj}$. To enumerate all forces in this sum, we need to write down all pairs

$$(1,2),(1,3),\cdots,(1,n), \qquad (2,1),(2,3),\cdots,(2,n), \qquad \cdots \qquad (n,1),(n,2),\cdots(n,n-1).$$

These pairs appear pairwise - if $(1,2)$ appears in the sum, then so does $(2,1)$. Therefore, the sum $(*)$ will involve individual pairwise sums such as

$$\boldsymbol{F}_{12} + \boldsymbol{F}_{21}.$$

But by Newton's third law, this is exactly zero. Since the sum $(*)$ is the sum over all such pairs, $(*)$ is actually zero, and thus

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \sum_{k=1}^{n} \boldsymbol{F}_{k,\mathrm{ext}}. \tag{8.4}$$

We define a closed system:

Definition 8.1 *A closed system is one where there are no external forces.*

Therefore, following on from Equation (8.4), we have the following result:

Theorem 8.1 *For a closed system,*

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = 0,$$

in other words, the total momentum of the system is a constant.

This is the **principle of conservation of momentum**. In the next section we look at some implications and applications of this principle.

8.4 Conservation of momentum – applications

We work through some implications and applications of the principle of conservation of momentum, re-expressed here in word form:

> In a closed system, the total momentum of the system is conserved.

Here, by 'conserved' we mean 'constant', or 'stays the same for all time.

We can in fact relax the assumption about the system being closed: if the external force is in a fixed direction then momentum conservation will still apply – but only in a plane perpendicular to that direction.

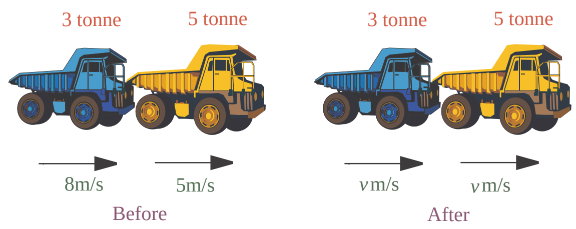
Example: Pool ball collisions conserve momentum in spite of the external force of gravity because that force is perpendicular to the plane of the pool table.

8.4.1 Worked examples

Example: A three-tonne truck is moving along a track at 8 m/s towards a five tonne truck travelling at 5 m/s on the same track.

If the trucks become coupled at impact, find the velocity at which they continue to move if
(a) they are travelling in the same direction,
(b) in opposite direction.

Solution – part (a):



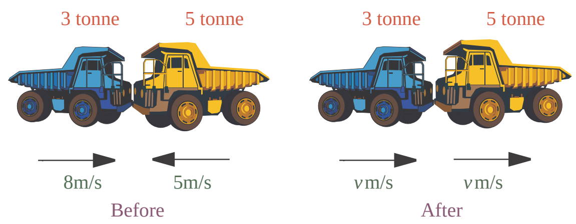
1 tonne = 1000 kg.

$$3000 \times 8 + 5000 \times 5 = 3000 \times v + 5000 \times v$$

$$24000 + 25000 = 8000v$$

$$8v = 49 \qquad \text{so} \qquad v = 6.125\,\text{m/s}.$$

Part (b):

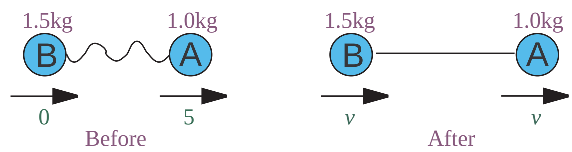


$$3000 \times 8 - 5000 \times 5 = 3000 \times v + 5000 \times v$$

$$-1000 = 8000 \times v$$

$$v = -1/8 = -0.125\,\text{m/s}.$$

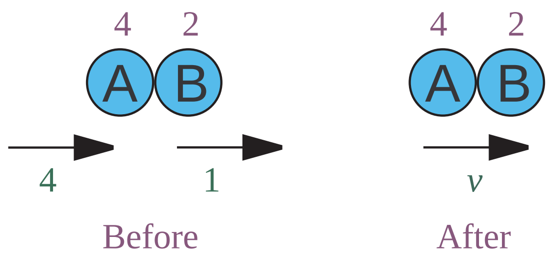
Example: Two particles A and B joined by a light inextensible string are lying together on a smooth horizontal plane. The masses of A and B are 1 kg and 1.5 kg respectively.
A is projected away from B with speed 5m/s.
Find the speed of each particle after the string jerks taut.



Solution:

$$\begin{aligned} 1.5 \times 0 + 1.0 \times 5 &= 1 \times v + 1.5v \\ 5 &= 2.5v \\ v &= 2\,\text{m/s}. \end{aligned}$$

Example: A body A of mass $4\,\mathrm{kg}$ travelling with velocity 4m/s collides directly with a body B of mass $2\,\mathrm{kg}$ moving with velocity $1\,\mathrm{m/s}$. They coalesce on impact.
Find the velocity with which the combined body moves.



Solution:

$$
\begin{aligned}
4 \times 4 + 2 \times 1 &= 4 \times v + 2 \times v, \\
18 &= 6v, \\
v &= 3\,\mathrm{m/s}.
\end{aligned}
$$

Chapter 9

Statics and Friction

Overview

We have already discussed the topic of **statics** briefly - without giving it a fancy name. The idea here is to look at situations where a system is in equilibrium and to fully characterize all of the forces acting on the system. The key of course is Newton's second law: since the system is in equilibrium, there is no acceleration and hence, the vector sum of the forces is zero. We also look at frictional forces in this context. The focus is on worked examples.

9.1 Review

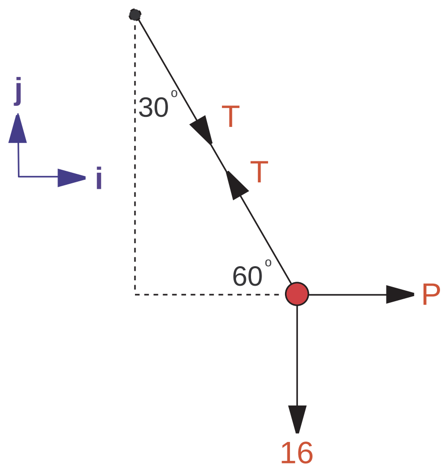
Recall, a body is said to be in equilibrium if the vector sum of forces on the body is zero:

$$\sum \boldsymbol{F} = 0 \qquad \text{(body in equilibrium)}.$$

By Newton's second law, the acceleration of a body in equilibrium is zero $\boldsymbol{a} = 0$.

> Example: A particle of weight 16 N is attached to one end of a light elastic string whose other end is fixed.
>
> The particle is pulled aside by a horizontal force until the string is at 30° to the vertical.
>
> Find the horizontal force and the tension in the string.

Solution: Draw the force diagram shown in the figure. We have:



$$\begin{aligned}\textbf{Resultant} &= P\mathbf{i} - 16\mathbf{j} + T\sin 60°\mathbf{j} - T\cos 60°\mathbf{i} \\ &= (P - \tfrac{1}{2}T)\mathbf{i} + (-16 + \tfrac{\sqrt{3}}{2}T)\mathbf{j} \\ &= \mathbf{0}.\end{aligned}$$

Therefore

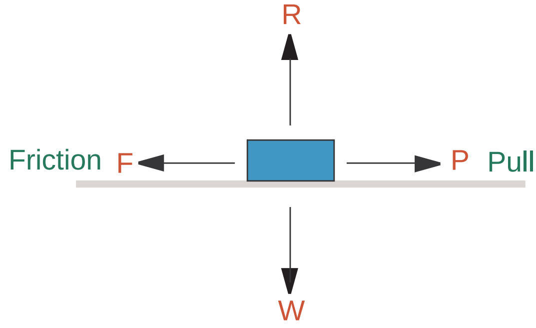
$$P - \tfrac{1}{2}T = 0, \qquad -16 + \tfrac{\sqrt{3}}{2}T = 0.$$

$$T = \tfrac{32}{\sqrt{3}}\text{N}, \qquad P = \tfrac{16}{\sqrt{3}}\text{N}.$$

9.2 Friction

Consider the schematic diagram in the Figure showing a block on a rough table. The block is pulled by a force P. The potential motion of the block due to this pulling force is opposed by friction:

- If the force P is small the block will not move.
- As the force P is increased the block will eventually move.
- As the force P increases, the friction increases to a maximum, beyond which it cannot increase.



When the block is on the point of moving friction is said to be **limiting**. The frictional force F has then reached its maximum value and the block is said to be in **limiting equilibrium**.

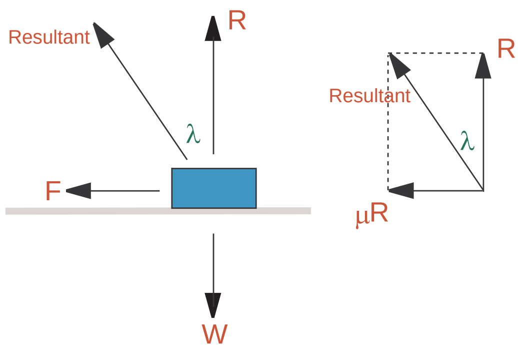
9.2.1 Coefficient of friction

Consider the figure on the right showing a block in limiting equilibrium.

Definition 9.1 *For a block in limiting equilibrium, the coefficient*

$$\mu = F/R$$

being the ratio of the frictional force to the reaction force, is called the **coefficient of friction**.



In general, μ will depend on the materials of the two contact surfaces. Also, the coefficient of friction will in general depend on whether there is relative motion between the two surfaces: in fact there are two coefficients of friction for static friction and dynamic friction. For the moment, we ignore this distinction.

9.3 Friction forces - properties

We begin with a definition:

Definition 9.2 *The result of the reaction force R and the friction force F (with $F = \mu R$) is called the* **resultant contact force**.

The vector sum of these two forces is

$$-F\mathbf{i} + R\mathbf{j},$$

and hence the magnitude of the resultant contact force is

$$\begin{aligned}\text{magnitude} &= \sqrt{F^2 + R^2}, \\ &= \sqrt{\mu^2 R^2 + R^2}, \\ &= R\sqrt{\mu^2 + 1}.\end{aligned}$$

Definition 9.3 *The angle λ between the resultant contact force and the reaction force is called the* **angle of friction**.

We have

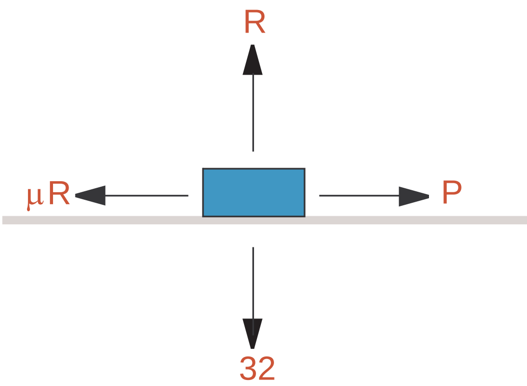
$$\tan\lambda = \frac{F}{R} = \frac{\mu R}{R} \implies \lambda = \tan^{-1}\mu.$$

All of the above can be summarized in the following **laws of friction**:

- When the surface of two objects are in rough contact, and have a tendency to move relative to each other, equal and opposite frictional forces act, one on each of the objects, so as to oppose the potential movement.
- Until it reaches its limiting value, the magnitude of the frictional force F is sufficient to prevent motion.
- When the limiting value is reached, $F = \mu R$, where R is the normal reaction between the surfaces and μ is the coefficient of friction for those two surfaces.
- For all rough contacts $0 < F \leq \mu R$
- If a contact is smooth $\mu = 0$.

9.3.1 Worked examples

Example: A block of weight 32 N is lying in rough contact on a horizontal plane. A horizontal force of P N is applied to the block until it is just about to move the block.
(a) If $P = 8$, find μ.
(b) If $\mu = 0.4$, find P.



Solution:

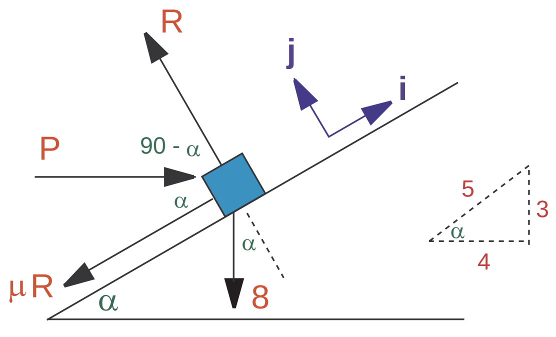
- Vertically: $R = 32$.
- Horizontally: $\mu R = P$.
- Therefore $P = 32\mu$.

Thus,

(a) If $P = 8$, $\mu = \frac{8}{32} = \frac{1}{4}$.

(b) If $\mu = 0.4$, find $P = 32 \times 0.4 = 12.8$N.

Example: Refer to the figure. A particle of weight 8 N rests in rough contact with a plane inclined at α to the horizontal, where $\tan\alpha = 3/4$. A horizontal force of P N is applied to the particle. When $P = 16$ the particle is on the point of slipping up the plane. Find μ.



Solution: We have

$$\begin{aligned} \mathbf{0} &= -\mu R\mathbf{i} + R\mathbf{j} + P\cos\alpha\mathbf{i} \\ &\quad -P\sin\alpha\mathbf{j} - 8\sin\alpha\mathbf{i} - 8\cos\alpha\mathbf{j} \\ &= (P\cos\alpha - \mu R - 8\sin\alpha)\mathbf{i} \\ &\quad +(R - P\sin\alpha - 8\cos\alpha)\mathbf{j} \end{aligned}$$

$$P\cos\alpha - \mu R - 8\sin\alpha = 0, \tag{9.1}$$

$$R - P\sin\alpha - 8\cos\alpha = 0. \tag{9.2}$$

Now, Equation (9.2) implies:

$$R = P\sin\alpha + 8\cos\alpha = 16 \times \tfrac{3}{5} + 8 \times \tfrac{4}{5} = 16\,\mathrm{N}.$$

Substituting into Equation (9.1):

$$16\mu = -8\sin\alpha + P\cos\alpha = -8 \times \tfrac{3}{5} + 16 \times \tfrac{4}{5} = 8,$$

$$\mu = \tfrac{8}{16} = \tfrac{1}{2}.$$

Example: Repeat the previous exercise for the case where the pushing force P is weak, such that the block is just on the verge of sliding **down** the plane.

Solution: In this situation, the friction force opposes the impending motion, which is down the plane. Therefore, the friction force points up. Thus, it is simply a matter of reversing the sign of the friction force from the previous example, by changing μR from the previous example to $-\mu R$:

$$P\cos\alpha + \mu R - 8\sin\alpha = 0, \tag{9.3}$$

$$R - P\sin\alpha - 8\cos\alpha = 0. \tag{9.4}$$

From Equation (9.3):

$$\tfrac{4}{5}P + \tfrac{1}{2}R - 8 \times \tfrac{3}{5} = 0,$$

$$8P + 5R - 48 = 0. \tag{9.5}$$

From Equation (9.4):

$$R - \tfrac{3}{5}P - 8 \times \tfrac{4}{5} = 0,$$

$$5R - 3P - 32 = 0. \tag{9.6}$$

Eliminating R between Equations (9.5) and (9.6):

$$11P = 16, \qquad P = \tfrac{16}{11}\,\mathrm{N}.$$

Chapter 10

Mechanical Work and Power

Overview

We introduce the concepts of mechanical work and mechanical power. We look at specific examples involving vehicle power, and a particularly nice example in cycling is illustrated.

10.1 Work

Definition 10.1 *When an object moves through a displacement* $\boldsymbol{s}$ *under the action of a* **constant** *force* $\boldsymbol{F}$*, we define the amount of work done* **by the force** *as*

$$\boxed{W = \boldsymbol{F} \cdot \boldsymbol{s}.} \tag{10.1}$$

The SI unit of work is the Joule:

Definition 10.2 *The work done by a constant force of 1 N is* 1 *Joule if the displacement is 1 m in the direction of the force.*

More generally, when the force is not constant, then the amount of work done during an infinitesimal displacement is

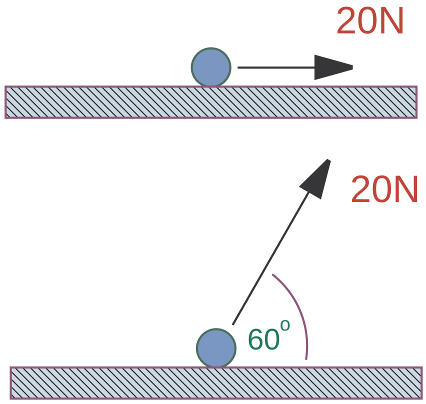
$$\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$$

Thus, the total work done in a net displacement $\boldsymbol{s}$ is

$$\boxed{W = \int_0^s \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}.} \tag{10.2}$$

10.1.1 Worked examples

> Example: Refer to the figure. A body resting in smooth contact with a horizontal plane moves 2.6 m along the plane under the action of a force of 20 N. Find the work done by a force applied:
> (a) horizontally,
> (b) at 60° to the plane.



Solution – part (a):

$$\text{Work done} = 20 \times 2.6 = 52 \text{ Joules.}$$

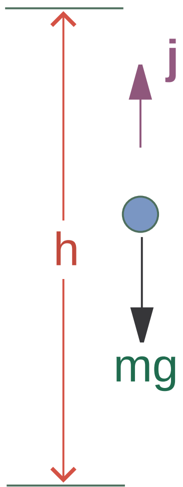
Part (b): Let

$$\boldsymbol{s} = (2.6\,\mathrm{m})\mathbf{i},$$
$$\boldsymbol{F} = (20\,\mathrm{N})\left(\cos 60°\mathbf{i} + \sin 60°\mathbf{j}\right).$$

Let W = work done. We have,

$$\begin{aligned} W &= \boldsymbol{F} \cdot \boldsymbol{s}, \\ &= [(2.6\,\mathrm{m})\mathbf{i},] \cdot [(20\,\mathrm{N})\left(\cos 60°\mathbf{i} + \sin 60°\mathbf{j}\right)], \\ &= 2.6 \times 20 \cos 60°\ \mathrm{J}, \\ &= 20 \times 2.6 \times \tfrac{1}{2}\ \mathrm{J}, \\ &= 26\,\mathrm{J}. \end{aligned}$$

> Example: Find the work done by the earth's gravitational force in lifting an object through a height h at constant speed.

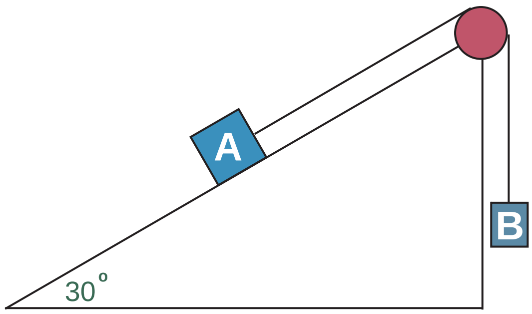


Solution: $\boldsymbol{F} = -mg\mathbf{j}$ and $\boldsymbol{s} = -h\mathbf{j}$ (the force and the displacement are in the same direction).

Thus, $W = \boldsymbol{F} \cdot \boldsymbol{s} = mgh$. This is a general result for the work done by the earth's gravitational field on a particle of mass m:

$$\boxed{W = mgh \qquad \text{(work done by earth's gravitational force on a particle of mass } m\text{).}} \qquad (10.3)$$

Example: Refer to the figure. Assume that the pulley is smooth, the mass of A is $(1/2)$kg, $\mu = 1/\sqrt{3}$ and B moves down with constant speed. Find:
(a) the frictional force on A,
(b) the tension in the string,
(c) the weight of B.
Also, when the particle B moves down 1 m find:
(d) the work done by gravity on particle B,
(e) the total work done by the string on the system.



Solution: Draw a force diagram as in Figure 10.1.

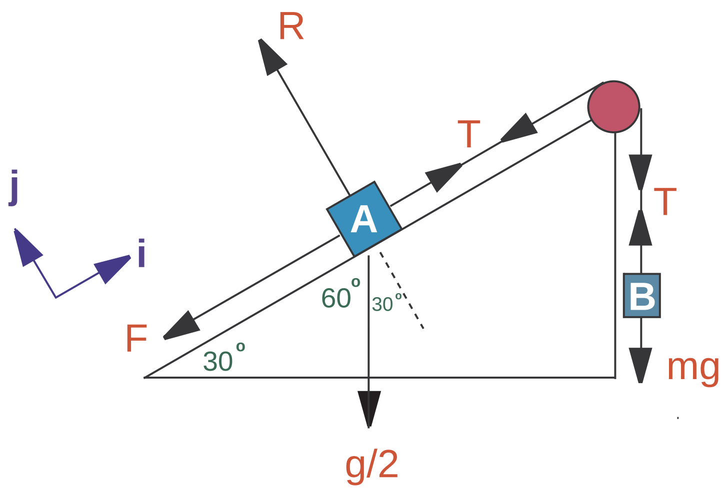


Figure 10.1: Force diagram for particles A and B

for particle A: $\quad T - F - \frac{1}{2}g\cos 60° = 0, \qquad T - F = \dfrac{1}{4}g.$

$$R - \tfrac{1}{2}g\cos 30° = 0, \qquad R = \frac{\sqrt{3}}{4}g.$$

Thus, for part (a):

$$F = \mu R = \frac{1}{4}g,$$

and for part (b):

$$T = F + \frac{1}{4}g = \mu R + \frac{1}{4}g = \frac{1}{4}g + \frac{1}{4}g = \frac{1}{2}g.$$

For part (c) we write down the force balance for particle B, which is simply $mg = T$, hence $mg = g/2$ from Equation (10.1.1).

For part (e) the work done by the string on the particle B is due to the tension force in the string:

$$W_{T,B} = (+T) \times (-1\,\mathrm{M}),$$

while the work done by the string on the particle A is

$$W_{T,A} = (+T) \times (+1\,\mathrm{M}),$$

hence

$$W_{T,B} + W_{T,A} = 0,$$

and the total work done by the string on the system of two particles is zero.

As the string is totally passive in this example, it acts only to transmit the force on one particle to the other, so we are looking at a situation in which it appears as though all work done by the string is done directly by one particle on another through internal forces. By Newton's third law, the sum of all such forces is zero, and thus, the total work done by the string is zero.

10.2 Power

Definition 10.3 *Power is the rate of doing work:*

$$\boxed{P = \frac{\mathrm{d}W}{\mathrm{d}t}.} \tag{10.4}$$

The SI unit of power is the Watt, with $1\,\mathrm{Watt} = 1\mathrm{Joule/second}$. We can also define the average power over a finite time interval Δt of doing work:

$$P_{\text{average}} = \frac{\text{Total work done in a time interval } \Delta t}{\Delta t}. \tag{10.5}$$

10.3 Application – vehicle power

The power of a vehicle is defined as the rate at which the driving force is working. Suppose D is the driving force and v is the speed. The distance moved per second is v. The Work done per second $= Dv$. Therefore, the Power H of the vehicle is given by

$$H = Dv \tag{10.6}$$

If the vehicle is stationary, $H = 0$

What is normally called the power of a vehicle is in fact the **maximum** power,

$$H \equiv P_{\max}.$$

Note that if a vehicle moves under its constant, maximum power $P_{\max}$ and a resistance force F_{res} then the equation of motion (Newton's Second Law) is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = D - F_{\text{res}} = \frac{P_{\max}}{v} - F_{\text{res}}$$

We'll solve equations like this later, but for the moment we simply note that starting from rest the vehicle will ultimately reach a maximum velocity (at maximum power !) given by

$$v_{\text{max}} = \frac{P_{\max}}{F_{\text{res}}}.$$

Example: On a level track, a train has a maximum speed of 50 m/s. The total resistance is 28 kN. Find the maximum power of the engine.

Solution: Start with Newtno's second law: Newton's Second Law:

$$Ma = D - F_{\text{res}}.$$

At maximum power $a = 0$. Therefore

$$D = F_{\text{res}} = 28{,}000.$$

But $P_{\max} = Dv_{\max}$, hence

$$\begin{aligned} P_{\max} = Dv_{\max} = 28{,}000 \times 50 &= 1{,}400{,}000\,\text{W} \\ &= 1400\,\text{kW} = 1.4\,\text{MW} \end{aligned}$$

Example: Following on from the previous exercise, suppose that the total resistance is reduced and the power needed to maintain the speed at 50 m/s is now 1250 kW. Find the new resistance.

Solution: We have

$$P_{\max} = 1250\,\text{kW} = Dv_{\max} = D \times (50\,\text{m/s}).$$

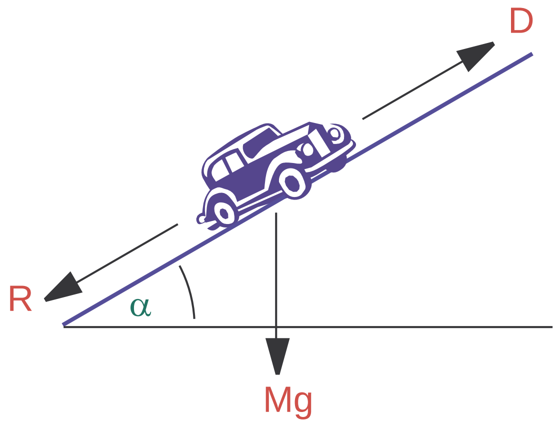
But

$$\frac{P_{\max}}{v_{\max}} = D \implies D = \frac{1250 \times 10^3}{50} = 25\,\text{kN}.$$

There is no acceleration, hence $F_{\text{res}} = D$, hence

$$F_{\text{res}} = 25\,\text{kN}.$$

> Example: When a car of mass 1200 kg is driving up a hill inclined at an angle α to the horizontal, with engine working at 32 kW, the maximum speed is 25 m/s.
> If $\alpha = \sin^{-1}\left(\frac{1}{16}\right)$, find the resistance to motion.



Solution: There is no acceleration. Therefore, the forces are in equilibrium. A force balance in the x-direction (i.e. in the direction along the incline) gives

$$D = F_{\text{res}} + W \sin\alpha.$$

But $P_{\max} = D v_{\max}$, hence

$$\frac{P_{\max}}{v_{\max}} = D,$$

hence

$$\frac{P_{\max}}{v_{\max}} = F_{\text{res}} + W \sin\alpha.$$

Re-arranging gives

$$F_{\text{res}} = \frac{P_{\max}}{v_{\max}} - W \sin\alpha.$$

Fill in the numbers:

$$\begin{aligned} F_{\text{res}} &= \frac{32,000}{25} - 1200 \times 9.1 \times \tfrac{1}{16}, \\ &= 545\,\text{N}. \end{aligned}$$

10.4 Cycling

> Example: A cyclist on a flat road moves rides at maximum power to obtain a speed of $v = 8\,\text{m/s}$. To maintain this speed, the cyclist must overcome two forces:
>
> - Rolling friction due to contact with the road:
>
> $$F_{\text{rolling friction}} = \mu W,$$
>
> where μ is the coefficient of rolling friction and W is the combined weight of the bike and

Quantity	Value
μ	0.006
m_{bike}	$10\,\text{kg}$
m_{rider}	$70\,\text{kg}$
A	$0.5\,\text{m}^2$
C_{w}	0.5
ρ	$1.226\,\text{kg}/\text{m}^3$
g	$9.8\,\text{m}/\text{s}^2$.

the rider.

- Air resistance ('drag'), with

$$F_{\text{drag}} = \tfrac{1}{2} A C_{\text{w}} \rho v^2,$$

where C_{w} is the drag coefficient, A is the frontal area of the rider, and ρ is the density of air.

With the data given in the table below, calculate the rider's maximum power.

Solution: the driving force is D and D balances with the retarding forces to give

$$D = \mu g (m_{\text{bike}} + m_{\text{rider}}) + \tfrac{1}{2} A C_{\text{w}} \rho v^2.$$

But $D = H/v = P_{\text{max}}/v$, hence

$$P_{\text{max}} = \mu g (m_{\text{bike}} + m_{\text{rider}}) v + \tfrac{1}{2} A C_{\text{w}} \rho v^3.$$

Fill in:

$$P_{\text{max}} = 0.006 \times 80 \times 9.8 \times 8 + (0.5) \times (0.5) \times (0.5) \times 1.226 \times 8^3 = 116\,\text{W}.$$

This is the rider's maximum power.

10.4.1 Road Cycling

In a Grand Tour such as the Tour de France, the mountain stages are key in determining the winner of the General Classification ('Yellow Jersey'). In the mountain stages, gravity provides the dominant contribution to the resistance, thus the power balance becomes

$$P_{\text{max}} \approx \mu g (m_{\text{bike}} + m_{\text{rider}}) v + \cancel{\tfrac{1}{2} A C_{\text{w}} \rho v^3}.$$

The bike is usually very light compared to the rider, so this can be further approximated as:

$$P_{\max} \approx \mu g m_{rider} v,$$

thus,

$$v \approx \frac{1}{\mu}\left(\frac{P_{\max}}{m_{rider} g}\right),$$

thus, the **power to weight ration** is the key determinant of the rider's speed and hence, performance. According to Team Sky, Chris Froome's power output for the last $15.3\,\mathrm{km}$ in the Mont Ventoux stage of the 2015 Tour de France was $5.78\,\mathrm{W/kg}$, which if correct is attainable for a 'clean' world-class athlete. In contrast, Lance Armstrong used to achieve $P_{\max}/m_{\text{rider}} > 7\,\mathrm{W/kg}$ over sustained race periods.

10.4.2 Track Cycling

In contrast, in a velodrome ('track cycling'), air resistance provides the dominant contribution to the resistance, thus the power balance becomes

$$P_{\max} \approx \cancel{\mu g(m_{\text{bike}} + m_{\text{rider}})v} + \tfrac{1}{2} A C_{\text{w}} \rho v^3,$$

hence

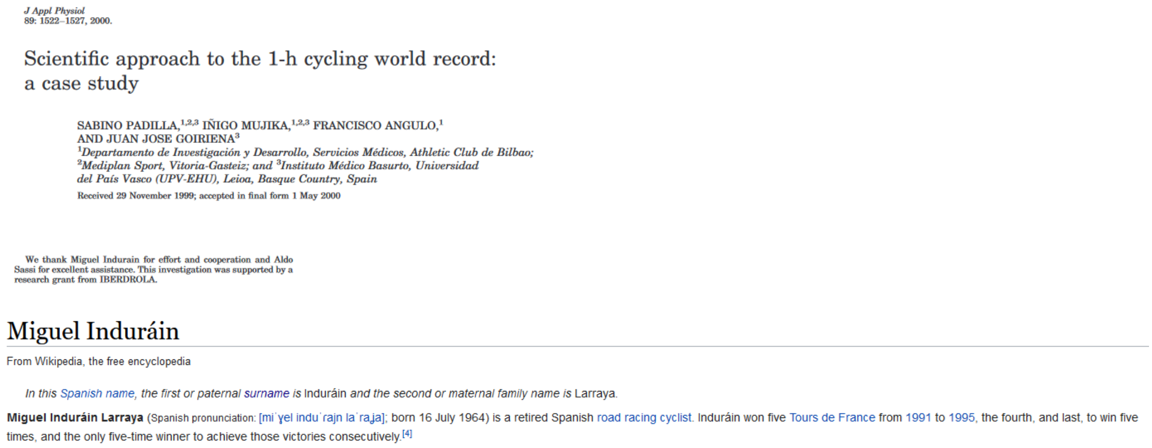
$$v^3 = \frac{2}{\rho C_w}\left(\frac{P_{\max}}{A}\right),$$

thus, the **power to surface area** is the key determinant of the rider's performance. This formula works! See Figure 10.2. This is a prompt to read a scientific article – a study of Miguel Indurain's setting of the **hour record** 1994. The article is full of data – I have extracted some of this data and examine the results graphically in Figure 10.3. The data for Boardman are an outlier – this record was set on a non-standard bike (Figure 10.4).

10.4.3 Current record-holders of the 1h event

- Women – Ellen van Dijk set a new UCI Women's hour record this year (2022), riding **49.254** km at the Velodrome Suisse in Grenchen, Switzerland. Van Dijk beat the existing record of 48.405 km set by Joss Lowden in 2021.

- Men – Victor Campenaertz set the UCI Men's hour record in 2019, riding **55.089** km in Mexico city. The record was set in a velodrome at an altitude of $2\,\mathrm{km}$ above sea level ($v^3_{max} \propto 1/\rho$). Canpenaertz beat the existing record of 54.526 km set by Bradley Wiggins in London in 2015.



J Appl Physiol
89: 1522–1527, 2000.

Scientific approach to the 1-h cycling world record: a case study

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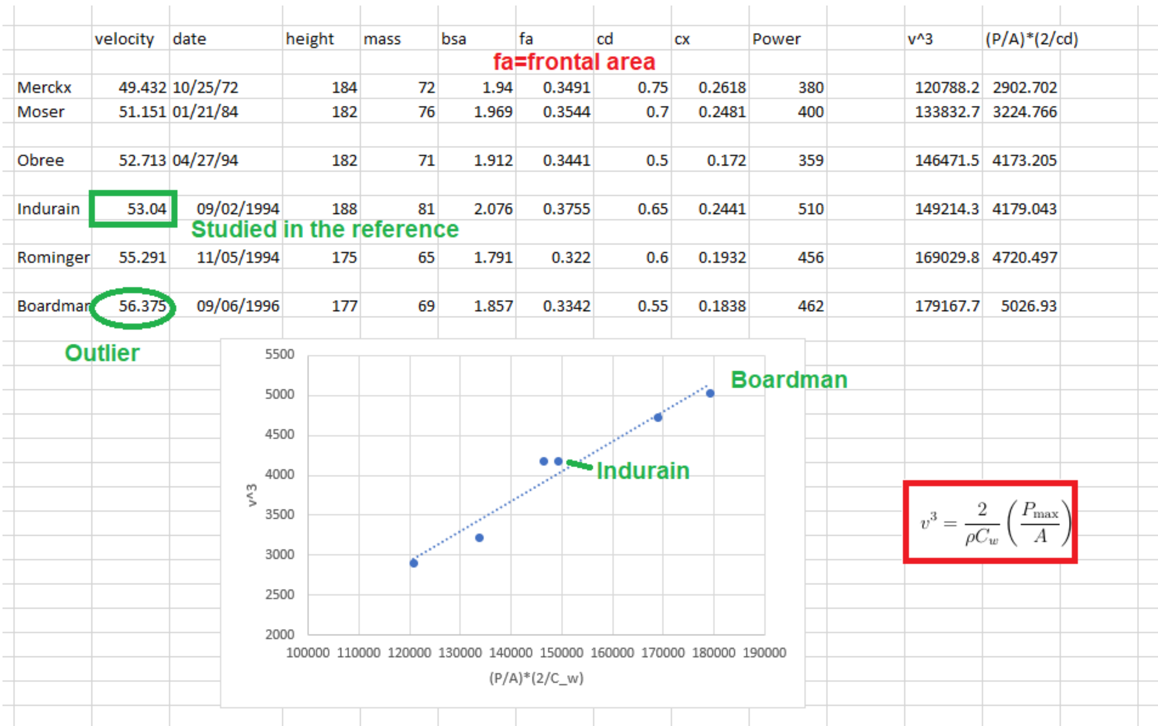
Miguel Induráin

From Wikipedia, the free encyclopedia

In this Spanish name, the first or paternal surname is Induráin *and the second or maternal family name is* Larraya.

Miguel Induráin Larraya (Spanish pronunciation: [miˈɣel induˈrajn laˈraʝa]; born 16 July 1964) is a retired Spanish road racing cyclist. Induráin won five Tours de France from 1991 to 1995, the fourth, and last, to win five times, and the only five-time winner to achieve those victories consecutively.[4]

Figure 10.2: Prompt to read a scientific paper – 'Scientific approach to the 1-h cycling world record: a case study', by Padilla, Mukika, Angulo, and Goriena (*J. Appl. Physio.* 89:1522-1527 (2000)). The case study refers to the setting of the hour record by Miguel Induráin in 1994 (53.040 km cycled in 1 hour on 2 September 1994).



	velocity	date	height	mass	bsa	fa	cd	cx	Power	v^3	(P/A)*(2/cd)
Merckx	49.432	10/25/72	184	72	1.94	0.3491	0.75	0.2618	380	120788.2	2902.702
Moser	51.151	01/21/84	182	76	1.969	0.3544	0.7	0.2481	400	133832.7	3224.766
Obree	52.713	04/27/94	182	71	1.912	0.3441	0.5	0.172	359	146471.5	4173.205
Indurain	53.04	09/02/1994	188	81	2.076	0.3755	0.65	0.2441	510	149214.3	4179.043
Rominger	55.291	11/05/1994	175	65	1.791	0.322	0.6	0.1932	456	169029.8	4720.497
Boardman	56.375	09/06/1996	177	69	1.857	0.3342	0.55	0.1838	462	179167.7	5026.93

Figure 10.3: Extraction of the data from the reference paper.

Chris Boardman took up the challenge using a modified version of the Lotus 110 bicycle, a successor to the earlier Lotus 108 bicycle he'd ridden to victory at the 1992 Olympic Games. South African company Aerodyne Technology built the frame. Boardman set the UCI Absolute record of 56.375 km (35.030 mi) in 1996, using another position pioneered by Obree, his arms out in front in a "Superman" position. This too was considered controversial by the UCI, and while the record was allowed to stand, the position was banned making Boardman's record set in 1996 effectively unbeatable using traditional bike position.[14]

The Lotus 108 bicycle, a forerunner to the Lotus 110 Chris Boardman used to set a new hour record of 56.375 km (35.030 mi) in 1996.

1997 UCI rule change [edit]

With the increasing gap between modern bicycles and what was available at the time of Merckx's record, the UCI established two records in 1997:

- *UCI Hour Record*: which restricted competitors to roughly the same equipment as Merckx, banning time trial helmets, disc or tri-spoke wheels, aerodynamic bars and monocoque frames.
- *Best Human Effort*: also known as the UCI "Absolute" Record[14] in which modern equipment was permitted.

As a result of the 1997 rule change, all records since 1972, including Boardman's 56.375 km (35.030 mi) in 1996, were moved to Best Human Effort and the distance of Eddy Merckx set in 1972 once more became the official UCI benchmark. In 2000, Boardman attempted the UCI record on a traditional bike, and rode 49.441 km (30.721 mi), topping Merckx by 10 metres (32.8 ft), an improvement of 0.02%.

In 2005, Ondřej Sosenka improved Boardman's performance at 49.700 km (30.882 mi) using a 54×13 gear. However, Sosenka failed a doping control in 2001 and then again in 2008, the latter resulting in a career-ending suspension which puts in doubt the validity of his record. All women's records from 1986 to 1996 were recategorized to Best Human Effort.

Figure 10.4: Follow-up information about the 'Boardman' data point in Figure 10.3.

Chapter 11

Mechanical Energy

Overview

The concept of mechanical energy is a very useful one in mechanics. In later modules you will see that energy is a much more fundamental idea than force. We introduce a framework in which mechanical energy can be defined. We then state the principle of **conservation of mechanical energy**. This can then be used to solve lots of different problems.

11.1 Energy

Having introduced the notion of work in a fairly rigorous way in Chapter 10, we can finally define energy:

Definition 11.1 *Energy is the capacity to do work.*

The SI unit of energy is the Joule (the same as work). Energy manifests itself in various ways - as light, heat, sound, and as mechanical energy. We deal with mechanical energy in this module:

Definition 11.2 Mechanical energy *is the capacity to do work as a result of position or motion.*

Furthermore, mechanical energy can be broken up into two parts:

- Kinetic energy – the capacity to do work as a result of motion.
- Potential energy – the capacity to do work as a result of position.

Thus, for a mechanical system, the total (mechanical) energy is

$$\text{total mechanical energy} = \text{kinetic energy} + \text{potential energy}.$$

11.1.1 Kinetic energy

Suppose that an object of mass m is accelerated from rest by a force $\boldsymbol{F}$. We have Newton's second law:

$$m\boldsymbol{a} = \boldsymbol{v} \implies m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{F}. \tag{11.1}$$

For the moment, we will work with one-dimensional motion, mindful that the following discussion can be extended to motion in higher dimensions. Thus, Equation (11.1) becomes

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F.$$

The work done by the force in moving the particle through an amount $\mathrm{d}x$ is

$$\mathrm{d}W = F\mathrm{d}x.$$

By Bewton's second law, this can be re-written as

$$\mathrm{d}W = m\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}x.$$

Yet, in a time $\mathrm{d}t$, the increment $\mathrm{d}x$ can be written as $v\,\mathrm{d}t$, hence the infinietesimal amount of work done is

$$\mathrm{d}W = m\frac{\mathrm{d}v}{\mathrm{d}t}v\,\mathrm{d}t.$$

Formally cancelling above and below by $\mathrm{d}t$ (chain rule), this is

$$\mathrm{d}W = mv\,\mathrm{d}v.$$

We may integrate this relation over the time required for the particle to start from rest and to attain a particular velocity v. This corresponds to integrating over all velocities, from $v = 0$ to the final particular velocity of interest v.

$$\int \mathrm{d}W = \int_0^v mv\,\mathrm{d}v = \tfrac{1}{2}mv^2. \tag{11.2}$$

This is the work done in accelerating the particle from rest to a new state where the particle has velocity v. This is work done due to motion – kinetic energy. Thus,

$$\text{kinetic energy} = \tfrac{1}{2}mv^2.$$

Kinetic energy is given the symbol T or K, and it is a fundamentally non-negative quantity. For motion in higher dimensions this result generalizes to

$$T = \tfrac{1}{2}m\boldsymbol{v}^2,$$

where $\boldsymbol{v}$ is the vector velocity and $\boldsymbol{v}^2 = \boldsymbol{v} \cdot \boldsymbol{v}$.

11.1.2 Work–energy theorem

More generally, we may suppose that a particle experiences a force to accelerate it from an initial velocity v_1 to a final velocity v_2, taking it from a position x_1 to a final position at x_2. Then, Equation (11.2) becomes

$$\int_{x_1}^{x_2} \mathrm{d}W = \int_{v_1}^{v_2} mv\,\mathrm{d}v = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2, \tag{11.3}$$

which can also be written as

$$W(x_1 \to x_2) = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2.$$

This result is the work-energy theorem:

> The work done in moving a particle through a displacement $x_1 \to x_2$ is equal to the corresponding change in the particle's kinetic energy.

11.1.3 Potential energy

In one dimension, **Potential energy** can be defined for forces that do not depend explicitly on time. Specifically, consider again Equation (11.3) and write $\mathrm{d}W = F(x)\mathrm{d}x$. Introduce

$$\mathcal{U}(x) = \int_a^x F(x)\,\mathrm{d}x,$$

where a is some arbitrary **reference level**.

Definition 11.3 *For a force in one dimension $F(x)$ that does not depend explicitly on time,*

$$\mathcal{U}(x) = -\int_a^x F(x)\,\mathrm{d}x, \tag{11.4}$$

is called the **potential energy***. Potential energy is defined with respect to an arbitrary reference level a.*

Definition 11.4 *A force is called* **conservative** *if it has a potential-energy function along the lines of Equation* (11.4)*.*

In one dimension, all forces that are a function only of position are obviously conservative. In higher dimensions, things are a bit trickier. In one dimension and higher, gravity is a conservative force, as we will demonstrate later. For a conservative force, we have

$$\begin{aligned} W(x_1 \to x_2) &= \int_{x_1}^{x_2} \mathrm{d}W, \\ &\overset{\text{conservative force}}{=} \int_{x_1}^{x_2} F(x)\mathrm{d}x, \\ &= \int_{x_1}^{a} F(x)\mathrm{d}x + \int_{a}^{x_2} F(x)\mathrm{d}x, \\ &= \int_{a}^{x_2} F(x)\mathrm{d}x - \int_{a}^{x_1} F(x)\mathrm{d}x, \\ &= -\mathcal{U}(x_2) + \mathcal{U}(x_1), \\ &\overset{\text{Work-energy theorem}}{=} \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2, \end{aligned}$$

Note that the precise value of the reference level a doesn't matter here because it appears twice in a self-cancelling way. Re-arranging these relations, we have

$$\tfrac{1}{2}mv_1^2 + \mathcal{U}(x_1) = \tfrac{1}{2}mv_2^2 + \mathcal{U}(x_2).$$

Thus, the mechanical energy is the same at positions x_1 and x_2 and hence, at all positions. This gives the following **principle of conservation of mechanical energy**:

For a conservative force $F(x)$, the total mechanical energy

$$\boxed{E = \tfrac{1}{2}mv^2 + \mathcal{U}(x), \qquad \mathcal{U}(x) = -\int_a^x F(x)\mathrm{d}x} \tag{11.5}$$

is conserved.

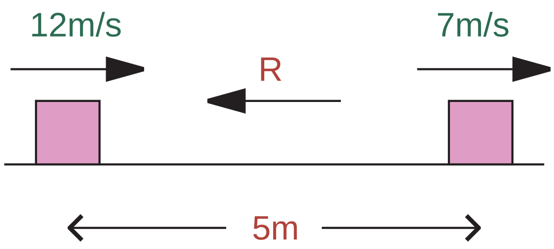
Here, by 'conserved' we mean 'constant' or 'not changing with time'.

11.1.4 Caution!

Caution: We have derived the principle of conservation of mechanical energy from (work-energy theorem)+(conervataive forces). The work-energy theorem is therefore much more general starting-point and holds for **all** forces.

The work-energy theorem holds for all forces and is very useful for solving problems - including those involving friction, as in the following example.

Example: A small block of mass 3 kg is moving on a horizontal plane against a constant resistance of R N. The speed of the block falls from 12 m/s to 7 m/s as the block moves 5 m. Find the magnitude of the resistance.



Solution: Use the work-energy theorem. The change in the kinetic energy is

$$\Delta T = \tfrac{1}{2}m(12^2 - 7^2\,\mathrm{m/s}^2) = 142.5\,\mathrm{J}.$$

This is equal to the work done by the resistance force on the block:

$$\Delta T = \int R\mathrm{d}x = Rx = R(5\,\mathrm{m}),$$

hence

$$R = \frac{\Delta T}{x} = \frac{142.5\,\mathrm{J}}{5\mathrm{m}} = 28.5\,\mathrm{N}.$$

11.1.5 Gravity

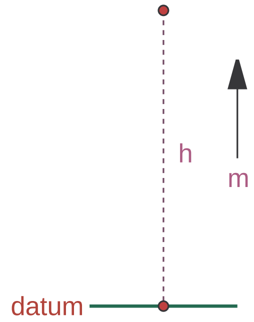
The gravitational force on a particle is $F = -mg$ (downward). This corresponds to a standard coordinate line x where $x > 0$ points upward and $x < 0$ points downward. The potential energy function is thus

$$\mathcal{U}(x) = -\int_a^x F\,\mathrm{d}x = +mg(x-a),$$

where a is the reference level. Calling $h = x - a$, the potential energy can be written as

$$\mathcal{U}(x) = mgh.$$

We are free to decide what level, in any given problem, corresponds to a. It need not be ground level. This level is called the **reference level** or datum.



Potential energy can be either negative or positive, and is only defined up to a constant (Figure 11.1):

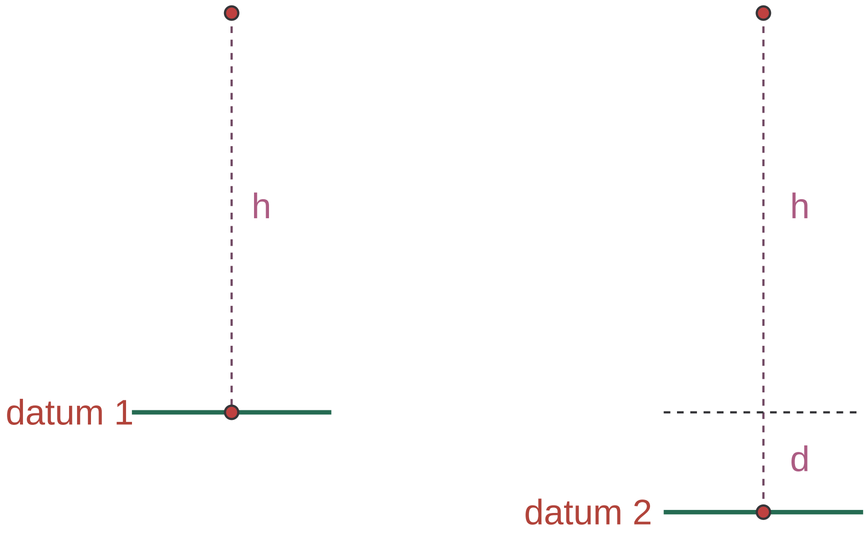
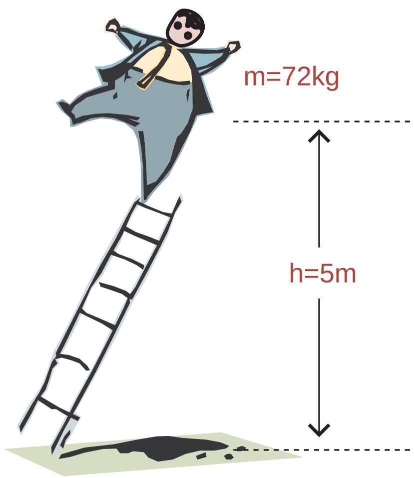


Figure 11.1: The reference level for gravitational potential energy is arbitrary



Example: A man of 72 kg climbs up a ladder to a height of 5 m. Assuming that the man can be treated as a particle(!), find the potential energy relative to the ground.

Solution:

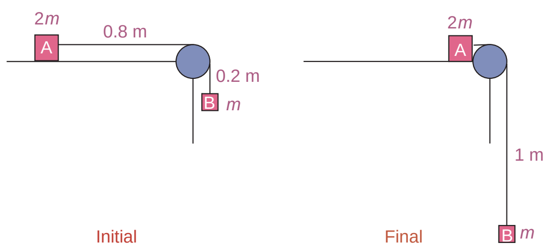
- Potential energy gained:

$$mgh = 72 \times 9.8 \times 5 = 3528 \, \text{Joules}.$$

- ~~Potential energy lost:~~

$$72 \times 9.8 \times 3 = 2117 \, \text{Joules}.$$

Example: A small block A of mass 2 m is lying in smooth contact with a table top. A light inextensible string of length 1 m is attached at one end to A, passes over a smooth pulley and carries a block of mass m hanging freely at the other end. Initially A is held at rest 0.8 m from the edge of the table.
If the system is released, find the speed of A when it reaches the edge.



Solution:

- Initial mechanical energy $E_1 = -mg \times 0.2 = -0.2mg$.
- Final mechanical energy:

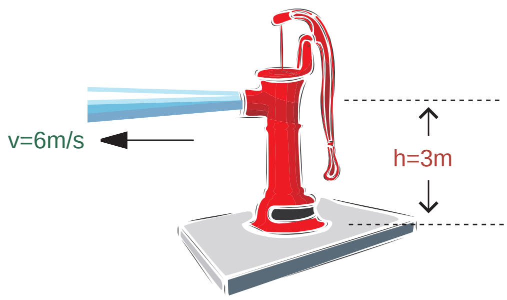
$$E_2 = \tfrac{1}{2} \times 2m \times v^2 + \tfrac{1}{2} \times m \times v^2 - mg \times 1 = \frac{3}{2}mv^2 - mg.$$

The mechanical energy is conserved: $E_1 = E_2$:

$$\begin{aligned}
-0.2mg &= \tfrac{3}{2}mv^2 - mg, \\
\tfrac{3}{2}mv^2 &= mg - 0.2mg, \\
\tfrac{3}{2}v^2 &= g - 0.2g = 0.8g, \\
v^2 &= \tfrac{2}{3} \times 0.8g = 5.227. \\
v &= 2.29\,\mathrm{m/s}.
\end{aligned}$$

11.2 More worked examples

Example: Water is being raised by a pump from a storage tank at ground level and ejected at 3 m above ground level though a pipe at 6 m/s. The water is delivered at a rate of 420 kg per second.
Find the mechanical energy supplied by the pump in one second (the power).



Solution: The energy at the reference level (ground level) is unknown and equal to E_1. The mass in question is

$$\Delta m = (420\,\mathrm{kg/sec}) \times 1\,\mathrm{sec} := Q\Delta t,$$

where $\Delta t = 1\,\mathrm{sec}$ and $Q = 420\,\mathrm{kg/sec}$. Thus, the energy at height $h = 3\,\mathrm{m}$ is

$$E_2 = \tfrac{1}{2}\Delta m v^2 + \Delta m g h = \Delta m \left(\tfrac{1}{2}v^2 + gh\right).$$

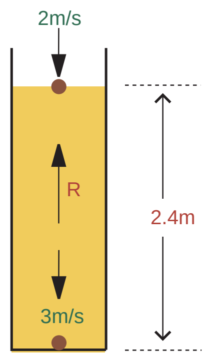
By conservation of mechanical energy, $E_1 = E_2$ and thus,

$$\begin{aligned}
P &= \frac{E_1}{\Delta t} = \frac{E_2}{\Delta t}, \\
&= \frac{\Delta m \left(\frac{1}{2}v^2 + gh\right)}{\Delta t}, \\
&= Q\left(\tfrac{1}{2}v^2 + gh\right).
\end{aligned}$$

Filling in, this is

$$P = 420\left(\tfrac{1}{2}6^2 + 9.8 \times 3\right) = 19908\,\mathrm{Joules/sec}.$$

> Example: A stone, of mass 4 kg, falls vertically downwards through a tank of viscous oil. The speed as the stone enters the oil is 2 m/s and, at the bottom of the tank, it is 3 m/s. The depth of the oil is 2.4 m.
> Find the oil resistance on the stone.



Solution: Resistance forces are a bit mysterious. They are part of a family of 'dissipative forces' where their functional form $F = F(\cdots)$ is a complicated function of velocity. They are therefore not conservative in general. Thus, we cannot assume here that energy is conserved. But we can still use the work-energy theorem! We have

$$\Delta T = \int F_{\text{net}} \mathrm{d}s,$$

where the net force is $mg - R$, hence

$$\tfrac{1}{2} m(v_2^2 - v_1^2) = (mg - R)h,$$

where $h = 2.4\text{m}$, where the subscript 1 denotes the top of the tank, and where the subscript 2 denotes the bottom of the tank. Fill in the values:

$$\tfrac{1}{2}(4\,\text{kg})\left(3^2 - 2^2\,\text{m/s}^2\right) = (mg - R)h,$$

hence

$$(mg - R)h = 10.$$

Thus,

$$R = mg - \frac{10}{h}.$$

Fill in the numbers:

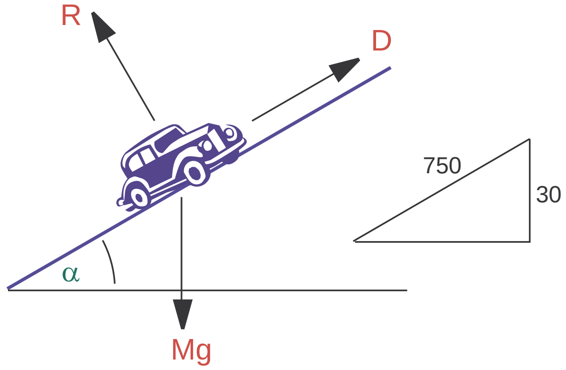
$$R = (4 \times 9.8) - (10/2.4) = 35.03\,\text{N}.$$

Example: A car of mass 1000 kg drives up a slope of length 750 m at an incline of 1 in 25. If the resistance is negligible, calculate the driving force of the engine if the speed at the foot of the slope is 25 m/s and the speed at the top is 20 m/s.
Take the incline[a] of $1/25$ to mean

$$\sin\alpha = \frac{1}{25}$$

.

[a]This is a rather unusual example as the incline usually refers to the $\tan$ of an angle. Because $1/25$ is so small, $\tan(\alpha) \approx \sin(\alpha) \approx \alpha$ in this example, so the unusual nature of the example is not important.



Solution: note that there is a change in the velocities, so there is an acceleration, hence the forces are unbalanced. A force diagram will therefore not help. Also, note that we are looking to solve for an unknown force, so the principle of conservation of mechanical energy – although applicable – will not help us here. The only thing to do then is to use the work-energy theorem again. We compute the change in kinetic energy, letting the subscript 2 denote the final situation (top of hill) and the subscript 1 denote the initial situation (bottom of hill):

$$\Delta T = \tfrac{1}{2}m(v_2^2 - v_1^2) = \tfrac{1}{2}(1000\,\mathrm{kg})\left(20^2 - 25^2\,\mathrm{m/s}^2\right) = -112500\,\mathrm{J}.$$

We, err, work out the work done by the forces on the car. We split the net force into the gravitational force and the driving force:

$$\boldsymbol{F} = \boldsymbol{F}_{\text{grav}} + \boldsymbol{F}_{\text{drive}}.$$

In the standard coordinate system,

$$\boldsymbol{F}_{\text{grav}} = -mg\left(\sin\alpha\mathbf{i} + \cos\alpha\mathbf{j}\right),$$
$$\boldsymbol{F}_{\text{drive}} = D\mathbf{i},$$

and the displacement is $\boldsymbol{s} = \Delta x\mathbf{i}$, with $\Delta x = 750\,\mathrm{m}$. Thus,

$$W = \boldsymbol{F}\cdot\boldsymbol{s} = (D - mg\sin\alpha)\Delta x.$$

We now apply the work-energy theorem:

$$\begin{aligned} \Delta T &= W, \\ -112500 &= (D - mg\sin\alpha)\Delta x, \end{aligned}$$

$$mg\sin\alpha\Delta x - 112500 = D\Delta x.$$

Thus,

$$D\Delta x = 181,500,$$

and the driving force is thus

$$D = \frac{181,500}{\Delta x} = \frac{181,500}{750} = 242\,\mathrm{N}.$$

Chapter 12

Conservation of momentum – application to particle collisions

Overview

We use the principles of conservation of momentum and conservation of mechanical energy to analyse collisions between particles.

12.1 General principles

Recall in Chapter 8 we defined the momentum of a system of n particles, each its own individual momentum $\boldsymbol{p}_i$:

$$\text{(Momentum of system:) } \boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \cdots \boldsymbol{p}_n$$

We also showed how the principle of conservation of momentum follows from Newton's second and third laws:

> In a closed system, the total momentum of the system is conserved,

where by 'conserved' we mean 'constant', or 'stays the same for all time. We also relaxed the assumption about the system being closed: if the external force is in a fixed direction then momentum conservation will still apply – but only in a plane perpendicular to that direction.

Consider also the mechanical energy of a closed system. Because there is no net external force on such a system, there principle of mechanical energy states that the kinetic energy will be conserved:

$$\text{K.E} = \text{const} \qquad \text{(closed system).}$$

12.2 Elastic collisions

Consider the collision shown in Figure 12.1

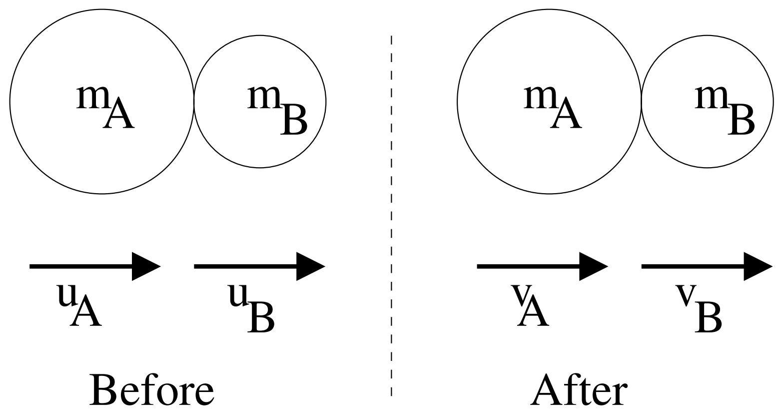


Figure 12.1: Elastic collision between two particles

Conservation of momentum gives

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2.$$

Conservation of kinetic energy gives

$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2.$$

Let $\mu = m_2/m_1$. Then, conservation of kinetic energy reduces to

$$\begin{aligned} u_1^2 + \mu u_2^2 &= v_1^2 + \mu v_2^2 \\ u_1^2 - v_1^2 &= \mu(v_2^2 - u_2^2) \\ (u_1 + v_1)(u_1 - v_1) &= \mu(v_2 + u_2)(v_2 - u_2). \end{aligned} \tag{12.1}$$

From conservation of momentum we have $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$, hence

$$u_1 - v_1 = \mu(v_2 - u_2) \tag{12.2}$$

Substitute Equation (12.2) into Equation (12.1) to obtain

$$v_1 - v_2 = -(u_1 - u_2).$$

Definition 12.1 *A collision in which the kinetic energy is conserved is called elastic. In an elastic* *collision between two bodies, the relative velocities before and after the collision are related via*

$$\boxed{v_1 - v_2 = -(u_1 - u_2).} \tag{12.3}$$

Note that in a closed system (no interaction between the particles and the environment) the collision will be necessarily elastic. In contrast, in an open system, things are a bit more complicated.

12.3 Inelastic collision between two particles

In practice, collisions are inelastic: energy is transferred from the closed system to the outside world via heat, sound, etc. However, in such scenarios, conservation of momentum still holds. We therefore need to reformulate our conservation laws slightly to solve problems for inelastic collisions. Therefore, for inelastic collisions, we have the following two principles:

- Conservation of momentum
- ~~Conservation of kinetic energy~~ Newton's law of restitution:

$$\text{(Relative speed after collision)} = -C_R \times \text{(Relative speed before collision)}$$

where C_R is the **coefficient of restitution.**

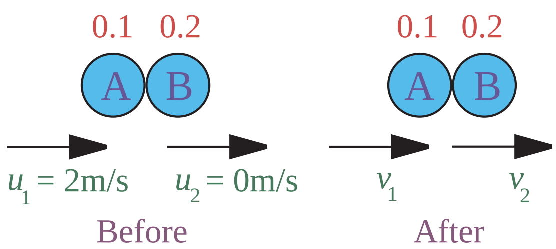
This can be thought of as a generalization of the previous analysis of elastic collisions: Newton's law of restitution can be thought of as a generalization of Equation (12.3), which now reads

$$v_1 - v_2 = -C_R(u_1 - u_2). \qquad \text{(inelastic collisions)}$$

Note that the special case of elastic collisions can always be recovered by setting $C_R = 1$.

12.4 Worked examples

Example: A particle A of mass 0.1 kg is moving with velocity 2 m/s towards another particle B of mass 0.2 kg which is at rest. Both particles are on a smooth horizontal table. If the coefficient of restitution between A and B is 0.8 find the velocity of each particle after collision.



Solution: Start with momentum conservation:

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2, \\ 0.1 \times 2 + 0.2 \times 0 &= 0.1 \times v_1 + 0.2 \times v_2, \\ 0.2 &= 0.1v_1 + 0.2v_2, \\ 2 &= v_1 + 2v_2. \end{aligned}$$

We also have Newton's law of restitution:

$$\begin{aligned} v_1 - v_2 &= -C_R(u_1 - u_2) \\ v_1 - v_2 &= -0.8(2 - 0) \\ v_1 - v_2 &= -1.6. \end{aligned}$$

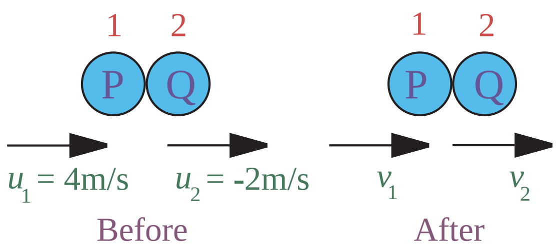
These two principles therefore provide simultaneous equations in v_1 and v_2:

$$\begin{aligned} v_1 + 2v_2 &= 2, \\ v_1 - v_2 &= -1.6. \end{aligned}$$

The solution:

$$3v_2 = 3.6, \quad v_2 = 1.2$$
$$v_1 = v_2 - 1.6 = -0.4.$$

Example: A particle P of mass 1 kg, moving with speed 4 m/s collides directly with another particle Q of mass 2 kg moving in the opposite direction with speed 2 m/s. The coefficient of restitution for these particles is 0.5.
Find
(a) the velocity of each particle after collision,
(b) the loss in kinetic energy.



Solution – part (a). Use the principle of conservation of momentum:

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2, \\ 1 \times 4 + 2 \times (-2) &= 1 \times v_1 + 2 \times v_2, \\ v_1 + 2v_2 &= 0. \end{aligned}$$

Again, we have Newton's law of restitution:

$$\begin{aligned} v_1 - v_2 &= -C_R(u_1 - u_2), \\ v_1 - v_2 &= -0.5(4 - (-2)) = -3, \\ v_1 - v_2 &= -3. \end{aligned}$$

We obtain two simultaneous equations:

$$\begin{aligned} v_1 + 2v_2 &= 0, \\ v_1 - v_2 &= -3, \end{aligned}$$

with solution

$$\begin{aligned} 3v_2 = 3, &\qquad v_2 = 1, \\ v_1 = v_2 - 3 &= 1 - 3 = -2, \end{aligned}$$

hence

$$v_1 = -2\,\mathrm{m/s}, \qquad v_2 = 1\,\mathrm{m/s}.$$

For part (b) we have

$$\begin{aligned} E_{\mathrm{init}} &= \tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2, \\ &= \tfrac{1}{2} \times 1 \times 16 + \tfrac{1}{2} \times 2 \times 4, \\ &= 12\,\mathrm{J}, \end{aligned}$$

and

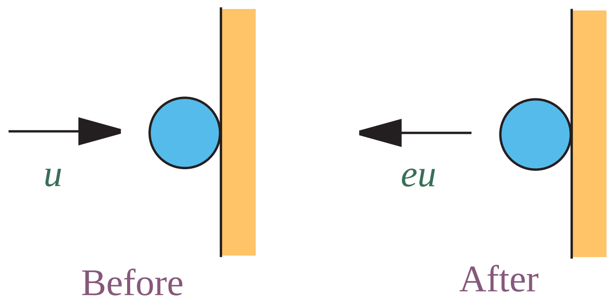
$$\begin{aligned} E_{\mathrm{fin}} &= \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2, \\ &= \tfrac{1}{2} \times 1 \times 4 + \tfrac{1}{2} \times 2 \times 1, \\ &= 3\,\mathrm{J}, \end{aligned}$$

The loss in kinetic energy is

$$E_{\mathrm{init}} - E_{\mathrm{fin}} = (12 - 3)\mathrm{J} = 9\mathrm{J}.$$

12.5 Collisions with a fixed wall

For collisions of a particle with a fixed wall, the principle of conservation of momentum is of no use. For example, consider a particle moving at right angles to a wall. The particle velocity changes sign after the collision, meaning that there is a change in the particle's momentum. Thus, the particle experiences an external force during the collision, namely the reaction force of the wall on the particle. In the collision, kinetic energy is not necessarily conserved either, so the only principle we can resort to is Newton's law of restitution. Thus, if a particle moving with speed u, perpendicular to a wall, collides with the wall, then the velocity is reversed and the speed is $C_R u$, where C_R is the coefficient of restitution between the wall and particle. This principle can be seen in the figure below:



$$[v_1 - v_2 = -C_R(u_1 - u_2)] \qquad \Longrightarrow \qquad [v_1 = -C_R u_1]$$

Chapter 13

Circular motion

Overview

In this chapter, we consider a particle constrained to move in a circle.

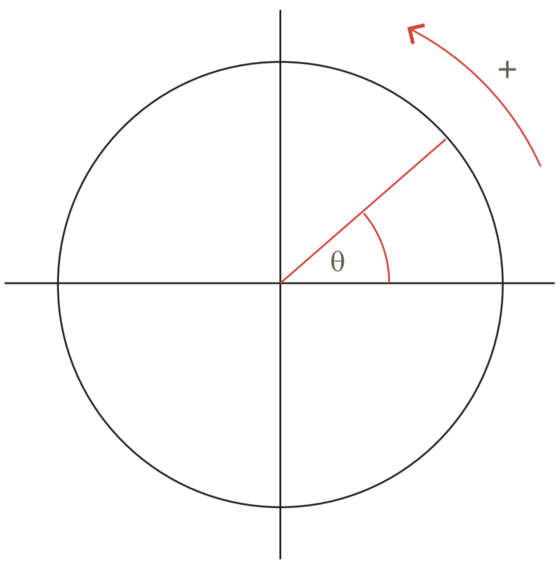
13.1 Introduction and definitions

In this section we consider a particle constrained to move in a circle. We begin with some basic definitions:

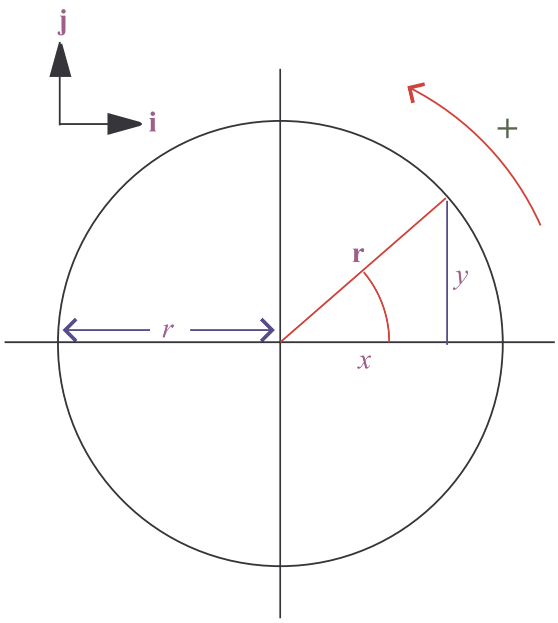
Definition 13.1 *Angular velocity is the rate of change of θ with time:*

$$\boxed{\omega = \frac{d\theta}{dt}.} \tag{13.1}$$

- measured in radians/second
- Beware units: revs/sec, revs/min, revs/hr (1 rev $= 2\pi$ radians)
- We take positive θ to be in the anticlockwise sense.
- Angular speed $= |d\theta/dt|$



Consider a particle moving with angular velocity ω on a circle of radius r. We can derive a simple relation between the linear velocity v (metres/sec) and ω.



Start with

$$x = r\cos\theta \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}\theta} = -r\sin\theta$$

$$y = r\sin\theta \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = r\cos\theta$$

Apply the chain rule:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\frac{\mathrm{d}x}{\mathrm{d}\theta} = -r\,\omega\sin\theta$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\frac{\mathrm{d}y}{\mathrm{d}\theta} = r\,\omega\cos\theta$$

Introduce the position vector $\boldsymbol{r}$:

$$\boldsymbol{r} = x\,\mathbf{i} + y\,\mathbf{j}.$$

Introduce also the velocity velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$. Since $\mathbf{i}$ and $\mathbf{j}$ are constants, we have

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}} = \frac{\mathrm{d}x}{dt}\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{j} = -r\,\omega\sin\theta\,\mathbf{i} + r\,\omega\cos\theta\,\mathbf{j}$$

We have

$$\begin{aligned} v^2 &= r^2\omega^2\sin^2\theta + r^2\omega^2\cos^2\theta, \\ &= r^2\omega^2(\sin^2\theta + \cos^2\theta), \\ &= r^2\omega^2, \end{aligned}$$

hence

$$\boxed{v = |\omega| r} \tag{13.2}$$

Example: Express

(a) 3 rev/min in rad/sec,

(b) 0.005 rad/sec in rev/hr.

Solution:

(a) 1 rev $= 2\pi$ radians. 3 rev/min $\to \frac{3\times 2\pi}{60}$ rad/sec $= \frac{\pi}{10}$ rad/sec.

(b) 1 rad $= \frac{1}{2\pi}$ rev. 0.005 rad/sec $\to \frac{0.005}{2\pi} \times 3600$ rev/hr $= \frac{9}{\pi}$ rev/hr.

Example: A point on the circumference of a disc is rotating at a constant speed of 3 m/s. If the radius of the disc is 0.24 m, find in rad/sec the angular speed of the disc.

Solution:

$$v = 3 \ \ \text{m/s}$$
$$r = 0.24 \ \ \text{m}$$
$$v = |\omega|\, r$$

Hence,

$$|\omega| = \frac{v}{r} = \frac{3}{0.24} = 12.5\,\mathrm{rad/sec.}$$

13.2 Centripetal Acceleration

When a particle follows a circular path at constant speed, the velocity vector changes with time as the particle turns. Therefore, the particle is accelerating, even though the speed remains constant since acceleration is the rate of change of velocity. This is known as **centripetal acceleration**. If the particle also speeds up or slows down, it has an extra contribution to the acceleration known as angular acceleration. The aim of this section is to derive a mathematical expression for the acceleration for a particle constrained to execute circular motion.

To do this, we introduce **polar coordinates**. We start by introducing the concepts of unit normal and unit tangent vectors to the circle, as shown in Figure 13.1.

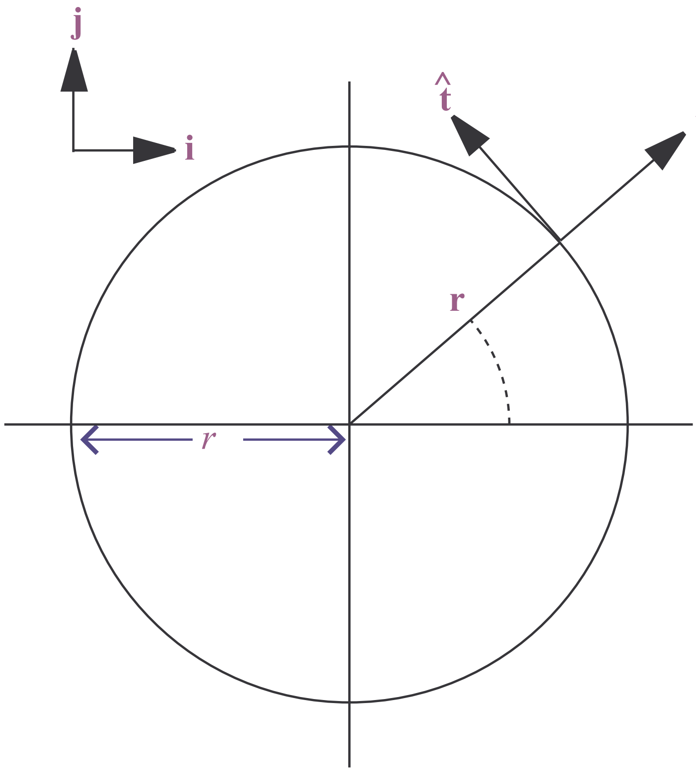


Figure 13.1: The unit normal and unit tangent vectors to the circle

Referring to the figure, we see that while the unit cartesian vectors $\mathbf{i}$ and $\mathbf{j}$ are fixed, both $\hat{\mathbf{n}}(t)$ and $\hat{\mathbf{t}}(t)$ are functions of time: they depend on the particle's instantaneous position. Like any vector, $\hat{\mathbf{n}}(t)$ and $\hat{\mathbf{t}}(t)$ can be written in terms of their cartesian components:

$$\begin{aligned} \hat{\boldsymbol{n}}(t) &= \quad \cos\theta(t)\,\mathbf{i} + \sin\theta(t)\,\mathbf{j} \\ \hat{\boldsymbol{t}}(t) &= -\sin\theta(t)\,\mathbf{i} + \cos\theta(t)\,\mathbf{j} \end{aligned}$$

A nice thing about these equations is that they can be put into matrix form:

$$\begin{pmatrix} \hat{\boldsymbol{n}} \\ \hat{\boldsymbol{t}} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix},$$

and the inverse is particularly simple: A nice thing about these equations is that they can be put into matrix form:

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{n}} \\ \hat{\boldsymbol{t}} \end{pmatrix};$$

in particular,

$$\mathbf{j} = \sin\theta\hat{\boldsymbol{n}} + \cos\theta\hat{\boldsymbol{t}}. \tag{13.3}$$

In any case, we can now write the displacement $(\mathbf{r})$, velocity $(\mathbf{v})$ and acceleration $(\mathbf{a})$ vectors in either Cartesian or polar coordinates.

- Displacement: We have

$$\begin{aligned} \boldsymbol{r} &= x\mathbf{i} + y\mathbf{j}, \\ &= r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}, \\ &= r(\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}), \\ &= r\hat{\mathbf{n}}. \end{aligned}$$

- Velocity: We have

$$\begin{aligned} \boldsymbol{v} &= -r\,\omega\sin\theta\,\mathbf{i} + r\,\omega\cos\theta\,\mathbf{j}, \\ &= r\omega(-\sin\theta\mathbf{i} + \omega\cos\theta\mathbf{j}), \\ &= r\omega\,\hat{\mathbf{t}}. \end{aligned}$$

  The velocity is always in the tangential direction.

We now work on the acceleration: We have

$$\boldsymbol{v}(t) = r\omega\,\hat{\mathbf{t}},$$

and

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}.$$

By the product rule, we have

$$\boldsymbol{a} = r\hat{\mathbf{t}}\frac{\mathrm{d}\omega}{\mathrm{d}t} + r\omega\frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}t}, \tag{13.4}$$

so it remains to work out $\mathrm{d}\hat{\mathbf{t}}/\mathrm{d}t$:

$$\begin{aligned} \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right), \\ &= -\mathbf{i}\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta + \mathbf{j}\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta, \\ &= -\omega\mathbf{i}\cos\theta - \omega\mathbf{j}\sin\theta, \\ &= -\omega\hat{\mathbf{r}}. \end{aligned}$$

Hence, Equation (13.4) becomes

$$\boldsymbol{a} = r\hat{\mathbf{t}}\frac{\mathrm{d}\omega}{\mathrm{d}t} - r\omega^2\hat{\mathbf{r}}.$$

This is an important result, so we put it in a nice box:

$$\boxed{\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = r\hat{\mathbf{t}}\frac{\mathrm{d}\omega}{\mathrm{d}t} - r\omega^2\hat{\mathbf{r}}.} \tag{13.5}$$

Note that the **radial component** of the acceleration is $-\omega^2 r\hat{\mathbf{n}}$ which has magnitude $\omega^2 r$ and is directed towards the centre of the circle:

> For a particle to move in a circle, it has to have an acceleration of $\omega^2 r$ towards the centre.

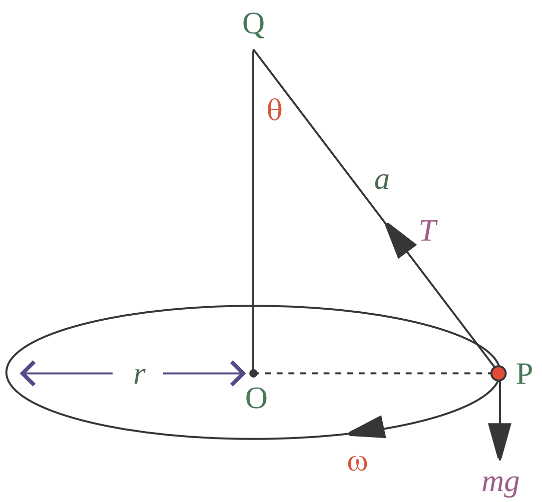
By Newton's second law, there must be a net force on the particle acting towards the centre of the circle. The magnitude of the force must be equal to $m\omega^2 r$. Taking Equations (13.2) and (13.5) the acceleration towards the centre (centripetal accelaration) can also be written as

$$\boxed{\text{Centripetal acceleration} = \omega^2 r = \frac{v^2}{r}.} \tag{13.6}$$

Caution! The expression Force $= m\omega^2 r$ can be misleading – even if it is very simple. It seems to state that the circular motion produces a force. This is mixing up cause and effect! Things are the other way around. Forces must be present to constrain the particle to do circular motion. The net result of these forces is the acceleration $m\omega^2 r$. The expression $m\omega^2 r$ is an effect of the forces. It is not the cause of the motion.

13.3 Example – the conical pendulum

Example: particle P of mass m is attached to one end of a light inextensible string of length a and describes a horizontal circle, centre O, with constant angular speed ω. The other end of the string is fixed to a point Q and, as P rotates, the string makes an angle θ with the vertical.
Show that:
(a) the tension in the string always exceeds the weight of the particle,
(b) the depth of O below Q is independent of the length of the string.



Solution – part (a): Force balance:

$$T\cos\theta = mg, \qquad T\sin\theta = ma = m\omega^2 r.$$

Solve for T:

$$T^2\cos^2\theta + T^2\sin^2\theta = (mg)^2 + (m\omega^2 r)^2,$$

hence

$$T = mg\sqrt{1 + (\omega^2 r/g)^2} > mg.$$

Part (b): We have $|OQ| = a\cos\theta$. From the force balance:

$$mg = T\cos\theta \implies \cos\theta = mg/T.$$

Also,

$$T\sin\theta = m\omega^2 r = m\omega^2 a\sin\theta,$$

hence

$$T = m\omega^2 a \implies a = T/(m\omega^2).$$

Putting these results together, we get

$$|OQ| = a\cos\theta = \left[T/(m\omega^2)\right] \times (mg/T) = g/\omega^2,$$

independent of a.

13.4 Motion in a vertical circle

We now consider motion in a vertical circle, and the consequence of this is that the particle speed is not constant:

- Particles slow down as they rise on a circular path
- They speed up as they fall.

In principle, we have to write the equation of motion (Newton's Second Law) in the tangential direction. This allows us to solve for the angular acceleration. However, in most cases of interest, the work done by the non-gravitational forces is zero. For example, a reaction force normal to the circular path does no work. We then have conservation of mechanical energy:

$$E = \frac{1}{2}mv^2 + mgh = \text{ constant}$$

This, together with the radial part of the equation of motion, allows us to solve the problem.

Example: One end of a light rod of length a metres is pivoted at a fixed point O. A particle of mass m kg is attached to the other end. The rod is hanging at rest.

The particle is given a blow and moves initially with velocity u m/s.

(a) Find the value of u if the rod comes to rest first when horizontal.
(b) Show that the particle will perform a complete circle if $u > 2\sqrt{ga}$.
(c) When $u = 2\sqrt{ga}$ find the force in the rod when the particle is at the highest point.
(d) If $u = \sqrt{3ga}$ find the height of the particle above the centre of the circle when the tension is zero.

Solution – part (a). Use conservation of mechanical energy:

$$E_{\text{init}} = \tfrac{1}{2}mu^2.$$

Where the reference level is the bottom of the circle.

$$E_{\text{final}} = mga.$$

$$E_{\text{init}} = E_{\text{final}} \implies u = \sqrt{2ga}.$$

Part (b): Again, by conservation of mechanical energy:

$$\tfrac{1}{2}mv^2 + mga(1-\cos\phi) = \tfrac{1}{2}mu^2,$$

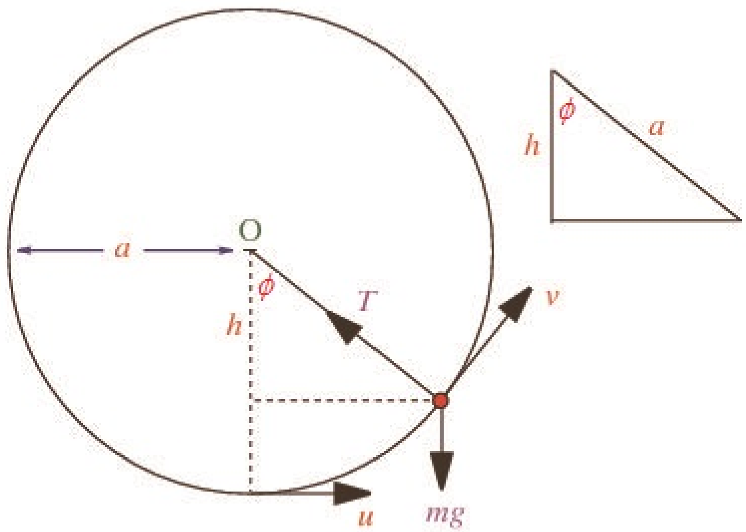


Figure 13.2: Force balance for the vertical circle

where $h = a(1-\cos\phi)$ is the height above the reference level. Re-arranging gives

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left[u^2 - 2ga(1-\cos\phi)\right].$$

We must have

$$\tfrac{1}{2}mv^2 > 0, \qquad \forall\phi$$

hence

$$u^2 - 2ga(1-\cos\phi) > 0 \qquad \forall\phi.$$

In the worst-case scenario ($\phi = \pi$, $\cos\phi = -1$), we must have

$$u^2 - 4ga > 0,$$

hence

$$u > 2\sqrt{ga}.$$

Part (c): When $u = 2\sqrt{ga}$ at the top of the circle, the particle is just barely doing circular motion. It follows that the net force on the particle at this point is zero, hence $T + mg = 0$, hence $T = -mg$.

Alternatively, we can do a force balance as a function of ϕ, as in Figure 13.4, to work out T as a function of ϕ. But we must be careful: we do the force balance in the radial and tangential directions, noting that there is acceleration in the radial direction (centripetal acceleration). In the radial direction,

- There is the force of tension, $-T\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is the unit vector in the normal direction from Section 13.2

- There is a component of the gravity force. Gravity is $-mg\mathbf{j}$. Referring back again to Section 13.2 and Equation (13.3) this is

$$-mg\mathbf{j} = -mg(\sin\theta\hat{\boldsymbol{n}} + \cos\theta\hat{\boldsymbol{t}}).$$

  We have to be careful not to mix up θ and ϕ: we have $\theta = \phi + (\pi/2)$, hence the gravitational force is

$$+mg(\cos\phi\hat{\boldsymbol{n}} - \sin\phi\hat{\boldsymbol{t}}).$$

Putting the radial force components together, these are

$$-T\hat{\boldsymbol{n}} + mg\cos\phi\hat{\boldsymbol{n}},$$

and these are equal to the mass times the centripetal acceleration:

$$-T\hat{\boldsymbol{n}} + mg\cos\phi\hat{\boldsymbol{n}} = -\frac{mv^2}{a}\hat{\boldsymbol{n}},$$

hence

$$T - mg\cos\phi = \frac{mv^2}{a}.$$

Finally,

$$T = mg\cos\phi + \frac{mv^2}{a}.$$

Additionally, conservation of energy gives

$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}mv^2 + mga(1-\cos\phi),$$

Cancel $(1/2)m$:

$$u^2 = v^2 + 2ga(1-\cos\phi),$$

Re-arrange:

$$v^2 = u^2 - 2ga(1-\cos\phi).$$

Hence,

$$T = mg\cos\phi + \frac{mu^2}{a} - 2mg(1-\cos\phi).$$

Hence,

$$T(\phi) = \frac{mu^2}{a} - mg\left(2 - 3\cos\phi\right). \tag{13.7}$$

Thus, for the problem in hand, set $u^2 = 4ga$ to obtain

$$T(\pi) = 4mg - mg(2+3) = -mg,$$

as obtained by the previous argument.

Part (d): Start with Equation (13.7) with $u = \sqrt{3ga}$ to obtain

$$T(\phi) = 3mg - 2mg + 3mg\cos\phi = mg + 3mg\cos\phi.$$

Set $T = 0$ to obtain

$$1 + 3\cos\phi = 0,$$

hence

$$\cos\phi = -\tfrac{1}{3}.$$

The height is $h = a(1 - \cos\phi)$, hence

$$h = a\left(1 + \tfrac{1}{3}\right) = \tfrac{4}{3}a.$$

Equivalently, the height is $(1/3)a$ above the centre of the circle.

Chapter 14

Simple harmonic motion

Overview

A particle of with displacement $x(t)$ from the origin is said to undergo simple harmonic motion if $x(t)$ satisfies the following differential equation:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x,$$

where ω is a real constant. In this section we show in what physical applications this happens, and we examine some properties of simple harmonic motion (SHM).

14.1 Elastic strings and springs

Definition 14.1

- *The* **natural length** a *of an elastic string is its unstretched length.*
- *The* **extension** x *is the difference between the stretched length and the natural length.*

An illustration of these definitions is provided in Figure 14.1.

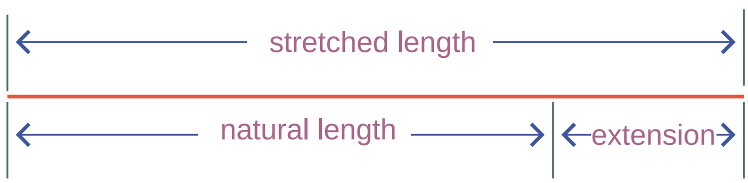


Figure 14.1: Strings and springs: natural length and extension

Hooke's law applies to stretched strings provided the extension is sufficiently small:

Definition 14.2 *Hooke's law states that the extension in a stretched elastic string is proportional to the tension T in the string:*

$$T \propto x \qquad \text{or} \qquad T = kx,$$

where k is a positive constant.

We usually write $k = \lambda/a$, where a is the natural length. Then, Hooke's law reads

$$\boxed{T = \lambda(x/a)} \tag{14.1}$$

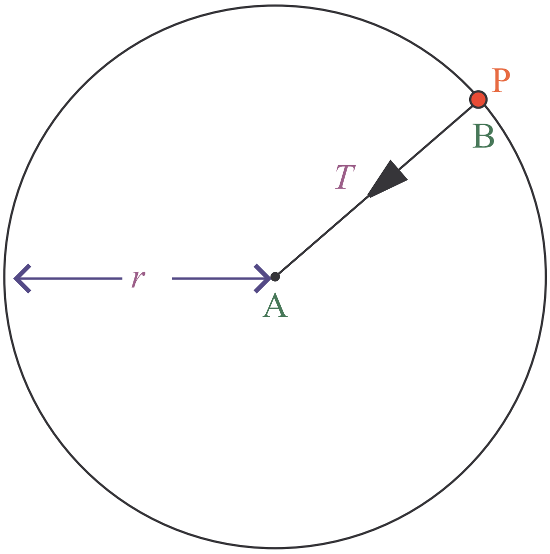
The constant $\lambda = ka$ is called the **modulus of elasticity**. Note that λ has the same units as force – in SI units, Newtons.

14.1.1 Limitation

Hooke's law holds only up to a point – for sufficiently small extensions. The definition of 'sufficiently small' is a bit circular here – by 'sufficiently small extension' we mean those extensions for which Hooke's law holds. But this is not facetious: this is a measurable limit called the **elastic limit**. Beyond the elastic limit the string will lose its shape once the extension in the string is shrunk back down to zero. This is **hysteresis**. The end of this regime corresponds to the **plastic limit**. Beyond the plastic limit the string will snap – this corresponds to very large extensions.

14.2 Elastic strings and circular motion!!

Example: One end of an elastic string AB is fixed to a point on a smooth table. A particle P is attached to the other end and moves in a horizontal circle with centre A and radius r. The elastic string AB has natural length 2.5 m and its modulus is 40 N. The mass of P is 5 kg.
If the string is extended by 0.5 m find the speed of P.



Solution:

$$\frac{mv^2}{r} = T,$$
$$T = \frac{\lambda x}{\ell},$$
$$\frac{mv^2}{r} = \frac{\lambda x}{\ell}.$$
$$v^2 = \frac{\lambda x r}{m\ell} = \frac{\lambda x(\ell + x)}{m\ell}$$
$$= \frac{40 \times 0.5 \times 3}{5 \times 2.5} = 4.8.$$

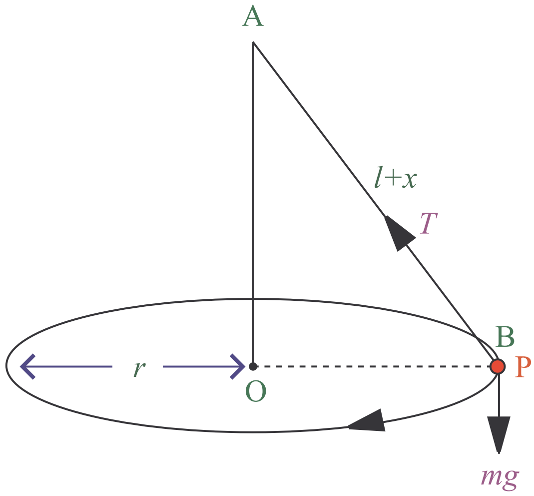
Hence, $v = 2.19\,\mathrm{m/s}$.

Example: If in the above example P has mass 1.5 kg and its speed is 6 m/s and the extension of the string is 0.4 m find its modulus given that its natural length is 2 m.

Solution:

$$\frac{mv^2}{r} = \frac{\lambda x}{\ell},$$
$$\lambda = \frac{mv^2\ell}{rx} = \frac{mv^2\ell}{(\ell + x)x}$$
$$= \frac{1.5 \times 36 \times 2}{2.4 \times 0.4} = 112.5\,\mathrm{N}.$$

Example: An elastic string of natural length ℓ and modulus of elasticity $2mg$ has one end fixed to a point A and has a particle P of mass m attached to the other end B.
P travels in a horizontal circle with angular speed ω. The elastic limit occurs when tension $T = 3mg$.
Find ω when this state is reached.



Solution: First, do a force balance:

- Vertical direction:

$$T\cos\theta = mg.$$

- Horizontal direction – circular motion:

$$T\sin\theta = m\omega^2 r,$$

where r is the radius of the circular motion.

But we are given that $T = 3mg$, hence

$$\omega^2 = \frac{3g\sin\theta}{r}.$$

We need to eliminate θ and r. From trigonometry,

$$\sin\theta = \frac{r}{\ell + x},$$

hence

$$\frac{\sin\theta}{r} = \frac{1}{\ell + x},$$

hence

$$\omega^2 = \frac{3g}{\ell + x} = \frac{3g/\ell}{1 + (x/\ell)}. \quad (14.2)$$

We eliminate x/ℓ via Hooke's law:

$$T = 3mg = \lambda(x/\ell) = 2mg(x/\ell) \implies (x/\ell) = \tfrac{3}{2}.$$

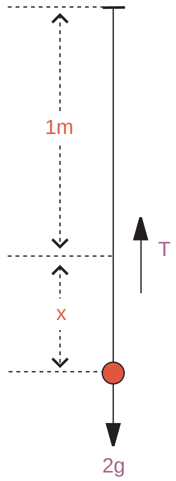
Substitute into Equation (14.2):

$$\omega^2 = \frac{3g/\ell}{1 + \frac{3}{2}} = \frac{6g}{5\ell},$$

hence

$$\omega = \sqrt{\frac{6g}{5\ell}}.$$

> Example: A light elastic string of natural length 1 m and modulus 35 N is fixed at one end, and a particle of mass 2 kg is attached to the other end.
> Find the length of the string at equilibrium.



Solution – we have $T = \lambda(x/a)$. In equilibrium,

$$T = mg \implies x = mga/\lambda.$$

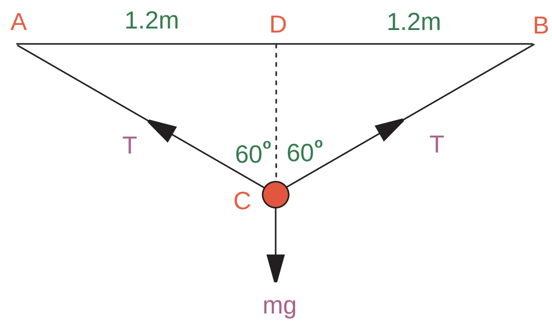
Filling in gives

$$x = 2 \times 9.8 \times 1/35 = 0.56\,\mathrm{m},$$

and hence, the length at equilibrium is

$$L = x + a = 1.56\,\mathrm{m}.$$

Example: The natural length of an elastic string AB is $2.4m$ and its modulus is $4g$ N. The ends A and B are attached to two points on the same level $2.4\,\mathrm{m}$ apart and a particle of mass m kg is attached to the mid-point C of the string. When the particle hangs in equilibrium each half of the string is at $60°$ to the vertical. Find the value of the mass m.



Solution: The force balance gives

$$2T\cos 60° = mg \implies T = mg.$$

We have $T = \lambda(x/a)$ with $\lambda = 4g$ and $a = |AB|$. The tension is

$$T = 4g\left(\frac{x}{(a/2)}\right) = 8g(x/a), \qquad \text{(note the denominator!)}$$

and thus,

$$m = 8x/a. \tag{14.3}$$

So it remains to work out x. Call $L = |AC|$. Clearly, $L = a/(2\cos 30°) = a/\sqrt{3}$. The extension is

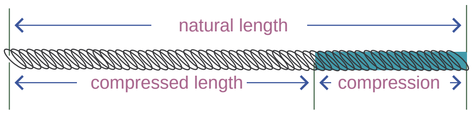
$$x = L - \tfrac{1}{2}a = a\left(\tfrac{1}{\sqrt{3}} - \tfrac{1}{2}\right).$$

Substitution into (14.3) gives

$$m = 8\left(\tfrac{1}{\sqrt{3}} - \tfrac{1}{2}\right) \approx 0.62\,\mathrm{kg}.$$

14.3 Springs

Hooke's law applies to **springs** as well, which can be compressed as well as stretched. Let a be the natural length and x the compression. When compressed, a spring exerts an outward push, or **thrust**:



$$\text{Thrust} = \lambda\frac{\text{compression}}{a} \qquad \text{or} \qquad T = \lambda\left(\frac{x}{a}\right).$$

14.3.1 The sign of the force in Hooke's law

We take $T \equiv F$ and $k = \lambda/a$ in Hooke's Law. Allowing for compression and expansion (i.e. springs), we have

$$F = -kx$$

for the properly-signed string/spring force before the onset of the plastic limit. We have

- For extension, $x > 0$, hence $F < 0$, and the spring is accelerated in a negative x-direction, i.e. back towards $x = 0$;
- For compression, $x < 0$, hence $F > 0$, and the spring is accelerated in a positive x-direction, i.e. back towards $x = 0$ again.

For this reason, the force in Hooke's law is called a **restoring force** because the force drives any deviation in x away from zero back towards the equilibrium position at $x = 0$.

14.4 Work done to stretch a spring

Recall, the work done by a force in moving a particle from x_1 to x_2 is

$$W = \int_{x_2}^{x_1} F \, \mathrm{d}x.$$

For Hooke's Law, $F = -kx$

$$\begin{aligned} W &= -\int_{x_2}^{x_1} kx \, \mathrm{d}x, \\ &= -\tfrac{1}{2}k(x_2^2 - x_1^2). \end{aligned}$$

Equally from before, the potential function is

$$\mathcal{U}(x) = -\int_{\mathrm{ref}}^{x} F(x) \, \mathrm{d}x.$$

Taking the reference position to be $x = 0$, this is

$$\mathcal{U}(x) = -\int_0^x (-kx)\mathrm{d}x = \tfrac{1}{2}kx^2.$$

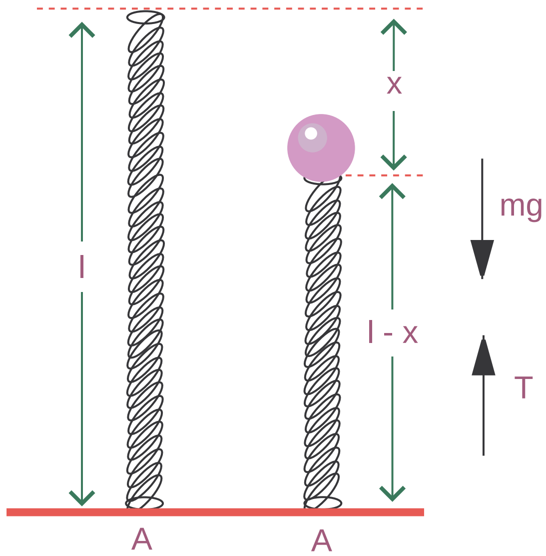
Thus, the principle of conservation of mechanical energy for the string/spring in the absence of gravity (i.e. string/spring aligned with earth's surface) is

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \mathrm{Const.}$$

More generally, for the string/spring aligned in the direction of earth's gravity, we have the following principle of conservation of mechanical energy:

> If a particle of mass m is moving with speed v is at a height h above the zero of gravitational P.E., and is also attached to an elastic spring stretched by an amount x, then the mechanical energy is
>
> $$E = \frac{1}{2}mv^2 + mgh + \tfrac{1}{2}kx^2.$$
>
> If no other forces are present, then the above expression for the mechanical energy is a conserved quantity.

14.4.1 Worked examples



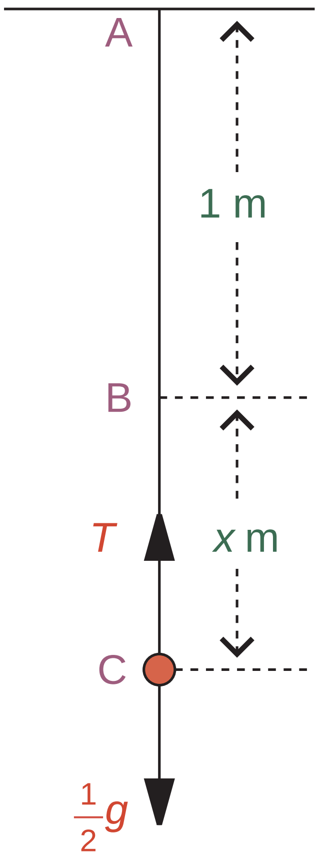
Example: One end of a spring, which is strong enough to stand vertically, is fixed to a point A on a horizontal plane. A particle of mass m is attached to the other end of the spring and is allowed to descend to the equilibrium position.
Find how much work is done.

Solution: Displacement:

$$\frac{\lambda}{a}x = mg, \qquad x = \frac{amg}{\lambda}.$$

Work done:

$$W = \frac{\lambda}{2a}x^2 = \frac{\lambda}{2a}\left(\frac{amg}{\lambda}\right)^2 = \frac{am^2g^2}{2\lambda}.$$



Example: A light elastic string ($\ell = 1$ m, $\lambda = 7$ N) has one end fixed to a point A. A particle of mass 0.5 kg hangs in equilibrium at a point C vertically below A.

(a) Find the distance AC.

The particle is raised to B, between A and C where AB is 1 m and is released from rest.

(b) Find the velocity v at point C.

(c) Find where the particle first comes to rest.

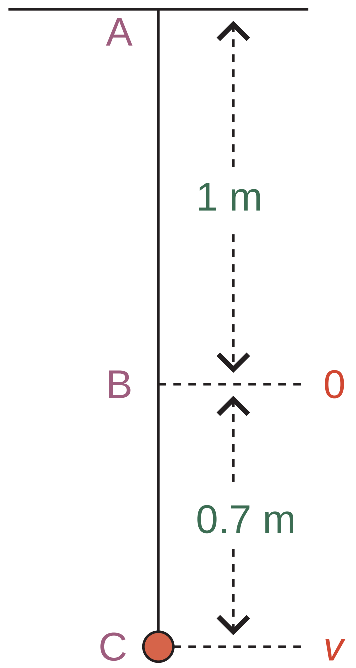
Solution – part (a):

$$\begin{aligned} T &= mg = \tfrac{1}{2}g. \\ T &= \frac{\lambda}{\ell}x = \frac{7}{1}x = 7x. \end{aligned}$$

Therefore

$$7x = \tfrac{1}{2}g, \qquad x = \tfrac{1}{14}g = 0.7, \qquad |AC| = 1.7\,\mathrm{m}.$$

Solution – part (b):



The reference level is fixed such that, at point B,

$$\text{Elastic P.E.} = 0, \qquad \text{Gravitational P.E.} = 0.$$

Also, the K.E. is zero.
At point C, with $|BC| = x$, the elastic P.E. is

$$\frac{\lambda}{2a}x^2 = \frac{7}{2\times 1}(0.7)^2 = 1.715.$$

Also, the gravitational P.E. is

$$-mgx = -0.5 \times 9.8 \times 0.7 = -3.41.$$

The kinetic energy to be determined is

$$\text{K.E.} = \tfrac{1}{2}mv^2 = 0.25v^2.$$

By conservation of mechanical energy, the energy at points B and C is the same, hence

$$0 = 1.715 - 3.41 + 0.25v^2,$$

and thus,

$$v^2 = 6.86, \qquad v = 2.619\,\mathrm{m/s}.$$

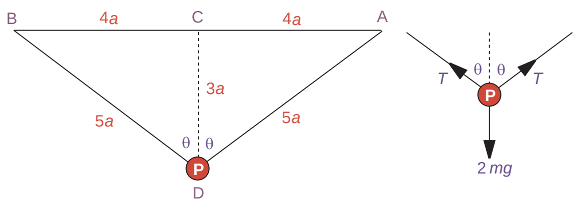
Solution – part (c): At the point D, with $|BD| = x$:

$$\begin{aligned} \text{Mechanical Energy} &= \frac{\lambda}{2a}x^2 - mgx \\ &= \tfrac{7}{2}x^2 - \tfrac{1}{2}gx = \tfrac{1}{2}x(7x - g). \end{aligned}$$

Therefore $\frac{1}{2}x(7x - g) = 0$, hence $x = \frac{1}{7}g = 1.4\,\mathrm{m}$. N.B. $x = 0$ corresponds to B.

Example: A particle P of mass $2m$ is fastened to the end of each of two identical elastic strings (equilibrium length $\ell = 3a$, $\lambda = 3\alpha mg$) and is in equilibrium as shown in the accompanying sketch.

(a) Find α.
(b) Find the mechanical energy of the system.
(c) The particle is now raised to C and released. Find v at D.



Solution – part (a). Tension:

$$T = \frac{\lambda}{3a}(5a - 3a) = \frac{2\lambda}{3} = \frac{2 \times 3mg\alpha}{3} = 2mg\alpha.$$

Hence,

$$\begin{aligned} 2T\cos\theta &= 2mg, \\ 4mg\alpha \times (3/5) &= 2mg, \\ \alpha &= \tfrac{10}{12} = \tfrac{5}{6}. \end{aligned}$$

Solution – part (b). Find the mechanical energy of the system:

$$\begin{aligned} \text{Elastic P.E.} &= 2 \times \frac{\lambda}{2 \times 3a} \times (2a)^2 \\ &= 2 \times \frac{3mg\alpha}{6a} \times 4a^2 \\ &= 4mga\alpha = \frac{10}{3}mga \end{aligned}$$

Gravitational P.E. $= -3a \times 2mg = -6mga$. Also, the kinetic energy is zero. Hence, the total mechanical energy is

$$\text{Mechanical Energy} = \tfrac{10}{3}mga - 6mga = -\tfrac{8}{3}mga.$$

Solution – part (c). Find v at D. We have:

$$\begin{aligned}\text{Elastic P.E. at C} &= 2\times\frac{\lambda}{2\times 3a}\times a^2\\ &= 2\times\frac{3mg\alpha}{6a}\times a^2\\ &= mga\alpha\\ &= \tfrac{5}{6}mga\end{aligned}$$

The gravitational potential energy and the kinetic energy are both zero at C, and the mechanical, so the total mechnical energy is got as follows:

- Elastic P.E. at C – $(5/6)mga$
- Gravitational P.E. at C – 0
- Kinertic energy at C – 0

for a total mechanical energy $(5/6)mga$.

Now, the mechnical energy at D is the mechanical energy for a particle at rest (see (b)) plus the kinetic energy:

$$\text{Mechnical energy} = -\tfrac{8}{3}mga+\tfrac{1}{2}(2mv^2) = -\tfrac{8}{3}mga+mv^2.$$

By conservation of mechanical energy,

$$\text{mechanical energy at C} = \text{mechnical energy at D},$$

hence

$$\begin{aligned}-\tfrac{8}{3}mga+mv^2 &= \tfrac{5}{6}mga,\\ mv^2 &= \tfrac{8}{3}mga+\tfrac{5}{6}mga\\ &= \tfrac{21}{6}mga = \tfrac{7}{2}mga.\\ v^2 &= \tfrac{7ga}{2},\\ v &= \sqrt{\frac{7ga}{2}}.\end{aligned}$$

14.5 Simple harmonic motion

For a particle attached to a spring obeying Hooke's Law, we have $F = -kx$. But by Newton's second law,

$$F = ma = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2},$$

hence

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = kx.$$

Dividing across by m and calling

$$\omega = \sqrt{k/m},$$

we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x.$$

This is precisely the equation of simple harmonic motion:

Definition 14.3 *A particle of with displacement $x(t)$ from the origin is said to undergo simple harmonic motion if $x(t)$ satisfies the following differential equation:*

$$\boxed{\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x,} \tag{14.4}$$

where ω is a real constant.

The differential equation (14.4) has a definite solution:

Theorem 14.1 *The solution to Equation* (14.4) *can be written as*

$$x(t) = x_0 \sin(\omega t + \phi), \tag{14.5}$$

where x_0 and ϕ are constants.

Proof: The trial solution (14.5) is plugged into the L.H.S. and the R.H.S. of Equation (14.4) and agreement between the two sides is sought:

$$\begin{aligned} \text{L.H.S} &= \frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} x_0 \sin(\omega t + \phi), \\ &= -\omega^2 x_0 \sin(\omega t + \phi). \end{aligned}$$

and

$$\text{R.H.S.} = -\omega^2 x = -x\omega^2 \sin(\omega t + \phi).$$

We obtain L.H.S = R.H.S. and thus the trial solution (14.5) is indeed a solution of the differential equation (14.4). In later modules you will find out that this trial solution is in fact the **unique general solution** of Equation (14.4). ∎

14.5.1 The period

We have

$$\begin{aligned} x(t+\frac{2\pi}{\omega}) &= x_0 \sin\left(\omega(t+(2\pi/\omega))+\phi\right), \\ &= x_0 \sin\left((\omega t+\phi)+2\pi\right), \\ &= x_0 \sin(A+2\pi), \qquad A=\omega t+\phi, \\ &= x_0 \sin(A), \\ &= x_0 \sin(\omega t+\phi). \end{aligned}$$

Thus, $x(t)$ repeats itself every $2\pi/\omega$ seconds.

Definition 14.4 *The time length $T = 2\pi/\omega$ is called the* **period** *of the simple harmonic motion.*

Referring to Figure 14.2

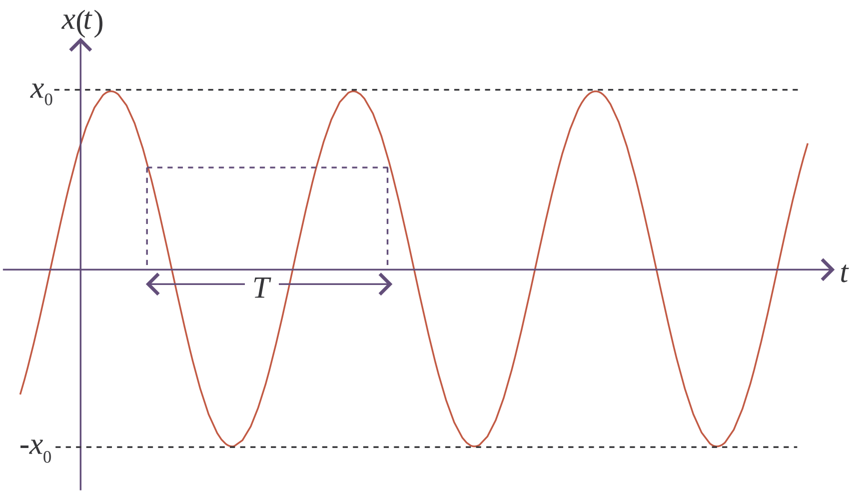
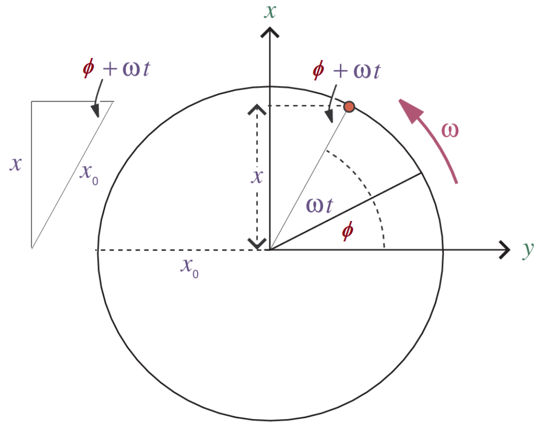


Figure 14.2: Graph of $x(t) = x_0 \sin(\omega t + \phi)$

for $x(t) = x_0 \sin(\omega t + \phi)$, we have

- T is the time from peak to peak or trough to trough.
- x_0 is called the **amplitude**.
- ω is called the **angular frequency**.
- $\nu = \frac{1}{T}$ is called the **frequency**.
- ϕ is called the **phase**.

Thus, Simple Harmonic Motion can be seen as the projection of a circular motion onto a diameter:



- amplitude: x_0
- angular frequency: ω
- phase: ϕ
- period: T

14.5.2 Speed and acceleration in Simple Harmonic Motion

We compute the speed in simple harmonic motion:

$$\begin{aligned} x &= x_0 \sin(\omega t + \phi) \\ v &= \frac{\mathrm{d}x}{\mathrm{d}t} = \omega x_0 \cos(\omega t + \phi) \\ v^2 &= \omega^2 x_0^2 \cos^2(\omega t + \phi) \\ &= \omega^2 x_0^2 (1 - \sin^2(\omega t + \phi)) \\ &= x_0^2 \omega^2 - \omega^2 x_0^2 \sin^2(\omega t + \phi) \\ &= \omega^2 x_0^2 - \omega^2 x^2 \\ &= (x_0^2 - x^2)\omega^2 = \omega^2 (x_0^2 - x^2). \end{aligned}$$

Similarly, the acceleration is

$$a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -x_0 \omega^2 \sin(\omega t + \phi).$$

Thus,

- Maximum speed:

$$\begin{aligned} v &= x_0 \omega \cos(\omega t + \phi), \\ |v| &= x_0 \omega |\cos(\omega t + \phi)|. \end{aligned}$$

The maximum speed occurs when $\cos(\omega t + \phi) = \pm 1$ and hence the **maximum speed** is

$$v_{\max} = x_0 \omega.$$

- Maximum acceleration:

$$a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x_0\omega^2\sin(\omega t + \phi),$$
$$|a| = x_0\omega^2|\sin(\omega t + \phi)|.$$

The maximum acceleration occurs when $\sin(\omega t + \phi) = \pm 1$ and thus the **maximum acceleration** is $x_0\omega^2$.

14.5.3 Worked examples

Example: A particle P is describing SHM with amplitude 2.5 m. When P is 2 m from the centre of the path, the speed is 3 m/s.

Find
(a) the period,
(b) the maximum speed,
(c) the maximum acceleration.

Solution: We have

$$x(t) = x_0\sin(\omega t + \phi),$$
$$v^2 = \omega^2(x_0^2 - x^2).$$
$$x_0 = 2.5\,\mathrm{m}.$$

When $x = 2$ and $v = 3$,

$$9 = \omega^2((2.5)^2 - 2^2) = 2.25\omega^2.$$
$$\omega^2 = \frac{9}{2.25} = 4, \qquad \omega = 2\,\mathrm{rad/s}.$$

Putting it all together, we have

Part (a) – $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi\,\mathrm{s}$.

Part (b) – $v_{\mathrm{max}} = x_0\omega = 2 \times 2.5 = 5\,\mathrm{m/s}$

Part (c) – $a_{\mathrm{max}} = x_0\omega^2 = 4 \times 2.5 = 10\,\mathrm{m/s}^2$.

Example: A, B and C in that order are three points on a straight line and a particle P is moving on that line with SHM. The velocities of P at A, B and C are 0 m/s, 2 m/s and -1 m/s respectively.

If AB = 1 m and AC = 4 m, find the amplitude and the period of the motion.

Solution: Refer to Figure 14.3.

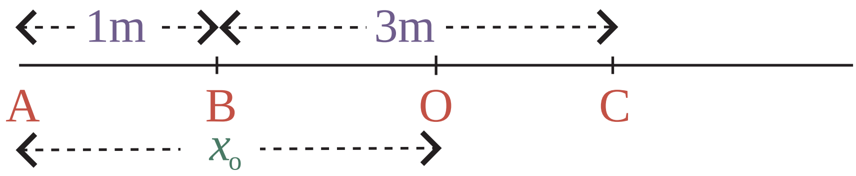


Figure 14.3:

Note that $v^2 = \omega^2(x_0^2 - x^2)$. Therefore $v = 0$ when $x = \pm x_0$. Thus A must be at $-x_0$.

- At A: $x = -x_0$.

- At B: $x = -x_0 + 1$, and

$$\begin{aligned} 2^2 &= \omega^2\big(x_0^2 - (-x_0 + 1)^2\big) \\ &= \omega^2\big(x_0^2 - x_0^2 + 2x_0 - 1\big), \\ 4 &= \omega^2\big(2x_0 - 1\big), \\ \omega^2 &= \frac{4}{2x_0 - 1}. \end{aligned}$$

- At C: $x = -x_0 + 4$, and

$$\begin{aligned}(-1)^2 &= \omega^2\big(x_0^2 - (-x_0+4)^2\big)\\ &= \omega^2\big(8x_0 - 16\big),\\ 1 &= 8\omega^2\big(x_0 - 2\big),\\ \omega^2 &= \frac{1}{8(x_0-2)}.\end{aligned}$$

$$\begin{aligned}\frac{4}{2x_0-1} &= \frac{1}{8(x_0-2)},\\ 32(x_0-2) &= 2x_0 - 1,\\ 30x_0 = 63, x_0 &= \frac{21}{10} = 2.1 \text{ m}.\end{aligned}$$

Putting it all together:

$$\omega^2 = \tfrac{4}{4.2-1} = \tfrac{4}{3.2} = \tfrac{5}{4}$$

hence

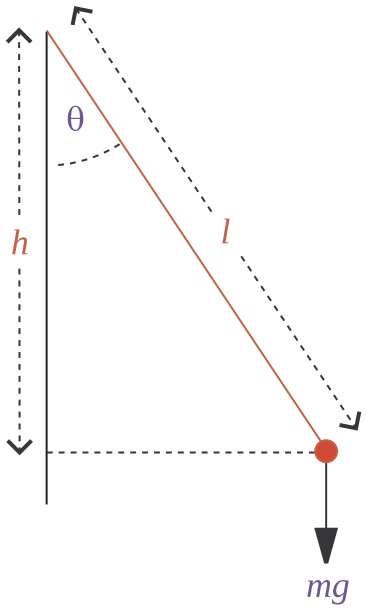
$$\omega = \sqrt{5/4} \approx 1.118\,\mathrm{rad/s},$$

and

$$T = 2\pi/\omega \approx 2\pi/1.118 \approx 5.619\,\mathrm{s}.$$

14.6 The simple pendulum

A particle, attached to a light rod, and free to swing in a plane, is called a **simple pendulum**.
We analyse the dynamics using two different methods.



14.6.1 Force balance

- Tangential acceleration: $\ell\ddot{\theta}$
- Tangential force: $-mg\sin\theta$
- Newton's second law:

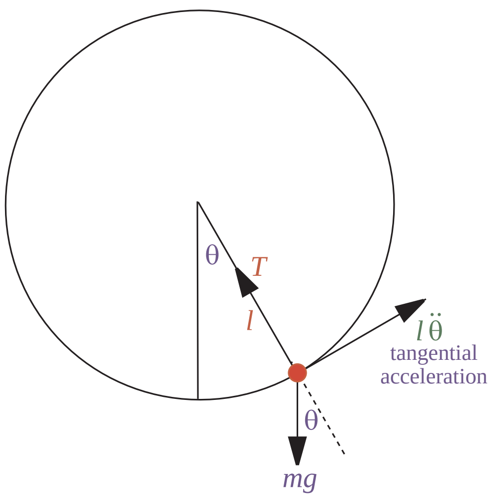
$$m\ell\ddot{\theta} = -mg\sin\theta$$

hence

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta \qquad (14.6)$$

where we have used the 'dot' notation for derivatives:

$$\dot{\theta} \equiv \frac{\mathrm{d}\theta}{\mathrm{d}t}, \qquad \ddot{\theta} \equiv \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}.$$



14.6.2 Conservation of mechanical energy

$$\begin{aligned}
\text{Speed of particle } &= \ell\dot{\theta}, \\
\text{K.E. } &= \tfrac{1}{2}m\ell^2\dot{\theta}^2, \\
\text{P.E. } &= -mgh = -mg\ell\cos\theta, \\
E = \text{ K.E. } + \text{ P.E. } &= \tfrac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta
\end{aligned}$$

The total mechanical energy is conserved:

$$E = \tfrac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta = \text{Const.}$$

Differentiating with respect to t:

$$\begin{aligned}
\frac{d}{dt}\left(\tfrac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta\right) &= 0 \\
\tfrac{1}{2}m\ell^2\frac{d}{dt}(\dot{\theta}^2) - mg\ell\frac{d}{dt}(\cos\theta) &= 0
\end{aligned}$$

$$\tfrac{1}{2}m\ell^2\frac{d}{dt}(\dot{\theta}^2) - mg\ell\frac{d}{dt}(\cos\theta) = 0$$

Now

$$\begin{aligned}
\frac{d}{dt}(\dot{\theta}^2) &= 2\dot{\theta}\frac{d\dot{\theta}}{dt} = 2\dot{\theta}\ddot{\theta}, \\
\frac{d}{dt}(\cos\theta) &= \dot{\theta}\frac{d(\cos\theta)}{d\theta} = -\dot{\theta}\sin\theta.
\end{aligned}$$

Therefore

$$m\ell^2\dot{\theta}\ddot{\theta} + mg\ell\dot{\theta}\sin\theta = 0.$$

Finally,

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta, \tag{14.7}$$

in agreement with Equation (14.6).

14.6.3 Small-amplitude approximation

For small θ, we use the limit

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1,\qquad \theta \text{ in radians}$$

to write

$$\sin\theta\approx\theta$$

for small θ. Thus, Equation (14.6) (or equivalently, Equation (14.7)) becomes

$$\boxed{\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=-\frac{g}{\ell}\theta.}\tag{14.8}$$

This is just the equation for Simple Harmonic Motion again!!

There is a corresponding approximation for $\cos\theta$: since $\sin\theta\approx\theta$ for θ small, we have

$$\begin{aligned}\sin\theta &\approx \theta,\\ -\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta &\approx \theta,\\ -\cos\theta &\approx \text{Const.}+\tfrac{1}{2}\theta^2,\\ \cos\theta &\approx \text{Const.}-\tfrac{1}{2}\theta^2.\end{aligned}$$

The constant must be equal to unity because $\cos(0)=1$, hence

$$\cos\theta\approx 1-\tfrac{1}{2}\theta^2,\qquad \theta \text{ small.}$$

Using this result, the expression for the energy simplifies to

$$E=\tfrac{1}{2}m\ell^2\dot{\theta}^2-mg\ell(1-\tfrac{1}{2}\theta^2).$$

This can be re-written as

$$E=\tfrac{1}{2}m\ell^2\dot{\theta}^2+\tfrac{1}{2}mg\ell\theta^2+\text{Const.},$$

where the constant is $-mg\ell$. As the energy is only defined up to a constant (set by the arbitrary reference level of the potential-energy function), an equally good expression for the energy is

$$E=\tfrac{1}{2}m\ell^2\dot{\theta}^2+\tfrac{1}{2}mg\ell\theta^2,$$

which in form is identical to the one already considered for Hooke's Law and Simple Harmonic Motion.

Now, the frequency of simple harmonic motion for the simple pendulum can be read off from

Equation (14.8): it is

$$\omega^2 = g/\ell.$$

The period is $2\pi/\omega$, hence

$$\boxed{T = 2\pi\sqrt{\frac{\ell}{g}}.} \tag{14.9}$$

14.6.4 Worked examples

Example: The period of a simple pendulum is 3τ. If the period is to be reduced to 2τ, find the percentage change in length.

Solution: Suppose the period is $n\tau$. Then

$$n\tau = T = 2\pi\sqrt{\frac{\ell}{g}},$$
$$n^2\tau^2 = 4\pi^2\frac{\ell}{g},$$
$$\ell = \frac{n^2\tau^2 g}{4\pi^2}.$$

We have

$$\ell = \frac{n^2\tau^2 g}{4\pi^2}$$

Let the length be ℓ_3 when the period is 3τ and ℓ_2 when the period is 2τ. Then

$$\ell_3 = \frac{9\tau^2 g}{4\pi^2} \qquad \text{and} \qquad \ell_2 = \frac{\tau^2 g}{\pi^2}.$$

The relative change in the length is

$$\frac{\ell_3 - \ell_2}{\ell_3} = \frac{5}{9} \approx 0.55$$

Convert into a percentage:

$$100 \times \left(\frac{\ell_3 - \ell_2}{\ell_3}\right) = 55\% \,.$$

Example: A simple pendulum is 2 m long and the time it takes to perform 50 oscillations is measured.

(a) On Earth, the time taken is 142 s. Find g on Earth.
(b) On the Moon, the time taken is 341 s. Find the acceleration due to gravity on the moon.

Solution– Part (a). On Earth:

$$\begin{aligned} T &= \frac{142}{50} = 2.84\,\mathrm{s}, \qquad T = 2\pi\sqrt{\frac{2}{g}}. \\ 2\pi\sqrt{\frac{2}{g}} &= 2.84, \\ \sqrt{\frac{g}{2}} &= \frac{2\pi}{2.84} = \frac{\pi}{1.42}. \end{aligned}$$

Thus,

$$g = 2\left(\frac{\pi}{1.42}\right)^2 = 9.78\,\mathrm{m/s^2}.$$

Solution– Part (b). On the Moon:

$$\begin{aligned} T' &= \frac{341}{50} = 6.82\,\mathrm{s}, \qquad T' = 2\pi\sqrt{\frac{2}{g'}}. \\ 2\pi\sqrt{\frac{2}{g'}} &= 6.82, \\ \sqrt{\frac{g}{2}} &= \frac{2\pi}{6.82} = \frac{\pi}{3.41}. \end{aligned}$$

Finally,

$$g = 2\left(\frac{\pi}{3.41}\right)^2 = 1.7\,\mathrm{m/s^2}.$$

Chapter 15

Advanced Topic: Electric and Magnetic Forces

Overview

We look at electric and magnetic forces. This brings together a number of topics looked at before – motion in a constant force field (parabolic trajectories), cross product, and Lorentz force. Specifically, we show that the trajectory of a charged particle experiencing the Lorentz force in a constant magnetic field is a helix. For a special initial condition this even reduces to circular motion! This is a nice way to wrap up the module as a number of key concepts are tied together.

15.1 Electrostatic Force – Constant Electric Field

The starting-point is to look back at Chapter 7, where we saw that a charged particle (charge q) experiencing an electric field $\boldsymbol{E}$ experiences an electrostatic force, which can be written as

$$\boldsymbol{F} = \boldsymbol{E}q,$$

where $\boldsymbol{E}$ is the electric field. In this section, we are going to look at the particular example of the constant electric field, and to explore the similarities of particle motion in a constant electric field with the motion of a particle in a constant gravitational field. We do this through a detailed example.

Example: A particle of charge q and mass m with an initial velocity $v_0\mathbf{i}$ enters an electric field $\boldsymbol{E} = -E\boldsymbol{j}$. We assume that $\boldsymbol{E}$ is uniform, i.e. its value is constant at all points in the

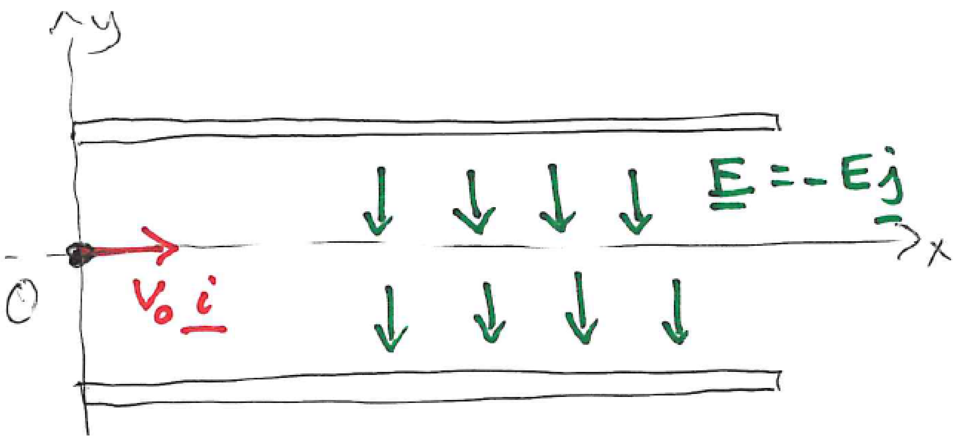


Figure 15.1:

region between two plates of length L (Figure 15.1). For definiteness, we also assume that the constant E is positive, such that the electric field points downwards. The aim of this example is to describe the motion of the particle.

We first of all write down the forces that act o n the particle in the x-y and y-direction is:

$$\boldsymbol{F} = q\boldsymbol{E} = -qE\mathbf{j}.$$

Hence, if $\boldsymbol{F} = F_x\mathbf{i} + F_y\mathbf{j}$, we have:

$$F_x = 0, \qquad F_y = -qE.$$

We next want to know if the force in the y-direction influences the x-component of the velocity. The answer is no! The reason is as follows – start with Newton's Law:

$$\begin{aligned} m\frac{\mathrm{d}v_x}{\mathrm{d}t} &= F_x = 0, \\ m\frac{\mathrm{d}v_y}{\mathrm{d}t} &= F_y = -qE. \end{aligned}$$

Thus, v_x remains constant for all time – anything that happens in the y-direction does not affect the fact that $v_x = \text{Const.}$.

We next solve for v_x and v_y as functions of time:

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = 0 \implies v_x = \text{Const.}$$

Thus, $v_x =$ Const. = Initial value of $v_x = v_0$:

$$v_x = v_0 \qquad \text{for all times.}$$

In the y-direction:

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -qE$$

Integrate:

$$\int_{v_{y0}}^{v_y} \mathrm{d}v_y = \int_0^t (-qE/m)\mathrm{d}t.$$

The qE/m term is constant, it comes outside the integral. Also, $v_{y0} = 0$, so we get:

$$v_y = -\frac{qE}{m}t.$$

Thus, in terms of the velocity vector, we get:

$$\boldsymbol{v} = v_0\mathbf{i} + \left(-\frac{qE}{m}t\right)\mathbf{j}.$$

We chose the origin to be the point of entry, and we conclude the analysis by computing the **trajectory**, that is, the functions $x(t)$ and $y(t)$. We have:

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = v_0$$

Integrate once, with $x(0) = 0$ to obtain:

$$x(t) = v_0 t.$$

Also,

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{qE}{m}t.$$

Integrate once, with $y(0) = 0$

$$y(t) = -\tfrac{1}{2}\frac{qE}{m}t^2.$$

Thus (like for projectile motion under gravity), the particle is deflected downwards ($q > 0$) in a parabolic path.

15.2 Lorentz Force

In this section we show that the trajectory of a charged particle in a constant magnetic field is a helix. This is an advanced topic, however, it is included here in this module because it introduces university-level mathematical techniques to students. The final answer is also quite neat!

The starting-point is to look back at Chapter 7, where we saw that a charged particle (charge q) moving with velocity $\boldsymbol{v}$ in a constant magnetic field $\boldsymbol{B}$ experiences the **Lorentz Force**:

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B},$$

where $\times$ is the vector cross product.

Let the trajectory of the particle be given by a vector $\boldsymbol{x}(t)$:

$$\boldsymbol{x}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Thus, the velocity of the particle is given by:

$$\begin{aligned} \boldsymbol{v}(t) &= \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}, \\ &= \frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\mathbf{k}, \\ &= v_x(t)\mathbf{i} + v_y(t)\mathbf{j} + v_z(t)\mathbf{k}. \end{aligned}$$

Suppose that the magnetic field $\boldsymbol{B}$ is constant and oriented in the z-direction, $\boldsymbol{B} = B\mathbf{k}$. Thus, the Lorentz force is given by:

$$\boldsymbol{F} = q \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix},$$

which works out to be:

$$\boldsymbol{F} = q\left(v_y B\mathbf{i} - v_x B\mathbf{j}\right).$$

Newton's Second Law for the particle then reads:

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qBv_y, \tag{15.1a}$$

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -qBv_x, \tag{15.1b}$$

$$m\frac{\mathrm{d}v_z}{\mathrm{d}t} = 0. \tag{15.1c}$$

We introduce a quantity ω, where

$$\omega = \frac{|q|B}{m}; \tag{15.2}$$

ω has units of Time^{-1}, since

$$[\omega] = \frac{\mathrm{Coulomb}}{\mathrm{kg}} \frac{\mathrm{kg}}{(\mathrm{Coulomb/s}) \cdot \mathrm{s}^2}$$

Definition 15.1 *ω in Equation* (15.2) *is called the* **Cyclotron Frequency***.*

Thus, Equation (15.1) becomes:

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \omega v_y, \tag{15.3a}$$
$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\omega v_x, \tag{15.3b}$$
$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = 0. \tag{15.3c}$$

Here – without loss of generality, we work with $q > 0$. The last equation gives $v_z = \mathrm{Const.}$ But $\mathrm{d}z/\mathrm{d}t = v_z$, hence

$$z(t) = v_z t + z_0,$$

where z_0 is the initial location of the particle in the z-direction. We can take this to be zero, to give

$$z(t) = v_z t.$$

For the first two equations in (15.3), we attempt a trial solution

$$v_x = v_0 \sin(\omega t), \qquad v_y = v_0 \cos(\omega t). \tag{15.4}$$

We can check that Equation (15.3)(a)–(b) is satisfied by this choice. E.g.

$$\begin{aligned} \text{LHS, Equation (15.3)(a): } \frac{\mathrm{d}v_x}{\mathrm{d}t} &= \omega v_0 \cos(\omega t), \\ \text{RHS, Equation (15.3)(a): } \omega v_y &= \omega v_0 \cos(\omega t). \end{aligned}$$

Thus, LHS = RHS, and Equation (15.3)(a) is satisfied by the trial solution. Equation (15.3)(b) is similar.

Integrating Equation (15.4) gives the position (not velocity) as a function of time, e.g.

$$\begin{aligned} v_x &= v_0 \sin(\omega t), \\ v_x &= \frac{\mathrm{d}x}{\mathrm{d}t}, \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= v_0 \sin(\omega t), \\ x(t) - x(0) &= \int_0^t v_0 \sin(\omega t) \mathrm{d}t, \end{aligned}$$

Hence,

$$x(t) = \left(x_0 + \frac{v_0}{\omega}\right) - \frac{v_0}{\omega}\cos(\omega t),$$

where $x_0 = x(0)$ is the initial position of the particle on the x-axis.

Similarly,

$$\begin{aligned} v_y &= v_0\cos(\omega t), \\ v_y &= \frac{\mathrm{d}y}{\mathrm{d}t}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= v_0\cos(\omega t), \\ y(t) - y(0) &= \int_0^t v_0\cos(\omega t)\mathrm{d}t, \end{aligned}$$

Hence,

$$x(t) = y_0 + \frac{v_0}{\omega}\sin(\omega t),$$

Thus, the solution to the Lorentz-force problem is:

$$\boldsymbol{x}(t) = \left(x_0 + \frac{v_0}{\omega}\right) - \frac{v_0}{\omega}\cos(\omega t)\mathbf{i} + \left(y_0 + \frac{v_0}{\omega}\sin(\omega t)\right)\mathbf{j} + v_z t\mathbf{k}.$$

This can also be written using the triplet notation for vectors:

$$\boldsymbol{x}(t) = \left(\left(x_0 + \frac{v_0}{\omega}\right) - \frac{v_0}{\omega}\cos(\omega t), y_0 + \frac{v_0}{\omega}\sin(\omega t), v_z t\right).$$

This result can be plotted – the result is a helix curve. The helix is in the positive z-direction if $v_z > 0$ and in the negative z-direction if $v_z < 0$ – see Figure 15.2.

Example: A Hydrogen Ion has mass $1.67 \times 10^{-27}\,\mathrm{kg}$, charge $1.60 \times 10^{-19}\,\mathrm{Coulomb}$, and moves in a magnetic field of strength $0.30\mathrm{T}$. Compute the Cyclotron frequency.

Solution:

$$\omega = \frac{1.60 \times 10^{-19} \times 0.30}{1.67 \times 10^{-27}} = 2.87 \times 10^7\mathrm{s}^{-1}.$$

Compute the time required by the charged particle to make one full turn around the helix.

Solution: One full turn is achieved in going from $t = 0$ to $t = T$, where $\omega T = 2\pi$, hence $T = 2\pi/\omega$, hence

$$T = 2.19 \times 10^{-7}\,\mathrm{s}.$$

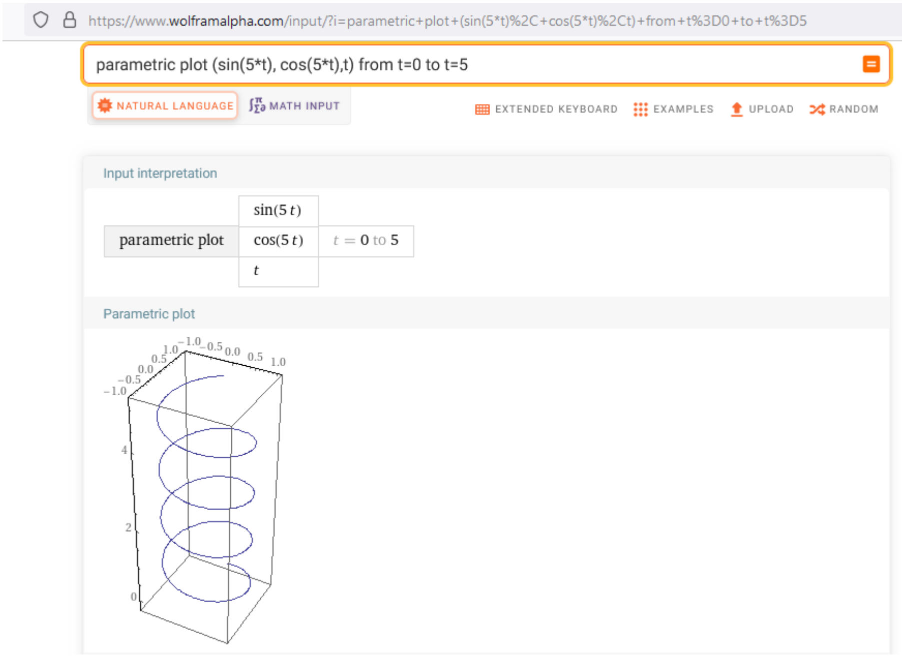


Figure 15.2: Representative helix curve, drawn using Wolfram Alpha

15.3 Lorentz Force and Circular Motion

We have just seen how Newton's Law for a charged particle in a magnetic field (Lorentz Force, $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$), gives rise to motion in a helix, specifically:

$$\boldsymbol{x}(t) = \left(\left(x_0 + \frac{v_0}{\omega}\right) - \frac{v_0}{\omega}\cos(\omega t), y_0 + \frac{v_0}{\omega}\sin(\omega t), v_z t\right).$$

If the initial velocity $v_z = 0$, this degenerates into

$$\boldsymbol{x}(t) = \left[\left(x_0 + \frac{v_0}{\omega}\right) - \frac{v_0}{\omega}\cos(\omega t)\right]\mathbf{i} + \left[y_0 + \frac{v_0}{\omega}\sin(\omega t)\right]\mathbf{j}.$$

The origin of the coordinate system can be chosen such that $x_0 = y_0 = 0$. Also, let

$$r = v_0/\omega.$$

Thus, the particle trajectory becomes:

$$\boldsymbol{x}(t) = -r\cos(\omega t)\mathbf{i} + r\sin(\omega t)\mathbf{j},$$

which is a circular motion that starts at $(-1, 0)$ (nine o'clock), and moves clockwise. The radius of the circular motion is simply $r = v_0/\omega$, and the velocity of the circular motion is identified with $v_0 = \omega r$.

Example: A Hydrogen Ion has mass $1.67 \times 10^{-27}\,\mathrm{kg}$, charge $1.60 \times 10^{-19}\,\mathrm{Coulomb}$, and moves in a magnetic field of strength $0.30\mathrm{T}$. Compute:

1. The force on the Hydrogen Ion due to the magnetic field.
2. The inward acceleration caused by the force.
3. The radius of the circular motion.

Take $v_0 = 7.59 \times 10^6\,\mathrm{m/s}$.

Solution: The magnitude of the Lorentz force is $F = qv_0B$, hence

$$F = qv_0B = (1.60 \times 10^{-19}) \times (7.59x106) \times 0.30 = 3.64 \times 10^{-13}\,\mathrm{N}.$$

The force produces an inward acceleration, hence $F = ma$, hence $a = F/m$, hence

$$a = \frac{F}{m} = \frac{3.64 \times 10^{-13}}{1.67 \times 10^{-27}} = 2.18 \times 10^{14}\,\mathrm{m/s^2}.$$

For the radius, use $a = v_0^2/r$, hence $r = v_0^2/a$, hence

$$r = \frac{v_0^2}{a} = \frac{(7.59 \times 10^6)^2}{2.18 \times 10^{14}} = 0.264\,\mathrm{m}.$$

As a consistency check, the cyclotron frequency is $\omega = qB/m$. This should be equal to $\omega = v_0/r$. We check:

$$\begin{aligned}
\omega &\overset{\text{Cyclotron Freq.}}{=} \frac{qB}{m}, \\
&= \frac{qv_0B}{v_0m}, \\
&= \frac{F}{v_0m}, \\
&= \frac{ma}{v_0m}, \\
&= \frac{v_0^2/r}{v_0}, \\
&= \frac{v_0}{r},
\end{aligned}$$

so the definition of the cyclotron frequency, $\omega = qB/m$, is consistent with the definition of frequency in circular motion, $\omega = v_0/r$.