Mechanics and Special Relativity (MAPH10030) Assignment 3

Issue Date: 03 March 2010
Due Date: 24 March 2010

In question 4, a numerical answer is required, with precision to three significant figures. Marks will be deducted for more or less precision. You may use $M_e = 5.97 \times 10^{24}\,\mathrm{kg}$**. In the other questions, a symbolic answer is fine.**

1. When an object is in circular orbit of radius r about the earth (mass M_e), the orbital period is
$$T = \frac{2\pi r^{3/2}}{\sqrt{GM_e}},$$
and the orbital velocity speed is
$$v = \sqrt{\frac{GM_e}{r}}.$$
Show that when the object is moved into a circular orbit of slightly larger radius $r + \Delta r$, where $\Delta r \ll r$, its new period is $T + \Delta T$ and its new orbital velocity is $v - \Delta v$, where
$$\Delta T = \frac{3\pi\Delta r}{v}, \qquad \Delta v = \frac{\pi\Delta r}{T}$$
[4 points].

Find dT/dr:
$$\frac{dT}{dt} = \tfrac{3}{2}\frac{2\pi}{\sqrt{GM_e}}r^{1/2} = \frac{3\pi}{v}.$$
Then integrate again:
$$\Delta T = \int_r^{r+\Delta r} \mathrm{d}T = 3\pi \int_r^{r+\Delta r} \frac{1}{v\,(r)}\mathrm{d}r.$$
Now $\Delta r/r \ll 1$, and v is a continuous function. So we can take the $1/v\,(r)$ outside the integral because it does not change very much over a small increment Δr. Hence,
$$\Delta T = \int_r^{r+\Delta r} \mathrm{d}T = \frac{3\pi}{v}\int_r^{r+\Delta r} \mathrm{d}r = \frac{3\pi\Delta r}{v}.$$
Similarly, compute dv/dr:
$$\frac{dv}{dr} = -\tfrac{1}{2}\sqrt{GM_e}r^{-3/2} = -\frac{\pi}{T}.$$

Thus,

$$-\Delta v = \int_r^{r+\Delta r} dv = -\pi \int_r^{r+\Delta r} \frac{1}{T(r)} \mathrm{d}r,$$

where we put a minus sign in front of the Δv because by our choice, Δv is inherently positive. Doing the same trick with the integral,

$$\Delta v = \frac{\pi \Delta r}{T},$$

as required.

2. See Fig. 1. A projectile of mass m is fired from the surface of the earth at an angle α from the vertical. The initial speed v_0 is equal to $\sqrt{GM_\mathrm{e}/R_\mathrm{e}}$. How high does the projectile rise? Neglect air resistance and the earth's rotation [4 points].

First, we use convervation of angular momentum. The initial angular momentum is

$$\boldsymbol{J}_{\mathrm{init}} = Rv_0 \sin\alpha \text{ into the page.}$$

The angular momentum at the maximum point is

$$\boldsymbol{J}_{\mathrm{top}} = r_{\max} v_1 \text{ into the page,}$$

and the velocity is purely tangent to the earth's surface at this point. Equating these quantities gives a formula for v_1 in terms of other things:

$$v_1 = Rv_0 \sin\alpha / r_{\max}.$$

Next, we use conservation of energy:

$$\tfrac{1}{2} v_0^2 - \frac{GM_\mathrm{e}}{R} = \tfrac{1}{2} v_1^2 - \frac{GM_\mathrm{e}}{r_{\max}}.$$

But $v_0^2 = GM_\mathrm{e}/R$. Hence,

$$-\tfrac{1}{2}\frac{GM_\mathrm{e}}{R} = \tfrac{1}{2} v_1^2 - \frac{GM_\mathrm{e}}{r_{\max}},$$

and

$$v_1^2 = \frac{R\sin^2\alpha GM_\mathrm{e}}{r_{\max}^2}.$$

So,

$$-\tfrac{1}{2}\frac{GM_\mathrm{e}}{R} = \tfrac{1}{2} R\sin^2\alpha GM_\mathrm{e} x^2 - GM_\mathrm{e} x, \qquad x = 1/r_{\max}.$$

Tidying up,

$$R\sin^2\alpha GM_\mathrm{e} x^2 - 2GM_\mathrm{e} x + (GM_\mathrm{e}/R) = 0,$$

with solution

$$x = \frac{GM_\mathrm{e} \pm \sqrt{G^2 M_\mathrm{e}^2 - G^2 M_\mathrm{e}^2 \sin^2\alpha}}{R\sin^2\alpha GM_\mathrm{e}}.$$

This simplifies further:

$$x = \frac{1 \pm \sqrt{1 - \sin^2 \alpha}}{R \sin^2 \alpha} = \frac{1 + \cos \alpha}{R \sin^2 \alpha}.$$

Finally,

$$r_{\rm max} = \frac{R \sin^2 \alpha}{1 \pm \cos \alpha}.$$

But which sign to choose? Note that if $\alpha = 0$, then the quadratic becomes degenerate and has solution $x = r_{\rm max}^{-1} = 1/(2R)$. We would like our formula to possess this behaviour: $r_{\rm max} \to 2R$ as $\alpha \to 0$. This suggests taking the MINUS sign. For, as $\alpha \to 0$, $\sin^2 \alpha \sim \alpha^2$, and $1 - \cos \alpha \sim \alpha^2/2$. Thus, the final answer is

$$r_{\rm max} = \frac{R \sin^2 \alpha}{1 - \cos \alpha}.$$

3. A satellite of mass m is in circular orbit about the earth. The radius of the orbit is r_0 and the mass of the earth is $M_{\rm e}$.

   (a) Find the total mechanical energy of the satellite [2 points].

   We use the ellipse formula:

$$E = -\frac{GM_{\rm e}m}{2a},$$

   where a is the semimajor axis. Now a circle is a degerate ellipse, whose semimajor and semiminor axes are equal. Thus, the energy of the orbit is

$$E = -\frac{GM_{\rm e}m}{2r_0}.$$

   (b) Now suppose that the satellite moves in the extreme upper atmosphere of the earth where it is retarded by a constant but small friction force f. The satellite will slowly spiral towards the earth. Since the friction force is weak, the change in radius will be very slow. Therefore, we assume that at any instant the satellite is effectively in a circular orbit of average radius r. Find the approximate change in radius per revolution of the satellite, Δr [2 points].

   We use the power equation derived in class:

$$\frac{dE}{dt} = \boldsymbol{f} \cdot \boldsymbol{v},$$

   where $\boldsymbol{f}$ is the frictional force and $\boldsymbol{v}$ is the velocity. Now friction opposes motion, so $\boldsymbol{f} = -f\,(\boldsymbol{v}/|\boldsymbol{v}|)$, and $\boldsymbol{f} \cdot \boldsymbol{v} = -fv$. Find dE/dt:

$$\frac{dE}{dt} = -\frac{d}{dt}\,(GM_{\rm e}m/2r) = \frac{GM_{\rm e}m}{2r^2}\frac{dr}{dt} = -fv.$$

   Hence,

$$\frac{GM_{\rm e}m}{2r^2}\frac{dr}{dt} = -fv.$$

But $dr/dt = (dr/d\theta)\,\dot{\theta}$, and

$$\frac{GM_{\mathrm{e}}m}{2r^2}\frac{dr}{d\theta} = -f\left(v/\dot{\theta}\right).$$

But the motion such that it is always circular, and $v = v_\theta = r\dot{\theta}$. Hence,

$$\frac{GM_{\mathrm{e}}m}{2r^2}\frac{dr}{d\theta} = -fr.$$

Thus,

$$\frac{dr}{d\theta} = -fr\left(\frac{2r^2}{GM_{\mathrm{e}}m}\right) = -\frac{2fr^3}{GM_{\mathrm{e}}m}.$$

Integrating over a revolution gives Δr:

$$\Delta r = -\frac{2f}{GM_{\mathrm{e}}m}\int_0^{2\pi} r\left(\theta\right)^3 \mathrm{d}\theta.$$

But the radius varies very slowly with θ, because $dr/\theta \propto f$. Therefore, we take the integrand out from under the integral sign, and

$$\Delta r = -\frac{2r^3 f}{GM_{\mathrm{e}}m} \times 2\pi = -\frac{4\pi r^3 f}{GM_{\mathrm{e}}m}.$$

Not surprisingly, the change in radius is negative, and the satellite spirals towards the earth.

(c) Find the approximate change in the kinetic energy of the satellite per revolution, ΔK [2 points].

We use the energy partition

$$-\frac{GM_{\mathrm{e}}m}{2r} = K - \frac{GM_{\mathrm{e}}m}{r},$$

hence

$$K = \frac{GM_{\mathrm{e}}m}{r} > 0.$$

Thus,

$$\frac{dK}{d\theta} = -\frac{GM_{\mathrm{e}}m}{2r^2}\frac{dr}{d\theta}.$$

But $dr/d\theta = -2fr^3/\left(GM_{\mathrm{e}}m\right)$, and

$$\frac{dK}{d\theta} = \frac{GM_{\mathrm{e}}m}{2r^2}\frac{2fr^3}{\left(GM_{\mathrm{e}}m\right)}.$$

Effecting the cancellations, this is

$$\frac{dK}{d\theta} = fr,$$

and the kinetic energy INCREASES, albeit slowly (because $dK/dr \propto f$). Integrating over a revolution and taking $r\left(\theta\right)$ outside the integrand gives

$$\Delta K = 2\pi fr.$$

4. A space vehicle is in circular orbit around the earth. The mass of the vehicle is $3,000\,\mathrm{kg}$ and the radius of the orbit is $2R_\mathrm{e} = 12,800\,\mathrm{km}$. It is desired to transfer the vehicle to a circular orbit or radius $4R_\mathrm{e}$.

   (a) What is the minimum energy expenditure required for the transfer?

   Since a circular orbit minimizes the effective potetnial, the minimum energy required for the transfer is associated with a transfer to a second circular orbit. Now

$$E_1 = -\frac{GM_\mathrm{e}m}{4R_\mathrm{e}},$$

   and

$$E_2 = -\frac{GM_\mathrm{e}m}{8R_\mathrm{e}}$$

   and the minimum energy is

$$\Delta E = E_2 - E_1 = -\frac{GM_\mathrm{e}m}{8R_\mathrm{e}} + \frac{GM_\mathrm{e}m}{4R_\mathrm{e}} = \frac{GM_\mathrm{e}m}{8R_\mathrm{e}}.$$

   We also need a numerical answer to three significant figures. Now

$$GM_\mathrm{e}m/R_\mathrm{e} = 1.8666e + 11$$

   hence $\Delta E = 2.33e + 10\,\mathrm{J}$.

   (b) An efficient way to accomplish the transfer is to use a semi-elliptical orbit (known as a Hohmann transfer orbit), shown in the figure. What velocity changes are required at the points of intersection, points A and B in Fig. 2 [3 points].

   The first transfer point involves a transition from an circular to an elliptic orbit, of semimajor axis $2a = 6R_\mathrm{e}$.

$$\begin{aligned} E_1 &= -\frac{GM_\mathrm{e}m}{4R_\mathrm{e}} = \tfrac{1}{2}mv_1^2 - \frac{GM_\mathrm{e}m}{2R_\mathrm{e}}, \\ E_{1t} &= -\frac{GM_\mathrm{e}m}{6R_\mathrm{e}} = \tfrac{1}{2}mv_{1t}^2 - \frac{GM_\mathrm{e}m}{2R_\mathrm{e}}, \end{aligned}$$

   Solving for the velocities, obtain

$$v_1 = \sqrt{\frac{GM_\mathrm{e}}{2R_\mathrm{e}}}, \qquad v_{1t} = \sqrt{\frac{2GM_\mathrm{e}}{3R_\mathrm{e}}}.$$

   Hence,

$$\Delta v_1 = v_{1t} - v_1 = \sqrt{GM_\mathrm{e}/R_\mathrm{e}}\left(\sqrt{2/3} - \sqrt{1/2}\right).$$

   Now

$$\sqrt{GM_\mathrm{e}/R_\mathrm{e}} = 7.8879e + 03,$$

   hence

$$\Delta v_1 = 863\,\mathrm{m/s}.$$

The second transfer point involves a transition from elliptic orbit of semimajor axis $2a = 6R_{\rm e}$, to a circular orbit, of radius $4R{\rm e}$.

$$
\begin{aligned}
E_{2t} &= -\frac{GM_{\rm e}m}{6R_{\rm e}} = \tfrac{1}{2}mv_{2t}^2 - \frac{GM_{\rm e}m}{4R_{\rm e}}, \\
E_2 &= -\frac{GM_{\rm e}m}{8R_{\rm e}} = \tfrac{1}{2}mv_2^2 - \frac{GM_{\rm e}m}{4R_{\rm e}},
\end{aligned}
$$

Solving for the velocities, obtain

$$
v_{2t} = \sqrt{\frac{GM_{\rm e}}{6R_{\rm e}}}, \qquad v_2 = \sqrt{\frac{GM_{\rm e}}{4R_{\rm e}}}.
$$

Hence,

$$
\Delta v_2 = v_2 - v_{2t} = \sqrt{GM_{\rm e}/R_{\rm e}}\left(\sqrt{1/4} - \sqrt{1/6}\right).
$$

Now

$$
\sqrt{GM_{\rm e}/R_{\rm e}} = 7.8879e + 03,
$$

hence

$$
\Delta v_2 = 724\,{\rm m/s}.
$$

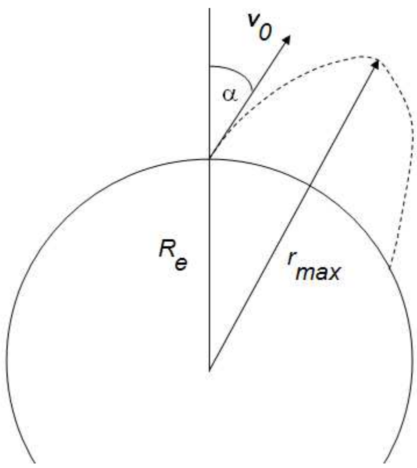

Figure 1: Problem 1

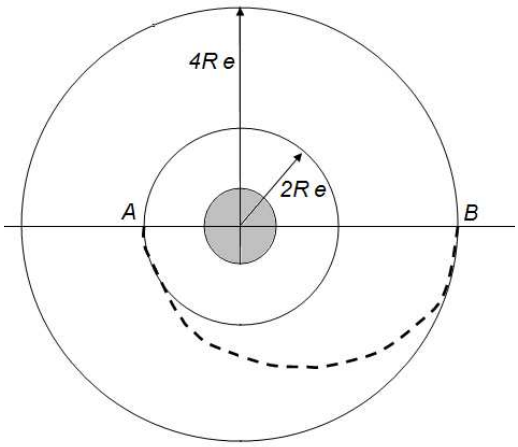

Figure 2: Problem 3