The SIR Model — CHAPTER 3

In this lecture, we introduce the simplest possible model in mathematical epidemiology — SIR.

Plan of lecture:

- Assumptions
- Parameters

We start with the assumptions (§ 3.1)

- Homogeneous population
- Fast-moving epidemic — no natural births/deaths
- Closed population

The population is divided into three compartments:

- Susceptible S
- Infected I
- Recovered R

Because of the assumptions, the sum

$$S + I + R$$

is constant, denoted by N.

The model is based on a conservation principle. Because there are no natural births/deaths/migration, the only way the population of any one compartment can change is if people move in or out of the can change is if people move in or out of the compartment.

We now look at the individual compartments in turn.

Susceptible : A person moves out of the susceptible compartment and into the infectious compartment if they encounter an infectious individual and are infected. As such, the probability Δp that a susceptible person becomes infected in a time interval from t to $t + \Delta t$ is proportional to:

$$\Delta p \propto S \cdot I \cdot \Delta t \, .$$

When a susceptible person becomes infected, they leave the susceptible compartment. The the rate at which individuals leave the susceptible compartment must be proportional to $\Delta p / \Delta t$, hence:

$$\frac{dS}{dt} = - \alpha S \cdot I$$

where

- α is a positive constant
- Minus sign because # susceptibles is decreasing.

Infectious : By conservation:

$$\frac{dI}{dt} = \begin{pmatrix} \text{Rate at which individuals are} \\ \text{entering I-compartment} \end{pmatrix} - \begin{pmatrix} \text{Rate at which individuals are} \\ \text{leaving I-compartment} \end{pmatrix} (R.R.)$$

$$= + \alpha S \cdot I - R.R.$$

The recovery rate R.R. is the rate at which people leave the infectious compartment and enter the recovered compartment. We must have:

$$R.R. = \gamma I$$

where γ is another constant. Hence,

$$\frac{dI}{dt} = \alpha S \cdot I - \gamma I.$$

Recovery: We assume that recovered individuals cannot become re-infected (no endemic states). So the only way in which the # people in the recovered compartment can change is by people entering:

$$\frac{dR}{dt} = + \gamma I.$$

Putting it all together, we have:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\alpha S I,$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \alpha S I - \gamma I,$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I.$$

Check:

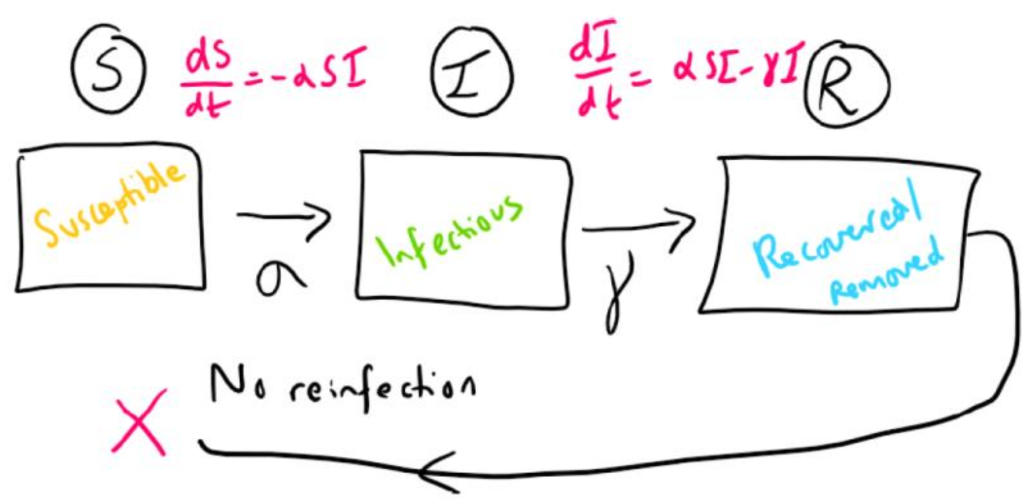
$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = -\alpha S I + (\alpha S I - \gamma I) + \gamma I = 0.$$

Hence,

$$\frac{d}{dt}(\underbrace{S + I + R}_{N}) = 0$$

So N = const., consistent with the model assumptions.

A graphical representation of the equations is shown in the figure just below ↴.



Dimensional Analysis and the parameter β.

From the equations, we recognize that γ has dimensions of $1/\text{Time}$, while α has dimensions of :

$$\frac{1}{[\#\text{ People} \times \text{Time}]}$$

So if we introduce :

$$\alpha = \beta / N$$

where β has "dimensions of a pure rate" ($1/\text{Time}$), then we can rewrite the SIR model as :

$$\boxed{\begin{aligned} \frac{dS}{dt} &= -\frac{\beta}{N} S I \\ \frac{dI}{dt} &= \frac{\beta}{N} S I - \gamma I \\ \frac{dR}{dt} &= \gamma I \end{aligned}} \qquad (1)$$

Interpretation of β : β is a pure rate, it can be further decomposed as :

$\beta = c \times p =$ (Number of contacts of an individual per unit time)

$\times$ (Probability that a contact leads to infection).

In some crude fashion, "non-pharmaceutical interventions" to moderate the spread of an infectious disease involve :

- Reducing contacts ($c \searrow 0$),
- Making contacts safer ($p \searrow 0$).

Initial Conditions : Eq. (1) is valid for $t > 0$. At $t = 0$, we must impose initial conditions. The obvious ones are:

$$S(t=0) = S_0, \quad I(t=0) = I_0, \quad R(t=0) = 0,$$

such that $S_0 + I_0 = N$. If $I_0 = 1$, then I_0 is "patient zero".