Mechanics and Special Relativity (ACM10030)
Assignment 3

Issue Date: 23 March 2010
Due Date: 30 March 2010

1. Refer to Fig. 1. A projectile of mass m is fired from the surface of the earth at an angle α from the vertical. The initial speed v_0 is equal to $\sqrt{GM_\mathrm{e}/R_\mathrm{e}}$. How high does the projectile rise? Neglect air resistance and the earth's rotation.

   Hint: Do not try to solve for the orbit! Instead, use the conservation laws directly. **[8 marks]**

2. Consider a particle of mass m in two dimensions, experiencing a central force $\boldsymbol{F} = -k\boldsymbol{r}$, where $\boldsymbol{r}$ is the radius vector of the particle relative to the force centre, and in an inertial frame. There are two ways of solving for the motion of such a system. The first way is as to write down the equations of motion in Cartesian form,

   $$m\ddot{x} + kx = 0, \qquad m\ddot{y} + ky = 0,$$

   and observe that the answer is two uncoupled SHM's, $x = A_x \cos(\omega t + \varphi_x)$, $y = A_y \cos(\omega t + \varphi_y)$, where A_x and A_y are constants. This solution pair satisfies the generic conic-section equation

   $$A\,(x/A_x)^2 + B\,(x/A_x)\,(y/A_y) + C\,(y/A_y)^2 + D\,(x/A_x) + E\,(y/A_y) + F = 0,$$

   where

   $$A = C = 1, \qquad D = E = 0,$$
   $$B = -2\cos\theta, \qquad F = -\sin^2\theta, \qquad \theta = \varphi_x - \varphi_y.$$

   Hence, $B^2 - 4AC = 4\,(\cos^2\theta - 1) < 0$, and the motion is an ellipse. This is the quick and easy answer. However, the assignment requires that you follow the class notes, and find the answer as follows:

   (a) What is the angular momentum $\boldsymbol{J}$ for the particle relative to the force centre? Show that this is conserved.

   (b) Write down the equations of motion in two dimensions, in polar coordinates. Hint: $F_r = -kr$, $F_\theta = 0$.

   (c) Reduce the system to a one-equation problem and identify the effective potential. Sketch the result.
   Answer: $m\ddot{r} = -\left(d\mathcal{U}_\mathrm{eff}/dr\right)$, $\mathcal{U}_\mathrm{eff} = [J^2/(2mr^2)] + (kr^2/2)$.

(d) Using the class notes as a hint, find the shape of the orbit of the particle. Are the orbits always closed?

Hint: Reduce the orbit problem to the integral

$$\theta - \theta_0 = \frac{J}{\sqrt{2m}} \int^r \frac{\mathrm{d}s}{s^2\sqrt{E - \frac{1}{2}\frac{J^2}{ms^2} - \frac{1}{2}ks^2}},$$

and explain why E is always positive. Then solve for the integral using

$$\mathcal{I} = \int \frac{\mathrm{d}s}{s\sqrt{-Cs^4 + Bs^2 - A}} = \underbrace{-\tfrac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{-Au^2 + Bu - C}}}_{u=1/s^2} = \tfrac{1}{2}\frac{1}{\sqrt{A}}\sin^{-1}\left[\frac{-2Au + B}{\sqrt{B^2 - 4AC}}\right],$$

where A, B, and C are positive constants.

(e) Would a solar system governed by such a force law make sense? [A few sentences should suffice] **[12 marks]**

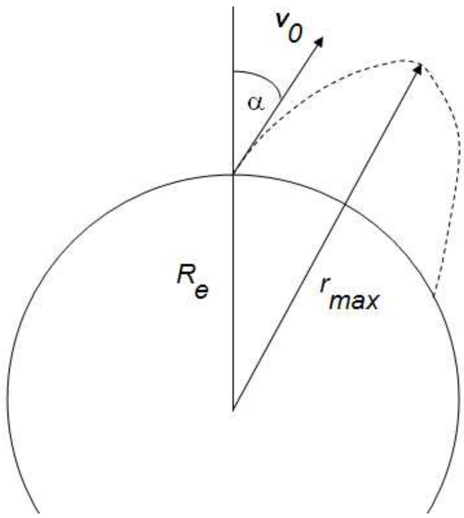

Figure 1: Definition sketch for problem 1