Mechanics and Special Relativity (MAPH10030) Assignment 1

Issue Date: 01 February 2010
Due Date: 08 February 2010

1. Recall the definition of the dot product for vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$:

$$\boldsymbol{a}\cdot\boldsymbol{b} = \sum_{i=1}^{3} a_i b_i, \qquad |\boldsymbol{a}|^2 = \sum_{i=1}^{3} a_i^2.$$

   (a) Show that $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ [2 Points].

   Since the dot product is rotation-invariant, we can rotate our coordinate system such that the vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ live in the x-y plane. Now refer to Fig. 1. Using the Law of Cosines,

$$L^2 = |\boldsymbol{x}_1|^2 + |\boldsymbol{x}_2|^2 - 2|\boldsymbol{x}_1||\boldsymbol{x}_2|\cos\theta.$$

   But

$$L^2 = |\overrightarrow{P_2P_1}|^2 = |\boldsymbol{x}_1 - \boldsymbol{x}_2|^2 = (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 = |\boldsymbol{x}_1|^2 + |\boldsymbol{x}_2|^2 - 2\boldsymbol{x}_1\cdot\boldsymbol{x}_2.$$

   Equating both expressions for L^2 gives

$$\boldsymbol{x}_1\cdot\boldsymbol{x}_2 = |\boldsymbol{x}_1||\boldsymbol{x}_2|\cos\theta,$$

   as required.

   (b) Show that if $|\boldsymbol{a}-\boldsymbol{b}| = |\boldsymbol{a}+\boldsymbol{b}|$, then $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ [2 Points].

   Square both sides.

$$\begin{aligned} LHS &= |\boldsymbol{x}-\boldsymbol{y}|^2 = (\boldsymbol{x}-\boldsymbol{y})^2 = \boldsymbol{x}^2 + \boldsymbol{y}^2 - 2\boldsymbol{x}\cdot\boldsymbol{y}, \\ RHS &= |\boldsymbol{x}+\boldsymbol{y}|^2 = (\boldsymbol{x}+\boldsymbol{y})^2 = \boldsymbol{x}^2 + \boldsymbol{y}^2 + 2\boldsymbol{x}\cdot\boldsymbol{y}. \end{aligned}$$

   Equating ($LHS = RHS$) gives

$$4\boldsymbol{x}\cdot\boldsymbol{y} = 0,$$

   hence $\boldsymbol{x}\cdot\boldsymbol{y} = 0$. According to (Q1(a)), it follows that $\cos\theta = \pi/2$, hence, the angle between the vectors is $\pi/2$, i.e., they are orthogonal.

2. Refer to Eqs. (1) and (2). A girl throws a water balloon at an angle α above the horizontal with a speed $|\boldsymbol{v}_0|$. The horizontal component of the balloon's velocity $u = |\boldsymbol{v}_0| \cos\alpha$ is directed towards a car that is approaching the girl with a constant speed V. If the balloon is to hit the car at the same height at which it leaves her hand, what is the maximum distance the car can be from the girl when the balloon is thrown?

   The answer, H, involves V, v_0, α, and g [4 Points].

   We are to consider the foremost tip of the car. We ask the question, at what time does the balloon hit this tip, assuming that the collision occurs at the launch height. The initial velocity of the balloon is $\boldsymbol{v}_0 = (u, v) = v_0 (\cos\alpha, \sin\alpha)$. We work in the FOR of the earth with a choice of origin $(x_0, y_0) = 0$. Hence,

$$x = v_0 \cos\alpha\, t, \qquad y = v_0 \sin\alpha\, t - \tfrac{1}{2} g t^2.$$

   The coordinate of the car in this frame is

$$x_{\rm car} = H - Vt,$$

   where the minus sign indicates that the car is approaching the girl, who is fixed to the FOR of the earth. To find the time of collision, form the following equality:

$$x = x_{\rm car} \implies v_0 \cos\alpha\, t = H - Vt \implies (v_0 \cos\alpha + V)\, t = H.$$

   Hence,

$$t_{\rm coll} = \frac{H}{v_0 \cos\alpha + V}.$$

   Rather unsurprisingly, the collision is hastened by the car's having a finite velocity in the girl's direction.

   Now we find H. At the collision time, $y = y_{\rm car}$ too. This location is at $y = 0$. Hence, $v_0 \sin\alpha t_{\rm c} - \frac{1}{2} g t_{\rm c}^2 = 0$. Assuming $t_{\rm c} \neq 0$, obtain

$$t_{\rm c} = \frac{2 v_0 \sin\alpha}{g}.$$

   Now we have two equations for $t_{\rm c}$. We equate them and solve for H,

$$t_{\rm c} = \frac{2 v_0 \sin\alpha}{g} = \frac{H}{v_0 \cos\alpha + V},$$

   hence

$$H = 2 v_0 \sin\alpha\, (V + v_0 \cos\alpha)\, g^{-1}.$$

3. Consider a particle experiencing the force $F = kx$, a repulsive spring force.

   (a) Write down the equation of motion and the energy [1 Point].
   Newton's equation: $m\ddot{x} = F = kx$. Identify $\sigma = \sqrt{k/m}$, hence

$$\ddot{x} - \sigma^2 x = 0.$$

(b) Reduce the motion to an integral using the energy [1 Point].
Multiply the equation of motion (EOM) by $\dot{x}$ and integrate w.r.t. time. The result is

$$\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\sigma^2 x^2 = \varepsilon = E/m,$$

a constant with units of $[\text{Energy}][\text{mass}]^{-1}$. Solve for dx/dt:

$$\left(\frac{dx}{dt}\right)^2 = \sigma^2 x^2 + 2\varepsilon \geq 0,$$

hence ε is required to be nonnegative. Inverting for dt/dx,

$$\begin{aligned} \frac{dx}{dt} &= \sqrt{2\varepsilon}\sqrt{1+\frac{\sigma^2}{2\varepsilon}x^2}, \\ \frac{dt}{dx} &= \frac{1}{\sqrt{2\varepsilon}}\frac{1}{\sqrt{1+\frac{\sigma^2}{2\varepsilon}x^2}}, \\ t &= \frac{1}{\sqrt{2\varepsilon}}\int_0^x \frac{\mathrm{d}x'}{\sqrt{1+\frac{\sigma^2}{2\varepsilon}x^2}}. \end{aligned}$$

(c) Solve this integral using any method you know [2 Points].
First, we transform to dimensionless variables: $s = \sigma x/\sqrt{2\varepsilon}$. The integral is thus

$$t = \frac{1}{\sigma}\int_{x_0}^{\sigma x/\sqrt{2\varepsilon}} \frac{\mathrm{d}s}{\sqrt{1+s^2}}.$$

Looking this integral up in the tables,

$$t = \frac{1}{\sigma}\left[\sinh^{-1}\frac{\sigma x}{\sqrt{2\varepsilon}} - \sinh^{-1}\frac{\sigma x_0}{\sqrt{2\varepsilon}}\right].$$

Define a constant of integration $\tilde{A}$,

$$\tilde{A} = \sinh^{-1}\frac{\sigma x_0}{\sqrt{2\varepsilon}}.$$

Hence,

$$\sigma t = \sinh^{-1}\frac{\sigma x}{\sqrt{2\varepsilon}} - \tilde{A} \iff x = \frac{\sqrt{2\varepsilon}}{\sigma}\sinh\left(\sigma t + \tilde{A}\right).$$

Defining a further constant of integration

$$\tilde{B} = \frac{\sqrt{2\varepsilon}}{\sigma},$$

the solution is

$$x = \tilde{B}\sinh\left(\sigma t + \tilde{B}\right).$$

Using $\sinh s = \left(e^s - e^{-s}\right)/2$, this is

$$x = \tfrac{1}{2}\tilde{B}e^{\tilde{A}}e^{\sigma t} - \tfrac{1}{2}\tilde{B}e^{-\tilde{A}}e^{-\sigma t}.$$

Defining further constants of integration $A = \tilde{B}e^{\tilde{A}}/2$ and $B = -\tilde{B}e^{-\tilde{A}}/2$, this is

$$x = Ae^{\sigma t} + Be^{-\sigma t},$$

and A and B can be fixed by the initial conditions on x and $\dot{x}$.

4. A particle with mass m moves in one dimension with the celebrated *Lennard–Jones potential*

$$\mathcal{U}(x) = 4\epsilon\left[\left(\frac{x_0}{x}\right)^{12} - \left(\frac{x_0}{x}\right)^{6}\right].$$

where ϵ and x_0 are positive constants.

(a) Construct a timescale based on the energy ϵ, the lengthscale x_0, and the mass m. Hence, write down the non-dimensional equation of motion [2 points].
Dimensions of energy: $\mathrm{ML}^2/\mathrm{T}^2$. Hence, a timescale T_0 is $\epsilon = mx_0^2/T^2$, or $T_0 = \sqrt{(mx_0^2)/\epsilon}$.
Equation of motion:

$$m\frac{d^2x}{dt^2} = -\frac{d\mathcal{U}}{dx}.$$

Dividing and multiplying by the fundamental scales makes no difference to the equation:

$$\frac{mx_0}{T_0^2}\frac{d^2(x/x_0)}{d(t/T_0)^2} = -\frac{\epsilon}{x_0}\frac{d(\mathcal{U}/\epsilon)}{d(x/x_0)}.$$

Introduce non-dimensional length $s = x/x_0$, non-dimensional time $\tau = t/T_0$, and nondimensional potential $\overline{\mathcal{U}} = \mathcal{U}/\epsilon$. Hence,

$$\frac{d^2s}{d\tau^2} = -\frac{\epsilon}{mx_0^2}T_0^2\frac{d\overline{\mathcal{U}}}{ds}.$$

But, by our choice of T_0,

$$\frac{d^2s}{d\tau^2} = -\frac{d\overline{\mathcal{U}}}{ds}, \qquad \overline{\mathcal{U}} = 4\left(s^{-12} - s^{-6}\right).$$

(b) From this, identify the non-dimensional potential function, $\overline{\mathcal{U}}(s)$. Evaluate any maxima, minima, and zeros of the function. Then plot it [2 points].
We have found

$$\overline{\mathcal{U}} = 4\left(s^{-12} - s^{-6}\right).$$

It has a zero at $s = 1$. It has a minimum at $\overline{\mathcal{U}}'(s) = 0$, i.e. $-(12/s^{13}) + (6/s^7) = 0$, i.e. $s = 2^{1/6}$. It's a minimum because

$$\overline{\mathcal{U}}''(s) = 4\times\left[\left(12.\cdot 13/s^{14}\right) - \left(6\cdot 7/s^8\right)\right] > 0$$

at $s = 2^{1/6}$. We also know the asymptotic behaviour of the function:

$$\begin{aligned}\overline{\mathcal{U}} &\sim +\infty, &&\text{as } s \to 0^+,\\ \overline{\mathcal{U}} &\sim -0, &&\text{as } s \to \infty.\end{aligned}$$

Now we can draw it (Fig. 2).

(c) What is the non-dimensional period of small oscillations around the stable minimum? What is the corresponding dimensional value? [2 points]

From class notes, the period of small oscillations around the well minimum is $\omega = \sqrt{\mathcal{U}''(x_{\mathrm{eq}})/m}$. But $T = 2\pi/\omega$, hence $T = 2\pi\sqrt{m/\mathcal{U}''(x_{\mathrm{eq}})}$.

Now $\overline{\mathcal{U}}''(2^{1/6}) = 4 \times 18/2^{1/3}$

In our case, the non-dimensional period is

$$\frac{T}{T_0} = 2\pi\sqrt{\frac{1}{(72/2^{1/3})}} = 2\pi\frac{2^{1/6}}{\sqrt{72}}.$$

hence

$$T = 2\pi\frac{2^{1/6}}{\sqrt{72}}T_0 = 2\pi\frac{2^{1/6}}{\sqrt{72}}\sqrt{mx_0^2/\epsilon}.$$

(d) If the particle starts from rest at non-dimensional distance $s = x/x_0 = 1$, what is its ultimate fate? [2 points]

The energy is conserved and equal to $E = \overline{\mathcal{U}}(s = 1) = 0$. Hence, the turning points are at $s = 1$ and $s = \infty$. The motion is therefore unbound: a particle starting from rest at $s = 1$ tends to infinity.

Recall the equations of motion for trajectory motion in a uniform gravitational field g:

$$x = x_0 + ut, \quad (1)$$
$$y = y_0 + vt - \tfrac{1}{2}gt^2. \quad (2)$$

where (x_0, y_0) is the initial location of the particle relative to a given inertial frame and (u, v) is the initial velocity. Neglect air resistance.

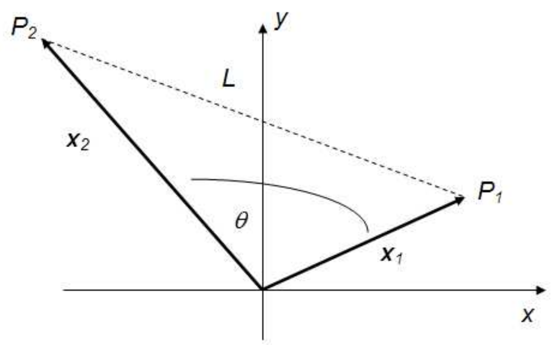

Figure 1: Dot-product calculation

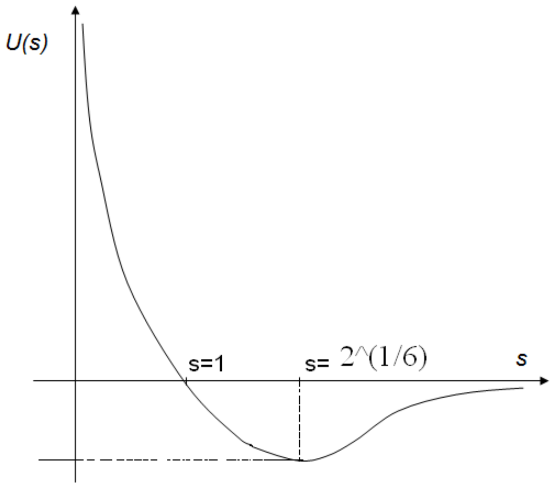

Figure 2: Lennard-Jones potential