Applied Statistical Modelling (STAT 40510)
Main Project
Task 1: Theoretical Characterization of the Models

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TBC

Format of the Project

The main project in Epidemiological Modelling in STAT 40510 will be made up of several tasks.

- Follow the online lectures independently, attend weekly office hours in Weeks 5-7.
- Over the same time period, complete (in a group) **Tasks 1 and 2** to test your knowledge of what you have learned.
- Again over the same time period, you will be assigned your most challenging task, **Task 3**. You should begin to do background reading to understand what is required here.
- In Week 8, you should present your work to date, the presentation should consist of:
  - The theoretical concepts you have learned in Tasks 1–2;
  - How you will apply these in Task 3.
- The final report (due towards the end of the trimester) will be based entirely on Task 3.

We start with a very general SEIR model which allows for natural births and deaths in a closed population:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mu N - \mu S - \frac{\beta I S}{N} \tag{1a}$$
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\beta I S}{N} - (\mu + a)E \tag{1b}$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = aE - (\gamma + \mu)I \tag{1c}$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I - \mu R. \tag{1d}$$

1. Show that $S + E + I + R = N$, but note that this is only constant because of the simplifying assumption that birth and death rates are equal; in general N is a variable.

2. Show that Equation (1) has two constant solutions, a disease-free equilibrium

   $$DFE = (N, 0, 0, 0),$$

   and an **endemic equilibrium**

   $$EE = (S_0, E_0, I_0, R_0),$$

   where all of the coefficients here are non-zero.

3. Compute the coefficients of the endemic equilibrium in terms of a, β, γ, μ.

4. Show that, for given initial conditions $S(0) > 0$, $E(0) = 0$, $I(0) > 0$, and $R(0) = 0$, the solution of Equation (1) remains inside the hypercube $[0, N]^4$ for all time.

   Hint: Assume for contradiction that $I(t_*) = 0$. By continuity, there is an interval of time $[0, t_*)$ where $I(t) > 0$. On this interval, show:

   - Use the integrating-factor method of ordinary-differential equations to write the solution of Equation (1)(a) as:

     $$S(t) = S(0)\mathrm{e}^{-\int_0^t [\mu + (\beta/N)I]\mathrm{d}t} + \cdots,$$

     where the $\cdots$ are to be filled in, hence conclude that $S(t) > 0$ for $t \in [0, t_*)$.
   - Use a similar approach to show that $E(t) > 0$ for $t \in [0, t_*)$.
   - Use

     $$\frac{\mathrm{d}I}{\mathrm{d}t} = aE - (\gamma + \mu) > -(\gamma + \mu)I.$$

     From this, we can use **Gronwall's inequality** to conclude that

     $$I(t) > I(0)\mathrm{e}^{-(\gamma+\mu)t}, \qquad t \in [0, t_*).$$

     In particular, $I(t) > I(0)\mathrm{e}^{-(\gamma+\mu)t} > 0$ as $t \to t_*$, which is a contradiction, since $I(t_*) = 0$.

- From this, conclude that $I(t) \geq 0$ for all $t \geq 0$ and hence,

$$S(t),\, I(t)\, E(t)\, R(t) \geq 0, \text{ for all } t \geq 0.$$

- But $S + E + I + R = N$, so ...

5. Equation (1) can be re-written as:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix} &= \begin{pmatrix} 0 \\ \beta IS/N \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\beta IS/N \\ -aE \\ aE - \gamma I \\ \gamma I \end{pmatrix}, \\
&= \begin{pmatrix} 0 \\ \beta IS/N \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \beta IS/N \\ aE \\ -aE + \gamma I \\ -\gamma I \end{pmatrix}, \\
&= \boldsymbol{F} - \boldsymbol{V},
\end{aligned}$$

where $\boldsymbol{F}$ represents the rate of generation of new infections and $\boldsymbol{V}$ represents the transfer of individuals from one compartment to the next. We **linearize** this equation around the constant disease-free state to obtain:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix} = &\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \beta S(0)/N & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix} \\
&- \begin{pmatrix} 0 & -\beta S(0)/N & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & -a & \gamma & 0 \\ 0 & 0 & -\gamma & 0 \end{pmatrix}\begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix}
\end{aligned}$$

Zoom in in the E and I-compartments, these are the **infected compartments**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} E \\ I \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \beta S(0)/N \\ 0 & 0 \end{pmatrix}}_{=F}\begin{pmatrix} E \\ I \end{pmatrix} - \underbrace{\begin{pmatrix} a & 0 \\ -a & \gamma \end{pmatrix}\begin{pmatrix} E \\ I \end{pmatrix}}_{=V}$$

FV^{-1} is the **next-generation matrix** for the SEIR model. In other, more complicated models, the next-generation matrix will be bigger than 2×2, and the steps in calculating it will be more involved. However, the principle will always be the same: zooming in on the infected compartments and breaking up the resulting rate equations into $\boldsymbol{F}$ and $\boldsymbol{V}$. Furthermore, $\mathcal{R}_0$ will always be defined as:

$$\mathcal{R}_0 = \max \operatorname{spec}(FV^{-1}),$$

that is, $\mathcal{R}_0$ is the **maximum eigenvalue of the next-generation matrix**.

For the SEIR model, show that:

$$\mathcal{R}_0 = \beta/\gamma.$$