Mechanics and Special Relativity (MAPH10030)
Assignment 2

Issue Date: 16 February 2010
Due Date: 23 February 2010

1. Consider a particle that is constrained on top of a semicircle (See Fig. 1). Gravity points downwards. Suppose that the particle starts from rest. At what angle does the particle fall off the semicircle? [4 points]

   Hint: Please give the solution in two forms: in terms of the angle ϕ, and the angle θ. The answer in the ϕ-angle is given in the e-book mentioned in Lecture 1.

   Work in the θ coordinates. In the absence of constraints, the EOM is

$$
\begin{aligned}
m\left(\ddot{r}-r\dot{\theta}^2\right) &= -\frac{\partial \mathcal{U}}{\partial r}, \\
m\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right) &= -\frac{1}{r}\frac{\partial \mathcal{U}}{\partial \theta},
\end{aligned}
$$

   where $\mathcal{U} = mgy = mgr\sin\theta$. Now the motion is constrained, $\dot{r}=0$, so we use the constrained EOM discussed in class

$$
\begin{aligned}
mr\dot{\theta}^2 &= N_r - mg\sin\theta, \\
mr\ddot{\theta} &= -mg\cos\theta.
\end{aligned}
$$

   Reduce the tangential equation to an energy-conservation law:

$$E = \tfrac{1}{2}m\dot{\theta}^2 + mgr\sin\theta = E = E\left(t=0\right) = mgr\sin\left(\pi/2\right) = mgr.$$

   Hence,

$$r\dot{\theta}^2 = 2g\left(1-\sin\theta\right).$$

   Insert this result into the radial EOM, obtain

$$N_r = -mg\sin\theta + mr\dot{\theta}^2 = g\left(2-3\sin\theta\right).$$

   The particle falls off the semicircle when the force constraining it to the surface vanishes, i.e. $N_r = 0$, or

$$\tfrac{2}{3} = \sin\theta.$$

   It is customary to measure the angle in this problem form the vertical, $\phi = \frac{1}{2}\pi - \theta$, hence $\cos\phi = \sin\theta$, and

$$\phi = \cos^{-1}\tfrac{2}{3}.$$

   Subtract one mark if the answer in decimal form, $\theta = 0.73\,\text{Rad}$ or $\theta \approx 0.73, \text{Rad}$, as both these answers are wrong.

2. One force acting on a machine part is $\boldsymbol{F} = (-5.00\,\mathrm{N})\,\hat{\boldsymbol{x}} + (4.00\,\mathrm{N})\,\hat{\boldsymbol{y}}$. The vector from the origin to the point where the force is applied is $\boldsymbol{r} = (-0.450\,\mathrm{m})\,\hat{\boldsymbol{x}} + (0.150\,\mathrm{m})\,\hat{\boldsymbol{y}}$.

    - In a sketch show $\boldsymbol{r}$, $\boldsymbol{F}$, and the origin [1 point].
    See Fig. 1 (b).
    - Use the right-hand rule to determine the direction of the torque. Then, compute the torque from the determinant definition. Make sure that the direction obtained in both calculations is the same [3 points].
    By the RHR, the direction of the torque is into the page. Using the determinant rule,

$$\begin{aligned}\boldsymbol{\tau} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ -0.450 & 0.150 & 0 \\ -5.00 & 4.00 & 0 \end{vmatrix}, \\ &= \hat{\boldsymbol{z}}\,(-0.450 \times 4.00 + 0.150 \times 5.00) = -1.05\hat{\boldsymbol{z}}.\end{aligned}$$

    Since the coordinate frame is right-handed, $\hat{\boldsymbol{z}}$ must point out of the page, hence $\boldsymbol{\tau}$ is into the page.

3. (a) Show that if the total linear momentum of a system of particles is zero, the angular momentum of the system is the same about all origins. [3 points]
    Given: $\sum_i \boldsymbol{p}_i = 0$. Angular momentum:

$$\boldsymbol{J} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i.$$

    A new system of axes: $\boldsymbol{r}'_i = \boldsymbol{r}_i + \boldsymbol{R}$, where $d\boldsymbol{R}/dt = 0$ because we are effecting an instantaneous shift in the axes. Hence, $\boldsymbol{p}'_i = \boldsymbol{p}_i$, and

$$\begin{aligned}\boldsymbol{J}' &= \sum_i \boldsymbol{r}'_i \times \boldsymbol{p}_i, \\ &= \sum_i (\boldsymbol{r}_i + \boldsymbol{R}) \times \boldsymbol{p}_i, \\ &= \sum_i (\boldsymbol{r}_i \times \boldsymbol{p}_i + \boldsymbol{R} \times \boldsymbol{p}_i)\,, \\ &= \boldsymbol{J} + \left(\boldsymbol{R} \times \sum_i \boldsymbol{p}_i\right), \\ &= \boldsymbol{J}.\end{aligned}$$

   (b) Show that if the total force on a system of particles is zero, the torque on the system is the same about all origins [3 points].
    Let $\boldsymbol{F}_i$ be the total force experienced by particle i. This can be decomposed into interactions and external parts, but that is not needed. Let us note however, that

$$\boldsymbol{F}_i = \sum_{i \neq j} \boldsymbol{F}_{ij}^{\text{interaction}} + \boldsymbol{F}_i^{\text{external}}.$$

Now, $\sum_i \boldsymbol{F}_i = 0$ in a particular system of axes, and $\boldsymbol{r}_i' = \boldsymbol{r}_i + \boldsymbol{R}$ represents an instantaneous shift in axes. The forces ought to be translation invariant, hence $\boldsymbol{F}_i' = \boldsymbol{F}_i$. Hence,

$$\begin{aligned}
\boldsymbol{\tau}' &= \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i', \\
&= \sum_i (\boldsymbol{r}_i + \boldsymbol{R}) \times \boldsymbol{F}_i, \\
&= \sum_i (\boldsymbol{r}_i \times \boldsymbol{F}_i + \boldsymbol{R} \times \boldsymbol{F}_i), \\
&= \boldsymbol{\tau} + \left(\boldsymbol{R} \times \sum_i \boldsymbol{F}_i \right), \\
&= \boldsymbol{\tau}.
\end{aligned}$$

4. Recall the law of gravity for point particles m_1 and m_2: the force on particle 1 due to particle 2 is given by

$$\boldsymbol{F}_{12} = -\frac{Gm_1 m_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^2} \left(\frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} \right). \tag{1}$$

In class, we stated that the same law holds for spherical bodies at finite separations, and that the proof of this statement follows by integration. In this problem we obtain a hint at how this integration might be done by considering the gravitational force exerted by a continuous line of particles on a point particle of mass m.

Consider the system shown in Fig. 2. A continuous line of particles extends from $x = -a$ to $x = a$, at $y = 0$. A point mass lies at $x = 0$, $y = L$.

(a) Show that the force on the particle due to a point-like mass $dm\,(x)$ extending from x to $x + \mathrm{d}x$ is

$$\mathrm{d}\boldsymbol{F}_{1,x} = -\frac{Gm\,\mathrm{d}m\,(x)}{(x^2 + L^2)^{3/2}} (L\hat{\boldsymbol{y}} - x\hat{\boldsymbol{x}}).$$

We use the point-mass formula because $\mathrm{d}m$ is an infinitesimal mass element. Let $\boldsymbol{r}$ be a vector from $P = (x, 0)$ to the point $M = (0, L)$. Then,

$$\boldsymbol{r} = \overrightarrow{OM} - \overrightarrow{OP} = L\hat{\boldsymbol{y}} - x\hat{\boldsymbol{x}}.$$

The gravitational force on m due to $\mathrm{d}m$ is directed along $-\boldsymbol{r}$ and the separation distance in the force formula is $r = |\boldsymbol{r}| = \sqrt{L^2 + x^2}$. Using the formula

$$\mathrm{d}\boldsymbol{F} = -Gm\,\mathrm{d}m \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3},$$

obtain

$$\mathrm{d}\boldsymbol{F} = -Gm\,\mathrm{d}m \frac{L\hat{\boldsymbol{y}} - x\hat{\boldsymbol{x}}}{(x^2 + L^2)^{3/2}}.$$

(b) Assume a linear mass density $\mathrm{d}m = \rho\,\mathrm{d}x$ and thus obtain the total force $\boldsymbol{F}_1$ on the point mass m. You might have to use your favour table of integrals to do this.

$$\begin{aligned}
\mathrm{d}\boldsymbol{F} &= -Gm\rho\,\mathrm{d}x\frac{L\hat{\boldsymbol{y}} - x\hat{\boldsymbol{x}}}{(x^2+L^2)^{3/2}} \\
\boldsymbol{F} &= \int_{x=-a}^{x=a}\left[-Gm\rho\frac{L\hat{\boldsymbol{y}} - x\hat{\boldsymbol{x}}}{(x^2+L^2)^{3/2}}\right]\mathrm{d}x \\
&= -Gm\rho L\hat{\boldsymbol{y}}\int_{-a}^{a}\frac{\mathrm{d}x}{(x^2+L^2)^{3/2}} + Gm\rho\hat{\boldsymbol{x}}\int_{-a}^{a}\frac{x\mathrm{d}x}{(x^2+L^2)^{3/2}}
\end{aligned}$$

The second integral is zero because it is an odd function integrated over a symmetric domain. Thus, the force is entirely directed in the y-direction, and equal to

$$\begin{aligned}
\boldsymbol{F} &= -Gm\rho L\hat{\boldsymbol{y}}\int_{-a}^{a}\frac{\mathrm{d}x}{(x^2+L^2)^{3/2}}, \\
&= -\frac{Gm\rho L}{L^2}\hat{\boldsymbol{y}}\int_{-a/L}^{a/L}\frac{\mathrm{d}s}{(1+s^2)^{3/2}}, \\
&= -\frac{Gm\rho L}{L^2}\hat{\boldsymbol{y}}\int_{-a/L}^{a/L}\frac{\partial}{\partial s}\frac{s}{\sqrt{1+s^2}}, \\
&= -\frac{Gm\rho L}{L^2}\hat{\boldsymbol{y}}\frac{2a/L}{\sqrt{1+(a/L)^2}}.
\end{aligned}$$

Tidying up the formula yields the final answer [full marks if student gets to here]:

$$\begin{aligned}
\boldsymbol{F} &= -\frac{2Gm\rho a}{L^2}\left[1+\left(\frac{a}{L}\right)^2\right]^{-2}\hat{y}, \\
&= -\frac{GmM}{L^2}\left[1+\left(\frac{a}{L}\right)^2\right]^{-2}\hat{y}
\end{aligned}$$

[Additional comment] For large separations L, the lowest-order contribution to the force is

$$\boldsymbol{F} = -\frac{GmM}{L^2}\hat{y} + O\left((a/L)^2\right),$$

and the point mass m 'sees' the rod as another point mass of mass M.

(c) How would the force distribution change if $\mathrm{d}m = \rho_0\left[1+\varepsilon\,(x/L)\right]\mathrm{d}x$?

Now, the force integral is

$$\boldsymbol{F} = -Gm\rho_0 L\hat{\boldsymbol{y}}\int_{-a}^{a}\frac{\left[1+\varepsilon\,(x/L)\right]\mathrm{d}x}{(x^2+L^2)^{3/2}} + Gm\rho_0\hat{\boldsymbol{x}}\int_{-a}^{a}\frac{\left[1+\varepsilon\,(x/L)\right]x\mathrm{d}x}{(x^2+L^2)^{3/2}}$$

Identify the odd integrals and set them to zero:

$$\boldsymbol{F} = -Gm\rho_0 L\hat{\boldsymbol{y}} \int_{-a}^{a} \frac{\mathrm{d}x}{\left(x^2+L^2\right)^{3/2}} + Gm\rho_0\hat{\boldsymbol{x}} \int_{-a}^{a} \frac{\varepsilon\left(x/L\right)x\mathrm{d}x}{\left(x^2+L^2\right)^{3/2}}$$

We have seen the integral for the y-direction before. To do it, let $\rho \to \rho_0$ in part (b). Now there is a contribution to the force in the x-direction too:

$$\begin{aligned}
\text{Contribution in the x-direction} &= Gm\rho_0 \int_{-a}^{a} \frac{\varepsilon\left(x/L\right)x\mathrm{d}x}{\left(x^2+L^2\right)^{3/2}} \\
&= \frac{Gm\rho_0}{L} \int_{-a/L}^{a/L} \frac{s^2\,\mathrm{d}s}{\left(1+s^2\right)^{3/2}}, \\
&= \frac{Gm\rho_0}{L} \left[\sinh^{-1} s - \frac{s}{\sqrt{1+s^2}}\right]_{-a/L}^{a/L}
\end{aligned}$$

This is

$$\frac{Gm\rho}{L}\left[2\sinh^{-1}\left(a/L\right) - \frac{2\left(a/L\right)}{\sqrt{1+\left(a/L\right)^2}}\right]$$

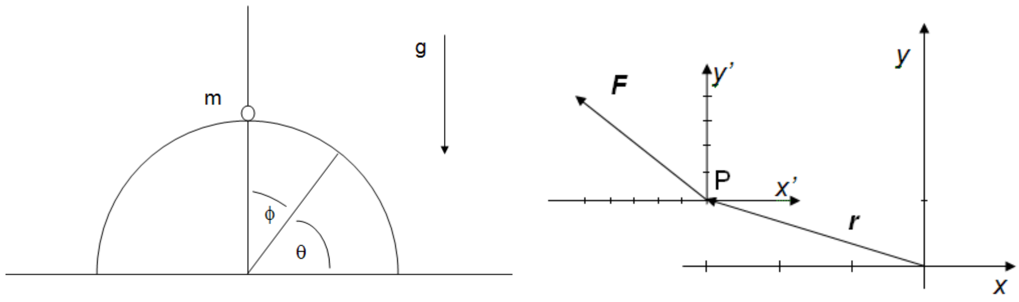
Therefore, the force is

$$\boldsymbol{F} = -\hat{\boldsymbol{y}}\frac{GMm}{L^2}\left[1+\left(\frac{a}{L}\right)^2\right]^{-2} + \hat{\boldsymbol{x}}\frac{GMm}{L^2}\left[\frac{L}{a}\sinh^{-1}\left(a/L\right) - \left[1+\left(\frac{a}{L}\right)^2\right]^{-2}\right].$$

[Full marks if the student gets this far.] (Note that $M = 2a\rho_0$ as before.) [Additional comment] A plot of the function

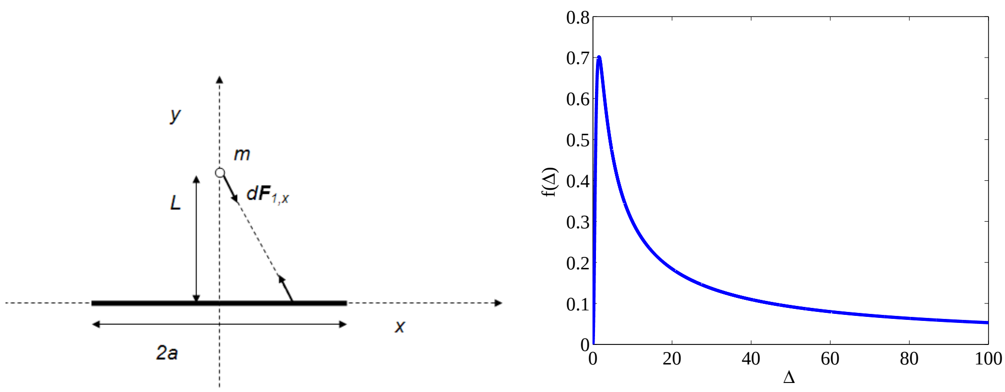
$$f\left(\Delta\right) = \frac{1}{\Delta}\sinh^{-1}\Delta - \left[1+\Delta^2\right]^{-2}.$$

shows that it is always positive (Fig. 2 (b)), and thus, the x-component of gravity is always in the positive x-direction. This makes sense: the most massive part of the rod is in the positive half-line, and these positive contributions to the total force dominate over contributions negative contributions from the negative half-line. Note, however, that there is an optimal a/L value that maximizes this force.



(a) Problem 1 (b) Problem 2

Figure 1: Sketches for problems 1 and 2



(a) Definition sketch (b) The function $f(\Delta)$

Figure 2: Problem 4