In this lecture, we look at the mathematical properties of the basic SIR model:

$$\boxed{\begin{aligned} \frac{dS}{dt} &= -\frac{\beta}{N} SI \\ \frac{dI}{dt} &= \frac{\beta}{N} SI - \gamma I \\ \frac{dR}{dt} &= \gamma I \qquad t>0 \end{aligned}} \qquad (1)$$

with I.Cs $S(t=0) = S_0$, $I(t=0) = I_0$, and $S_0 + I_0 = N$. This is taken from §3.2 and §3.3 of the typed notes.

Reduction to a single ODE (§3.2). A very unusual feature of the coupled three-equation model in (1) is that it can be reduced down to a single ODE. Here's how.

Step 1 : Look at the S- and R-equations:

$$\frac{dS}{dt} = -\frac{\beta}{N} SI, \qquad \frac{dR}{dt} = \gamma I$$

Combine:

$$\frac{dS}{dt} = -\frac{\beta}{N} S \left(\frac{1}{\gamma} \frac{dR}{dt} \right)$$

The dt's formally cancel:

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$$dS = -\frac{\beta}{N} \cdot \frac{1}{\gamma} \cdot S \cdot dR$$

Divide out by S:

$$\frac{dS}{S} = -\left(\frac{\beta}{\gamma} \cdot \frac{1}{N}\right) dR$$

Integrate from $t=0$ ($S(0)=S_0$, $R(0)=R_0$) to a finite time ($S(t)=S$, $R(t)=R$):

$$\int_{S_0}^{S} \frac{dS}{S} = -\left(\frac{\beta}{\gamma} \cdot \frac{1}{N}\right) \int_0^R dR$$

Hence:

$$\log S - \log S_0 = -\left(\frac{\beta}{\gamma} \cdot \frac{1}{N}\right) R$$

$$\Rightarrow \quad S = S_0 \, e^{-\underbrace{\left(\frac{\beta}{\gamma} \cdot \frac{1}{N}\right) R}_{u}}$$

$$\boxed{u = \left(\frac{\beta}{\gamma} \cdot \frac{1}{N}\right) R, \qquad S = S_0 e^{-u}.} \qquad (2)$$

<u>Step 2</u>: Use $S + I + R = N = \text{const.}$

From (2):

$$I = N - S - R$$
$$= N - S_0 e^{-u} - R.$$

Sub into dR/dt :

$$\frac{dR}{dt} = \gamma\left[N - S_0 e^{-u} - R\right].$$

We rescale this equation:

$$\frac{\beta}{\gamma}\cdot\frac{1}{N}\frac{dR}{dt} = \cancel{\gamma}\cdot\cancel{N}\frac{\beta}{\cancel{\gamma}}\cdot\frac{1}{\cancel{N}} - \cancel{\gamma} S_0 \frac{\beta}{\cancel{\gamma}}\cdot\frac{1}{N} e^{-u}$$

$$- \gamma\left[\frac{\beta}{\gamma}\cdot\frac{1}{N} R\right]$$

Group terms:

$$\frac{du}{dt} = \beta - \frac{S_0}{N}\cdot\beta e^{-u} - \gamma u$$

Divide out by $\frac{S_0}{N}\cdot\beta$:

$$\frac{1}{\left(S_0\cdot\beta/N\right)}\frac{du}{dt} = \frac{\cancel{\beta} N}{\cancel{\beta} S_0} - e^{-u} - \frac{\gamma N}{\beta S_0} u$$

Identify :

- Rescaled time : $d\tau = (S_0/N)\cdot\beta\cdot dt$
- Parameter $a = N/S_0 > 1$.
- Parameter $\boxed{R_0 = \frac{\beta}{\gamma}\frac{S_0}{N}}$ (3)

So the reduced equation becomes:

$$\boxed{\frac{du}{d\tau} = a - \frac{1}{R_0}u - e^{-u}} \qquad (4)$$

Why?? This exercise enables us to learn some key concepts in mathematical epidemiology. For this, we look at § 3.3 of the typed notes.

We analyze Eqn (4); it's better to write it as:

$$\frac{du}{d\tau} = f(u), \qquad f(u) = a - \frac{1}{R_0}u - e^{-u}. \qquad (5)$$

We look at the equilibria of (5), that is, constant solutions, $u = \text{const.} = u_*$. We must have

$$\frac{du_*}{dt} = 0 \quad \ldots \quad u_* = \text{const.}$$

$$\parallel$$

$$f(u_*) = 0.$$

So the equilibrium solution(s) satisfy:

$$f(u_*) = 0. \qquad \text{(ROOTS)}$$

We find the roots of $f(u) = 0$ by a graphical method.

* Behaviour of $f(u)$ near $u = 0$:

* Behaviour of $f(u)$ near $u=0$:

$$f(u) \simeq f(0) + f'(0)u$$
$$= \underbrace{(a-1)}_{a>1} + \left(1 - \frac{1}{R_0}\right)u$$

$$\boxed{f(u) = a - \frac{1}{R_0}u - e^{-u}}$$

We see that $f(0) > 0$.

* Behaviour of $f(u)$ as $u \to \infty$:

$$f(u) \sim -\frac{1}{R_0}u \quad \text{as } u \to \infty,$$

so $f(u) \to -\infty$ as $u \to \infty$.

* Maxima and minima: We look at solutions of $f'(u) = 0$, i.e.:

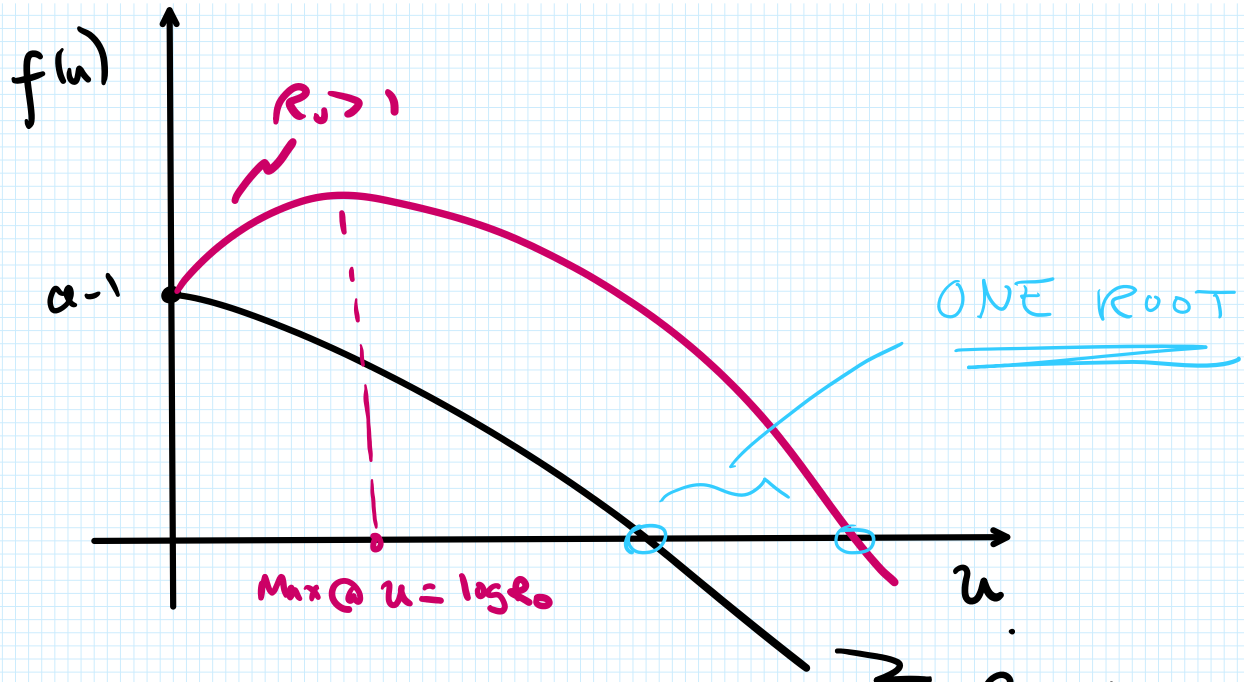
$$\frac{1}{R_0} = e^{-u} \implies u = +\log R_0$$

But u is a (scaled) population, so we require u to be positive, so for a max/min to exist we require $R_0 > 1$. In this case, we look at

$$f''(u) = -e^{-u} < 0.$$

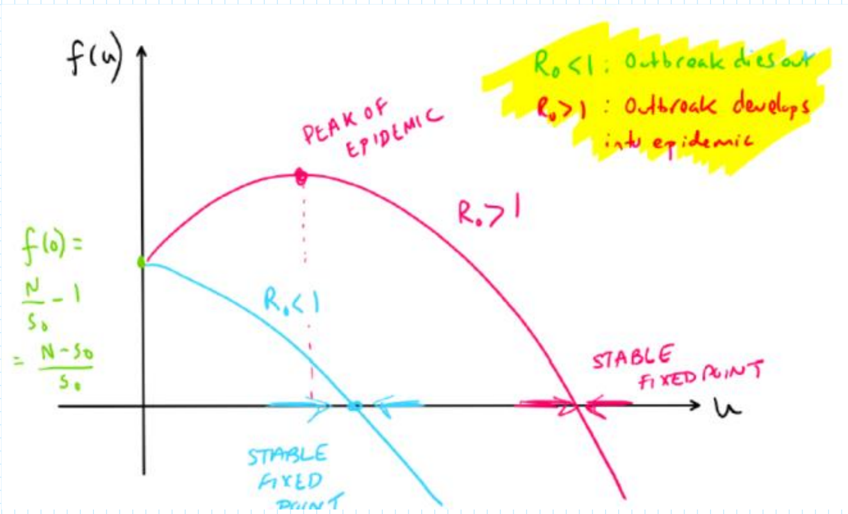
So $u = \log R_0$ is a MAX.

We now sketch $f(u)$:



In both cases, we see there is only one root (FIXED POINT / EQ$^{\underline{M}}$ SOLUTION).

The same idea again is shown in the figure below.



In the case with $R_0 > 1$, $f(u)$ has a peak. At peak, $f'(u) = 0$. But

$$\frac{dR}{dt} \propto f.$$

$$\frac{d^2R}{dt^2} \propto f'(u)\frac{du}{dt} \propto f'(u)f(u).$$

$\therefore \; d^2R/dt^2 = 0$ @ peak.

So, for $R_0 > 1$:

- Starting with an initial condition $u_0 = 0$ (no recovered people at $t = 0$), the number of recovered people accelerates, with $\mathrm{d}^2u/\mathrm{d}t^2 = f'(u)f(u) > 0$.
- At peak, the acceleration stops, $f'(u) = 0$.
- After peak, the number of recovered people decelerates, with $\mathrm{d}^2u/\mathrm{d}t^2 < 0$.

On the other hand, for $R_0 < 1$:

- $\mathrm{d}^2u/\mathrm{d}t^2 = f'(u)f(u) < 0$ always.
- The number of recovered people decelerates from the start.

Key observation:

The parameter-value $R_0 = 1$ is therefore called the **threshold** value: If $R_0 > 1$ the ~~epidemic~~ outbreak accelerates before slowing down, while if $R_0 < 1$, the ~~epidemic~~ outbreak decelerates from the start.

Case #1 is a true epidemic.

3.3.3 One more way of looking at things

Consider again :

$$R_0 = \frac{\beta}{\gamma} \frac{S_0}{N},$$

require $R_0 > 1$ for an epidemic. So, take either:

- β/γ large, meaning infectious person stays infectious for long enough to infect many others;
- S_0/N large, meaning there is a large pool of susceptible people present at $t = 0$.