Mechanics and Special Relativity (ACM10030)
Assignment 2

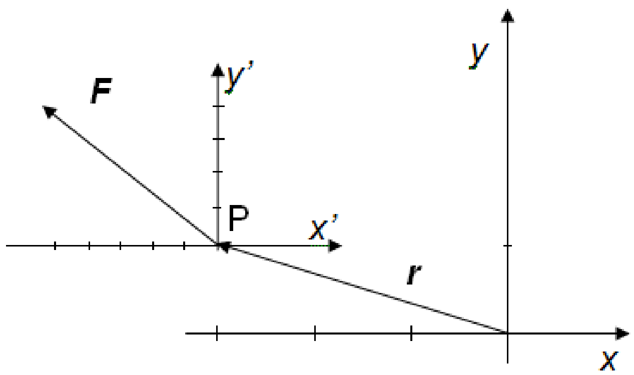
Issue Date: 28th February 2011
Due Date: 21st March 2011

1. **Force and torque** One force acting on a machine part is $\boldsymbol{F} = (-5.00\,\mathrm{N})\,\hat{\boldsymbol{x}} + (4.00\,\mathrm{N})\,\hat{\boldsymbol{y}}$. The vector from the origin to the point where the force is applied is $\boldsymbol{r} = (-0.450\,\mathrm{m})\,\hat{\boldsymbol{x}} + (0.150\,\mathrm{m})\,\hat{\boldsymbol{y}}$.

   (a) In a sketch show $\boldsymbol{r}$, $\boldsymbol{F}$, and the origin. You must show $\boldsymbol{r}$ and $\boldsymbol{F}$ on different sets of axes because they have different physical units.

   (b) Use the right-hand rule to determine the direction of the torque. Then, compute the torque from the determinant definition. Make sure that the direction obtained in both calculations is the same.

See below for the sketch. By the RHR, the direction of the torque is into the page.



Using the determinant rule,

$$
\begin{aligned}
\boldsymbol{\tau} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ -0.450 & 0.150 & 0 \\ -5.00 & 4.00 & 0 \end{vmatrix}, \\
&= \hat{\boldsymbol{z}}\,(-0.450 \times 4.00 + 0.150 \times 5.00) = -1.05\hat{\boldsymbol{z}}.
\end{aligned}
$$

Since the coordinate frame is right-handed, $\hat{\boldsymbol{z}}$ must point out of the page, hence $\boldsymbol{\tau}$ is into the page.

2. **Angular momentum and torque**

   (a) Show that if the total linear momentum of a system of particles is zero, the angular momentum of the system is the same about all origins.

   (b) Show that if the total force on a system of particles is zero, the torque on the system is the same about all origins.

(a) Given: $\sum_i \boldsymbol{p}_i = 0$. Angular momentum:

$$\boldsymbol{J} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i.$$

A new system of axes: $\boldsymbol{r}_i' = \boldsymbol{r}_i + \boldsymbol{R}$, where $d\boldsymbol{R}/dt = 0$ because we are effecting an instantaneous shift in the axes. Hence, $\boldsymbol{p}_i' = \boldsymbol{p}_i$, and

$$\begin{aligned} \boldsymbol{J}' &= \sum_i \boldsymbol{r}_i' \times \boldsymbol{p}_i, \\ &= \sum_i (\boldsymbol{r}_i + \boldsymbol{R}) \times \boldsymbol{p}_i, \\ &= \sum_i (\boldsymbol{r}_i \times \boldsymbol{p}_i + \boldsymbol{R} \times \boldsymbol{p}_i), \\ &= \boldsymbol{J} + \left(\boldsymbol{R} \times \sum_i \boldsymbol{p}_i \right), \\ &= \boldsymbol{J}. \end{aligned}$$

(b) Let $\boldsymbol{F}_i$ be the total force experienced by particle i. This can be decomposed into interactions and external parts, but that is not needed. Note however, that

$$\boldsymbol{F}_i = \sum_{i \neq j} \boldsymbol{F}_{ij}^{\text{interaction}} + \boldsymbol{F}_i^{\text{external}}.$$

Now, $\sum_i \boldsymbol{F}_i = 0$ in a particular system of axes, and $\boldsymbol{r}_i' = \boldsymbol{r}_i + \boldsymbol{R}$ represents an instantaneous shift in axes. Galilean invariance implies that the forces are translation-invariant: $\boldsymbol{F}_i' = \boldsymbol{F}_i$. Hence,

$$\begin{aligned} \boldsymbol{\tau}' &= \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i', \\ &= \sum_i (\boldsymbol{r}_i + \boldsymbol{R}) \times \boldsymbol{F}_i, \\ &= \sum_i (\boldsymbol{r}_i \times \boldsymbol{F}_i + \boldsymbol{R} \times \boldsymbol{F}_i), \\ &= \boldsymbol{\tau} + \left(\boldsymbol{R} \times \sum_i \boldsymbol{F}_i \right), \\ &= \boldsymbol{\tau}. \end{aligned}$$

3. **Gravitational forces on extended bodies** Recall the law of gravity for point particles m_1 and m_2: the force on particle 1 due to particle 2 is given by

$$\boldsymbol{F}_{12} = -\frac{Gm_1m_2}{|\boldsymbol{x}_1-\boldsymbol{x}_2|^2}\left(\frac{\boldsymbol{x}_1-\boldsymbol{x}_2}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}\right). \tag{1}$$

In class, we stated that the same law holds for spherical bodies at finite separations, and that the proof of this statement follows by integration. In this problem we obtain a hint at how this integration might be done by considering the gravitational force exerted by a continuous line of particles on a point particle of mass m.

Consider the system shown in Fig. 1(a). A continuous line of particles extends from $x=-a$ to $x=a$, at $y=0$. A point mass lies at $x=0$, $y=L$.

(a) Show that the force on the particle due to a point-like mass $\mathrm{d}m\,(x)$ extending from x to $x+\mathrm{d}x$ is

$$\mathrm{d}\boldsymbol{F}_{1,x} = -\frac{Gm\,\mathrm{d}m\,(x)}{(x^2+L^2)^{3/2}}\left(L\hat{\boldsymbol{y}}-x\hat{\boldsymbol{x}}\right).$$

(b) Assume a linear mass density $\mathrm{d}m=\rho\,\mathrm{d}x$ ($\rho=$ Const.) and thus obtain the total force $\boldsymbol{F}_1$ on the point mass m. Use a table of integrals if necessary.

(a) We use the point-mass formula because $\mathrm{d}m$ is an infinitesimal mass element. Let $\boldsymbol{r}$ be a vector from $P=(x,0)$ to the point $M=(0,L)$. Then,

$$\boldsymbol{r} = \overrightarrow{OM}-\overrightarrow{OP} = L\hat{\boldsymbol{y}}-x\hat{\boldsymbol{x}}.$$

The gravitational force on m due to $\mathrm{d}m$ is directed along $-\boldsymbol{r}$ and the separation distance in the force formula is $r=|\boldsymbol{r}|=\sqrt{L^2+x^2}$. Using the formula

$$\mathrm{d}\boldsymbol{F} = -Gm\,\mathrm{d}m\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3},$$

obtain

$$\mathrm{d}\boldsymbol{F} = -Gm\,\mathrm{d}m\frac{L\hat{\boldsymbol{y}}-x\hat{\boldsymbol{x}}}{(x^2+L^2)^{3/2}}.$$

(b) We have,

$$\begin{aligned}
\mathrm{d}\boldsymbol{F} &= -Gm\rho\,\mathrm{d}x\frac{L\hat{\boldsymbol{y}}-x\hat{\boldsymbol{x}}}{(x^2+L^2)^{3/2}}\\
\boldsymbol{F} &= \int_{x=-a}^{x=a}\left[-Gm\rho\frac{L\hat{\boldsymbol{y}}-x\hat{\boldsymbol{x}}}{(x^2+L^2)^{3/2}}\right]\mathrm{d}x\\
&= -Gm\rho L\hat{\boldsymbol{y}}\int_{-a}^{a}\frac{\mathrm{d}x}{(x^2+L^2)^{3/2}}+Gm\rho\hat{\boldsymbol{x}}\int_{-a}^{a}\frac{x\mathrm{d}x}{(x^2+L^2)^{3/2}}
\end{aligned}$$

The second integral is zero because it is an odd function integrated over a symmetric domain. Thus, the force is entirely directed in the y-direction, and equal to

$$\begin{aligned}
\boldsymbol{F} &= -Gm\rho L\hat{\boldsymbol{y}}\int_{-a}^{a}\frac{\mathrm{d}x}{(x^2+L^2)^{3/2}},\\
&= -\frac{Gm\rho L}{L^2}\hat{\boldsymbol{y}}\int_{-a/L}^{a/L}\frac{\mathrm{d}s}{(1+s^2)^{3/2}},\\
&= -\frac{Gm\rho L}{L^2}\hat{\boldsymbol{y}}\int_{-a/L}^{a/L}\frac{\partial}{\partial s}\frac{s}{\sqrt{1+s^2}},\\
&= -\frac{Gm\rho L}{L^2}\hat{\boldsymbol{y}}\frac{2a/L}{\sqrt{1+(a/L)^2}}.
\end{aligned}$$

Tidying up the formula yields the final answer [full marks if student gets to here]:

$$\begin{aligned}
\boldsymbol{F} &= -\frac{2Gm\rho a}{L^2}\left[1+\left(\frac{a}{L}\right)^2\right]^{-1/2}\hat{y},\\
&= -\frac{GmM}{L^2}\left[1+\left(\frac{a}{L}\right)^2\right]^{-1/2}\hat{y}
\end{aligned}$$

[Additional comment] For large separations L, the lowest-order contribution to the force is

$$\boldsymbol{F} = -\frac{GmM}{L^2}\hat{y} + O\left((a/L)^2\right),$$

and the point mass m 'sees' the rod as another point mass of mass M.

4. **Gravitational self-energy** Consider a solid sphere of uniform density ρ, radius R_0, and mass M.

   (a) Explain why the gravitational interaction between a mass element $\mathrm{d}m$ and a solid sphere of radius r and constant density ρ – where the mass element sits on the surface of the sphere – is given by

   $$\mathrm{d}\mathcal{U} = -G\mathrm{d}m\left(\tfrac{4}{3}\pi r^3\rho/r\right).$$

   (b) By integrating over all such mass elements that sit in a shell of thickness $\mathrm{d}r$ on the surface of the sphere in part (a), show that the gravitational interaction between the shell and the sphere is

   $$\mathrm{d}\mathcal{U} = -\tfrac{16}{3}\pi^2 G\rho^2 r^4\,\mathrm{d}r.$$

   (c) Do one final integration to show that the gravitational self-energy of the sphere is

   $$\mathcal{U} = -\tfrac{3}{5}\frac{GM^2}{R_0}.$$

(a) For this part, consider an interaction between a point-like mass $\mathrm{d}m$ and an extended spherical body of radius R. The force exerted by the body on the point-like mass acts through the centre of the sphere. Thus, if r is the separation of the two objects, the interaction potential is

$$\mathrm{d}\mathcal{U} = -\frac{G\mathrm{d}m}{r} \times \text{Mass of extended body}$$

and

$$\text{Mass of extended body} = \rho \times \text{Volume} = \tfrac{4}{3}\pi r^3 \rho,$$

hence

$$\mathrm{d}\mathcal{U} = -\frac{G\mathrm{d}m}{r}\left(\tfrac{4}{3}\pi r^3 \rho\right),$$

as required.

(b) Since the mass $\mathrm{d}m$ and the extended body are composed of the same material, we have

$$\mathrm{d}m = \rho \times \text{Volume of infinitesimal element.}$$

If we integrate over all such elements in a shell of thickness $\mathrm{d}r$, we would obtain the result

$$\mathrm{d}m = \rho \times (\text{Surface area of sphere of radius } r) \times \mathrm{d}r = 4\pi r^2 \rho \mathrm{d}r.$$

More precisely, we have

$$\begin{aligned}
\mathrm{d}m &= \rho \times \text{Volume of infinitesimal element,} \\
&= \rho r^2 \mathrm{d}r \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi, \\
\mathrm{d}m_r &= \int_\Omega \rho r^2 \mathrm{d}r \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi, \\
&= \rho r^2 \mathrm{d}r \int_0^\pi \mathrm{d}\theta \, \sin\theta \int_0^{2\pi} \mathrm{d}\varphi, \\
&= 4\pi \rho r^2 \mathrm{d}r.
\end{aligned}$$

where Ω is the sphere of radius 1, θ is the polar angle, and φ is the azimuthal angle of spherical polar coordinates (these steps are not necessary to the proof).

Thus, we have

$$\begin{aligned}
\mathrm{d}\mathcal{U} &= -\frac{G\mathrm{d}m}{r}\left(\tfrac{4}{3}\pi r^3 \rho\right), \\
\mathrm{d}m &= \rho 4\pi r^2 \mathrm{d}r, \\
\Longrightarrow \mathrm{d}\mathcal{U} &= -\tfrac{4}{3}\rho G \pi r^2 \left(4\pi r^2 \rho \mathrm{d}r\right),
\end{aligned}$$

or

$$\mathrm{d}\mathcal{U} = -\tfrac{16}{3}\pi^2 G \rho^2 r^4 \mathrm{d}r.$$

(c) Integrate the last expression from $r = 0$ to $r = R$, the sphere radius. We have,

$$\begin{aligned} \mathcal{U} &= -\tfrac{16}{3}\pi^2 G\rho^2 \int_0^r r^4 \mathrm{d}r, \\ &= -\tfrac{16}{3}\pi^2 G\rho^2 \left(\tfrac{1}{5}R^5\right), \\ &= -\frac{3G}{5R}\tfrac{16}{3.3}\pi^2 R^6 \rho^2, \\ &= -\frac{3G}{5R}\left(\tfrac{4}{3}\rho\pi R^3\right)^2, \\ &= -\frac{3G}{5R}M^2, \end{aligned}$$

as required.

5. **Bonus problem: This question is not mandatory, but can be used to top up the marks on the other questions, for a maximum of five top-up marks.** Consider a particle that is constrained on top of a semicircle (Fig. 1(b)). Gravity points downwards. Suppose that the particle starts from rest. At what angle does the particle fall off the semicircle?

   Give the solution in two forms: in terms of the angle ϕ, and the angle θ.

Work in the θ coordinate (standard coords from class notes). In the absence of constraints, the EOM is

$$\begin{aligned} m\left(\ddot{r} - r\dot{\theta}^2\right) &= -\frac{\partial \mathcal{U}}{\partial r}, \\ m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) &= -\frac{1}{r}\frac{\partial \mathcal{U}}{\partial \theta}, \end{aligned}$$

where $\mathcal{U} = mgy = mgr\sin\theta$. Now the motion is constrained, $\dot{r} = 0$, so we use the constrained EOM discussed in class

$$\begin{aligned} mr\dot{\theta}^2 &= N_r - mg\sin\theta, \\ mr\ddot{\theta} &= -mg\cos\theta. \end{aligned}$$

Reduce the tangential equation to an energy-conservation law:

$$E = \tfrac{1}{2}mr^2\dot{\theta}^2 + mgr\sin\theta = E = E\left(t=0\right) = mgr\sin\left(\pi/2\right) = mgr.$$

Hence,

$$r\dot{\theta}^2 = 2g\left(1 - \sin\theta\right).$$

Insert this result into the radial EOM, obtain

$$N_r = +mg\sin\theta - mr\dot{\theta}^2 = g\left(3\sin\theta - 2\right).$$

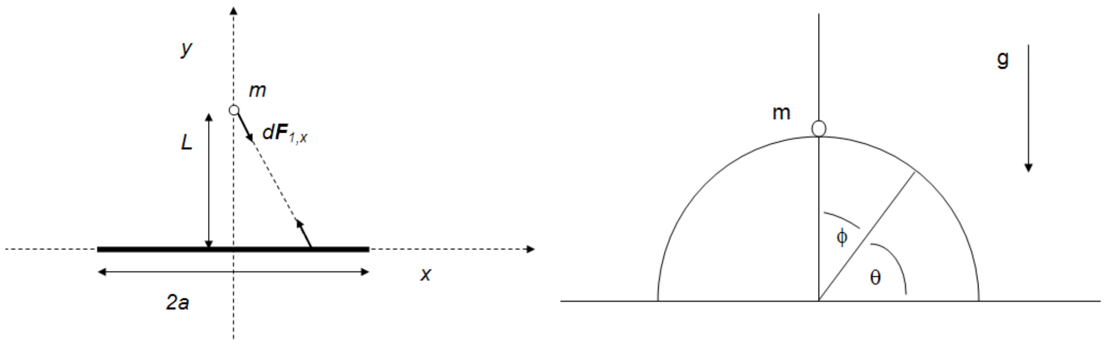
The particle falls off the semicircle when the force constraining it to the surface vanishes, i.e. $N_r = 0$, or

$$\tfrac{2}{3} = \sin\theta.$$

It is customary to measure the angle in this problem form the vertical, $\phi = \frac{1}{2}\pi - \theta$, hence $\cos\phi = \sin\theta$, and

$$\phi = \cos^{-1}\tfrac{2}{3}.$$

[Subtract two marks if the answer in decimal form, $\theta = 0.73\,\mathrm{Rad}$ or $\theta \approx 0.73\,\mathrm{Rad}$, as both these answers are wrong.]



(a) Gravitational interaction between a particle and a rod.

(b) Sketch for bonus problem.

Figure 1: