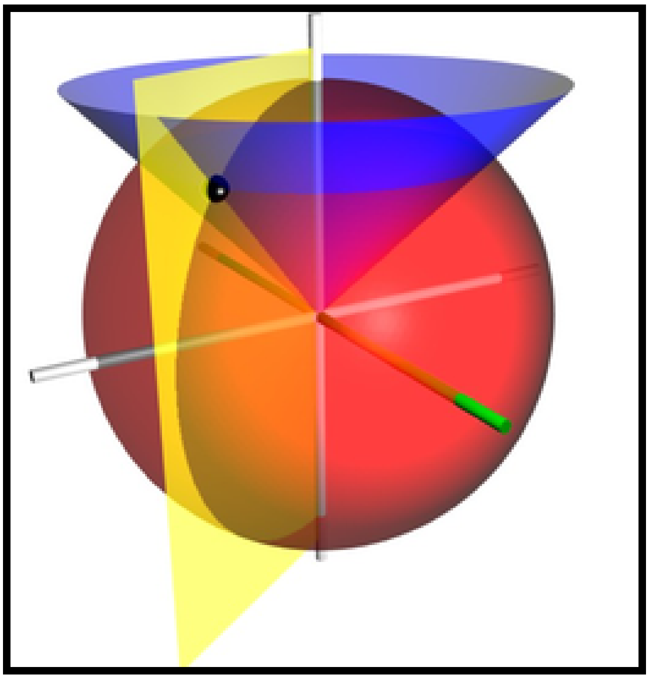
University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

School of Mathematical Sciences
Scoil na nEolaíochtaí Matamaitice
Vector Integral and Differential Calculus (ACM 20150)



Dr Lennon Ó Náraigh

Lecture notes in Vector Calculus, September 2013

Vector Integral and Differential Calculus (ACM20150)

- Subject: Applied and Computational Mathematics
- School: Mathematical Sciences
- Module coordinator: Dr Lennon Ó Náraigh
- Credits: 5
- Level: 2
- Semester: First

This module introduces the fundamental concepts and methods in the differentiation and integration of vector-valued functions and also provides an introduction to the Calculus of Variations.

Fundamentals Vectors and scalars, the dot and cross products, the geometry of lines and planes, **Curves in three-dimensional space** Differentiation of curves, the tangent vector, the Frenet-Serret formulas, key examples of Frenet-Serret systems to include two-dimensional curves, and the helix, **Partial derivatives and vector fields** Introduction to partial derivatives, scalar and (Cartesian) vector fields, the operators div, grad, and curl in the Cartesian framework, applications of vector differentiation in electromagnetism and fluid mechanics, **Mutli-variate integration** Area and volume as integrals, integrals of vector and scalar fields, Stokes's and Gauss's theorems (statement and proof), **Consequences of Stokes's and Gauss's theorems** Green's theorems, the connection between vector fields that are derivable from a potential and irrotational vector fields, **Curvilinear coordinate systems** Basic concepts, the metric tensor, scale factors, div, grad, and curl in a general orthogonal curvilinear system, special curvilinear systems including spherical and cylindrical polar coordinates, **The Calculus of variations** Derivation of the Euler-Lagrange equation, applications in geometry, optics and mechanics

Further topics may include: Introduction to differential forms, exact and inexact differential forms, **Advanced integration** Integrating the Gaussian function using polar coordinates, the gamma function, the volume of a four-ball by appropriate coordinate parameterization, the volume of a ball in an arbitrary (finite) number of dimensions using the gamma function, **Fluid mechanical application** Incompressible flow over a wavy boundary, **Calculus of variations** Constrained variations.

What will I learn?

On completion of this module students should be able to

1. Write down parametric equations for lines and planes, and perform standard calculations based on these equations (e.g. points/lines of intersection, condition for lines to be skew);
2. Compute the Frenet-Serret vectors for an arbitrary differentiable curve;
3. Differentiate scalar and vector fields expressed in a Cartesian framework;
4. Perform operations involving div, grad, and curl;
5. Perform line, surface, and volume integrals. The geometric objects involved in the integrals may be lines, arbitrary curves, simple surfaces, and simple volumes, e.g. cubes, spheres, cylinders, and pyramids;
6. State precisely and prove Gauss's and Stokes's theorems;
7. Derive corollaries of these theorems, including Green's theorems and the necessary and sufficient condition for a vector field to be derivable from a potential;
8. Compute the scale factors for arbitrary orthogonal curvilinear coordinate systems;
9. Apply the formulas for div, grad, and curl in arbitrary orthogonal curvilinear coordinate systems;
10. Derive the Euler-Lagrange equations;
11. Apply the Euler-Lagrange equations in simple mechanical and optical problems.

Editions

First edition: September 2010

Second edition: September 2011

Third edition: September 2012

This edition: September 2013

Contents

Chapter 1

Introduction

1.1 Overview

Here is the executive summary of the module:

> This module involves the study of vector and scalar fields in two- and three-dimensional space. A *field* is an object that assigns a vector or a scalar to each point in space. We need to find out how to integrate and differentiate these things, hence vector integral and differential calculus.

In more detail, a field is a map that assigns either a scalar or a vector to each point in the map domain, giving scalar- and vector-fields, respectively. We study this concept in depth:

1. We formulate the derivative of a scalar field, based on the *gradient* operator;
2. We learn how to differentiate vector fields, using the *divergence* and *curl* operators;
3. We define line and area integrals – generalization of integration on $\mathbb{R}$.
4. We state and prove two fundamental theorems of vector integration – Gauss' and Stokes' theorems.[1] These can be crudely thought of as generalizations of integration by parts.
5. These topics are formulated against the backdrop of Cartesian space (that is, a triple $(x, y, z) \in \mathbb{R}^3$ labels points in space). However, the integration theorems enable us to generalize div, grad, and curl to differentiation on curved surfaces ('manifolds').
6. Lastly, we shall switch focus and derive the Euler–Lagrange equations, a technique for solving extremization problems involving *functionals* – maps from spaces of functions to the real line.

[1] Sir George Gabriel Stokes F.R.S. Born in Skreen Co. Sligo, 1829, died in Cambridge, England, 1903.

1.2 Learning and Assessment

Learning:

- Thirty six classes, three per week.
- In some classes, we will solve problems together or look at supplementary topics.
- To develop an ability to *solve problems autonomously*, you will be given homework exercises, and it is recommended that you do *independent study*. Supplementary problems are available in the Schaum's textbook (see below).

Assessment:

- Three homework assignments, for a total of 20%;
- **One** in-class tests, for a total of 20%;
- One end-of-semester exam, 60%

Policy on late submission of homework:

The official university policy concerning late submission of homework **in the absence of extenuating circumstances** is followed strictly in this module: homework that is late by up to one week will have the grade awarded reduced by two grade points; homework that is late by more than one week is dealt with similarly (UCD Science undergraduate student handbook, p. 10).

Office hours

I do not keep specific office hours. If you have a question, you can visit me whenever you like – from 09:00-18:00 I am usually in my office if not lecturing. It is a bit hard to get to. The office number, building name, and location are indicated on a map at the back of this introductory chapter.

Otherwise, email me:

`onaraigh@maths.ucd.ie`

Textbooks

- Lecture notes will be put on the web. These are self-contained. They will be available *before* class. It is anticipated that you will print them and bring them with you to class. You can then annotate them and follow the proofs and calculations done on the board. Thus, you are still expected to attend class, and I will occasionally deviate from the content of the notes, give hints about solving the homework problems, or give a revision tips for the final exam.

- There are some books for extra reading, if desired:
  - *Vector analysis and an introduction to tensor analysis*, M. R. Spiegel, Schaum's Outline Series, McGraw–Hill (Five copies in library, 515).
  - *Mathematical methods for physicists*, G. B. Arfken, H. J. Weber, and F. Harris, Wiley, Fifth Edition (One copy of third edition in library, 510).
  - *Vectors, tensors and the basic equations of fluid mechanics*, R. Aris, Dover (One copy in library, 532; also available for $£8.00$ on Amazon.co.uk).

1.3 A modern perspective on vector calculus

Before beginning the lecture course, let us discuss a contemporary problem that uses the techniques of vector calculus.

The advection-diffusion equation: The concentration C of a chemical in the atmosphere, a pollutant on the sea-surface, or of a blob of dye in a container of fluid is a function of space and time:

$$C = C\left(\boldsymbol{x}, t\right), \qquad \boldsymbol{x} = \begin{cases} (x, y) \in \Omega \subset \mathbb{R}^2, & \text{or} \\ (x, y, z) \in \Omega \subset \mathbb{R}^3. \end{cases}$$

This concentration is stirred around by the flow field

$$\boldsymbol{u} = \left(u\left(\boldsymbol{x}, t\right), v\left(\boldsymbol{x}, t\right)\right)$$

in two dimensions, or

$$\boldsymbol{u} = \left(u\left(\boldsymbol{x}, t\right), v\left(\boldsymbol{x}, t\right), w\left(\boldsymbol{x}, t\right)\right)$$

in three dimensions. The flow is assumed to be incompressible: this means that density is conserved along streamlines; mathematically,

$$\nabla \cdot \boldsymbol{u} = 0.$$

At the same time, the concentration is 'diffused', so that regions where the concentration possesses high gradients are smoothed out, on a timescale

$$T = \left[\text{Length scale of variation}\right]^2 / D$$

where D is the *diffusion coefficient*. The law that expresses these two processes is called the *advection-diffusion equation*

$$\underbrace{\frac{\partial C}{\partial t}}_{\text{Instantaneous changes in concentration}} + \underbrace{\boldsymbol{u}\cdot\nabla C}_{\text{Stirring by the flow}} = \underbrace{D\nabla^2 C}_{\text{Diffusion}}. \tag{1.1}$$

The integral theorems discussed previously can be used to show that

$$\frac{d}{dt}\int_\Omega C\left(\boldsymbol{x},t\right)\mathrm{d}^n x = 0 + \text{Boundary terms},$$

hence, the total amount of chemical is *conserved*. If we multiply Eq. (1.1) by $C\left(\boldsymbol{x},t\right)$ and integrate over the flow domain Ω, we obtain, using the same integral theorems as before,

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}C^2\left(\boldsymbol{x},t\right)\mathrm{d}^n x = -D\int_\Omega |\nabla C\left(\boldsymbol{x},t\right)|^2 \mathrm{d}^n x + \text{Boundary terms}.$$

If the flow and the concentration gradients satisfy certain conditions on the boundary, the last term in this equation vanishes, and we are left with

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}C^2\left(\boldsymbol{x},t\right)\mathrm{d}^n x = -D\int_\Omega |\nabla C\left(\boldsymbol{x},t\right)|^2 \mathrm{d}^n x,$$

and the *variance* in the concentration, away from its mean value, decays to zero. Thus, the chemical becomes better and better mixed, over time.

The question of *how fast the mixing is* depends on the character of the flow. You will no doubt be aware of a certain experiment involving coffee and milk: if you add a drop of milk to a cup of black coffee and do not stir, the two components will eventually mix, but over a long interval. If you add the milk and then stir, the homogenization is faster. Mixing times therefore depend on the flow. It turns out that if the flow $\boldsymbol{u}$ is *chaotic* (in a sense described below), then the mixing is as close to optimal as can be imagined. A flow $\boldsymbol{u}$ is chaotic if two initially neighbouring fluid particles separate away from each other exponentially fast in time, under the influence of the flow. The average rate of separation is called the *Lyapunov exponent*, Λ_0.

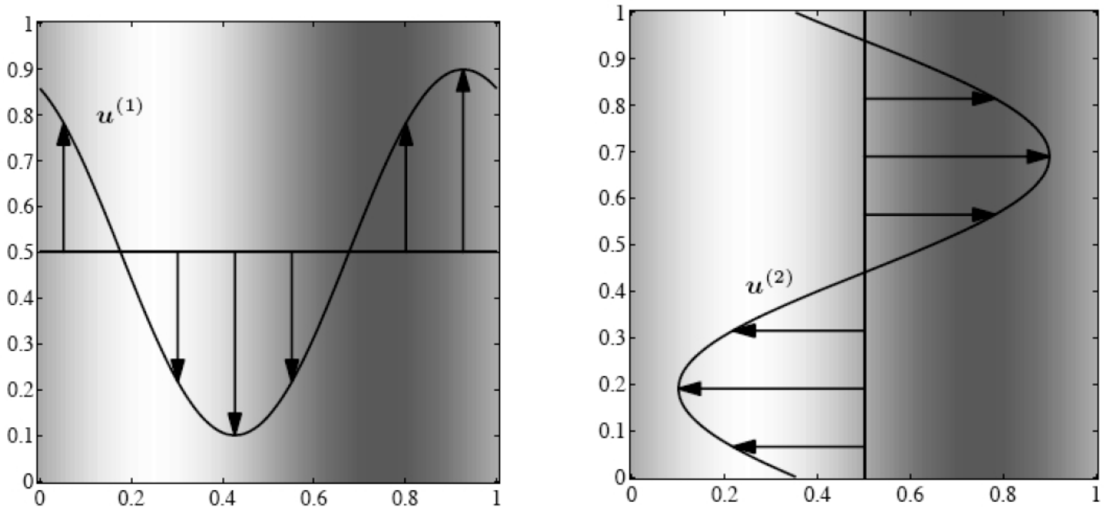
One popular model of mixing in two dimensions is the *random-phase sine flow*, which is a succession of undirectional quasi-periodic 'whisking' motions:

$$u = A_0 \sin\left(ky + \phi_j\right), \qquad v = 0, \tag{1.2}$$

in the first half-period of the flow, and

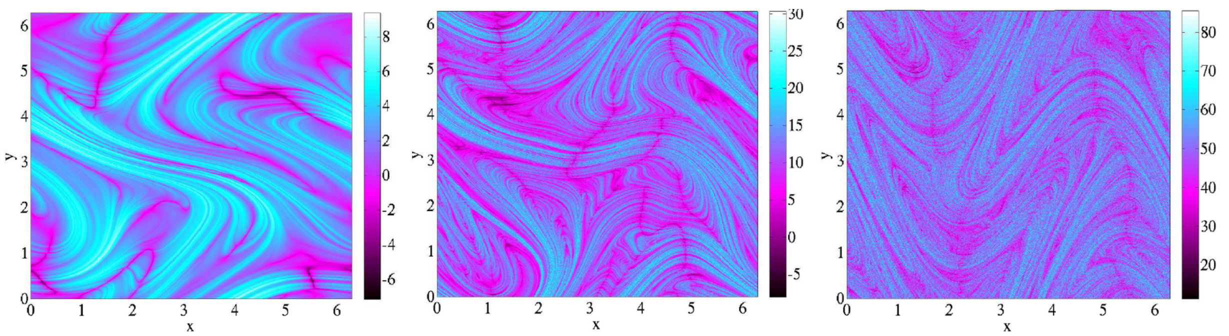
$$u = 0, \qquad v = A_0 \sin\left(kx + \psi_j\right), \tag{1.3}$$

in the second (See Fig. 1.1). Here ϕ_j and ψ_j are random phases that change after each whisking



(a) First half-period (b) Second half-period

Figure 1.1: Schematic description of the random-phase sine flow in each quasi-period.



(a) $A_0 = 0.5$ (b) $A_0 = 1$ (c) $A_0 = 2$

Figure 1.2: The Lyapunov exponent $\Lambda_0(\boldsymbol{x})$ for different trajectories (the consant Λ_0 is the average over all trajectories, and is positive). The larger the value of A_0, the larger the values taken by $\Lambda_0(\boldsymbol{x})$.

motion and A_0 and k are positive constants ('amplitude' and 'wavenumber' respectively). Particles drawn along by this flow satisfy the trajectory equation

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{x}, t),$$

and can be tracked numerically. The time-averaged rate of separation along trajectories gives rise to the Lyapunov exponent $\Lambda_0(\boldsymbol{x})$ (Fig. 1.2), which varies in space but not in time (the spatial variations label the trajectories). The decay rate of the concentration can also be measured (it is exponential). The energy of the flow is the space-time average

$$E = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t \int_\Omega \mathrm{d}^2 x |\boldsymbol{u}(\boldsymbol{x}, t)|^2.$$

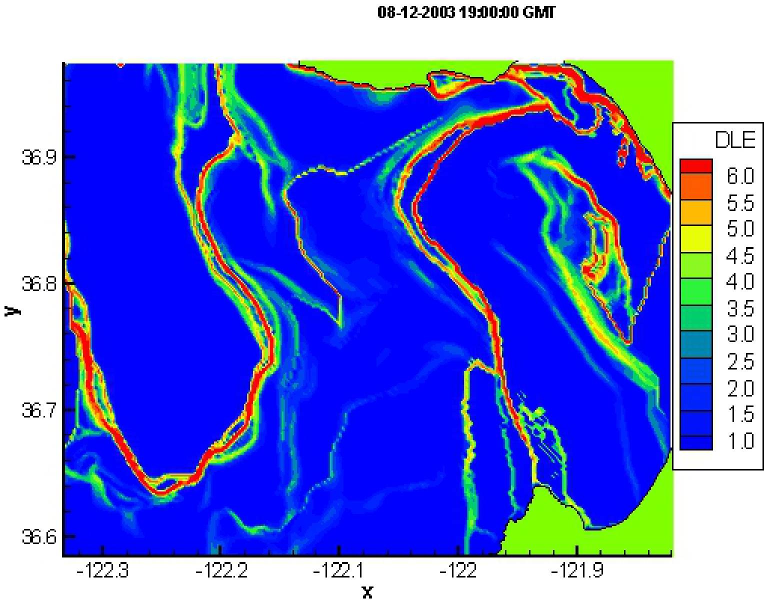
By the end of this course, you should be able to see that

$$E = \tfrac{1}{2}A_0^2,$$

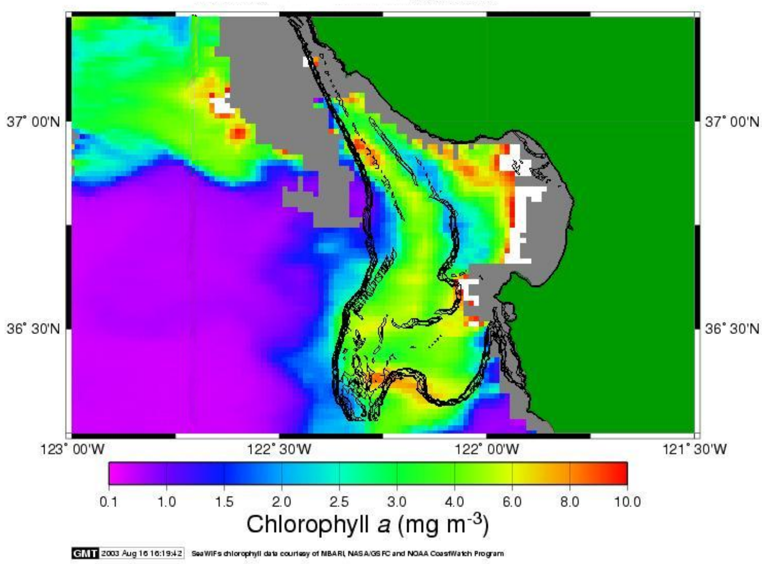
independent of wavenumber. Referring to Fig. 1.2, the more energy you put into the flow, the better mixed it becomes. This in part answers the question about stirring the cup of coffee: stirring, that is inputting mechanical energy (in the correct, chaotic, fashion), increases the Lyapunov exponent, and hence promotes mixing.

Note, from Fig. 1.2 that the Lyapunov exponent $\Lambda_0(\boldsymbol{x})$ can be calculated numerically for a given flow, and in fact is an *averaged separation rate*, averaged over an infinitely long time interval. There is a finite-time analogue, the finite-time Lyapunov exponent, when the temporal averaging is over a finite interval τ, and denoted by $\Lambda_0(\boldsymbol{x};\tau)$. Ridges in this quantity are called *Lagrangian coherent structures* [2]. A *ridge* is a local maximum *in only one direction*. Just as a ridge in a mountain range is a barrier to transport, so too is a ridge in the FTLE: particles cannot flow through them. Ridges can be found in the ocean and act as barriers to pollution dispersal, or to the uniform distribution of micro-organisms (as in Fig. 1.3). Before people discovered Lagrangian coherent structures, they thought tides would wash away pollution. However, these structures persist through tides, and they represent a permanent barrier.

[2] See 'Finding Order in the Apparent Chaos of Currents', New York Times, 28 September 2009.

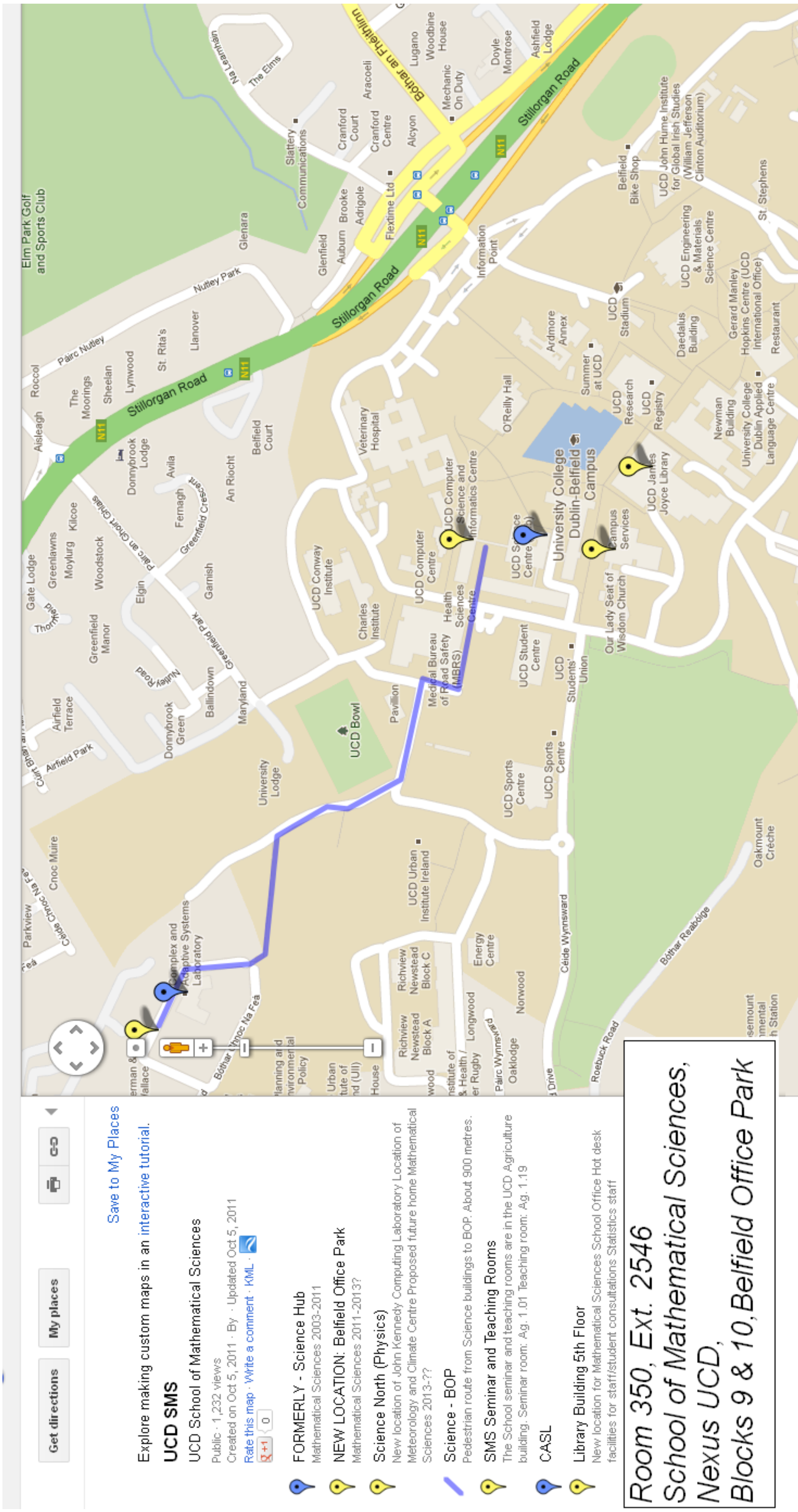


(a)



(b)

Figure 1.3: (a) A *snapshot* of the FTLE in Monterey Bay, CA, *at a particular point in time* (they can evolve in time); (b) A snapshot of the distribution of sea-surface chlorphyll: this very clearly is contained within the transport barriers represented by the ridges in the FTLE. (Taken from the webpage http://www.cds.caltech.edu/˜shawn)/LCS-tutorial/).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Science - BOP
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SMS Seminar and Teaching Rooms
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Nexus UCD,
Blocks 9 & 10,Belfield Office Park

Chapter 2

Vectors – revision

Overview

We review some basics of vector algebra that have already been covered in MATH 10270 or MATH 10280.

2.1 The connection between vectors and Cartesian coordinates

A vector is a quantity with magnitude and direction. A point P in space can be labelled by coordinates (a_1, a_2, a_3) with respect to some Cartesian coordinate frame with origin O. The distance from O to P is thus

$$|OP| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Associated with O and P is a direction – from O to P. Thus, we identify $\overrightarrow{OP}$ as a vector with direction from O to P, with magnitude $\sqrt{a_1^2 + a_2^2 + a_3^2}$. We can also identify the vector by its coordinates, writing

$$\overrightarrow{OP} \equiv (a_1, a_2, a_3) \equiv \boldsymbol{a}.$$

Two vectors, $\overrightarrow{OP} = \boldsymbol{a} = (a_1, a_2, a_3)$ and $\overrightarrow{OQ} = \boldsymbol{b} = (b_1, b_2, b_3)$ can be added together in an obvious way:

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

This is consistent with the parallelogram law of vector addition – see Fig. 2.1. We also have the notion of scalar multiplication: if $\lambda \in \mathbb{R}$, and if $\overrightarrow{OP} = \boldsymbol{a} = (a_1, a_2, a_3)$, then

$$\lambda\left(\overrightarrow{OP}\right) = \lambda\boldsymbol{a} = \lambda(a_1, a_2, a_3) \stackrel{\text{def}}{=} (\lambda a_1, \lambda a_2, \lambda a_3).$$

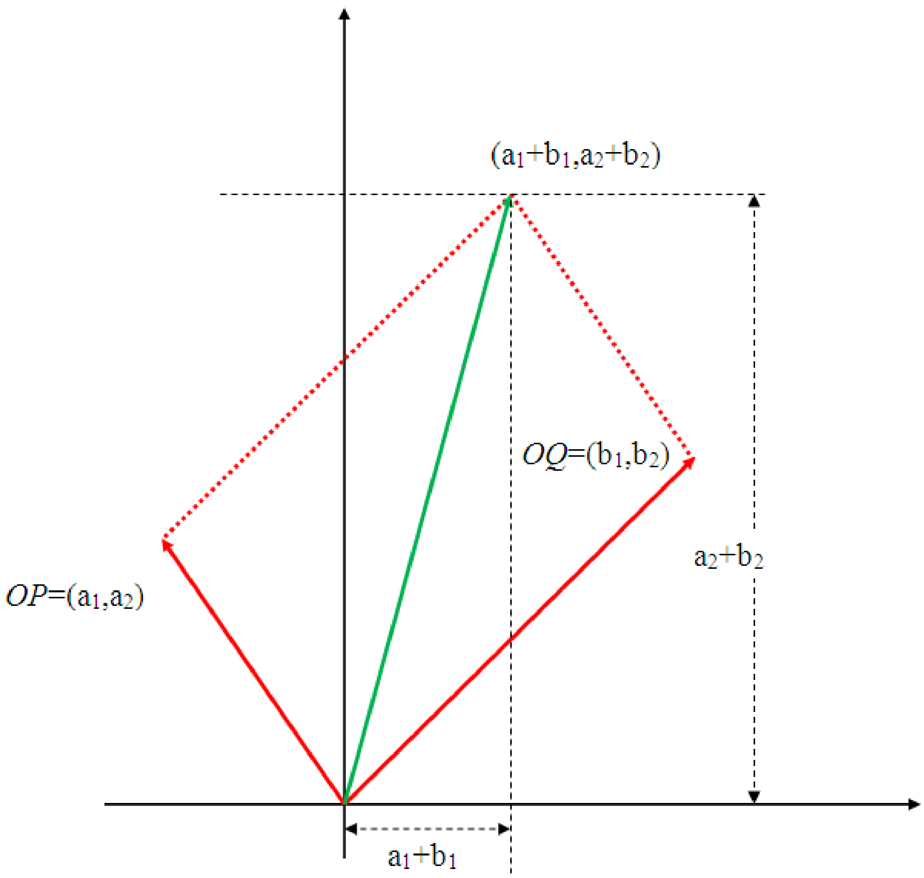

Figure 2.1: Parallelogram law for vector addition

In this way, we identify **unit vectors** (vectors of length one) that point along the three distinguished, mutually perpendicular directions of the Cartesian frame:

$$\hat{\boldsymbol{x}} = (1, 0, 0), \qquad \hat{\boldsymbol{y}} = (0, 1, 0), \qquad \hat{\boldsymbol{z}} = (0, 0, 1).$$

This introduces further consistency to the identification of triples (e.g. (a_1, a_2, a_3)) with vectors, since

$$\overrightarrow{OP} = \boldsymbol{a} = (a_1, a_2, a_3) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) = a_1\hat{\boldsymbol{x}} + a_2\hat{\boldsymbol{y}} + a_3\hat{\boldsymbol{z}}.$$

2.2 The dot product

Take two vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$. The dot product is a combination of these two vectors that returns a scalar, and is defined as follows:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The dot product inherits many of the usual properties of ordinary multiplication:

1. Commutative: $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}$,
2. Distributive: $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$. Also, $(\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{c}=\boldsymbol{a}\cdot\boldsymbol{c}+\boldsymbol{b}\cdot\boldsymbol{c}$,

for all $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in $\mathbb{R}^3$. Here $\mathbb{R}^3$ denotes all triples (x,y,z), where x, y, and z are real numbers; equivalently, it denotes all points in three-dimensional space.

The dot product can also be used to compute the length (magnitdue) of a vector, as

$$\mathrm{mag}(\boldsymbol{a})=\sqrt{a_1^2+a_2^2+a_3^2}=\sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}.$$

Henceforth, we denote $\mathrm{mag}(\boldsymbol{a})$ as $|\boldsymbol{a}|$.

Using the properties of dot-product multiplication, we can prove the following theorem:

Theorem 2.1 *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors in $\mathbb{R}^3$. Then*

$$\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\theta,$$

where $0\le\theta\le\pi$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

Proof: Introduce $\boldsymbol{c}:=\boldsymbol{a}-\boldsymbol{b}$. We apply the laws of dot-product multiplication to obtain

$$\begin{aligned}\boldsymbol{c}\cdot\boldsymbol{c} &= (\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}),\\ &= \boldsymbol{a}\cdot\boldsymbol{a}-2\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{b},\\ |\boldsymbol{c}|^2 &= |\boldsymbol{a}|^2-2\boldsymbol{a}\cdot\boldsymbol{b}+|\boldsymbol{b}|^2. \qquad (*)\end{aligned}$$

However, we refer to the triangle in Fig. 2.2, and we apply the cosine rule, to obtain

$$|\boldsymbol{c}|^2=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2-2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta. \qquad (**)$$

Equating (*) and (**), we obtain

$$\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\theta,$$

as required.

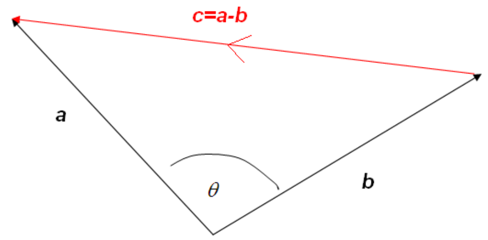

Figure 2.2: Sketch for applying the cosine rule to the dot-product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

Corollary: Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **orthogonal** (perpendicular) if and only if

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0.$$

As an example, consider

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} = (1, 0, 0) \cdot (0, 1, 0) = 1 \times 0 + 0 \times 0 + 0 \times 0 = 0.$$

Not surprisingly, $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ have zero dot product (and hence, are orthogonal), as they point along different mutually-perpendicular axes. Also, $\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} = 1$ &c. The vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are called an **orthonormal triad**.

2.3 The vector or cross product

Given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have seen how to form a scalar. We can also form a third vector from these two vectors, using the cross or vector product:

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \\ &= \hat{\boldsymbol{x}}\,(a_2 b_3 - a_3 b_2) - \hat{\boldsymbol{y}}\,(a_1 b_3 - a_3 b_1) + \hat{\boldsymbol{z}}\,(a_1 b_2 - a_2 b_1)\,, \\ &= \hat{\boldsymbol{x}}\,(a_2 b_3 - a_3 b_2) + \hat{\boldsymbol{y}}\,(a_3 b_1 - a_1 b_3) + \hat{\boldsymbol{z}}\,(a_1 b_2 - a_2 b_1)\,,. \end{aligned} \tag{2.1}$$

Properties of the vector or cross product:

1. Skew-symmetry: $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$,

2. Linearity: $(\lambda \boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (\lambda \boldsymbol{b}) = \lambda\,(\boldsymbol{a} \times \boldsymbol{b})$, for $\lambda \in \mathbb{R}$.

3. Distributive: $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$.

These results readily follow from the determinant definition. Result (1) is particularly weird. Note:

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{a} &= -\boldsymbol{a} \times \boldsymbol{a}, \qquad \text{Result (1)}, \\ 2\boldsymbol{a} \times \boldsymbol{a} &= 0, \\ \boldsymbol{a} \times \boldsymbol{a} &= 0. \end{aligned}$$

Numerical examples

1. Let

$$\boldsymbol{a} = \hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}, \qquad \boldsymbol{b} = 2\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} + 2\hat{\boldsymbol{z}}.$$

Then

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 1 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\hat{\boldsymbol{x}} - 7\hat{\boldsymbol{z}}.$$

2. The orthonormal triad $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ satisfy

$$\begin{aligned} \hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} &= \hat{\boldsymbol{z}}, \\ \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} &= \hat{\boldsymbol{x}}, \\ \hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}} &= \hat{\boldsymbol{y}}. \end{aligned} \tag{2.2}$$

2.4 Geometrical treatment of cross product

So far, our treatment of the cross product has been in terms of a particular choice of Cartesian axes. However, the definition of the cross product is in fact independent of any choice of such axes. To demonstrate this, we re-construct the cross product.

Step 1: Finding the length of $\boldsymbol{a} \times \boldsymbol{b}$ Note that

$$\begin{aligned} |\boldsymbol{a} \times \boldsymbol{b}|^2 + (\boldsymbol{a} \cdot \boldsymbol{b})^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\ &+ (a_1 b_1 + a_2 b_2 + a_3 b_3)^2, \\ &= \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right), \\ &= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2. \end{aligned}$$

Hence,

$$
\begin{aligned}
|\boldsymbol{a} \times \boldsymbol{b}|^2 &= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2 , \\
&= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \left(1 - \cos^2 \theta\right) , \\
&= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \sin^2 \theta
\end{aligned}
$$

and

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \theta,$$

where $0 \leq \theta \leq \pi$, such that the relation $|\boldsymbol{a} \times \boldsymbol{b}| \geq 0$ is satisfied.

Step 2: Finding the direction of $\boldsymbol{a} \times \boldsymbol{b}$ Note that

$$
\begin{aligned}
\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) &= a_1 \left(a_2 b_3 - a_3 b_2\right) + a_2 \left(a_3 b_1 - a_1 b_3\right) + a_3 \left(a_1 b_2 - a_2 b_1\right) , \\
&= 0.
\end{aligned}
$$

Similarly, $\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = 0$. Hence, $\boldsymbol{a} \times \boldsymbol{b}$ is a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. It remains to find the sense of $\boldsymbol{a} \times \boldsymbol{b}$. Indeed, this is arbitrary and must be fixed. We fix it such that we have a right-handed system, and such that the following rule-of-thumb is satisifed (Fig. 2.3).

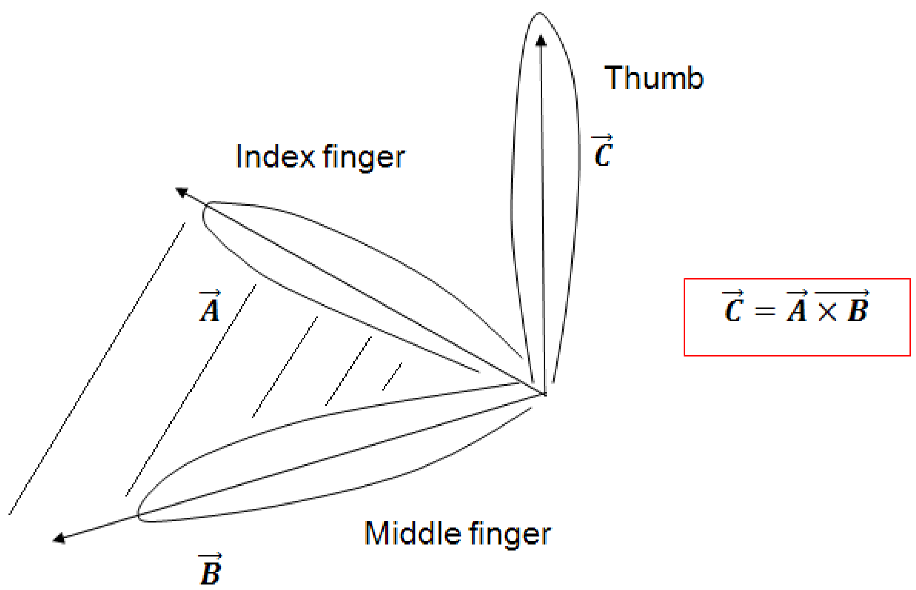

Figure 2.3: The right-hand rule.

Choosing a right-hand rule means that relations (2.2) are satisfied ($\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ form a 'right-handed' system). This also corresponds to putting a plus sign in front of the determinant in the original definition of the cross product.

In summary, $\boldsymbol{a} \times \boldsymbol{b}$ is a vector of magnitude $|\boldsymbol{a}||\boldsymbol{b}| \sin\theta$, that is normal to both $\boldsymbol{a}$ and $\boldsymbol{b}$, and whose sense is determined by the right-hand rule.

The cross product as an area: Consider a parallelogram, whose two adjacent sides are made up of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (Fig. 2.4). The area of the parallelogram is

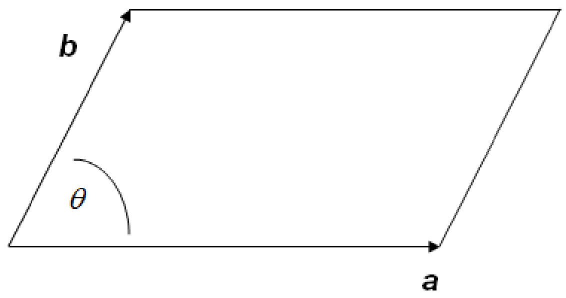

Figure 2.4: The cross product as an area

$$\begin{aligned} A &= (\text{base length}) \times (\text{perpendicular height}), \\ &= (\text{base length})\, |\boldsymbol{b}| \sin\theta, \\ &= |\boldsymbol{a}||\boldsymbol{b}| \sin\theta, \\ &= |\boldsymbol{a} \times \boldsymbol{b}|. \end{aligned}$$

The scalar triple product and volume: We can form a scalar from the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ by combining the operations just defined:

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})\,. \tag{2.3}$$

This is the so-called 'scalar triple product'.

Theorem 2.2 *The scalar triple product $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ is identically equal to*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Proof: By brute force,

$$\begin{aligned} &(a_1\hat{\boldsymbol{x}} + a_2\hat{\boldsymbol{y}} + a_3\hat{\boldsymbol{z}}) \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ = \;& (a_1\hat{\boldsymbol{x}} + a_2\hat{\boldsymbol{y}} + a_3\hat{\boldsymbol{z}}) \cdot [(b_2c_3 - c_2b_3)\,\hat{\boldsymbol{x}} + (b_3c_1 - b_1c_3)\,\hat{\boldsymbol{y}} + (b_1c_2 - b_2c_1)\,\hat{\boldsymbol{z}}] \\ = \;& a_1\,(b_2c_3 - c_2b_3) + a_2\,(b_3c_1 - b_1c_3) + a_3\,(b_1c_2 - b_2c_1)\,, \end{aligned}$$

which is the determinant of the theorem.

Now consider a parallelepiped spanned by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ (Fig. 2.5)

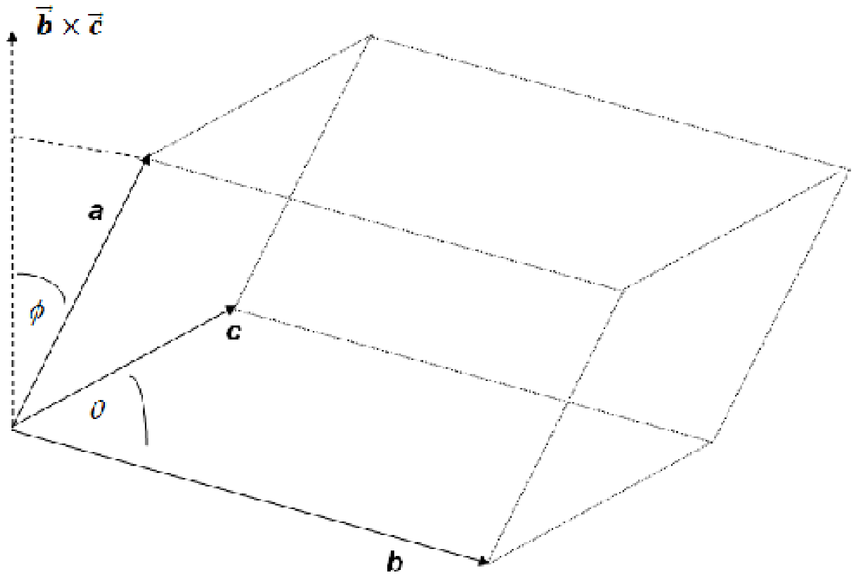

Figure 2.5: The scalar triple product as a volume

$$\begin{aligned} \text{Volume of parallelepiped} &= (\text{Perpendicular height}) \times (\text{Base area}) \\ &= (|\boldsymbol{a}|\cos\varphi) \times (|\boldsymbol{b}||\boldsymbol{c}|\sin\theta)\,, \\ &= (|\boldsymbol{a}|\cos\varphi)\,(|\boldsymbol{b}\times\boldsymbol{c}|)\,, \\ &= \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})\,. \end{aligned}$$

Corollary: Three nonzero vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are coplanar if and only if $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = 0$.

Proof: The volume of the parallelepiped spanned by the three vectors is zero iff the perpendicular height is zero, iff the three vectors are coplanar.

2.5 The vector triple product

Given three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, we can form yet another vector,

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})\,. \tag{2.4}$$

The brackets are important because the cross product is not associative, e.g.

$$\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}}) = \hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}} = -\hat{\boldsymbol{y}},$$

but

$$(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{x}}) \times \hat{\boldsymbol{y}} = 0 \times \hat{\boldsymbol{y}} = 0.$$

Theorem 2.3 *The vector triple product satisfies*

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b})\,, \tag{2.5}$$

a result that can be recalled by the mnemonic 'BAC minus CAB'.

Proof: Without loss of generality, we prove the result in a frame wherein the x- and y-axes of our frame lie in the plane generated by $\boldsymbol{b}$ and $\boldsymbol{c}$. In fact, we may take

$$\begin{aligned} \boldsymbol{c} &= \hat{\boldsymbol{x}} c_1, \\ \boldsymbol{b} &= b_1 \hat{\boldsymbol{x}} + b_2 \hat{\boldsymbol{y}}, \end{aligned}$$

and

$$\boldsymbol{a} = a_1 \hat{\boldsymbol{x}} + a_2 \hat{\boldsymbol{y}} + a_3 \hat{\boldsymbol{z}}.$$

The result then follows by a brute-force calculation of the LHS and the RHS of Eq. (2.5).

Chapter 3

The geometry of lines and planes

Overview

In this section we show how vector operations can be used to describe lines and planes in three-dimensional space. Some of this material will have been covered already in MATH 10270 or MATH 10280 but it is of vital importance to this module, so it is repeated here. The reason why these ideas are so important is that they carry over to general (smooth) curves and surfaces, which can be approximated to arbitrary precision by collections of line segments and planar surfaces.

3.1 The equation of a line

Find the equation of a straight line which passes through two given points A and B having position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ w.r.t. an origin O.

Let $\boldsymbol{r}$ be the position vector of any point P on the line through A and B. From Fig. 3.1

$$\overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OP} \implies \boldsymbol{a} + \overrightarrow{AP} = \boldsymbol{r} \implies \overrightarrow{AP} = \boldsymbol{r} - \boldsymbol{a},$$

and

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \implies \boldsymbol{a} + \overrightarrow{AB} = \boldsymbol{b} \implies \overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}.$$

But $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are colinear, hence

$$\overrightarrow{AP} = t\overrightarrow{AB} = t\,(\boldsymbol{b} - \boldsymbol{a})\,,$$

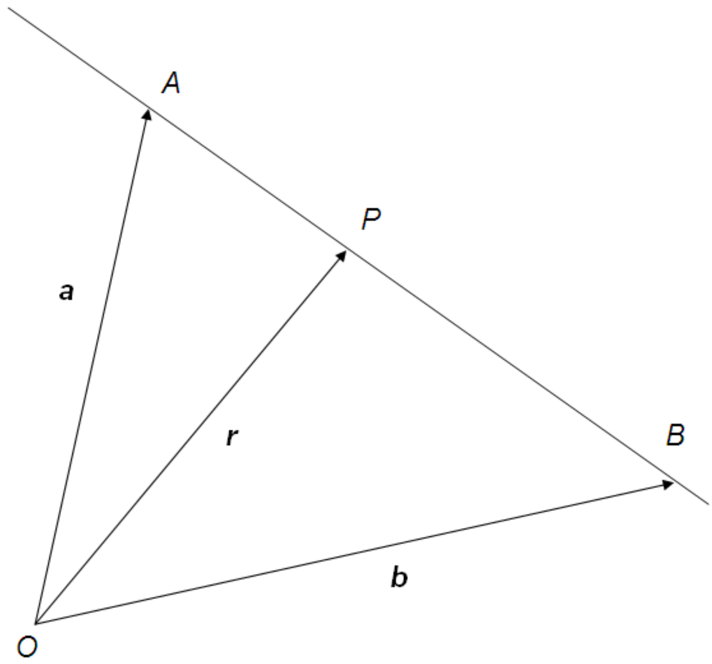

Figure 3.1: The equation of a line

where t is some real number. Putting these equations together gives

$$\boldsymbol{r} = \boldsymbol{a} + t\,(\boldsymbol{b} - \boldsymbol{a})\,.$$

Thus, two vectors are sufficient to specify a line in space: a vector $\boldsymbol{r}_0 := \boldsymbol{a}$ whose tip lies on the line, and a vector $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{a}$ that lies along the line. We therefore write

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{e}.$$

A straight line is thus a **one-parameter curve**.

Now let's go over to the Cartesian form of the line:

$$\begin{aligned} x &= x_0 + te_x, \\ y &= y_0 + te_y, \\ z &= z_0 + te_z. \end{aligned}$$

Eliminating t between the equations (if possible) gives

$$\frac{x - x_0}{e_x} = \frac{y - y_0}{e_y} = \frac{z - z_0}{e_z}.$$

If the line lies entirely in the x-y plane, then $z = 0$ and the elimination is carried out on only the x-

and y-variables:

$$\frac{x-x_0}{e_x}=\frac{y-y_0}{e_y} \implies y=y_0+\frac{e_y}{e_x}\left(x-x_0\right),$$

which is the standard equation of the line in a plane with slope $m=e_y/e_x$.

3.2 The perpendicular distance between a point and a line

Let $\boldsymbol{r}\left(t\right)=\boldsymbol{r}_0+t\boldsymbol{e}$ be the equation of a straight line L, and let P be a point with position vector $\boldsymbol{a}$ (henceforth written as $P(\boldsymbol{a})$). Find the shortest distance between the line and the point.

The shortest distance between the point P and the line L is in fact the perpendicular distance. Suppose that a perpendicular dropped from P to L intersects L at position vector $\boldsymbol{r}_1$, such that

$$\boldsymbol{r}_1=\boldsymbol{r}_0+t_1\boldsymbol{e}. \qquad (*)$$

(Refer to Fig. 3.2.) By construction of this perpendicular, the line $\boldsymbol{a}-\boldsymbol{r}_1$ is perpendicular to the

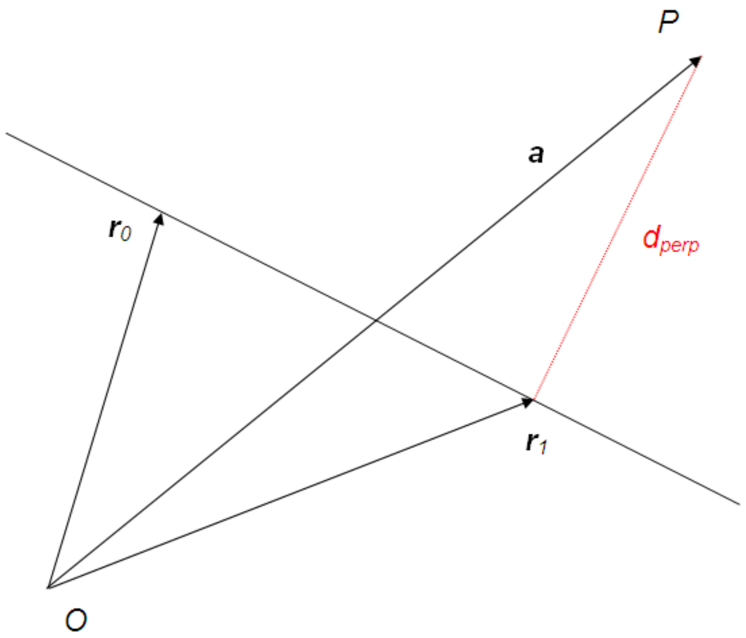

Figure 3.2: The shortest distance between a point and a line

line, or

$$\boldsymbol{e}\cdot\left(\boldsymbol{a}-\boldsymbol{r}_1\right)=0.$$

Hence,

$$\boldsymbol{e}\cdot\boldsymbol{a}=\boldsymbol{e}\cdot\boldsymbol{r}_1=\boldsymbol{e}\cdot\left(\boldsymbol{r}_0+t_1\boldsymbol{e}\right)=\boldsymbol{e}\cdot\boldsymbol{r}_0+t_1\boldsymbol{e}^2.$$

Solving for t_1,

$$t_1 = \frac{\boldsymbol{e}\cdot(\boldsymbol{a}-\boldsymbol{r}_0)}{|\boldsymbol{e}|^2}.$$

Substitute this expression into (*):

$$\boldsymbol{r}_1 = \boldsymbol{r}_0 + \boldsymbol{e}\frac{\boldsymbol{e}\cdot(\boldsymbol{a}-\boldsymbol{r}_0)}{|\boldsymbol{e}|^2} := \boldsymbol{r}_0 + \hat{\boldsymbol{e}}\left[\hat{\boldsymbol{e}}\cdot(\boldsymbol{a}-\boldsymbol{r}_0)\right],$$

where $\hat{\boldsymbol{e}} = \boldsymbol{e}/|\boldsymbol{e}|$ is a unit vector along the line L. The perpendicular distance is thus

$$\begin{aligned} d_\perp &= |\boldsymbol{a}-\boldsymbol{r}_1|, \\ &= |(\boldsymbol{a}-\boldsymbol{r}_0) - [\hat{\boldsymbol{e}}\cdot(\boldsymbol{a}-\boldsymbol{r}_0)]\,\hat{\boldsymbol{e}}|, \\ &= |(\boldsymbol{r}_0-\boldsymbol{a}) + [\hat{\boldsymbol{e}}\cdot(\boldsymbol{a}-\boldsymbol{r}_0)]\,\hat{\boldsymbol{e}}|, \\ &= |(\boldsymbol{r}_0-\boldsymbol{a}) - [\hat{\boldsymbol{e}}\cdot(\boldsymbol{r}_0-\boldsymbol{a})]\,\hat{\boldsymbol{e}}|. \end{aligned}$$

This is a valid final answer. However, a little more manipulation yields

$$\begin{aligned} d_\perp^2 = (\boldsymbol{r}_0-\boldsymbol{a})^2 - |\hat{\boldsymbol{e}}\cdot(\boldsymbol{r}_0-\boldsymbol{a})|^2 = (\boldsymbol{r}_0-\boldsymbol{a})^2\left(1-\cos^2\theta\right) \\ = (\boldsymbol{r}_0-\boldsymbol{a})^2\sin^2\theta = |(\boldsymbol{r}_0-\boldsymbol{a})\times\hat{\boldsymbol{e}}|^2, \end{aligned}$$

hence

$$d_\perp = |(\boldsymbol{r}_0-\boldsymbol{a})\times\hat{\boldsymbol{e}}|.$$

Now suppose the line lies in the x-y plane only. The equation of the line is $\alpha x + \beta y + \gamma = 0$, with slope $m = -\alpha/\beta$. But $e_y/e_x = m = m/1$, hence

$$\boldsymbol{e} = (e_x, e_y) = (1, m), \qquad \hat{\boldsymbol{e}} = \frac{(1,m)}{\sqrt{1+m^2}} = \frac{(1,-\alpha/\beta)}{\sqrt{1+\alpha^2/\beta^2}} = \frac{(\beta,-\alpha)}{\sqrt{\alpha^2+\beta^2}}$$

Thus,

$$\begin{aligned} d_\perp^2 &= \frac{1}{\alpha^2+\beta^2}\left|\underbrace{(x_0,y_0,0)\times(\beta,-\alpha,0)}_{=\boldsymbol{r}_0\times\hat{\boldsymbol{e}}} - \underbrace{(a_1,a_2,0)\times(\beta,-\alpha,0)}_{=\boldsymbol{a}\times\hat{\boldsymbol{e}}}\right|^2, \\ &= \frac{1}{\alpha^2+\beta^2}|-(y_0\beta+x_0\alpha)+(a_2\beta+a_1\alpha)|^2, \\ &= \frac{1}{\alpha^2+\beta^2}|\gamma+(a_2\beta+a_1\alpha)|^2, \\ &= \frac{|a_1\alpha+a_2\beta+\gamma|^2}{\alpha^2+\beta^2}, \end{aligned}$$

which is the old Leaving Cert. formula.

3.3 The equation of a plane

Find the equation of a plane which passes through three given points A, B, and C having position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ w.r.t. an origin O.

By construction, the vectors

$$\boldsymbol{v}_1 := \overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b},$$
$$\boldsymbol{v}_2 := \overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b},$$

lie in the plane we call Π (Fig. 3.3). A normal to this plane is

$$\boldsymbol{n} = \boldsymbol{v}_2 \times \boldsymbol{v}_1 = (\boldsymbol{c} - \boldsymbol{b}) \times (\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a}.$$

The equation of a plane is subset of all vectors $\boldsymbol{r} = (x, y, z)$ in $\mathbb{R}^3$, such that the vector $\boldsymbol{r} - \boldsymbol{b}$ is perpendicular to $\boldsymbol{n}$:

$$\Pi = \left\{\boldsymbol{r} \in \mathbb{R}^3 | \, (\boldsymbol{r} - \boldsymbol{b}) \cdot (\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a}) = 0\right\}.$$

Simplifying, the general vector $\boldsymbol{r}$ lies in the plane Π if and only if

$$\boldsymbol{r} \cdot \underbrace{(\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a})}_{\text{normal vector}} = \underbrace{\boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a})}_{\text{a constant}}.$$

This is the final answer.

Note that the general equation of a plane in three dimensions is

$$\Pi(\boldsymbol{n}, \boldsymbol{r}_0) = \left\{\boldsymbol{r} \in \mathbb{R}^3 | \, (\boldsymbol{r} - \boldsymbol{r}_0) \cdot \boldsymbol{n} = 0\right\}.$$

The plane is thus parametrized by the normal vector $\boldsymbol{n}$ and a reference vector $\boldsymbol{r}_0$ whose tip lies in the plane (Fig. 3.4). If $n_z \neq 0$, we have the Cartesian expression

$$z = z_0 + (y_0 - y)\frac{n_y}{n_z} + (x_0 - x)\frac{n_x}{n_z}.$$

Thus, a point $z = z(x, y)$ on a surface is labelled by two parameters, x and y. A plane is therefore a **two-parameter** object, just as a line was a one-parameter curve.

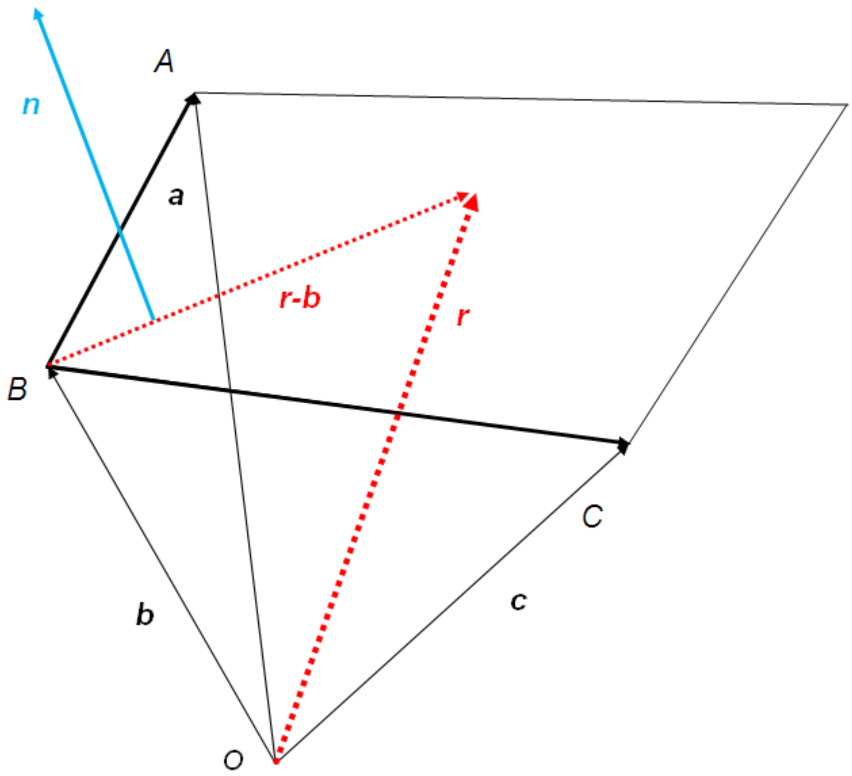

Figure 3.3: The equation of a plane

3.4 Skew lines and intersecting lines in three dimensions

Skew lines are a very nice application of three-dimensional geometry. In two dimensions, two non-parallel lines definitely intersect. However, in three dimensions, they need not intersect: they can "go around" one another. We start by considering two *intersecting lines*:

$$\begin{aligned} \boldsymbol{r}_L(t) &= \boldsymbol{r}_0 + t\boldsymbol{e}, \\ \boldsymbol{r}_M(u) &= \boldsymbol{s}_0 + u\boldsymbol{f}, \end{aligned}$$

and we show that their point of intersection $\boldsymbol{r}_L(t_0) = \boldsymbol{r}_M(u_0)$ is given by the solution of the equation

$$\begin{pmatrix} (\boldsymbol{r}_0 - \boldsymbol{s}_0)\cdot\boldsymbol{e} \\ (\boldsymbol{r}_0 - \boldsymbol{s}_0)\cdot\boldsymbol{f} \end{pmatrix} = \begin{pmatrix} -|\boldsymbol{e}|^2 & \boldsymbol{e}\cdot\boldsymbol{f} \\ -\boldsymbol{e}\cdot\boldsymbol{f} & |\boldsymbol{f}|^2 \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix}.$$

By assumption, the point of intersection exists. Hence,

$$\boldsymbol{r}_0 + t_0\boldsymbol{e} = \boldsymbol{s}_0 + u_0\boldsymbol{f},$$

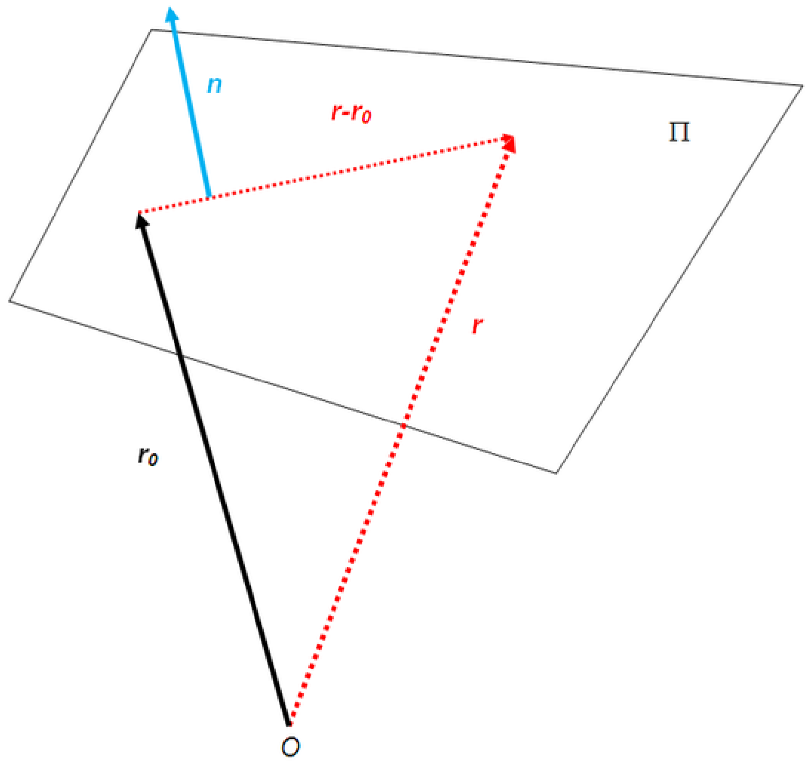

Figure 3.4: Figure for the general equation of a plane

for parameter values t_0 and u_0. Re-arrange,

$$\boldsymbol{r}_0 - \boldsymbol{s}_0 = u_0\boldsymbol{f} - t_0\boldsymbol{e}.$$

Take the scalar product of this equation with $\boldsymbol{e}$:

$$(\boldsymbol{r}_0 - \boldsymbol{s}_0)\cdot\boldsymbol{e} = u_0\boldsymbol{f}\cdot\boldsymbol{e} - t_0|\boldsymbol{e}|^2;$$

do the same thing with vector $\boldsymbol{f}$:

$$(\boldsymbol{r}_0 - \boldsymbol{s}_0)\cdot\boldsymbol{f} = u_0|\boldsymbol{f}|^2 - t_0\boldsymbol{e}\cdot\boldsymbol{f}.$$

Gather up:

$$\begin{aligned}(\boldsymbol{r}_0 - \boldsymbol{s}_0)\cdot\boldsymbol{e} &= -t_0|\boldsymbol{e}|^2 + u_0\boldsymbol{f}\cdot\boldsymbol{e},\\ (\boldsymbol{r}_0 - \boldsymbol{s}_0)\cdot\boldsymbol{f} &= -t_0\boldsymbol{e}\cdot\boldsymbol{f} + u_0|\boldsymbol{f}|^2,\end{aligned}$$

which is the required result:

$$\begin{pmatrix} (\boldsymbol{r}_0-\boldsymbol{s}_0)\cdot\boldsymbol{e} \\ (\boldsymbol{r}_0-\boldsymbol{s}_0)\cdot\boldsymbol{f} \end{pmatrix} = \begin{pmatrix} -|\boldsymbol{e}|^2 & \boldsymbol{e}\cdot\boldsymbol{f} \\ -\boldsymbol{e}\cdot\boldsymbol{f} & |\boldsymbol{f}|^2 \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix}.$$

In the previous example, we were told that the point of intersection exists. It was then fairly straightforward to compute that point. **We now formulate a general condition for the point of intersection to exist**. We start with the two lines

$$\begin{aligned} \boldsymbol{r}_L(t) &= \boldsymbol{r}_0 + t\boldsymbol{e}, \\ \boldsymbol{r}_M(u) &= \boldsymbol{s}_0 + u\boldsymbol{f}. \end{aligned}$$

From the first part, a candidate point (t_0, u_0) for the intersection is the solution of the equation

$$\begin{pmatrix} (\boldsymbol{r}_0-\boldsymbol{s}_0)\cdot\boldsymbol{e} \\ (\boldsymbol{r}_0-\boldsymbol{s}_0)\cdot\boldsymbol{f} \end{pmatrix} = \begin{pmatrix} -|\boldsymbol{e}|^2 & \boldsymbol{e}\cdot\boldsymbol{f} \\ -\boldsymbol{e}\cdot\boldsymbol{f} & |\boldsymbol{f}|^2 \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix},$$

provided the solution exists. Now the determinant of this matrix is

$$-|\boldsymbol{e}|^2|\boldsymbol{f}|^2 + (\boldsymbol{e}\cdot\boldsymbol{f})^2 = -|\boldsymbol{e}\times\boldsymbol{f}|^2.$$

Thus, if $\boldsymbol{e}\times\boldsymbol{f}\neq 0$, the point

$$\begin{pmatrix} t_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} -|\boldsymbol{e}|^2 & \boldsymbol{e}\cdot\boldsymbol{f} \\ -\boldsymbol{e}\cdot\boldsymbol{f} & |\boldsymbol{f}|^2 \end{pmatrix}^{-1} \begin{pmatrix} (\boldsymbol{r}_0-\boldsymbol{s}_0)\cdot\boldsymbol{e} \\ (\boldsymbol{r}_0-\boldsymbol{s}_0)\cdot\boldsymbol{f} \end{pmatrix}$$

is certainly a candidate solution. In plugging the solution of the matrix equation back into the intersection condition

$$\boldsymbol{r}_0 - \boldsymbol{s}_0 = u_0\boldsymbol{f} - t_0\boldsymbol{e},$$

we must be very careful: the only way to go from the solution of the matrix equation to the intersection condition is if $\boldsymbol{r}_0 - \boldsymbol{s}_0$ lies entirely in the plane generated by $\boldsymbol{e}$ and $\boldsymbol{f}$. However, in general,

$$\boldsymbol{r}_0 - \boldsymbol{s}_0 = \alpha\boldsymbol{e} + \beta\boldsymbol{f} + \gamma\boldsymbol{e}\times\boldsymbol{f}.$$

Thus, we require $\gamma = 0$, or

$$(\boldsymbol{r}_0 - \boldsymbol{s}_0)\cdot(\boldsymbol{e}\times\boldsymbol{f}) = 0.$$

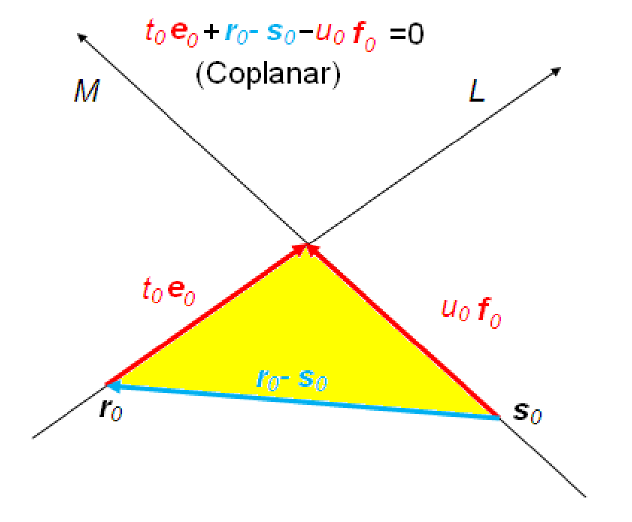

Figure 3.5: Vectors coplanar

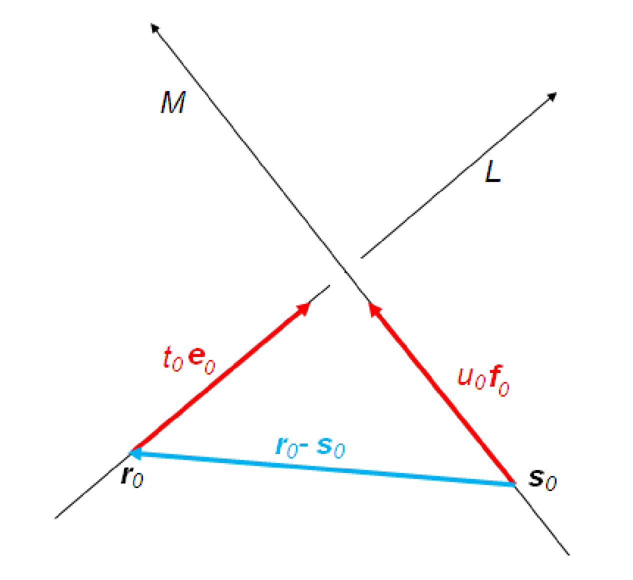

Figure 3.6: Skew lines: Vectors non-coplanar

Therefore, a set of sufficient conditions for the lines to intersect is the following:

$$\boldsymbol{e} \times \boldsymbol{f} \neq 0 \quad \text{AND} \tag{3.1}$$

$$(\boldsymbol{r}_0 - \boldsymbol{s}_0) \cdot (\boldsymbol{e} \times \boldsymbol{f}) = 0. \tag{3.2}$$

Condition (3.2) states that $\boldsymbol{r}_0 - \boldsymbol{s}_0$ lies entirely in the plane generated by $\boldsymbol{e}$ and $\boldsymbol{f}$ and thus, $(\boldsymbol{r}_0 - \boldsymbol{s}_0) \cdot \boldsymbol{e}$ and $(\boldsymbol{r}_0 - \boldsymbol{s}_0) \cdot \boldsymbol{f}$ can not both be zero. In geometrical language, the condition is that $\boldsymbol{e}$ and $\boldsymbol{f}$ must be non-parallel AND the difference $\boldsymbol{r}_0 - \boldsymbol{s}_0$ must lie entirely in the plane generated by $\boldsymbol{e}$ and $\boldsymbol{f}$. Lines that satisfy the first condition but not the second are called **skew lines**.

Chapter 4

Ordinary derivatives of vectors

Overview

In many applications, we must consider a vector in $\mathbb{R}^3$ that varies continuously as a single parameter is varied. In particular, in mechanics, the position $\boldsymbol{x}$ of a particle is a function of time. Such a situation is called a curve: a curve γ is a map

$$\begin{aligned} \gamma : \mathbb{R} &\to \mathbb{R}^3, \\ t &\to \boldsymbol{x}_\gamma(t) = (x_\gamma(t), y_\gamma(t), z_\gamma(t)). \end{aligned}$$

Here $x_\gamma(\cdot)$, $y_\gamma(\cdot)$, and $z_\gamma(\cdot)$ are functions of time that give the Cartesian coordinates of the particle. Although not technically correct, in this section we drop the curve label γ and write $\boldsymbol{x}_\gamma(t) = \boldsymbol{x}(t) = (x(t), y(t), z(t))$. Such sloppiness even has a formal name: it is called *an abuse of notation*.

4.1 Definitions and properties

Let $\boldsymbol{x}(t)$ be a curve parametrized by time t. The **derivative of the curve w.r.t. time** is defined as

$$\frac{d\boldsymbol{x}}{dt} = \lim_{\Delta t \to 0} \frac{\boldsymbol{x}(t + \Delta t) - \boldsymbol{x}(t)}{\Delta t},$$

provided the limit exists. A similar definition holds for the higher derivatives. Since

$$\boldsymbol{x}(t) = (x(t), y(t), z(t)) = \hat{\boldsymbol{x}}x(t) + \hat{\boldsymbol{y}}y(t) + \hat{\boldsymbol{z}}z(t),$$

where $\hat{\boldsymbol{x}}$ &c. are constant vectors, this derivative can also be written as

$$\frac{d\boldsymbol{x}}{dt} = \hat{\boldsymbol{x}}\frac{dx}{dt} + \hat{\boldsymbol{y}}\frac{dy}{dt} + \hat{\boldsymbol{z}}\frac{dz}{dt}.$$

It should be clear that curves inherit all the properties of real-valued functions. In particular,

Theorem 4.1 *The following properties are satisfied, for arbitrary differentiable curves* $\boldsymbol{A}(t)$*,* $\boldsymbol{B}(t)$*, and* $\boldsymbol{C}(t)$*:*

1. $$\frac{d}{dt}\left[\boldsymbol{A}(t)+\boldsymbol{B}(t)\right]=\frac{d\boldsymbol{A}}{dt}+\frac{d\boldsymbol{B}}{dt},$$

2. $$\frac{d}{dt}\left[\boldsymbol{A}(t)\cdot\boldsymbol{B}(t)\right]=\boldsymbol{A}(t)\cdot\frac{d\boldsymbol{B}}{dt}+\boldsymbol{B}(t)\cdot\frac{d\boldsymbol{A}}{dt},$$

3. $$\frac{d}{dt}\left[\boldsymbol{A}(t)\times\boldsymbol{B}(t)\right]=\boldsymbol{A}(t)\times\frac{d\boldsymbol{B}}{dt}+\frac{d\boldsymbol{A}}{dt}\times\boldsymbol{B}(t),$$

   (note the order!)

4. *For a scalar function* $f(t)$*,*

   $$\frac{d}{dt}\left[f(t)\boldsymbol{A}(t)\right]=f(t)\frac{d\boldsymbol{A}}{dt}+\boldsymbol{A}\frac{df}{dt},$$

5. $$\frac{d}{dt}\left[\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})\right]=\boldsymbol{A}\cdot\left(\boldsymbol{B}\times\frac{d\boldsymbol{C}}{dt}\right)+\boldsymbol{A}\cdot\left(\frac{d\boldsymbol{B}}{dt}\times\boldsymbol{C}\right)+\frac{d\boldsymbol{A}}{dt}\cdot(\boldsymbol{B}\times\boldsymbol{C}),$$

6. $$\frac{d}{dt}\left[\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})\right]=\boldsymbol{A}\times\left(\boldsymbol{B}\times\frac{d\boldsymbol{C}}{dt}\right)+\boldsymbol{A}\times\left(\frac{d\boldsymbol{B}}{dt}\times\boldsymbol{C}\right)+\frac{d\boldsymbol{A}}{dt}\times(\boldsymbol{B}\times\boldsymbol{C}),$$

   Here we move the derivative 'operator' sequentially through the product.

The proofs are straightforward because the vectors $\boldsymbol{A}=A_1\hat{\boldsymbol{x}}+A_2\hat{\boldsymbol{y}}+A_3\hat{\boldsymbol{y}}:=(A_1,A_2,A_3)$ &c.

inherit their differentiability properties from their components. For example,

$$\begin{aligned}
\frac{d}{dt}\left(\boldsymbol{A}(t)\cdot\boldsymbol{B}(t)\right) &= \frac{d}{dt}\sum_{i=1}^{3}A_i(t)B_i(t),\\
&= \sum_{i=1}^{3}\frac{d}{dt}\left[A_i(t)B_i(t)\right],\\
&= \sum_{i=1}^{3}\left[A_i(t)\frac{dB_i}{dt}+\frac{dA_i}{dt}B_i\right],\\
&= \boldsymbol{A}\cdot\frac{d\boldsymbol{B}}{dt}+\frac{d\boldsymbol{A}}{dt}\cdot\boldsymbol{B}.
\end{aligned}$$

Theorem 4.2 *Let $\boldsymbol{x}(t)$ be a curve in $\mathbb{R}^3$. Then $d\boldsymbol{x}(t)/dt$ is everywhere tangent to the curve.*

Proof: Take a point $\boldsymbol{x}(t)$ on the curve and a neighbouring point $\boldsymbol{x}(t+\Delta t)$, also on the curve, where Δt is small. Form the difference

$$\frac{\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t)}{\Delta t}.$$

As the interval Δt is made smaller, the difference $\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t)$ comes to lie parallel to the curve (Fig. 4.1), hence

$$\left(\frac{\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t)}{\Delta t}\right)\cdot\boldsymbol{n}\to 0,\ \text{ as } \Delta t\to 0,$$

where $\boldsymbol{n}$ is a unit normal vector to the curve at the point $\boldsymbol{x}(t)$. In other words,

$$\frac{d\boldsymbol{x}}{dt}\cdot\boldsymbol{n}=0,$$

and the vector $d\boldsymbol{x}/dt$ is therefore everywhere tangent to the curve $\boldsymbol{x}$. Thus, $d\boldsymbol{x}/dt$ is often called the **tangent vector** or the **velocity vector**.

4.2 Frenet–Serret frame

We introduce the notion of **arc length**. Consider a curve $\boldsymbol{x}(t)$. Along the curve, a small line element has length

$$\mathrm{d}s^2=\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2.$$

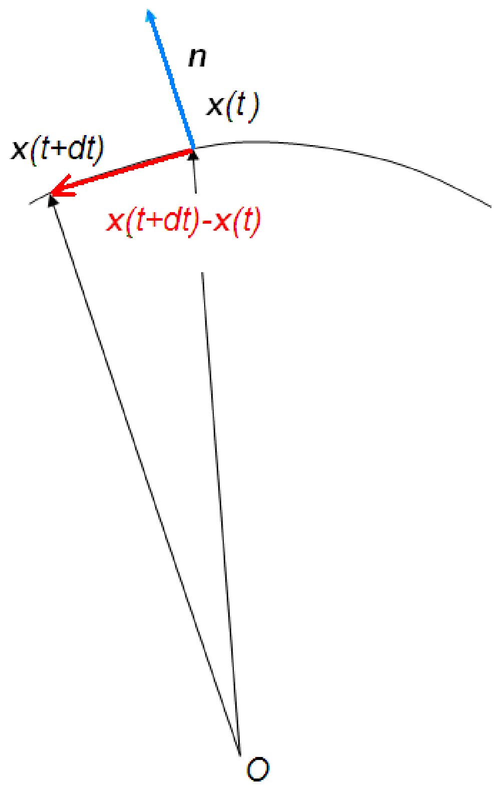

Figure 4.1: The difference $\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t)$ is tangent to the curve at $\boldsymbol{x}(t)$, in the limit as $\Delta t \to 0$.

Hence, the **arc length** along the curve, measured from a reference value $\boldsymbol{x}_0 = \boldsymbol{x}(t=0)$ is

$$s(t) = \int_0^{s(t)} \mathrm{d}s = \int_0^{s(t)} \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2} =$$

$$\int_0^t \sqrt{\left(\frac{dx}{dt'}\right)^2 + \left(\frac{dy}{dt'}\right)^2 + \left(\frac{dz}{dt'}\right)^2}\,\mathrm{d}t' = \int_0^t \left|\frac{d\boldsymbol{x}}{dt'}\right| \mathrm{d}t'.$$

This is a straightforward integration because $|d\boldsymbol{x}(t)/dt|$ is a simple function of time. Moreover,

$$\frac{ds}{dt} = \left|\frac{d\boldsymbol{x}}{dt}\right| \geq 0,$$

and the arclength is an **increasing function of time**. There is thus an inverse function $t = t(s)$, enabling a **reparametrization** of the curve according to arclength:

$$\tilde{\boldsymbol{x}}(s) = \boldsymbol{x}(t(s)).$$

Hence,

$$\frac{d\tilde{\boldsymbol{x}}}{ds} = \frac{d\boldsymbol{x}}{dt}\frac{dt}{ds} = \frac{d\boldsymbol{x}}{dt}\frac{1}{\left|\frac{d\boldsymbol{x}}{dt}\right|}, \qquad \text{(Chain Rule)}$$

and $d\tilde{\boldsymbol{x}}/ds$ is a unit vector tangent to the curve:

$$\boldsymbol{T} = \frac{d\tilde{\boldsymbol{x}}}{ds}.$$

Now

$$\boldsymbol{T}\cdot\boldsymbol{T} = 1,$$

hence

$$0 = \boldsymbol{T}\cdot\frac{d\boldsymbol{T}}{ds} + \frac{d\boldsymbol{T}}{ds}\cdot\boldsymbol{T} \implies \boldsymbol{T}\cdot\frac{d\boldsymbol{T}}{ds} = 0,$$

and $d\boldsymbol{T}/ds$ is perpendicular to the tangent vector $\boldsymbol{T}$. We therefore define a new unit vector $\boldsymbol{N} \propto d\boldsymbol{T}/ds$ that is normal to the tangent:

$$\frac{d\boldsymbol{T}}{ds} = \kappa(s)\boldsymbol{N},$$

and $\boldsymbol{N}$ is the **principal normal** to the curve and κ is the **curvature**.

Now our goal should be clear: we are deriving a triple of axes that move with the curve. $\boldsymbol{T}$ defines an axis everywhere parallel to the curve; $\boldsymbol{N}$ defines an axis that is everywhere perpendicular to the curve. In three dimensions, three axes are necessary: we therefore form a third unit vector

$$\boldsymbol{B} := \boldsymbol{T}\times\boldsymbol{N}.$$

The triple $(\boldsymbol{T},\boldsymbol{N},\boldsymbol{B})$ of axes along the curve $\tilde{\boldsymbol{x}}(s)$ parametrized by the arclength s is called the **Frenet–Serret frame**.

Note:

$$\begin{aligned}\frac{d\boldsymbol{B}}{ds} &= \boldsymbol{T}\times\frac{d\boldsymbol{N}}{ds} + \frac{d\boldsymbol{T}}{ds}\times\boldsymbol{N},\\ &= \boldsymbol{T}\times\frac{d\boldsymbol{N}}{ds} + \kappa\boldsymbol{N}\times\boldsymbol{N},\\ &= \boldsymbol{T}\times\frac{d\boldsymbol{N}}{ds}\end{aligned}$$

Hence

$$\boldsymbol{T}\cdot\left(\frac{d\boldsymbol{B}}{ds}\right) = \boldsymbol{T}\cdot\left(\boldsymbol{T}\times\frac{d\boldsymbol{N}}{ds}\right) = 0,$$

and $\boldsymbol{T}$ is perpendicular to $d\boldsymbol{B}/ds$. But $\boldsymbol{B}\cdot\boldsymbol{B}=1$, hence

$$\boldsymbol{B}\cdot\left(\frac{d\boldsymbol{B}}{ds}\right)=0.$$

Thus, $d\boldsymbol{B}/ds$ is perpendicular to $\boldsymbol{T}$ and $\boldsymbol{B}$, and must therefore lie along $\boldsymbol{N}$:

$$\frac{d\boldsymbol{B}}{ds}\propto\boldsymbol{N}.$$

We write

$$\frac{d\boldsymbol{B}}{ds}=-\tau(s)\boldsymbol{N},$$

where τ is the **torsion**. Finally, since $(\boldsymbol{T},\boldsymbol{N},\boldsymbol{B})$ form a right-handed system (by construction), and since $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}$, we may perform a cyclic permutation and obtain

$$\boldsymbol{N}=\boldsymbol{B}\times\boldsymbol{T}.$$

Operating on this with d/ds, we obtain

$$\begin{aligned}\frac{d\boldsymbol{N}}{ds}&=\boldsymbol{B}\times\frac{d\boldsymbol{T}}{ds}+\frac{d\boldsymbol{B}}{ds}\times\boldsymbol{T},\\&=\boldsymbol{B}\times(\kappa\boldsymbol{N})-\tau(\boldsymbol{N}\times\boldsymbol{T}),\\&=-\kappa\boldsymbol{T}+\tau\boldsymbol{B}.\end{aligned}$$

Let us assemble our results:

- $\boldsymbol{T}$ – unit tangent vector to curve $\boldsymbol{x}(s)$ parametrized by arclength s;
- $\boldsymbol{N}$ – unit vector normal to $\boldsymbol{T}$;
- $\boldsymbol{B}$ – a second unit vector normal to $\boldsymbol{T}$, $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}$.

$$\begin{aligned}\frac{d\boldsymbol{T}}{ds}&=\kappa(s)\boldsymbol{N},\\\frac{d\boldsymbol{B}}{ds}&=-\tau(s)\boldsymbol{N},\\\frac{d\boldsymbol{N}}{ds}&=\tau\boldsymbol{B}-\kappa\boldsymbol{T}.\end{aligned}$$

This framework is summarized graphically in Fig. 4.2.

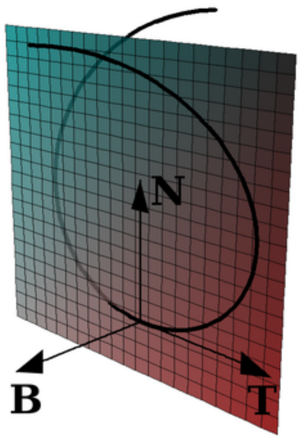

Figure 4.2: The Frenet–Serret frame along a curve. The plane shown the *osculating plane*, and this is the plane normal to the vector $\boldsymbol{B}$. From http://en.wikipedia.org/wiki/Frenet-Serret_formulas (3^{rd} August 2010)

4.3 Worked examples

1. **Curves in two dimensions:** As we know from school, a curve in two dimensions can always be written in the form

$$y = f(x).$$

In other words,

$$\boldsymbol{x} = (x, f(x)). \tag{4.1}$$

Now here, x is simply a label, which indicates that the first variable in the bracket pair $(x, f(x))$ ranges over the whole real line (or some interval thereof). Thus, we can re-write the curve (4.1) as

$$\boldsymbol{x} = (t, f(t)).$$

The unit tangent vector is available immediately as

$$\boldsymbol{T} = \dot{\boldsymbol{x}}/|\dot{\boldsymbol{x}}|,$$

where

$$\dot{\boldsymbol{x}} := \frac{d\boldsymbol{x}}{dt} = (1, f'(t)), \qquad |\dot{\boldsymbol{x}}| = \sqrt{1 + f'(t)^2}.$$

Henceforth, to save chalk/ink/typing we write f instead of $f(t)$ &c, the functional dependence of f on t being understood. Hence,

$$\boldsymbol{T} = \frac{(1, f')}{\sqrt{1 + f'^2}}. \tag{4.2}$$

To find the principal normal vector, we are going to have to differentiate Eq. (4.2):

$$\begin{aligned}
\frac{d\boldsymbol{T}}{dt} &= \left(\frac{d}{dt}(1+f'^2)^{-1/2}, \frac{d}{dt}\frac{f'}{(1+f'^2)^{1/2}}\right), \\
&= \left(-\frac{f'f''}{(1+f'^2)^{3/2}}, \frac{(1+f'^2)^{1/2}f'' - f'(1+f'^2)^{-1/2}f'f''}{1+f'^2}\right), \\
&= \left(-\frac{f'f''}{(1+f'^2)^{3/2}}, \frac{(1+f'^2)f'' - f'f'f''}{(1+f'^2)^{3/2}}\right), \\
&= \left(-\frac{f'f''}{(1+f'^2)^{3/2}}, \frac{f''}{(1+f'^2)^{3/2}}\right), \\
&= \frac{f''}{(1+f'^2)^{3/2}}\left(-f', 1\right).
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{d\boldsymbol{T}}{ds} &= \frac{d\boldsymbol{T}}{dt}\Big/\left|\frac{d\boldsymbol{x}}{dt}\right|, \\
&= \frac{f''}{(1+f'^2)^{3/2}}\frac{(-f',1)}{\sqrt{1+f'^2}}, \\
&= \kappa\boldsymbol{N}.
\end{aligned}$$

Actually, there was some ambiguity in our identification of the curvature in the derivation of the FS formulae – there are separate notions of **signed** and **unsigned** curvature. Here, we identify

$$\kappa_s := \frac{f''}{(1+f'^2)^{3/2}}$$

as the **signed curvature** of the curve (since it can take either sign). Also, we identify

$$\boldsymbol{N}_s := \frac{(-f',1)}{\sqrt{1+f'^2}}$$

as the **signed principal normal vector**. The unsigned curvature is $\kappa_{us} := |\kappa_s|$, such that

$$\kappa_s\boldsymbol{N}_s = |\kappa_s|\mathrm{sign}(\kappa_s)\boldsymbol{N}_s = \kappa_{us}\mathrm{sign}(\kappa_s)\boldsymbol{N}_s.$$

This gives an unsigned normal vector,

$$\boldsymbol{N}_{us} = \mathrm{sign}(\kappa_s)\boldsymbol{N}_s,$$

such that

$$\frac{d\boldsymbol{T}}{ds} = \kappa_{us}\boldsymbol{N}_{us}, \qquad \kappa_{us} \geq 0.$$

To confuse matters more, there is further ambiguity in our choice of $(\boldsymbol{N}_s, \kappa_s)$: we can have either

$$\kappa_s = \pm \frac{f''}{(1+f'^2)^{3/2}}, \qquad \boldsymbol{N}_s = \pm \frac{(-f', 1)}{\sqrt{1+f'^2}}.$$

Choosing the positive sign means that the definition of (signed) curvature agrees with the ordinary notion of curvature, as being a quantity proportional to the second derivative of the curve.

Because $(\boldsymbol{T}, \boldsymbol{N})$ live in the x-y plane for all time, it follows that $\boldsymbol{B}$ is in the z-direction:

$$\boldsymbol{B} = \hat{\boldsymbol{z}}.$$

Now

$$\tau \propto \frac{d\boldsymbol{B}}{dt},$$

hence

$$\tau = 0.$$

This makes sense: the torsion is actually a measure of how much the curve "twists" out of the plane generated by $(\boldsymbol{T}, \boldsymbol{N})$. Since the curve lies in this plane for all time, it is impossible for it to "twist" out of this plane, hence $\tau = 0$:

$\tau = 0$ for a curve that lives entirely in the x-y plane.

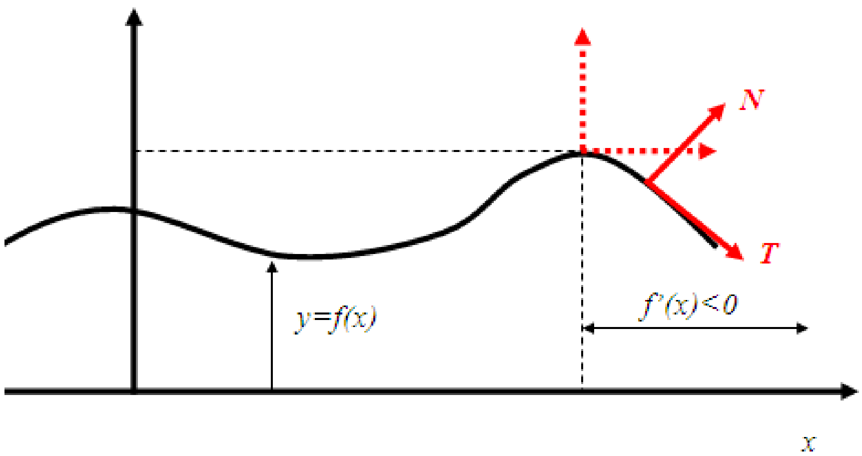

Figure 4.3: Normal and tangent vectors for a two-dimensional curve.

2. **A right-handed helix:** Consider the parametric equations

$$\begin{aligned} x(t) &= r\cos t, && (4.3a)\\ y(t) &= r\sin t, && (4.3b)\\ z(t) &= vt, \qquad t\in[0,\infty), \qquad r,v>0. && (4.3c) \end{aligned}$$

Graphically this corresponds to a right-handed helix. For, imagine a particle that follows the path (4.3). The particle does circular motion in the x-y plane and, at the same time, it moves up the z-axis. Moreover, if you coil your four fingers in the sense of the circular motion, your thumb points in the positive z-direction – the same direction of travel as the particle. Thus, the trajectory satisfies the right-hand rule.

First, we compute the tangent vector:

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \frac{d}{dt}\left(r\cos t, r\sin t, vt\right),\\ &= \left(-r\sin t, r\cos t, v\right),\\ \left|\frac{d\boldsymbol{x}}{dt}\right| &= \sqrt{r^2+v^2},\\ \frac{d\boldsymbol{x}}{dt}\Big/\left|\frac{d\boldsymbol{x}}{dt}\right| &= \frac{\left(-r\sin t, r\cos t, v\right)}{\sqrt{r^2+v^2}}. \end{aligned}$$

Hence

$$\boldsymbol{T} = \frac{\left(-r\sin t, r\cos t, v\right)}{\sqrt{r^2+v^2}}.$$

Also,

$$\frac{d\boldsymbol{T}}{dt} = \frac{\left(-r\cos t, -r\sin t, v\right)}{\sqrt{r^2+v^2}},$$

and

$$\begin{aligned} \frac{d\boldsymbol{T}}{dt}\Big/\left|\frac{d\boldsymbol{x}}{dt}\right| &= \frac{\left(-r\cos t, -r\sin t, 0\right)}{r^2+v^2},\\ &= \frac{r}{r^2+v^2}\left(-\cos t, -\sin t, 0\right),\\ &= \frac{d\boldsymbol{T}}{ds},\\ &= \kappa_s\boldsymbol{N}_s. \end{aligned}$$

Hence,

$$\boldsymbol{N}_s = \pm\left(-\cos t, -\sin t, 0\right), \qquad \kappa_s = \pm\frac{r}{r^2+v^2}.$$

Here, by taking the positive sign, the unsigned and signed curvatures agree:

$$\kappa_{us} = \kappa_s = \frac{r}{r^2+v^2} := \kappa;$$

hence, the signed and unsigned normal vectors also agree:

$$\boldsymbol{N}_{us} = \boldsymbol{N}_s = -\left(\cos t, \sin t, 0\right) := \boldsymbol{N}. \tag{4.4}$$

This means that $\boldsymbol{N}$ is an inward-pointing unit normal (the sign choice here is free and arbitrary choice). See Fig. 4.4 for more details. Here, the binormal points in the direction of motion (increasing z), which is a consequence of our having chosen the principal normal vector to be inward-pointing.

Next, we compute the torsion. We have,

$$\begin{aligned}
\boldsymbol{B} &= \boldsymbol{T}\times\boldsymbol{N},\\
&= \frac{1}{\sqrt{r^2+v^2}}\begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ -r\sin t & r\cos t & v \\ -\cos t & -\sin t & 0 \end{vmatrix},\\
&= \frac{1}{\sqrt{r^2+v^2}}\left(v\sin t, -v\cos t, r\right).
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{d\boldsymbol{B}}{dt} &= \frac{1}{\sqrt{r^2+v^2}}\left(v\cos t, v\sin t, 0\right),\\
\frac{d\boldsymbol{B}}{dt}\Big/\left|\frac{d\boldsymbol{x}}{dt}\right| &= \frac{v}{r^2+v^2}\left(\cos t, \sin t, 0\right),\\
&= -\frac{v}{r^2+v^2}\left(-\cos t, -\sin t, 0\right),\\
\frac{d\boldsymbol{B}}{ds} &= -\tau\boldsymbol{N}.
\end{aligned}$$

Hence,

$$\tau = \frac{v}{r^2+v^2}.$$

Thus, the conventional minus sign in the formula $d\boldsymbol{B}/ds = -\tau N$ conspires to make the torsion of a right-handed helix positive. Note also that the torsion remains positive regardless of whether we take $(+\boldsymbol{N}, +\kappa)$ or $(-\boldsymbol{N}, -\kappa)$ to be the normal-curvature pair.

Note finally that for a helix,

$$\frac{\tau}{\kappa} = \frac{v}{r}.$$

Hence,

The ratio of the torsion to the curvature is constant (t-independent) for a helix.

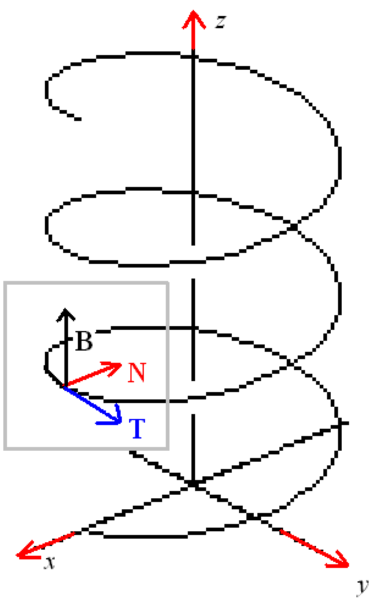

Figure 4.4: Frenet–Serret frame for a right-handed helix.

3. **A general curve:** We have, $x = t - t^3/3,\ y = t^2,\ z = t + t^3/3$.

We have,

$$
\begin{aligned}
\boldsymbol{x} &= (t - t^3/3)\hat{\boldsymbol{x}} + t^2\hat{\boldsymbol{y}} + (t + t^3/3)\hat{\boldsymbol{z}}, \\
\frac{d\boldsymbol{x}}{dt} &= (1 - t^2)\hat{\boldsymbol{x}} + 2t\hat{\boldsymbol{y}} + (1 + t^2)\hat{\boldsymbol{z}}, \\
\left|\frac{d\boldsymbol{x}}{dt}\right|^2 &= (1 - t^2)^2 + 4t^2 + (1 + t^2)^2, \\
&= 2(1 + 2t^2 + t^4) = 2(1 + t^2)^2. \\
\frac{ds}{dt} &= \left|\frac{d\boldsymbol{x}}{dt}\right| = \sqrt{2}(1 + t^2). \\
\boldsymbol{T} &= \frac{d\boldsymbol{x}}{ds} = \frac{d\boldsymbol{x}}{dt}\Big/\frac{ds}{dt} = \frac{(1 - t^2)\hat{\boldsymbol{x}} + 2t\hat{\boldsymbol{y}} + (1 + t^2)\hat{\boldsymbol{z}}}{\sqrt{2}(1 + t^2)}.
\end{aligned}
$$

Next,

$$\begin{aligned}
\frac{d\boldsymbol{T}}{dt} &= \frac{\{-2t\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + 2t\hat{\boldsymbol{z}}\}(1+t^2) - 2t\{(1-t^2)\hat{\boldsymbol{x}} + 2t\hat{\boldsymbol{y}} + (1+t^2)\hat{\boldsymbol{z}}\}}{\sqrt{2}(1+t^2)^2}, \\
&= \frac{-4t\hat{\boldsymbol{x}} + 2(1-t^2)\hat{\boldsymbol{y}}}{\sqrt{2}(1+t^2)^2}. \\
\frac{d\boldsymbol{T}}{ds} &= \frac{d\boldsymbol{T}}{dt}\Big/\frac{ds}{dt}, \\
&= \frac{-4t\hat{\boldsymbol{x}} + 2(1-t^2)\hat{\boldsymbol{y}}}{2(1+t^2)^3} = \frac{-2t\hat{\boldsymbol{x}} + (1-t^2)\hat{\boldsymbol{y}}}{(1+t^2)^3}.
\end{aligned}$$

Using the second FS equation,

$$\left(\frac{d\boldsymbol{T}}{ds}\right)^2 = \kappa^2\boldsymbol{N}^2 = \kappa^2,$$

$$\kappa^2 = \left|\frac{d\boldsymbol{T}}{ds}\right|^2 = \frac{4t^2 + (1-t^2)^2}{(1+t^2)^6} = \frac{(1+t^2)^2}{(1+t^2)^6} = \frac{1}{(1+t^2)^4}.$$

Again, we take $\kappa_s = \kappa_{us} = 1/(1+t^2)^2$: because the unsigned curvature is positive definite, there is no need for the labels 's' and 'us'. Thus, we unambiguously use the formula

$$\boldsymbol{N} = \frac{1}{\kappa}\frac{d\boldsymbol{T}}{ds}$$

and compute

$$\boldsymbol{N} = \frac{(1+t^2)^2\{-2t\hat{\boldsymbol{x}} + (1-t^2)\hat{\boldsymbol{y}}\}}{(1+t^2)^3} = \frac{-2t\hat{\boldsymbol{x}} + (1-t^2)\hat{\boldsymbol{y}}}{1+t^2}.$$

Furthermore,

$$\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N},$$

hence

$$\begin{aligned}
\boldsymbol{B} &= \frac{1}{\sqrt{2}(1+t^2)^2}\begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ (1-t^2) & 2t & (1+t^2) \\ -2t & (1-t^2) & 0 \end{vmatrix} \\
&= \frac{-(1-t^2)(1+t^2)\hat{\boldsymbol{x}} - 2t(1+t^2)\hat{\boldsymbol{y}} + (1+t^2)^2\hat{\boldsymbol{z}}}{\sqrt{2}(1+t^2)^2} \\
&= \frac{(t^2-1)\hat{\boldsymbol{x}} - 2t\hat{\boldsymbol{y}} + (1+t^2)\hat{\boldsymbol{z}}}{\sqrt{2}(1+t^2)}.
\end{aligned}$$

Next, we compute

$$\begin{aligned}
\frac{d\boldsymbol{B}}{dt} &= \frac{(1+t^2)\{2t\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} + 2t\hat{\boldsymbol{z}}\} - 2t\{(t^2-1)\hat{\boldsymbol{x}} - 2t\hat{\boldsymbol{y}} + (1+t^2)\hat{\boldsymbol{z}}\}}{\sqrt{2}(1+t^2)^2} \\
&= \frac{\sqrt{2}\{2t\hat{\boldsymbol{x}} - (1-t^2)\hat{\boldsymbol{y}}\}}{(1+t^2)^2}. \\
\frac{d\boldsymbol{B}}{ds} &= \frac{d\boldsymbol{B}}{dt} \Big/ \frac{ds}{dt}, \\
&= \frac{2t\hat{\boldsymbol{x}} - (1-t^2)\hat{\boldsymbol{y}}}{(1+t^2)^3}, \\
&= -\frac{1}{(1+t^2)^2}\frac{-2t\hat{\boldsymbol{x}} + (1-t^2)\hat{\boldsymbol{y}}}{(1+t^2)}, \\
&= -\frac{\boldsymbol{N}}{(1+t^2)^2}.
\end{aligned}$$

Now

$$\frac{d\boldsymbol{B}}{ds} = -\tau\boldsymbol{N}.$$

Therefore

$$\tau = \frac{1}{(1+t^2)^2}.$$

In conclusion, the space curve $x = t - t^3/3,\ y = t^2,\ z = t + t^3/3$ has curvature

$$\kappa = \frac{1}{(1+t^2)^2}$$

and torsion

$$\tau = \frac{1}{(1+t^2)^2}.$$

Chapter 5

Partial derivatives and fields

Overview

In this section we formulate the theory of scalar functions of several variables and learn how to differentiate such functions. We introduce the **gradient and curl operators**. Then, we introduce vector fields and learn how to differentiate them. First, we focus on partial derivatives. Although elementary partial differentiation is covered elsewhere (e.g. MATH 20060), it is repeated briefly here: it is important to get it right!

5.1 Partial derivatives

A function $\phi\left(x_1, x_2, \cdots, x_n\right)$ of n variables is a map from a subset of $\mathbb{R}^n$ to $\mathbb{R}$:

$$\begin{aligned} \phi : (\Omega \subset \mathbb{R}) &\rightarrow \mathbb{R} \\ (x_1, x_2, \cdots, x_n) &\rightarrow \phi\left(x_1, x_2, \cdots, x_n\right). \end{aligned}$$

Examples:

- The elevation above sea level at any point in Ireland is a function of latitude and longitude;
- The pressure of an ideal gas is a function of temperature and density (Boyle's Law);
- The quantity theory of money says that the GDP of an economy is a function of the velocity of money and the quantity of (broad) money in circulation.

The function ϕ assigns to each point $(x_1, x_2, \cdots x_n) \in \Omega$ a real number (scalar), and is therefore called a **scalar field**.

In this section, we shall consider functions of two variables (x, y); the generalization to three or more variables is straightforward. When we are given such a function, it is natural to ask how the function varies as x changes, and as y changes. Equivalently, we want to know how the function changes as we move in the 'x-direction', and in the 'y'-direction'. Thus, we make small variations in the x-coordinate, keeping y fixed:

$$\phi\left(x + \delta x, y\right).$$

Then, we form the quotient

$$\frac{\phi\left(x + \delta x, y\right) - \phi\left(x, y\right)}{\delta x}.$$

Taking $\delta x \to 0$, we obtain the *partial derivative of ϕ w.r.t. x (keeping y fixed)*:

$$\frac{\partial \phi}{\partial x}\left(x, y\right) = \lim_{\delta x \to 0} \frac{\phi\left(x + \delta x, y\right) - \phi\left(x, y\right)}{\delta x}.$$

Similarly, we have a partial derivative with w.r.t. y keeping x fixed: First, we form the quotient

$$\frac{\phi\left(x, y + \delta y\right) - \phi\left(x, y\right)}{\delta y},$$

then we take the limit as $\delta y \to 0$:

$$\frac{\partial \phi}{\partial y}\left(x, y\right) = \lim_{\delta y \to 0} \frac{\phi\left(x, y + \delta y\right) - \phi\left(x, y\right)}{\delta y}.$$

Thus, to form a partial derivative in the x-direction, you treat y as a constant and do ordinary differentiation on the x-variable.

Examples

1. The function $\phi\left(x, y\right) = x^2 + y^2$. Let us hold y fixed and differentiate w.r.t. x:

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2\right) = \frac{\partial}{\partial x}\left(x^2 + \text{Const.}\right) = \frac{\partial}{\partial x}\left(x^2\right) = 2x.$$

   Now hold x fixed and differentiate w.r.t. y:

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2\right) = \frac{\partial}{\partial y}\left(\text{Const.} + y^2\right) = \frac{\partial}{\partial y}\left(y^2\right) = 2y.$$

2. The function $\phi(x,y) = x/y$. Let us hold y fixed and differentiate w.r.t. x:

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\frac{x}{y} = \frac{\partial}{\partial x}\frac{x}{\text{Const.}} = \frac{1}{\text{Const.}} = \frac{1}{y}$$

Now hold x fixed and differentiate w.r.t. y:

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\frac{x}{y} = \frac{\partial}{\partial y}\frac{\text{Const.}}{y} = -\frac{\text{Const.}}{y^2} = -\frac{x}{y^2}$$

3. The **function of three variables** $\phi(x,y,z) = 1/\sqrt{x^2+y^2+z^2}$. Let us hold y **and** z fixed and differentiate w.r.t. x:

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= \frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+\text{Const.}}} = \frac{\partial}{\partial x}\left(x^2+\text{Const.}\right)^{-1/2} \\ &= -\tfrac{1}{2}\left(x^2+\text{Const.}\right)^{-3/2}(2x) = -\frac{x}{\left(x^2+\text{Const.}\right)^{3/2}} = -\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\end{aligned}$$

Now hold x **and** z fixed and differentiate w.r.t. y:

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{\partial}{\partial x}\frac{1}{\sqrt{\text{Const.}+y^2}} = \frac{\partial}{\partial x}\left(\text{Const.}+y^2\right)^{-1/2} \\ &= -\tfrac{1}{2}\left(\text{Const.}+y^2\right)^{-3/2}(2y) = -\frac{y}{\left(\text{Const.}+y^2\right)^{3/2}} = -\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\end{aligned}$$

Lastly, we hold x and y fixed and differentiate w.r.t. z:

$$\begin{aligned}\frac{\partial \phi}{\partial z} &= \frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{\partial}{\partial z}\frac{1}{\sqrt{\text{Const.}+z^2}} = \frac{\partial}{\partial z}\left(\text{Const.}+z^2\right)^{-1/2} \\ &= -\tfrac{1}{2}\left(\text{Const.}+z^2\right)^{-3/2}(2z) = -\frac{z}{\left(\text{Const.}+z^2\right)^{3/2}} = -\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\end{aligned}$$

Pedantic notation

1. When the function ϕ is in fact a function of a single variable only ($\phi = \phi(x)$, say) there is no difference between $\partial/\partial x$ and d/dx. In that case, $\partial\phi/\partial x = d\phi/dx = \phi'(x)$.

2. To save chalk, we will sometimes write $\partial\phi/\partial x$ as $\partial_x\phi$ or even ϕ_x. A similar notation holds for partial derivatives w.r.t. y and z.

5.2 The gradient operator in three dimensions

Let ϕ be a function of three variables,

$$\begin{aligned} \phi : \left(\Omega \subset \mathbb{R}^3\right) &\to \mathbb{R} \\ (x, y, z) &\to f(x, y, z). \end{aligned}$$

Then the gradient operator acting on ϕ is a vector with the following form:

$$\mathrm{grad}\,\phi := \hat{\boldsymbol{x}}\frac{\partial \phi}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial \phi}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial \phi}{\partial z}.$$

In class, we will write this vector as $\nabla\phi$, and call it 'grad ϕ' or 'nabla ϕ'.

Examples

1. The function $f(x, y, z) = x^2 + y^2 + z^2$. We know that $\partial_x \phi = 2x$, $\partial_y \phi = 2y$, and $\partial_z \phi = 2z$. Hence,

$$\nabla\phi = \hat{\boldsymbol{x}}\partial_x\phi + \hat{\boldsymbol{y}}\partial_y\phi + \hat{\boldsymbol{z}}\partial_z\phi = 2\hat{\boldsymbol{x}}x + 2\hat{\boldsymbol{y}}y + 2z\hat{\boldsymbol{z}} = 2(x, y, z) = 2\boldsymbol{x},$$

where $\boldsymbol{x}$ is a position vector.

2. The function $f(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$. We know that

$$\begin{aligned} \partial_x\phi &= -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \\ \partial_y\phi &= -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \\ \partial_z\phi &= -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}. \end{aligned}$$

Hence,

$$\begin{aligned} \nabla\phi = \hat{\boldsymbol{x}}\left(-\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right) + \hat{\boldsymbol{y}}\left(-\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right)\hat{\boldsymbol{z}}\left(-\frac{z}{(x^2 + y^2 + z^2)^{3/2}}\right) \\ = -\frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3}. \end{aligned}$$

5.3 The physical meaning of the gradient

In three dimensions, the surface is specified by an equation of the type

$$\phi(x, y, z) = 0.$$

This is the generic equation for a surface because if ϕ is sufficiently smooth, it can be inverted in the neighbourhood of a given point and an expression of the kind $z = z(x, y)$ can be found, which gives a surface (Fig. 5.1). Suppose that $\boldsymbol{x} = (x, y, z)$ satisfies $\phi = 0$. Then $\nabla\phi$ evaluated at $\boldsymbol{x}$ is normal to the surface. To prove this, we take $\boldsymbol{x} + \delta\boldsymbol{x}$, a neighbouring point of $\boldsymbol{x}$ that still resides on the surface. We form the difference

$$\begin{aligned} 0 &= 0 - 0, \\ &= \phi(\boldsymbol{x} + \delta\boldsymbol{x}) - \phi(\boldsymbol{x}), \\ &= \phi(x + \delta x, y + \delta y, z + \delta z) - \phi(x, y, z), \\ &= \frac{\partial\phi}{\partial x}(x, y, z)\delta x + \frac{\partial\phi}{\partial y}(x, y, z)\delta y + \frac{\partial\phi}{\partial z}(x, y, z)\delta z + \text{H.O.T.}, \\ &= \nabla\phi \cdot \delta\boldsymbol{x}. \end{aligned}$$

But $\boldsymbol{x}$ and $\boldsymbol{x} + \delta\boldsymbol{x}$ are vectors whose tip lies on the surface (Fig. 5.2). Hence, $\delta\boldsymbol{x}$ is tangent to the surface, and $\nabla\phi \cdot \delta\boldsymbol{x} = 0$, so $\nabla\phi(x, y, z)$ is normal to the surface.

The directional derivative

Suppose we have a scalar field $\phi(x, y, z)$ and we want to know how it changes in a given, fixed direction $\boldsymbol{e}$. The way to do this is to form the difference

$$\delta\phi = \phi(\boldsymbol{x} + t\boldsymbol{e}) - \phi(\boldsymbol{x}),$$

where t is a parameter that takes all real values. In particular, let t be small. Then we have

$$\begin{aligned} \delta\phi &= \phi(x + te_x, y + te_y z, z + te_z) - \phi(x, y, z), \\ &= \frac{\partial\phi}{\partial x} te_x + \frac{\partial\phi}{\partial y} te_y + \frac{\partial\phi}{\partial z} te_z, \\ &= (\nabla\phi) \cdot \boldsymbol{e}t, \\ &= \boldsymbol{e} \cdot \nabla\phi\, t. \end{aligned}$$

Hence,

$$\frac{\delta\phi}{t} = \boldsymbol{e} \cdot \nabla\phi.$$

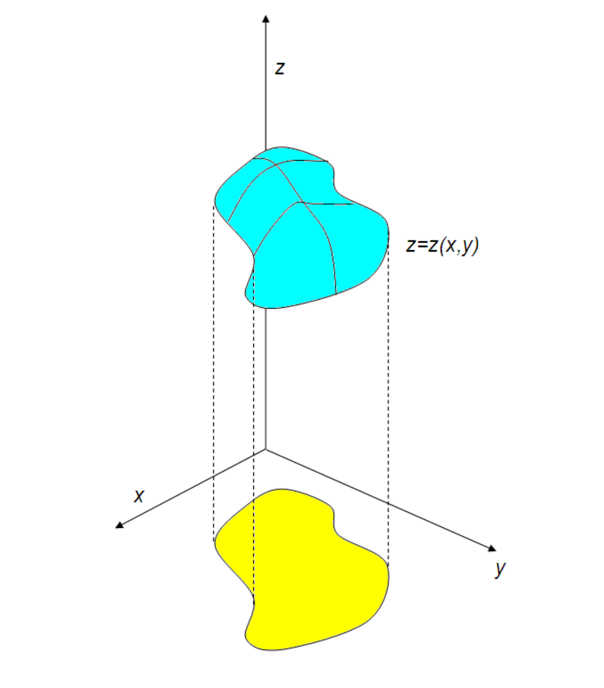

Figure 5.1: A schematic description of a surface described by $z = z(x, y)$.

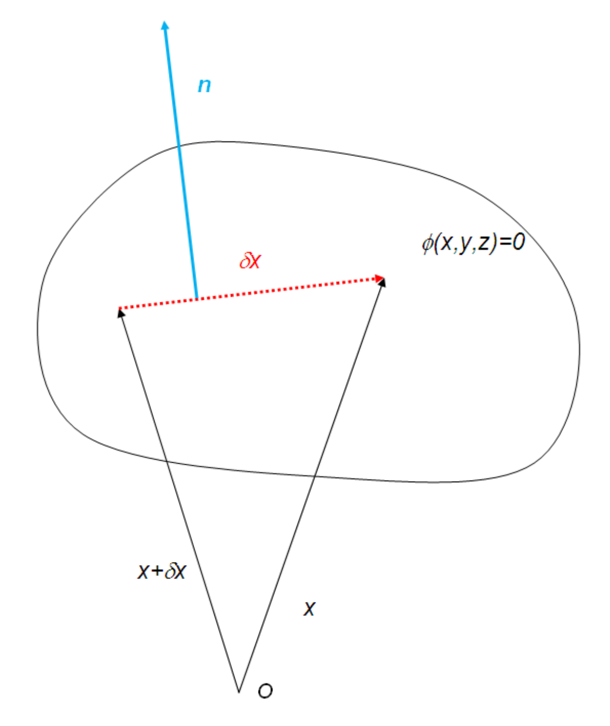

Figure 5.2: A schematic description of the normal to a surface, $\boldsymbol{n} = \nabla\phi$.

The Taylor approximation becomes exact when $t \to 0$:

$$\boldsymbol{e} \cdot \nabla\phi = \lim_{t\to 0} \frac{\delta\phi}{t}.$$

This is **the directional derivative in the direction** $\boldsymbol{e}$:

$$\frac{d\phi}{de} := \boldsymbol{e} \cdot \nabla\phi.$$

5.4 Vector fields and the divergence operator

A **vector field** $\boldsymbol{v}(x, y, z)$ in $\mathbb{R}^3$ is a map that assigns to each element of its domain $\Omega \subset \mathbb{R}^3$ a uniquely determined vector, also in $\mathbb{R}^3$. In map language,

$$\begin{aligned} \boldsymbol{v} : \left(\Omega \subset \mathbb{R}^3\right) &\to \mathbb{R}^3, \\ (x, y, z) &\to \boldsymbol{v}(x, y, z). \end{aligned}$$

Since $\boldsymbol{v}(x, y, z)$ is a vector, we can write

$$\begin{aligned}\boldsymbol{v}(x,y,z) &= \hat{\boldsymbol{x}} v_1(x,y,z) + \hat{\boldsymbol{y}} v_2(x,y,z) + \hat{\boldsymbol{z}} v_3(x,y,z) \\ &= = (v_1(x,y,z), v_2(x,y,z), v_3(x,y,z)).\end{aligned}$$

Example: if $\phi(x, y, z)$ is a scalar field, then

$$\nabla\phi = \hat{\boldsymbol{x}}\partial_x\phi + \hat{\boldsymbol{y}}\partial_y\phi + \hat{\boldsymbol{z}}\partial_z\phi$$

is a vector field.

The **divergence** of a vector field $\boldsymbol{v}(x, y, z)$ is a scalar computed as follows:

$$\operatorname{div}\boldsymbol{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

Formally, this is like 'dotting' ∇ with $\boldsymbol{v}$, so we write

$$\operatorname{div}\boldsymbol{v} = \nabla\cdot\boldsymbol{v}.$$

There is one crucial difference between ordinary dot products and the divergence: for ordinary vectors $\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{A}$; for vector fields, $\nabla\cdot\boldsymbol{v}$ is NOT equal to $\boldsymbol{v}\cdot\nabla$.

Examples

1. If $\boldsymbol{v}(x, y, z) = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z$, then

$$\operatorname{div}\boldsymbol{v} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

2. Consider a vector field

$$v_1 = \frac{\partial\psi}{\partial y},\ v_2 = -\frac{\partial\psi}{\partial x},\ v_3 = 0.$$

Then

$$\begin{aligned}\operatorname{div}\boldsymbol{v} &= \frac{\partial}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial z}0, \\ &= \frac{\partial}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\psi}{\partial x}.\end{aligned}$$

We now use the remarkable fact that **the partial derivatives of smooth functions com-**

mute, $\partial_{xy}\psi = \partial_{yx}\psi$ to obtain

$$\operatorname{div} \boldsymbol{v} = 0.$$

A vector field whose divergence is zero is called **incompressible**.

3. Consider the vector field

$$\boldsymbol{v} = \hat{\boldsymbol{x}}\partial_x\phi + \hat{\boldsymbol{y}}\partial_y\phi + \hat{\boldsymbol{z}}\partial_z\phi,$$

where $\phi(x, y, z)$ is some scalar field. The divergence of $\boldsymbol{v}$ is

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= \partial_x v_1 + \partial_y v_2 + \partial_z v_3, \\ &= \partial_x\partial_x\phi + \partial_y\partial_y\phi + \partial_z\partial_z\phi, \\ &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \end{aligned}$$

This particular operation on the scalar field ϕ is quite common in physics and is therefore given its own name: it is called the **Laplacian**, and given the notation ∇^2 (or Δ)

$$\nabla^2\phi \text{ or } \Delta\phi := \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

5.5 The physical meaning of the divergence

Consider a fluid that flows in a three-dimensional container. We take a small cuboid of sides of length Δx, Δy and Δz as a control volume, one of whose vertices lies at (x, y, z). Fluid flows into and out of the cuboid with velocity $\boldsymbol{v}(x, y, z, t)$ (the t is for time). The amount of mass leaving the system through the x-direction per unit time is

$$\begin{aligned} \frac{\text{Mass Out} - \text{Mass In}}{\text{Time}} \text{ in the } x \text{ direction} &= (\rho v_1)(x + \Delta x, y, z)\,\Delta y\Delta z - (\rho v_1)(x, y, z)\,\Delta y\Delta z, \\ &= \frac{\partial}{\partial x}(\rho v_1)\bigg|_{(x,y,z)} \Delta x\Delta y\Delta z + \text{H.O.T.}, \end{aligned}$$

where $\rho(x, y, z, t)$ is the scalar **fluid density** (Fig. 5.3). Similarly,

$$\begin{aligned} \frac{\text{Mass Out} - \text{Mass In}}{\text{Time}} \text{ in the } y \text{ direction} &= \frac{\partial}{\partial y}(\rho v_2)\bigg|_{(x,y,z)} \Delta x\Delta y\Delta z + \text{H.O.T.}, \\ \frac{\text{Mass Out} - \text{Mass In}}{\text{Time}} \text{ in the } z \text{ direction} &= \frac{\partial}{\partial z}(\rho v_3)\bigg|_{(x,y,z)} \Delta x\Delta y\Delta z + \text{H.O.T.}, \end{aligned}$$

Adding them,

$$\begin{aligned}
\frac{\text{Total Mass Out} - \text{Total Mass In}}{\text{Time}} &= \left[\frac{\partial}{\partial x}(\rho v_1) + \frac{\partial}{\partial y}(\rho v_3) + \frac{\partial}{\partial z}(\rho v_3)\right]_{(x,y,z)} \Delta x \Delta y \Delta z \\
&= \nabla \cdot (\rho \boldsymbol{v})\Big|_{(x,y,z)} \Delta x \Delta y \Delta z.
\end{aligned}$$

Now in this control volume, matter is not created or destroyed, so the change in the mass in the control volume over time must be balanced by changes in the density over time:

$$\begin{aligned}
\frac{\text{Total Mass Out} - \text{Total Mass In}}{\text{Time}} &= \frac{\text{Change in mass}}{\text{Time}} \\
&= -\frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z).
\end{aligned}$$

Why is there a minus sign here? Well, if Total Mass Out $>$ Total Mass in, then the LHS will be a positive quantity. At the same time, $\partial \rho/\partial t$ will be negative (the box is losing mass). Therefore, in order for the signs to balance, we need

$$\text{sign}(\text{LHS}) = +1 = \text{sign}(\text{RHS}) = \text{sign}\left[(-1)\partial\rho/\partial t\right].$$

Finally, we equate these two identical changes and take the constant volume element $\Delta x \Delta y \Delta z$ outside the time derivative:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0.$$

This result is called the **continuity equation** and holds whenever some continuum quantity is conserved (e.g. mass or charge).

5.6 The curl of a vector

Let $\boldsymbol{v}(x,y,z) = \hat{\boldsymbol{x}} v_1(x,y,z) + \hat{\boldsymbol{y}} v_2(x,y,z) + \hat{\boldsymbol{z}} v_3(x,y,z)$ be a vector field. The curl of $\boldsymbol{v}$ is a new vector field formed as follows

$$\begin{aligned}
\text{curl}\,\boldsymbol{v} &:= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ v_1 & v_2 & v_3 \end{vmatrix}, \\
&= \hat{\boldsymbol{x}}(\partial_y v_3 - \partial_z v_2) + \hat{\boldsymbol{y}}(\partial_z v_1 - \partial_x v_3) + \hat{\boldsymbol{z}}(\partial_x v_2 - \partial_y v_1). \qquad (5.1)
\end{aligned}$$

Because this is like the ordinary cross product of two vectors, we write

$$\text{curl}\,\boldsymbol{v} = \nabla \times \boldsymbol{v}.$$

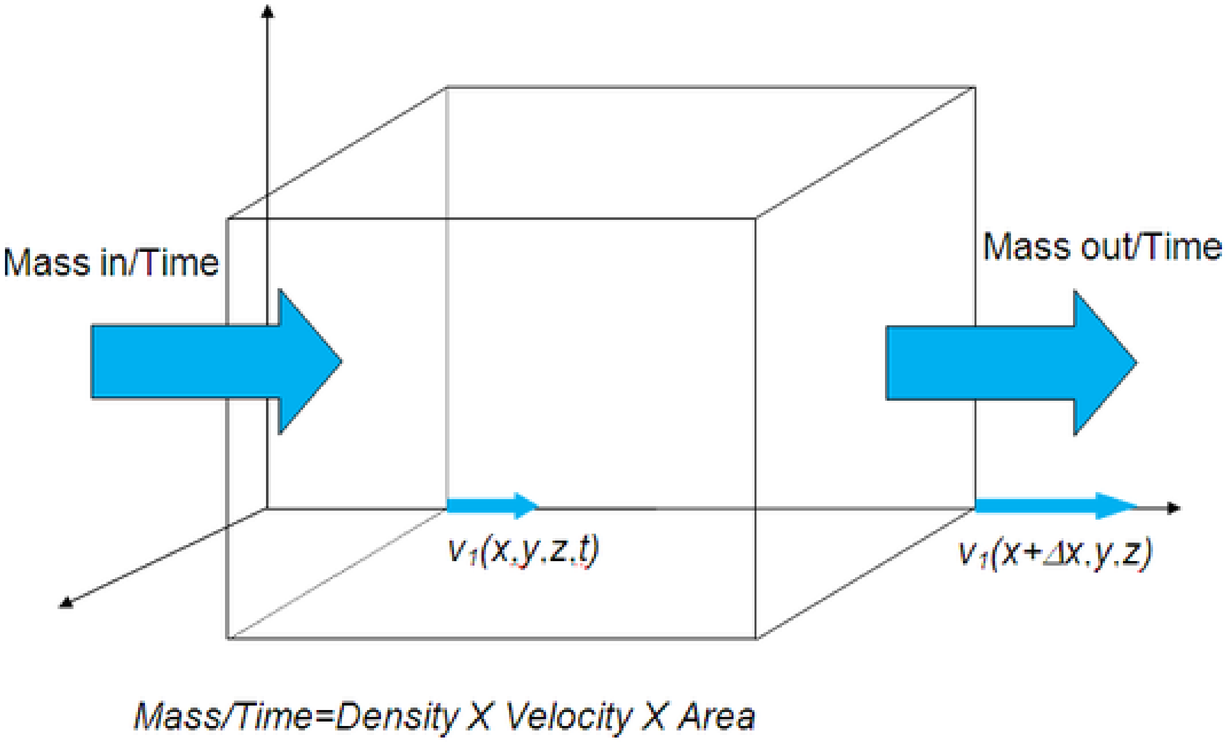

Figure 5.3: The physical meaning of the divergence

There is one crucial difference between ordinary cross products and the curl: for ordinary vectors $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$; for vector fields, $\nabla \times \boldsymbol{v}$ is NOT equal to $-\boldsymbol{v} \times \nabla$.

5.7 The physical meaning of the curl

Consider the following vector field:

$$\boldsymbol{v}(x, y, z) = y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}}.$$

Imagine that this represents the velocity of a fluid in a container. We can plot the vector field by drawing a little arrow at random points (x, y) in two-dimensional space (Fig. 5.4). The arrow should have length $\sqrt{x^2 + y^2}$ and point in the direction $y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}}$. Simply by inspection, we see that the field is rotating. If we stick a paddle wheel anywhere in the fluid, it will be carried by the flow and rotate clockwise. Using the right-hand rule, we expect the curl to be into the page. If we are to keep a right-handed coordinate system, the negative z-direction must point into the page. Moreover, we can apply the formula

$$\omega = \frac{v}{r}$$

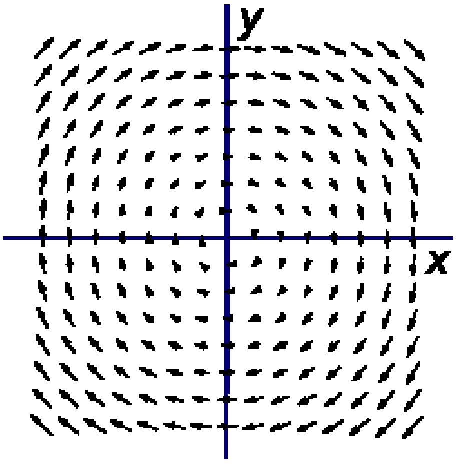

Figure 5.4: The vector field $\boldsymbol{v} = y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}}$.

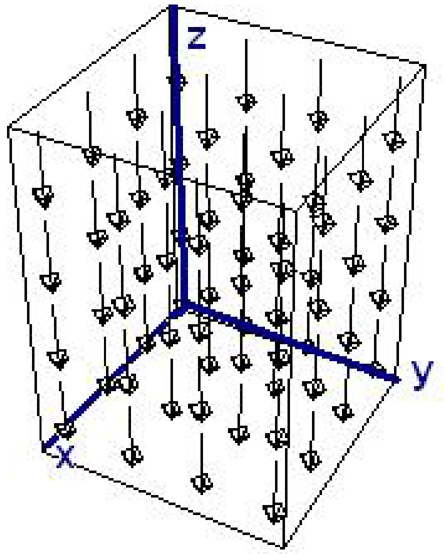

Figure 5.5: The curl of the vector field $\boldsymbol{v} = y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}}$.

for ordinary circular motion, to the vector field $\boldsymbol{v}$, giving

$$\omega = \frac{|y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}}|}{\sqrt{x^2 + y^2}} = 1$$

Thus, the 'amount of rotation' in the vector field is constant (independent of position), and the sense of rotation is into the page.

Now, we calculate the curl:

$$\nabla \times \boldsymbol{v} = 0\hat{\boldsymbol{x}} + 0\hat{\boldsymbol{y}} + \left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}y\right]\hat{\boldsymbol{z}} = -2\hat{\boldsymbol{z}};$$

it is indeed in the negative z-direction. It is also a constant! Thus, the curl corresponds to our intuitive idea about the amount of rotation in a vector field.

Plotting the curl of $\boldsymbol{v}$ is not very interesting (Fig. 5.5). Nevertheless, we see the physical meaning of curl: **it tells us by how much a vector field is rotating, and in what sense**.

5.8 Formulas involving div, grad, and curl

Let $\phi(x, y, z)$ and $\psi(x, y, z)$ be differentiable scalar fields and let $\boldsymbol{u}(x, y, z)$ and $\boldsymbol{v}(x, y, z)$ be differentiable vector fields. Then the following identities hold:

1. $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$;

2. $\nabla \cdot (\boldsymbol{u} + \boldsymbol{v}) = \nabla \cdot \boldsymbol{u} + \nabla \cdot \boldsymbol{v}$;
3. $\nabla \times (\boldsymbol{u} + \boldsymbol{v}) = \nabla \times \boldsymbol{u} + \nabla \times \boldsymbol{v}$;
4. $\nabla \cdot (\phi \boldsymbol{u}) = (\nabla \phi) \cdot \boldsymbol{u} + \phi(\nabla \cdot \boldsymbol{u})$;
5. $\nabla \times (\phi \boldsymbol{u}) = (\nabla \phi) \times \boldsymbol{u} + \phi(\nabla \times \boldsymbol{u})$;
6. $\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v})$;
7. $\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla) \, \boldsymbol{u} - \boldsymbol{v} \, (\nabla \cdot \boldsymbol{u}) - (\boldsymbol{u} \cdot \nabla) \, \boldsymbol{v} + \boldsymbol{u} \, (\nabla \cdot \boldsymbol{v})$;
8. $\nabla \, (\boldsymbol{u} \cdot \boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla) \, \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \, \boldsymbol{v} + \boldsymbol{v} \times (\nabla \times \boldsymbol{u}) + \boldsymbol{u} \times (\nabla \times \boldsymbol{v})$;
9. $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla \, (\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}$.

Properties 1–3 are obvious; the others are tricky and some of them will appear as exercises. Note that if λ and μ are a scalars (*constant* real numbers), then

$$\nabla \, (\lambda \phi + \mu \psi) = \lambda \nabla \phi + \mu \nabla \psi,$$

and similarly, for vector fields,

$$\nabla \cdot (\lambda \boldsymbol{u} + \mu \boldsymbol{v}) = \lambda \nabla \cdot \boldsymbol{u} + \mu \nabla \cdot \boldsymbol{v},$$

and

$$\nabla \times (\lambda \boldsymbol{u} + \mu \boldsymbol{v}) = \lambda \nabla \times \boldsymbol{u} + \mu \nabla \times \boldsymbol{v}.$$

This is the property of **linearity**. The operations div, grad, and curl thus take vector or scalar fields and map them linearly to other vector or scalar fields. They are thus called **linear operators**. In the next chapter we will gain more proficiency in handling these operators.

Chapter 6

Techniques in vector differentiation

Overview

In this section we gain more familiarity with the vector operators div, grad, and curl by doing a number of examples.

6.1 Worked example

1. If $\boldsymbol{u} = 2x^2\hat{\boldsymbol{x}} - 3yz\hat{\boldsymbol{y}} + xz^2\hat{\boldsymbol{z}}$ and $\phi = 2z - x^3y$, find $\boldsymbol{u} \cdot \nabla\phi$ at the point $(1, -1, 1)$ and $\boldsymbol{u} \times \nabla\phi$ at the point $(1, -1, 1)$.

   We have

$$
\begin{aligned}
\boldsymbol{u} &= 2x^2\hat{\boldsymbol{x}} - 3yz\hat{\boldsymbol{y}} + xz^2\hat{\boldsymbol{z}}, \\
\phi &= 2z - x^3y, \\
\operatorname{grad}\phi &= -3x^2y\,\hat{\boldsymbol{x}} - x^3\,\hat{\boldsymbol{y}} + 2\,\hat{\boldsymbol{z}}.
\end{aligned}
$$

   At $(1, -1, 1)$,

$$
\begin{aligned}
\boldsymbol{u} &= 2\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}, \\
\operatorname{grad}\phi &= 3\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} + 2\hat{\boldsymbol{z}}, \\
\boldsymbol{u} \cdot \operatorname{grad}\phi &= 6 - 3 + 2 = 5.
\end{aligned}
$$

Also,

$$\boldsymbol{u} \times \operatorname{grad}\phi = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= 7\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} - 11\ \hat{\boldsymbol{z}}.$$

2. If $\nabla\phi = 2xyz^3\hat{\boldsymbol{x}} + x^2z^3\hat{\boldsymbol{y}} + 3x^2yz^2\hat{\boldsymbol{z}}$, find $\phi(x, y, z)$ if $\phi(1, -2, 2) = 4$.

   We have

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= 2xyz^3, \\
\phi &= x^2yz^3 + f(y, z), \\
\frac{\partial\phi}{\partial y} &= x^2z^3, \\
\phi &= x^2yz^3 + g(z, x), \\
\frac{\partial\phi}{\partial z} &= 3x^2yz^2, \\
\phi &= x^2yz^3 + h(x, y).
\end{aligned}$$

   Therefore

$$f(y, z) = g(z, x) = h(x, y) = c \text{ (constant)}$$

   and

$$\phi(x, y, z) = x^2yz^3 + c.$$

$$\phi(1, -2, 2) = -16 + c = 4, \qquad c = 20,$$

$$\phi(x, y, z) = x^2yz^3 + 20.$$

3. Find the unit outward drawn normal to the surface $(x-1)^2 + y^2 + (z+2)^2 = 9$ at the point $(3, 1, -4)$.

   Solution: Let $\phi = (x-1)^2 + y^2 + (z+2)^2 - 9$. Then

$$\operatorname{grad}\phi = 2(x-1)\,\hat{\boldsymbol{x}} + 2y\,\hat{\boldsymbol{y}} + 2(z+2)\,\hat{\boldsymbol{z}}.$$

   At $(3, 1, -4)$, $\operatorname{grad}\phi = 4\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} - 4\hat{\boldsymbol{z}}$.

   Unit outward drawn normal:

$$\boldsymbol{n} = \frac{\operatorname{grad}\phi}{|\operatorname{grad}\phi|} = \frac{4\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} - 4\hat{\boldsymbol{z}}}{\sqrt{16+4+16}} = \frac{2\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - 2\hat{\boldsymbol{z}}}{3}.$$

4. Find the equation for the tangent plane and the equation (not just the direction) of the normal line to the surface $z = x^2 + y^2$ at the point $(2, -1, 5)$

   Solution: Let $\phi = x^2 + y^2 - z$. Then

$$\operatorname{grad}\phi = 2x\hat{\boldsymbol{x}} + 2y\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}.$$

   At $(2, -1, 5)$, $\operatorname{grad}\phi = 4\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}$.

   Normal (not necessarily a unit normal):

$$\boldsymbol{n} = \operatorname{grad}\phi = 4\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}.$$

   Let $\boldsymbol{r}_0 = 2\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}$. The tangent plane at $\boldsymbol{r}_0$ is given by

$$\begin{gathered}
(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \boldsymbol{n} = 0, \\
4(x-2) - 2(y+1) - (z-5) = 0, \\
4x - 2y - z = 5.
\end{gathered}$$

   Normal line:

$$\begin{gathered}
\boldsymbol{r} = \boldsymbol{r}_0 + \lambda\boldsymbol{n}, \\
\boldsymbol{r} - \boldsymbol{r}_0 = \lambda\boldsymbol{n}, \\
(x-2)\,\hat{\boldsymbol{x}} + (y+1)\,\hat{\boldsymbol{y}} + (z-5)\,\hat{\boldsymbol{z}} = \lambda(4\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}), \\
x - 2 = 4\lambda, \qquad y + 1 = -2\lambda, \qquad z - 5 = -\lambda, \\
\frac{x-2}{4} = \frac{y+1}{-2} = \frac{z-5}{-1} \ (= -\lambda).
\end{gathered}$$

6.2 Worked example

Show that $\boldsymbol{u} = (6xy + z^3)\hat{\boldsymbol{x}} + (3x^2 - z)\hat{\boldsymbol{y}} + (3xz^2 - y)\hat{\boldsymbol{z}}$ is irrotational ($\nabla \times \boldsymbol{u} = 0$). Find ϕ such that $\boldsymbol{u} = \nabla\phi$.

We have,

$$\begin{aligned}
\operatorname{curl}\boldsymbol{u} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\[2em] \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\[2em] 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} \\
&= (-1+1)\,\hat{\boldsymbol{x}} - (3z^2 - 3z^2)\,\hat{\boldsymbol{y}} + (6x - 6x)\,\hat{\boldsymbol{z}} = \mathbf{0}.
\end{aligned}$$

Suppose $\boldsymbol{u} = \operatorname{grad}\phi$.

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= 6xy + z^3, \\
\phi &= 3x^2y + xz^3 + f(y,z), \\
\frac{\partial\phi}{\partial y} &= 3x^2 - z, \\
\phi &= 3x^2y - yz + g(z,x), \\
\frac{\partial\phi}{\partial z} &= 3xz^2 - y, \\
\phi &= xz^3 - yz + h(x,y).
\end{aligned}$$

Therefore

$$f(y,z) = -yz + c, \qquad g(z,x) = xz^3 + c, \qquad h(x,y) = 3x^2y + c.$$

and

$$\phi(x,y,z) = 3x^2y + xz^3 - yz + c.$$

Caution: In this example, the final answer is of the form $\phi(x,y,z) = f(y,z) + g(z,x) + h(x,y)$. However, this is not true in general, e.g. $\boldsymbol{u} = \nabla\phi$, where $\phi = z\mathrm{e}^{xy}$ does not decompose into a sum like the one in this example.

6.3 Proofs

Show that $\nabla \times (\nabla \phi) = 0$, for any differentiable scalar field $\phi(x, y, z)$.

We have

$$\begin{aligned} \nabla \times (\nabla \phi) &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ \partial_x \phi & \partial_y \phi & \partial_z \phi \end{vmatrix} \\ &= \hat{\boldsymbol{x}} \left(\partial_x \partial_y \phi - \partial_y \partial_x \phi \right) - \hat{\boldsymbol{y}} \left(\partial_x \partial_z \phi - \partial_z \partial_x \right) + \hat{\boldsymbol{z}} \left(\partial_x \partial_y \phi - \partial_y \partial_x \phi \right). \end{aligned}$$

Since the scalar field is smooth, the partial derivatives commute, and this sum is zero. This exercise shows the implication

$$\boldsymbol{u} = \nabla \phi \implies \nabla \times \boldsymbol{u} = 0.$$

In exercise 6.2 we had $\nabla \times [(6xy + z^3)\hat{\boldsymbol{x}} + (3x^2 - z)\hat{\boldsymbol{y}} + (3xz^2 - y)\hat{\boldsymbol{z}}] = 0 \implies \exists \phi = 3x^2 y + xz^3 - yz$ such that

$$\nabla \phi = \left[(6xy + z^3)\hat{\boldsymbol{x}} + (3x^2 - z)\hat{\boldsymbol{y}} + (3xz^2 - y)\hat{\boldsymbol{z}} \right] = \boldsymbol{u}.$$

In fact, the implication always goes both ways:

> A vector field $\boldsymbol{u}(x, y, z)$ is irrotational if and only if it can be written as the gradient of a scalar field,
>
> $$\boldsymbol{u}(x, y, z) = \nabla \phi(x, y, z).$$

We shall prove this statement later in the course using Stokes' Theorem.

6.4 Further proofs

Prove that

$$\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u},$$

for any differentiable vector field $\boldsymbol{u}(x, y, z)$.

$$\mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{u}) = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial u_3}{\partial y} - \dfrac{\partial u_2}{\partial z} & \dfrac{\partial u_1}{\partial z} - \dfrac{\partial u_3}{\partial x} & \dfrac{\partial u_2}{\partial x} - \dfrac{\partial u_1}{\partial y} \end{vmatrix}.$$

Let $\boldsymbol{B} = \mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{u})$. Then

$$\begin{aligned} B_1 &= \frac{\partial^2 u_2}{\partial x \partial y} - \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_3}{\partial x \partial z} - \frac{\partial^2 u_1}{\partial z^2} \\ &= \frac{\partial}{\partial x}\left(\underbrace{\frac{\partial u_1}{\partial x}}_{***} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right) - \left(\underbrace{\frac{\partial^2 u_1}{\partial x^2}}_{***} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} \right) \\ &= \frac{\partial}{\partial x}(\mathrm{div}\,\boldsymbol{u}) - \mathrm{grad}\,^2 u_1. \end{aligned}$$

Similarly

$$B_2 = \frac{\partial}{\partial y}(\mathrm{div}\,\boldsymbol{u}) - \mathrm{grad}\,^2 u_2 \quad \text{and} \quad B_3 = \frac{\partial}{\partial z}(\mathrm{div}\,\boldsymbol{u}) - \mathrm{grad}\,^2 u_3.$$

Therefore

$$\begin{aligned} \mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{u}) &= \boldsymbol{B} = B_1\,\hat{\boldsymbol{x}} + B_2\,\hat{\boldsymbol{y}} + B_3\,\hat{\boldsymbol{z}} \\ &= \left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \right)(\mathrm{div}\,\boldsymbol{u}) - \mathrm{grad}\,^2(u_1\,\hat{\boldsymbol{x}} + u_2\,\hat{\boldsymbol{y}} + u_3\,\hat{\boldsymbol{z}}) \\ &= \mathrm{grad}\,(\mathrm{div}\,\boldsymbol{u}) - \mathrm{grad}\,^2\boldsymbol{u}. \end{aligned}$$

6.5 Physical application: fluid flow in two dimensions

In a general three-dimensional setting, a vector field $\boldsymbol{u}(x,y,z)$ (and possibly time) describes the velocity of a fluid at location $\boldsymbol{x} = (x,y,z)$. The **vorticity** $\boldsymbol{\omega}(x,y,z)$ measures the amount of rotation in the fluid, and its sense:

$$\omega = \nabla \times \boldsymbol{u} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ u_1 & u_2 & u_3 \end{vmatrix}$$

1. Prove that in a two-dimensional fluid, where

$$\boldsymbol{u}(x,y,z) = (u_1(x,y), u_2(x,y), 0)$$

the vorticity is given by

$$\boldsymbol{\omega}(x,y)=\omega(x,y)\hat{\boldsymbol{z}},\qquad \omega(x,y)=\partial_x u_2-\partial_y u_1.$$

2. The two-dimensional fluid is incompressible if $\nabla\cdot\boldsymbol{u}=0$, i.e.

$$\partial_x u_1+\partial_y u_2=0.$$

Prove that the necessary and sufficient condition for the fluid to be incompressible is the existence of a **streamfunction** $\psi(x,y)$, such that

$$u_1=\partial_y\psi,\qquad u_2=-\partial_x\psi.$$

3. Prove that

$$\nabla^2\psi=-\omega.$$

Hence, demonstrate that in an irrotational fluid, $\nabla^2\psi=0$.

1. We have

$$\begin{aligned}\boldsymbol{\omega} &= \begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \partial_x & \partial_y & \partial_z\\ u_1(x,y) & u_2(x,y) & 0\end{vmatrix},\\ &= \hat{\boldsymbol{x}}\left(\partial_y 0-\partial_z u_2(x,y)\right)-\hat{\boldsymbol{y}}\left(\partial_x 0-\partial_z u_1(x,y)\right)+\hat{\boldsymbol{z}}\left(\partial_x u_2-\partial_y u_1\right)\\ &= \hat{\boldsymbol{z}}\left(\partial_x u_2-\partial_y u_1\right),\end{aligned}$$

as required. The vorticity has magnitude $\omega=\partial_x u_2-\partial_y u_1$ and points in the z-direction.

2. Necessity: Assume that the flow is incompressible, $\partial_x u_1+\partial_y u_2=0$. We show that a streamfunction exists by construction: Let

$$\psi(x,y)=\int_a^y u_1(x,\lambda)\,\mathrm{d}\lambda-\int_b^x u_2(\mu,a)\mathrm{d}\mu,$$

where a is an arbitrary y-value in the domain of the fluid and λ is a dummy variable of integration. Similarly, b is an arbitrary x-value and μ is a dummy variable. By construction, and by the Fundamental Theorem of Calculus,

$$\frac{\partial\psi}{\partial y}=u_1(x,y)\,.$$

Now

$$\begin{aligned}
\frac{\partial\psi}{\partial x} &= \frac{\partial}{\partial x}\int_a^y u_1(x,\lambda)\,\mathrm{d}\lambda - u_2(x,a),\\
&= \int_a^y \frac{\partial u_1}{\partial x}(x,\lambda)\,\mathrm{d}\lambda - u_2(x,a),\\
&= -\int_a^y \frac{\partial u_2}{\partial \lambda}(x,\lambda)\,\mathrm{d}\lambda - u_2(x,a), \qquad \text{(By incompressibility)}\\
&= -\left[u_2(x,y) - u_2(x,a)\right] - u_2(x,a),\\
&= -u_2(x,y).
\end{aligned}$$

Hence, ψ is a streamfunction because the flow $(u_1(x,y), u_2(x,y))$ can be derived from it.

Sufficiency: Assume that the streamfunction exists. Then

$$u_1 = \frac{\partial\psi}{\partial y},\ u_2 = -\frac{\partial\psi}{\partial x},$$

and

$$\begin{aligned}
\operatorname{div}\boldsymbol{u} &= \frac{\partial}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right),\\
&= \frac{\partial}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\psi}{\partial x}.
\end{aligned}$$

Using the fact that the partial derivatives of smooth functions commute, $\partial_{xy}\psi = \partial_{yx}\psi$, we obtain

$$\operatorname{div}\boldsymbol{u} = 0.$$

Thus, a two-dimensional flow is incompressible if and only if it has a streamfunction.

3. We have

$$\begin{aligned}
\omega &= \partial_x u_2 - \partial_y u_1,\\
&= \partial_x\left(-\partial_x\psi\right) - \partial_y\left(+\partial_y\psi\right),\\
&= -\left(\partial_x^2 + \partial_y^2\right)\psi,
\end{aligned}$$

and $\omega = -\nabla^2\psi$. If the flow is irrotational, its curl is zero, and $\omega = 0$. Hence, in an irrotational flow, $\nabla^2\psi = 0$.

No streamfunction exists in three-dimensional flows and this simple analysis no longer holds there.

Chapter 7

Vector integration

Overview

Logically, the next step after differentiating vector and scalar fields is to integrate them. We start with line integrals and then proceed to surface and volume integrals. In each case, we reduce the integration problem to a series of ordinary integrations which are elementary.

7.1 Line integrals

Formally, we have the following small increment of displacement:

$$\mathrm{d}\boldsymbol{x} = \hat{\boldsymbol{x}}\mathrm{d}x + \hat{\boldsymbol{y}}\mathrm{d}y + \hat{\boldsymbol{z}}\mathrm{d}z,$$

which gives rise to the following possible integrals for a scalar field $\phi(x, y, z)$ and a vector field $\boldsymbol{v}(x, y, z)$:

$$\int_C \phi(x, y, z)\mathrm{d}\boldsymbol{x},$$
$$\int_C \boldsymbol{v}(x, y, z) \cdot \mathrm{d}\boldsymbol{x},$$
$$\int_C \boldsymbol{v}(x, y, z) \times \mathrm{d}\boldsymbol{x},$$

where C denotes a **contour**, that is, a curve $\boldsymbol{x}_C(t) : \mathbb{R} \to \mathbb{R}^3$. Let us work with the first kind of integral and introduce the formal definition (generalising to the other integrals will be left to exercises and examples).

Let $\boldsymbol{x}_C : [t_1, t_2] \to \mathbb{R}^3$ be some **piecewise smooth** curve. Then the line integral $\int_C \phi(x,y,z)\mathrm{d}\boldsymbol{x}$ along the curve C is defined as follows:

$$\begin{aligned}\int_C \phi(x,y,z)\mathrm{d}\boldsymbol{x} &:= \int_{t_1}^{t_2} \phi\left(\boldsymbol{x}_C(t)\right) \frac{d\boldsymbol{x}_C}{dt}\mathrm{d}t, \\ &= \hat{\boldsymbol{x}} \int_{t_1}^{t_2} \phi\left(\boldsymbol{x}_C(t)\right) \frac{dx_C}{dt}\mathrm{d}t + \hat{\boldsymbol{y}} \int_{t_1}^{t_2} \phi\left(\boldsymbol{x}_C(t)\right) \frac{dy_C}{dt}\mathrm{d}t + \hat{\boldsymbol{z}} \int_{t_1}^{t_2} \phi\left(\boldsymbol{x}_C(t)\right) \frac{dz_C}{dt}\mathrm{d}t\end{aligned}$$

7.2 Worked Examples

1. Let $\phi(x,y) = x^2 + y^2$. Compute the line integral along the curve

$$\begin{aligned} C_1 &: (0,0) \to (1,0), && \text{in a straight line,} \\ C_2 &: (1,0) \to (1,1), && \text{in a straight line.} \end{aligned}$$

(See Fig. 7.1.) Break up the integration into two parts. In the first part, the curve is

$$\boldsymbol{x}_{C1}(t) = (t,0), \qquad t \in [0,1], \qquad \frac{d\boldsymbol{x}_{C1}}{dt} = (1,0).$$

Hence,

$$\int_{C1} \phi(x,y,z)\mathrm{d}\boldsymbol{x} = \hat{\boldsymbol{x}} \int_0^1 t^2 \mathrm{d}t + \hat{\boldsymbol{y}} \int_0^1 0\,\mathrm{d}t = \tfrac{1}{3}\hat{\boldsymbol{x}}.$$

In the second part, the curve is

$$\boldsymbol{x}_{C2}(t) = (1,t), \qquad t \in [0,1], \qquad \frac{d\boldsymbol{x}_{C1}}{dt} = (0,1).$$

Hence,

$$\int_{C2} \phi(x,y,z)\mathrm{d}\boldsymbol{x} = \hat{\boldsymbol{y}} \int_0^1 (1+t^2)\,\mathrm{d}t = \tfrac{4}{3}\hat{\boldsymbol{y}}.$$

Putting them together,

$$\int_C \phi(x,y,z)\mathrm{d}\boldsymbol{x} = \int_{C1} \phi(x,y,z)\mathrm{d}\boldsymbol{x} + \int_{C2} \phi(x,y,z)\mathrm{d}\boldsymbol{x} = \tfrac{1}{3}\hat{\boldsymbol{x}} + \tfrac{4}{3}\hat{\boldsymbol{y}}.$$

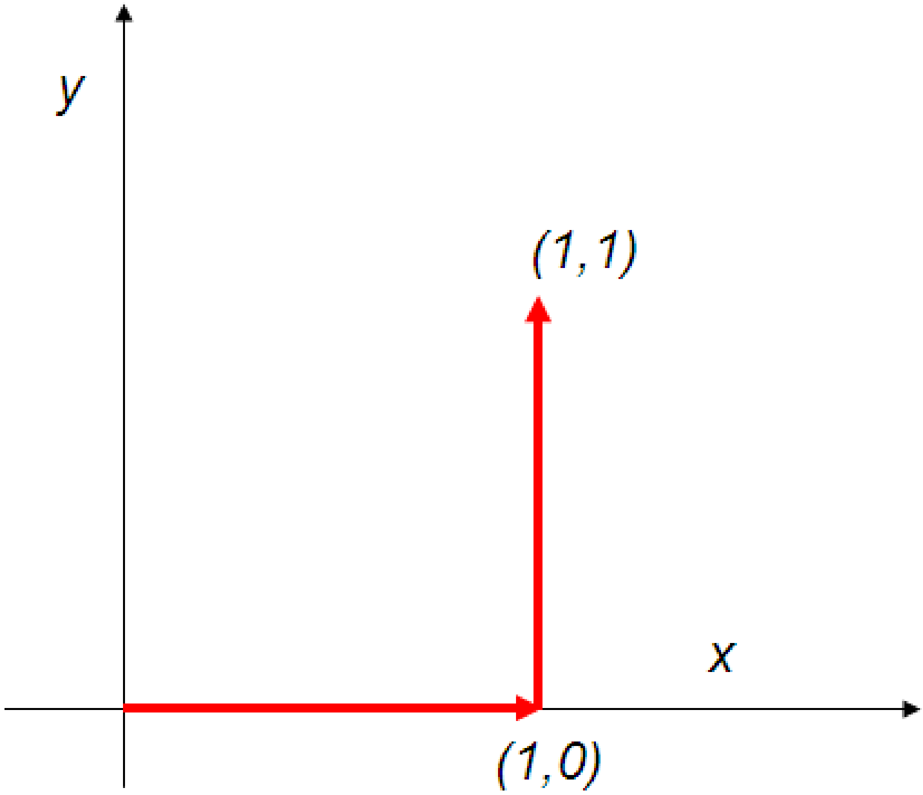

Figure 7.1: The path $(0,0) \to (1,0) \to (1,1)$.

2. The most common line integrals in physics are of the form $\int_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}$. Consider a vector field

$$\boldsymbol{v} = 3xy\hat{\boldsymbol{x}} - y^2\hat{\boldsymbol{y}}$$

integrated along the curve $y = 2x^2$, from the origin $(0,0)$ to the point $(1,2)$. Now the curve has the parametric form

$$\boldsymbol{x}_C(t) = \left(t, 2t^2\right), \qquad t \in [0,1] \qquad \frac{d\boldsymbol{x}_C}{dt} = (1, 4t)\,.$$

We compute

$$\begin{aligned}
\int_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} &= \int_C \left[v_1(x,y)\mathrm{d}x + v_2(x,y)\mathrm{d}y\right], \\
&= \int_C v_1(\boldsymbol{x}_C(t))\frac{\mathrm{d}x_C}{dt}\mathrm{d}t + \int_C v_2(\boldsymbol{x}_C(t))\frac{\mathrm{d}y_C}{dt}\mathrm{d}t, \\
&= \int_0^1 3(t)(2t^2)(1)\,\mathrm{d}t + \int_0^1 [-(2t^2)^2](4t)\,\mathrm{d}t, \\
&= \int_0^1 6t^3\,\mathrm{d}t - \int_0^1 16t^5\,\mathrm{d}t, \\
&= -\tfrac{7}{6}.
\end{aligned}$$

3. In mechanics, there is the notion of force. Suffice to say, force is a vector field in two or three dimensions, $\boldsymbol{F}(x,y)$, or $\boldsymbol{F}(x,y,z)$. The **work done**, W, as a particle is moved along a trajectory $\boldsymbol{x}_C(t)$ through the force field $\boldsymbol{F}(\boldsymbol{x})$ is the line integral of the force field along the trajectory:

$$W = \int_C \boldsymbol{F}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{x}.$$

Consider a force

$$\boldsymbol{F} = -k\left(\hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y\right).$$

Compare the work done moving against this force field when going from $(1,1)$ to $(4,4)$ along the following straight-line paths:

$$\begin{aligned}
(1,1) &\rightarrow (4,1) \rightarrow (4,4), \\
(1,1) &\rightarrow (1,4) \rightarrow (4,4), \\
(1,1) &\rightarrow (4,4), \text{ along } x = y.
\end{aligned}$$

For example, consider the third path. The curve is

$$\boldsymbol{x}_C(t) = (1+t, 1+t), \qquad t \in [0,3], \qquad \frac{d\boldsymbol{x}_C}{dt} = (1,1),$$

and the integral is

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} &= \int_C \left[F_1(x,y)\mathrm{d}x + F_2(x,y)\mathrm{d}y\right], \\
&= \int_0^3 F_1(\boldsymbol{x}_C(t))\frac{dx_C}{dt}\mathrm{d}t + \int_0^3 F_2(\boldsymbol{x}_C(t))\frac{dy_C}{dt}\mathrm{d}t, \\
&= -2k\int_0^3 (1+t)\mathrm{d}t, \\
&= -15k.
\end{aligned}$$

The other two cases are left as an exercise but you should get the same answer in all three cases. Here is why. The force field $\boldsymbol{F} = -k(\hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y)$ can be written as

$$\boldsymbol{F} = -k\left(\hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y\right) = -\nabla\left[\frac{k}{2}\left(x^2+y^2\right)\right] := -\nabla\mathcal{U}(x,y).$$

Thus, along any path C,

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} &= -\int_C \nabla\mathcal{U} \cdot \mathrm{d}\boldsymbol{x}, \\
&= -\int_{t_1}^{t_2} \nabla\mathcal{U}\left(\boldsymbol{x}_C(t)\right) \cdot \frac{\mathrm{d}\boldsymbol{x}_C}{dt}\mathrm{d}t,
\end{aligned}$$

$$\begin{aligned}\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{x} &= -\int_{t_1}^{t_2} \frac{d}{dt}\mathcal{U}\left(\boldsymbol{x}_C(t)\right)\mathrm{d}t,\\ &= -\left[\mathcal{U}(\boldsymbol{x}_C(t_2)) - \mathcal{U}(\boldsymbol{x}_C(t_1))\right],\end{aligned}$$

and the line integral is independent of the path and depends only on the initial and final points. Recall from previous lectures that a vector field $\boldsymbol{F}$ is irrotational if and only if it can be written in the form $\boldsymbol{F} = -\nabla\mathcal{U}$. Thus, we have the following string of statements:

A vector field $\boldsymbol{F}$ is irrotational if and only if

- $\nabla \times \boldsymbol{F} = 0$ if and only if
- $\boldsymbol{F} = -\nabla\mathcal{U}$ if and only if
- The line integral $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{x}$ depends only on the initial and final points of the path C and is independent of the details of the path between these terminal points.

Consider also a closed path C, for which $\boldsymbol{x}_C(t_2) = \boldsymbol{x}_C(t_1)$. For an irrotational vector field $\boldsymbol{u}(\boldsymbol{x})$ integrated over such a path,

$$\begin{aligned}\int_C \boldsymbol{u}(\boldsymbol{x})\cdot \mathrm{d}\boldsymbol{x} &:= \oint_C \boldsymbol{u}(\boldsymbol{x})\cdot \mathrm{d}\boldsymbol{x}\\ &= -\int_C \nabla\mathcal{U}\cdot \mathrm{d}\boldsymbol{x}\\ &= -\left[\mathcal{U}(\boldsymbol{x}_C(t_2)) - \mathcal{U}(\boldsymbol{x}_C(t_1))\right]\\ &= -\left[\mathcal{U}(\boldsymbol{x}_C(t_2)) - \mathcal{U}(\boldsymbol{x}_C(t_2))\right]\\ &= 0.\end{aligned}$$

4. In contrast, consider the force

$$\boldsymbol{G} = -k\left(\hat{\boldsymbol{x}}y - \hat{\boldsymbol{y}}x\right).$$

The curl of the force is

$$\nabla\times\boldsymbol{G} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ -ky & kx & 0 \end{vmatrix} = 2k\hat{\boldsymbol{z}}.$$

Let's integrate along the paths

$$\begin{aligned} C &: (1,1)\to(4,1)\to(4,4),\\ D &: (1,1)\to(1,4)\to(4,4).\end{aligned}$$

First path, first component:

$$\boldsymbol{x}_{C1}(t) = (1+t, 1)\,, \qquad t \in [0,3], \qquad \frac{d\boldsymbol{x}_C}{dt} = (1,0),$$

and

$$\boldsymbol{G} \cdot \mathrm{d}\boldsymbol{x} = \boldsymbol{G} \cdot \frac{d\boldsymbol{x}_C}{dt}\mathrm{d}t = G_x \mathrm{d}t = -ky\mathrm{d}t = -k\mathrm{d}t.$$

Integrating gives $-3k$. First path, second component:

$$\boldsymbol{x}_{C2}(t) = (4, 1+t)\,, \qquad t \in [0,3], \qquad \frac{d\boldsymbol{x}_C}{dt} = (0,1),$$

and

$$\boldsymbol{G} \cdot \mathrm{d}\boldsymbol{x} = \boldsymbol{G} \cdot \frac{d\boldsymbol{x}_C}{dt}\mathrm{d}t = G_y \mathrm{d}t = +kx\mathrm{d}t = 4k\mathrm{d}t.$$

Integrating gives $12k$. Adding up gives

$$\int_C \boldsymbol{G} \cdot \mathrm{d}\boldsymbol{x} = 9k.$$

Second path, first component:

$$\boldsymbol{x}_{D1}(t) = (1, 1+t)\,, \qquad t \in [0,3], \qquad \frac{d\boldsymbol{x}_C}{dt} = (0,1),$$

and

$$\boldsymbol{G} \cdot \mathrm{d}\boldsymbol{x} = \boldsymbol{G} \cdot \frac{d\boldsymbol{x}_D}{dt}\mathrm{d}t = G_y \mathrm{d}t = +kx\mathrm{d}t = k\mathrm{d}t.$$

Integrating gives $+3k$. Second path, second component:

$$\boldsymbol{x}_{D2}(t) = (1+t, 4)\,, \qquad t \in [0,3], \qquad \frac{d\boldsymbol{x}_C}{dt} = (1,0),$$

and

$$\boldsymbol{G} \cdot \mathrm{d}\boldsymbol{x} = \boldsymbol{G} \cdot \frac{d\boldsymbol{x}_D}{dt}\mathrm{d}t = G_x \mathrm{d}t = -ky\mathrm{d}t = -4k\mathrm{d}t.$$

Integrating gives $-12k$. Adding up gives

$$\int_C \boldsymbol{G} \cdot \mathrm{d}\boldsymbol{x} = -9k.$$

and the two paths differ.

Chapter 8

Integrals over surfaces and volumes

Overview

In this section we focus on computing the area and volume of irregular (i.e. non-cuboid) shapes in two and three dimensions. These involve integrals of the form

$$\int\int_{\text{Area enclosed by some curve}} \mathrm{d}x\,\mathrm{d}y, \qquad \int\int_{\text{Volume enclosed by some surface}} \mathrm{d}x\,\mathrm{d}y\mathrm{d}z.$$

The most novel feature of these problems for the class is the appearance of **non-constant limits of integration**. We will find out how to deal with these limits in the following problems.

Many of you will not have seen so-called **Riemann integration** in either one or several dimensions before. For that reason, a discussion of this topic is included in Appendix B.

8.1 The limits of integration are not constants any more

Imagine an evil genius who is rubbish at elementary maths but a wizard at calculus. She wants to compute the area of a right-angle triangle. She would proceed as follows.

Vertices at $(0,0)$, $(a,0)$, $(0,b)$, where a and b are positive constants. The triangle is thus bounded by the lines $y=0$, $x=0$, and $y=mx+b$, where $m=-b/a$. An element of area in the x-y plane is

$$\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y.$$

Hence,

$$\text{Area of triangle} = \int_{\text{Region bounded by three lines mentioned}} \mathrm{d}x\mathrm{d}y.$$

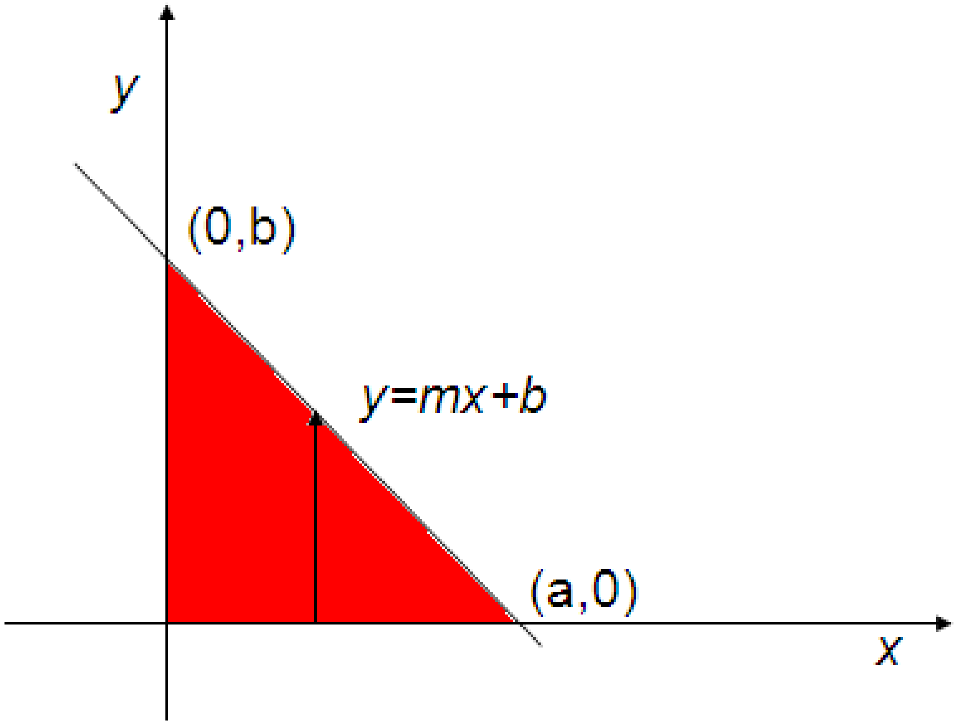

Figure 8.1: Integration domain to compute the area of a right-angled triangle.

Now the variable x is allowed to run between 0 and a, while the variable y is allowed to run between 0 and $mx+b$. Hence,

$$\begin{aligned}
\text{Area of triangle} &= \int_0^a \mathrm{d}x \int_0^{mx+b} \mathrm{d}y, \\
&= \int_0^a \mathrm{d}x y\big|_0^{mx+b}, \\
&= \int_0^a \mathrm{d}x \left(mx+b\right), \\
&= \int_0^a \mathrm{d}x \frac{1}{2m}\frac{d}{dx}\left(mx+b\right)^2, \\
&= \tfrac{1}{2m}\left(mx+b\right)^2\big|_0^a, \\
&= -\frac{1}{2m}b^2, \\
&= \frac{1}{2}ab.
\end{aligned}$$

8.2 Density integrals

Suppose that the integration domain in Fig. 8.1 instead represents a thin sheet of metal, whose density varies as

$$\rho(x,y) = 1 + \epsilon y \cos(2\pi x/a) = \frac{\text{Mass}}{\text{Unit area}}.$$

Compute the mass of the sheet.

We have,

$$\begin{aligned}
\mathrm{d}m &= \rho(x,y)\mathrm{d}x\,\mathrm{d}y,\\
m &= \int_{\text{Triangle}} \rho(x,y)\mathrm{d}x\,\mathrm{d}y,\\
&= \int_0^a \mathrm{d}x \int_0^{b-(b/a)x} \mathrm{d}y\left[1+\epsilon y\cos(2\pi x/a)\right],\\
&= \int_0^a \mathrm{d}x\left[y+\tfrac{1}{2}y^2\epsilon\cos(2\pi x/a)\right]_{y=0}^{y=b-(b/a)x},
\end{aligned}$$

Hence,

$$\begin{aligned}
m &= \int_0^a \left[b-(b/a)x\right]\mathrm{d}x + \tfrac{1}{2}\epsilon b^2\int_0^a \left[b-(b/a)x\right]^2\cos(2\pi x/a)\mathrm{d}x,\\
&= \tfrac{1}{2}ab + \tfrac{1}{2}\epsilon\int_0^a \cos\left(2\pi x/a\right)\mathrm{d}x - \epsilon(b/a)\int_0^a x\cos(2\pi x/a)\mathrm{d}x\\
&\qquad + \tfrac{1}{2}\epsilon(b/a)^2\int_0^a x^2\cos(2\pi x/a)\mathrm{d}x,\\
&= \tfrac{1}{2}ab + \tfrac{1}{2}\epsilon b^2 I_1 + (-\epsilon b/a)I_2 + \left(\epsilon b^2/2a^2\right)I_3.
\end{aligned}$$

Now

$$\begin{aligned}
I_1 &= \int_0^a \cos(2\pi x/a)\mathrm{d}x,\\
&= \frac{a}{2\pi}\sin(2\pi x/a)\Big|_0^a,\\
&= \frac{a}{2\pi}\left(\sin(2\pi)-\sin(0)\right) = 0;
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_0^a x\cos(2\pi x/a)\mathrm{d}x,\\
&= \frac{a^2}{4\pi^2}\int_0^{2\pi} s\cos(s)\mathrm{d}s,\\
&= \frac{a^2}{4\pi^2}\left[s\sin(s)+\cos(s)\right]_0^{2\pi} = 0.
\end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^a x^2 \cos(2\pi x/a)\mathrm{d}x, \\ &= \frac{a^3}{8\pi^3}\int_0^{2\pi} s^2\cos(s)\mathrm{d}s, \\ &= \frac{a^3}{8\pi^3}\left[(s^2-2)\sin(s)+2s\cos(s)\right]_0^{2\pi}, \\ &= \frac{a^3}{8\pi^3}\left[2(2\pi)\cos(2\pi)\right], \\ &= \frac{a^3}{8\pi^3}4\pi, \\ &= \frac{a^3}{2\pi^2}. \end{aligned}$$

Hence,

$$\begin{aligned} m &= \tfrac{1}{2}ab + \frac{\epsilon b^2}{2a^2}\frac{a^3}{2\pi^2}, \\ &= \tfrac{1}{2}ab + \frac{\epsilon}{4\pi^2}b^2 a. \end{aligned}$$

8.3 Volume integrations

Buoyed by her success, the evil genius of Sec. 8.1 decides to compute the volume of a a certain pyramid. The pyramid has three edges that come together at right angles; the edges have lengths a, b, and c (tetrahedron). We of course know that the final answer must be $abc/6$. Here is how our friend would proceed:

The three edges come together to form a right angle at the vertex $(0,0,0)$. The other extremities of the three edges are at $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$. The pyramid is thus bounded by the planes $x=0$, $y=0$, $z=0$, and by a certain other plane, which our friend must work out.

The points $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$ live in this fourth bounding plane. Hence, the vectors

$$\boldsymbol{v}_1 = (0,0,c)-(a,0,0) = (-a,0,c), \qquad \boldsymbol{v}_2 = (0,b,0)-(a,0,0) = (-a,b,0)$$

are parallel to the plane and have the same base at the point $(a,0,0)$. Thus,

$$\boldsymbol{n} = \boldsymbol{v}_2\times\boldsymbol{v}_1 = \hat{\boldsymbol{x}}bc + \hat{\boldsymbol{y}}ac + \hat{\boldsymbol{z}}ab = (bc, ac, ab)$$

is normal to the plane, and the plane is therefore defined by

$$[\boldsymbol{x}-(a,0,0)]\cdot\boldsymbol{n} = 0,$$

or

$$z = c - \frac{c}{a}x - \frac{c}{b}y.$$

In summary, the four bounding planes are

$$x = 0, \qquad y = 0, \qquad z = 0, \qquad z = c - \frac{c}{a}x - \frac{c}{b}y.$$

The volume element is

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Hence,

$$\text{Volume of pyramid} = \int_{\text{Region bounded by four planes}} \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Now the variable x is allowed to run between 0 and a, while the variable y is allowed to run between 0 and $y = b - x(b/a)$. This is because, in the x-y plane, the integration reduces to the triangle integration in Ex. 8.1. Finally, the variable z is allowed to run between 0 and $z = c - (c/a)x - (c/b)y$. Thus,

$$\begin{aligned}
\text{Volume of pyramid} &= \int_0^a \mathrm{d}x \int_0^{b-x(b/a)} \mathrm{d}y \int_0^{c-x(c/a)-y(c/b)} \mathrm{d}z, \\
&= \int_0^a \mathrm{d}x \int_0^{b-x(b/a)} \mathrm{d}y\, z\Big|_0^{c-x(c/a)-y(c/b)}, \\
&= \int_0^a \mathrm{d}x \int_0^{b-x(b/a)} \mathrm{d}y \left[c - x(c/a) - y(c/b)\right], \\
&= \int_0^a \mathrm{d}x \int_0^{b-x(b/a)} \mathrm{d}y \left(-\frac{b}{2c}\right) \frac{\partial}{\partial y} \left[c - x(c/a) - y(c/b)\right]^2, \\
&= -\frac{b}{2c} \int_0^a \mathrm{d}x \int_0^{b-x(b/a)} \mathrm{d}y \frac{\partial}{\partial y} \left[c - x(c/a) - y(c/b)\right]^2, \\
&= -\frac{b}{2c} \int_0^a \mathrm{d}x \left[c - x(c/a) - y(c/b)\right]^2 \Big|_{y=0}^{y=b-x(b/a)}, \\
&= +\frac{b}{2c} \int_0^a \mathrm{d}x \left[c - x(c/a)\right]^2, \\
&= \frac{b}{2c} \int_0^a \mathrm{d}x \left(-\frac{a}{3c}\right) \frac{\partial}{\partial x} \left[c - x(c/a)\right]^3, \\
&= -\frac{ab}{6c^2} \left[c - x(c/a)\right]^3 \Big|_0^a, \\
&= +\frac{ab}{6c^2} c^3, \\
&= \frac{abc}{6}.
\end{aligned}$$

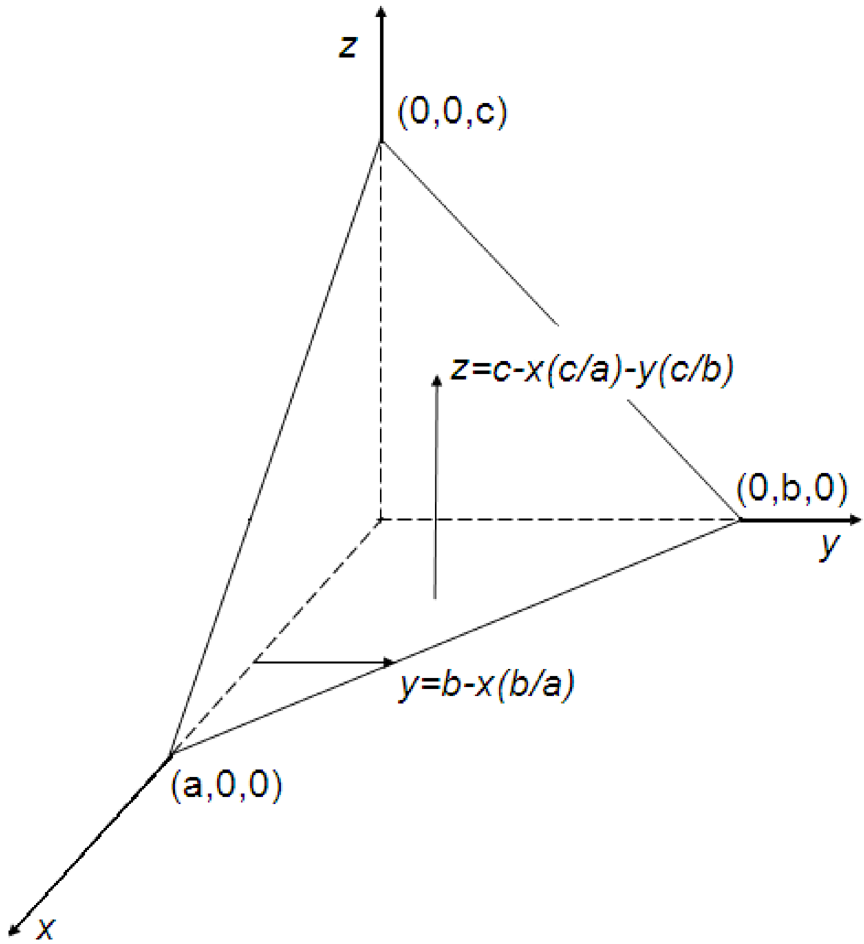

Figure 8.2: Integration domain for a tetrahedron.

Chapter 9

Integrals over surfaces and volumes, continued

Overview

In this section we devise a general method for computing area integrals. It holds for arbitrary shapes.

9.1 Parametrization of surface integrals

We focus on surface integrals in three dimensions. That is, we are to integrate a vector field $\boldsymbol{v}(x, y, z)$ over a surface S. The element of surface area actually has an orientation:

$$\mathrm{d}\boldsymbol{S} = \hat{\boldsymbol{n}}\,\mathrm{d}S,$$

where $\hat{\boldsymbol{n}}$ is normal to the surface at location $\boldsymbol{x}$ on the surface. By convention, we choose $\hat{\boldsymbol{n}}$ to be the outward-pointing normal. We focus on the most commonly-encountered integral:

$$\int_S \boldsymbol{v}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{S}.$$

To do line integrals along a curve, we had to introduce a **parametrization** of the curve. We must do a similar thing here: We parametrize the surface S as follows:

$$S = \{\boldsymbol{x} \in \mathbb{R}^3 | \boldsymbol{x} = \boldsymbol{x}_S(s,t), (s,t) \in \Omega_S\}, \qquad \Omega_S = \text{Some subset of } \mathbb{R}^2.$$

Thus, a curve in three dimensions is a one-parameter set, and a surface is a two-parameter set.

Refer to Figure 9.1 and consider the points

$$\boldsymbol{x}_S(s,t), \qquad \boldsymbol{x}_S(s,t+\mathrm{d}t), \qquad \boldsymbol{x}_S(s+ds,t), \qquad \boldsymbol{x}_S(s+ds,t+dt),$$

vectors whose tips all lie in the surface S. Form the differences

$$\boldsymbol{x}_S(s,t+\mathrm{d}t)-\boldsymbol{x}_S(s,t)=\frac{\partial \boldsymbol{x}_S}{\partial t}\mathrm{d}t,$$

and

$$\boldsymbol{x}_S(s+\mathrm{d}s,t)-\boldsymbol{x}_S(s,t)=\frac{\partial \boldsymbol{x}_S}{\partial s}\mathrm{d}s.$$

These are small vectors that lie in the surface and form the two lengths of a parallelogram. The area described by the four points $\boldsymbol{x}_S(s,t)$,...,$\boldsymbol{x}_S(s+\mathrm{d}s,t+\mathrm{d}t)$ is thus

$$\mathrm{d}\boldsymbol{S}=\frac{\partial \boldsymbol{x}_S}{\partial s}\times\frac{\partial \boldsymbol{x}_S}{\partial t}\mathrm{d}s\,\mathrm{d}t.$$

If the parameters s and t take values in a set Ω_S, then the surface integral $\int_S \boldsymbol{v}(\boldsymbol{x})\cdot \mathrm{d}\boldsymbol{S}$ is

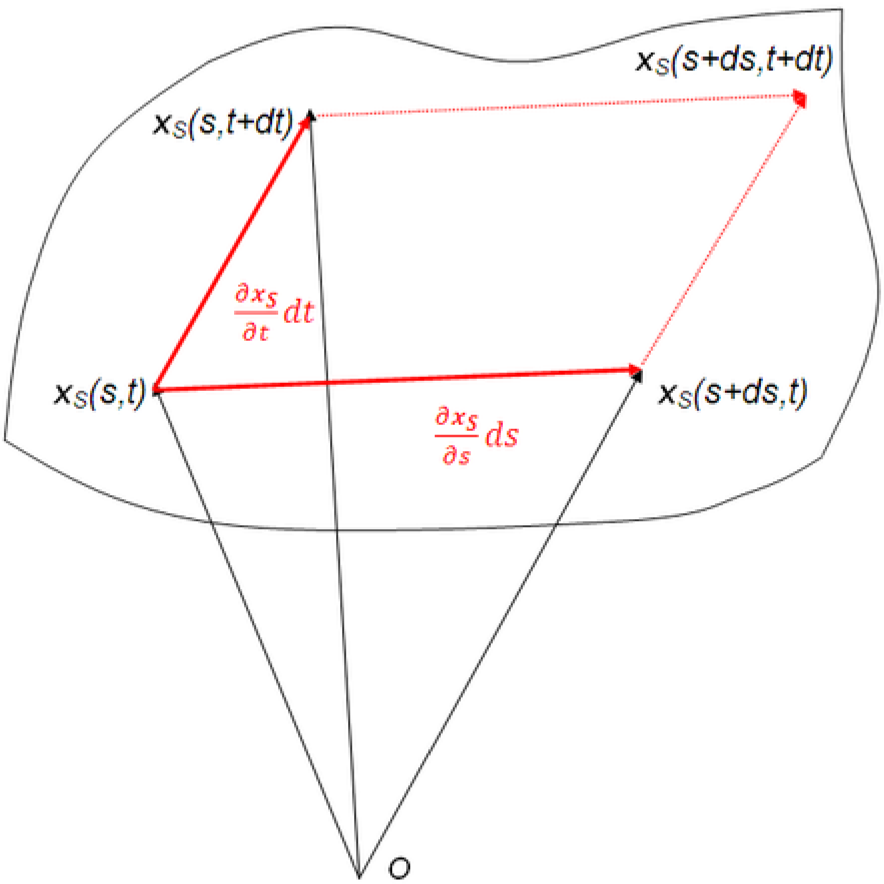

Figure 9.1: Parametrization of a surface

$$\int_S \boldsymbol{v}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{S} = \int\int_{\Omega_S} \boldsymbol{v}(\boldsymbol{x}_S(s,t)) \cdot \left(\frac{\partial \boldsymbol{x}_S}{\partial t} \times \frac{\partial \boldsymbol{x}_S}{\partial s}\right) \mathrm{d}t\, \mathrm{d}s$$

9.2 Worked examples

1. If $\boldsymbol{v} = 2y\hat{\boldsymbol{x}} - z\hat{\boldsymbol{y}} + x^2\hat{\boldsymbol{z}}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first (positive) octant bounded by the planes $y = 4$ and $z = 6$, evaluate $\int_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$.

   Let us compute the surface in parametric form. The parametric form of the curve is

$$\begin{aligned} y_S(s,t) &= s, \\ z_S(s,t) &= t, \\ x_S(s,t) &= s^2/8. \end{aligned}$$

   where $0 \le s \le 4$ and $0 \le t \le 6$. Hence,

$$\boldsymbol{x}_S(s,t) = (s^2/8, s, t).$$

$$\frac{\partial \boldsymbol{x}_S}{\partial s} = (s/4, 1, 0), \qquad \frac{\partial \boldsymbol{x}_S}{\partial t} = (0,0,1)$$

   and

$$\mathrm{d}\boldsymbol{S} = \left(\frac{\partial \boldsymbol{x}_S}{\partial s} \times \frac{\partial \boldsymbol{x}_S}{\partial t}\right) \mathrm{d}s\, \mathrm{d}t = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ s/4 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \mathrm{d}s\mathrm{d}t = [\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}(s/4)]\, \mathrm{d}s\, \mathrm{d}t$$

   Hence

$$\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \left(2y\hat{\boldsymbol{x}} - z\hat{\boldsymbol{y}} + x^2\hat{\boldsymbol{z}}\right) \cdot (\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}(s/4))\, \mathrm{d}s\, \mathrm{d}t = (2y + zs/4)\, \mathrm{d}s\, \mathrm{d}t.$$

   But $y = s$ and $z = t$, hence

$$\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = (2s + ts/4)\, \mathrm{d}s\, \mathrm{d}t.$$

   We let $0 \le s \le 4$ and $0 \le t \le 6$ and integrate. We make use of the following remarkable fact:

$$\int_{s_1}^{s_2} \mathrm{d}s \int_{t_1}^{t_2} \mathrm{d}t\, \phi(s,t) = \int_{t_1}^{t_2} \mathrm{d}t \int_{s_1}^{s_2} \mathrm{d}s\, \phi(s,t),$$

   that is, the order of integration can be reversed, for suitable functions ϕ. Such a reversal cannot be done if, in the first integral, the limits t_1 and t_2 depend on s. Here, however, the

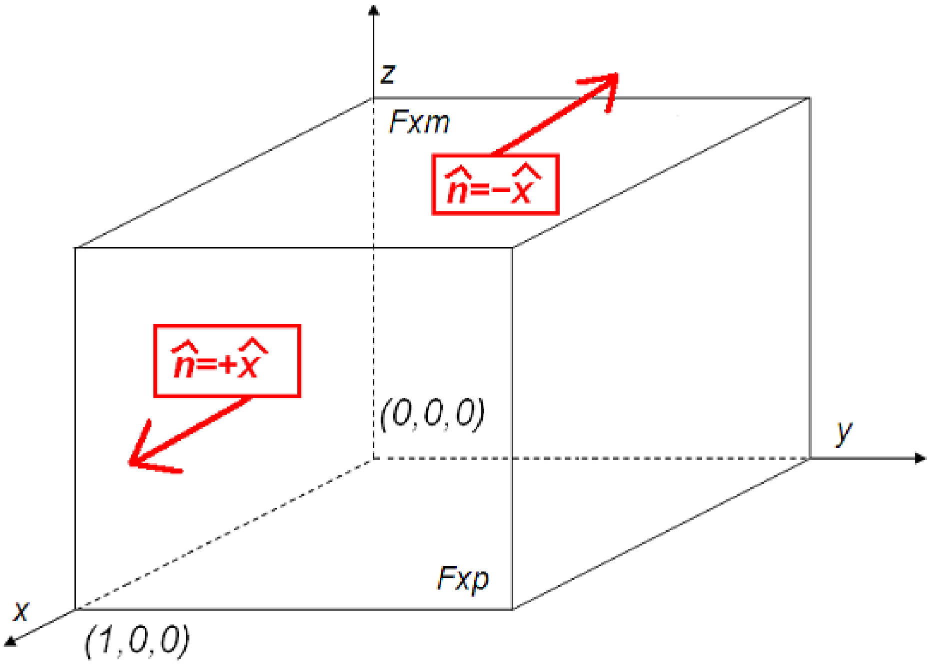

Figure 9.2: Integration over a cuboid.

limits are constants.

$$\begin{aligned}
\int_{s=0}^{s=4}\int_{t=0}^{t=6}(2s+ts/4)\,\mathrm{d}s\,\mathrm{d}t &= \int_{s=0}^{s=4}\int_{t=0}^{t=6} 2s\mathrm{d}s\,\mathrm{d}t + \tfrac{1}{4}\int_{s=0}^{s=4}\int_{t=0}^{t=6} ts\mathrm{d}s\,\mathrm{d}t,\\
&= \int_{t=0}^{t=6}\left(\int_{s=0}^{s=4} 2s\mathrm{d}s\right)\mathrm{d}t + \tfrac{1}{4}\left(\int_{s=0}^{s=4} s\mathrm{d}s\right)\left(\int_{t=0}^{t=6} t\mathrm{d}t\right),\\
&= 16\times 6 + \left(\tfrac{1}{4}\times\tfrac{1}{4}\times 16\times 36\right),\\
&= 132.
\end{aligned}$$

2. One particularly easy case involves surface integrals over cuboids. Let us consider such an example now: If $\boldsymbol{v} = x\hat{\boldsymbol{x}} + 2y\hat{\boldsymbol{y}} + 3z\hat{\boldsymbol{z}}$ and S is the unit cube with a vertex at $(0,0,0)$ and situated in the positive octant, compute $\int_S \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$.

   Refer to Figure 9.2. We divide the area S into its six faces, Fxp, Fxm, Fyp, Fym, Fzp, Fzm. Consider the face Fxp. This is the face contained entirely in a $y-z$ plane, with unit normal $+\hat{\boldsymbol{x}}$, and such that $x=1$. Consider also Fxm. Again, this face is contained entirely in a $y-z$ plane, with unit normal $-\hat{\boldsymbol{x}}$, and with $x=0$. Along Fxp,

$$\mathrm{d}\boldsymbol{S} = \mathrm{d}y\,\mathrm{d}z\hat{\boldsymbol{x}},$$

   and

$$\boldsymbol{v}\cdot\boldsymbol{S} = x\mathrm{d}S = x\mathrm{d}y\,\mathrm{d}z = \mathrm{d}y\,\mathrm{d}z, \qquad x=1.$$

   Along Fxm, $\mathrm{d}\boldsymbol{S} = -\mathrm{d}y\mathrm{d}z\hat{\boldsymbol{x}}$ and $\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S} = -x\mathrm{d}S = -x\mathrm{d}y\,\mathrm{d}z = 0$, since $x=0$ on this face.

Hence,

$$\int_{Fxm} + \int_{Fxp} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \int_0^1 \mathrm{d}y \int_0^1 1 \, \mathrm{d}z = 1.$$

Similarly,

$$\int_{Fym} + \int_{Fyp} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \int_0^1 \mathrm{d}x \int_0^1 2 \, \mathrm{d}z = 2,$$

and

$$\int_{Fzm} + \int_{Fzp} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \int_0^1 \mathrm{d}z \int_0^1 3 \, \mathrm{d}x = 3,$$

Putting it all together,

$$\int_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \left[\int_{Fxm} + \int_{Fxp} + \int_{Fym} + \int_{Fyp} + \int_{Fzm} + \int_{Fzp} \right] \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = 6.$$

9.3 Volume integrals

Volume integrals are much simpler than the other two, since the volume element $\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$ is a scalar. For a scalar field $\phi(x, y, z)$, the volume integral

$$\int_\Omega \phi(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

is the ordinary triple integral over the domain $\Omega \subset \mathbb{R}^3$. For a vector field $\boldsymbol{v}(x, y, z)$, the associated volume integral can be broken up into three scalar integrals:

$$\int_\Omega \boldsymbol{v}(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \hat{\boldsymbol{x}} \int_\Omega v_1(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z + \hat{\boldsymbol{y}} \int_\Omega v_2(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z + \hat{\boldsymbol{z}} \int_\Omega v_3(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

since the unit vectors $\hat{\boldsymbol{x}}$ &c. are constants and can be taken outside the integrals.

Example: If $\boldsymbol{v} = (2x^2 - 3z)\hat{\boldsymbol{x}} - 2xy\hat{\boldsymbol{y}} - 4x\hat{\boldsymbol{z}}$, evaluate

$$\int_\Omega \nabla \cdot \boldsymbol{v} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

where Ω is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Notice that

$$\nabla \cdot \boldsymbol{v} = 4x - 2x = 2x.$$

To find out where the plane $2x+2y+z = 4$ intersects the x and y axes, let $z = 0$. Then $2x+2y = 4$, and the plane intersects the x-axis when y=0, i.e. $x = 2$. Thus, in order for all values in the domain Ω to be included in the integration,

- x must vary between 0 and 2;

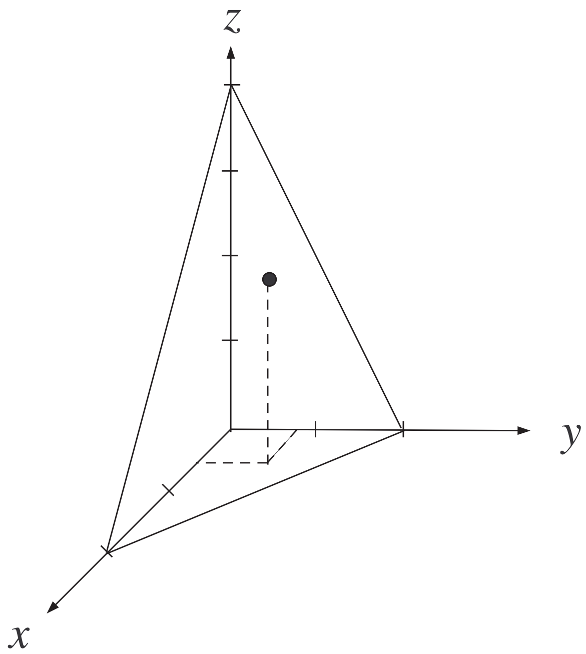

Figure 9.3: Area integration over a volume Ω bounded by three planes.

- y must vary between 0 and $y = 2 - x$;
- z must vary between 0 and $z = 4 - 2x - 2y$.

See Figure 9.3. Hence,

$$
\begin{aligned}
\int_V \operatorname{div} \boldsymbol{v} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z &= 2\int_0^2 \mathrm{d}x \int_0^{2-x} \mathrm{d}y \int_0^{4-2x-2y} \mathrm{d}z \, x \\
&= 2\int_0^2 \mathrm{d}x \, x \int_0^{2-x} \mathrm{d}y z\Big|_0^{4-2x-2y}, \\
&= 2\int_0^2 \mathrm{d}x \, x \int_0^{2-x} \mathrm{d}y(4-2x-2y), \\
&= 2\int_0^2 \mathrm{d}x \, x \int_0^{2-x} \mathrm{d}y(4-2x) - 4\int_0^2 \mathrm{d}x \, x \int_0^{2-x} \mathrm{d}y \, y, \\
&= 2\int_0^2 \mathrm{d}x \, x(4-2x)y\Big|_0^{2-x} - 2\int_0^2 \mathrm{d}x \, xy^2\Big|_0^{2-x}, \\
&= 2\int_0^2 \mathrm{d}x \, x \left[2\left(2-x\right)\left(2-x\right) - \left(2-x\right)^2\right], \\
&= 2\int_0^2 \mathrm{d}x \, x \left(2-x\right)^2, \\
&= 2\int_0^2 \mathrm{d}x \, \left(4x - 4x^2 + x^3\right), \\
&= 2\left(2x^2 - \tfrac{4}{3}x^3 + \tfrac{1}{4}x^4\right)_0^2 = 8/3.
\end{aligned}
$$

Pedantic note Sometimes, instead of the notation $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ for the volume element, we will write $\mathrm{d}V$, but we mean the same thing. The notation V will sometimes be used to denote a volume or domain in $\mathbb{R}^3$. Thus, it is not unusual to write

$$\int_V \phi(\boldsymbol{x})\,\mathrm{d}V$$

to denote the integration of the scalar field $\phi(\boldsymbol{x})$ over the domain $V \subset \mathbb{R}^3$.

Chapter 10

Stokes's and Gauss's Theorems

Overview

In ordinary calculus, recall the rule of integration by parts:

$$\int_a^b u\,\mathrm{d}v = (uv)\,|_a^b - \int_a^b v\,\mathrm{d}u.$$

That is, a difficult integral $u\,\mathrm{d}v$ can be split up into an easier integral $v\,\mathrm{d}u$ and a 'boundary term' $u(b)v(b) - u(a)v(a)$. In this section we do something similar for vector integrals.

10.1 Gauss's Theorem (or the Divergence Theorem)

Theorem 10.1 *Let V be a region in space bounded by a closed surface S, and let $\boldsymbol{v}(\boldsymbol{x})$ be a vector field with continuous first derivatives. Then*

$$\int_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V = \int_S \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S},$$

where $\mathrm{d}\boldsymbol{S}$ is outward-pointing surface-area element associated with the surface S.

Proof: First, consider a parallelepiped of sides of length Δx, Δy, and Δz, with one vertex positioned at (x, y, z) (Figure 10.1). As in previous exercises, label the faces Fxp, Fxm, Fyp, Fym, Fzp, and Fzm. We compute

$$\sum_{\text{all faces}} \boldsymbol{v}\cdot\Delta\boldsymbol{S},$$

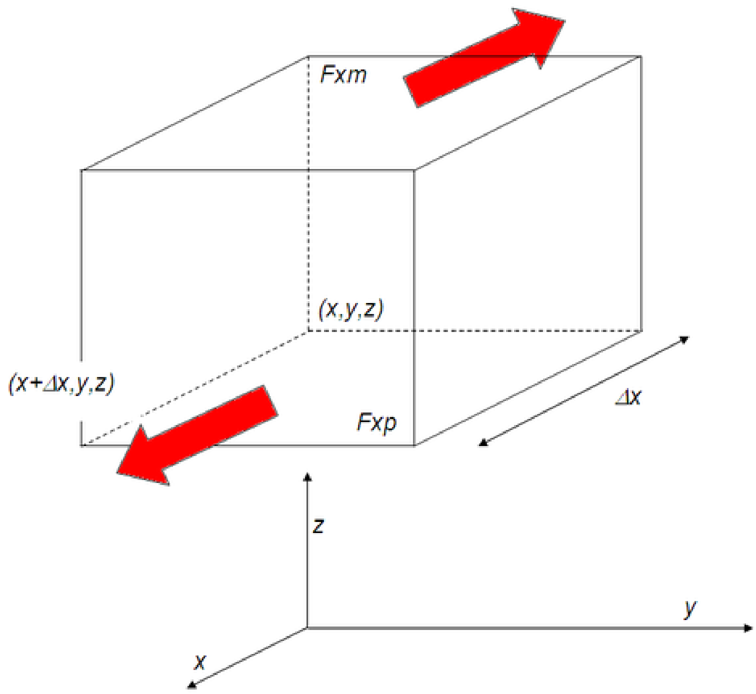

Figure 10.1: Area integration over a parallelepiped, as applied to Gauss's theorem.

where $\Delta\boldsymbol{S}$ is the area element on each face. For example, in the x-direction, we have a positive contribution from Fxp and a negative one from Fxm, to give

$$-v_1(x,y,z)\Delta y\Delta z + v_1(x+\Delta x,y,z)\Delta y\Delta z.$$

We immediately write down the other contributions: From Fyp and Fym, we have

$$-v_2(x,y,z)\Delta x\Delta z + v_2(x,y+\Delta y,z)\Delta x\Delta z,$$

and from Fzp and Fzm, we have

$$-v_3(x,y,z)\Delta x\Delta y + v_2(x,y,z+\Delta z)\Delta x\Delta y.$$

Summing over all six contributions (i.e. over all six faces), we have

$$\sum_{\text{all faces}} \boldsymbol{v}\cdot\Delta\boldsymbol{S} =$$
$$v_1(x+\Delta x,y,z)\Delta y\Delta z - v_1(x,y,z)\Delta y\Delta z + v_2(x,y+\Delta y,z)\Delta x\Delta z - v_2(x,y,z)\Delta x\Delta z +$$
$$v_3(x,y,z+\Delta z)\Delta x\Delta y - v_3(x,y,z)\Delta x\Delta y.$$

We apply Taylor's theorem to these increments, and omit terms that are $O(\Delta x^2, \Delta y^2, \Delta z^2)$. This becomes rigorous in the limit when the parallelepiped volume go to zero. In this way, we obtain

$$\sum_{\text{all faces}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \nabla \cdot \boldsymbol{v}\, \mathrm{d}V.$$

For the second and final step, consider an arbitrary shape of volume V in three dimensions. We break this volume up into many infinitesimally small parallelepipeds. By the previous result, we have

$$\sum_{\text{all parallelepipeds}} \nabla \cdot \boldsymbol{v}\, \mathrm{d}V = \sum_{\text{all parallelepipeds}} \left(\sum_{\text{all faces}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} \right). \tag{10.1}$$

Consider, however, two neighbouring parallelepipeds (Figure 10.2). Call them A and B These will share a common face, F, with normal vector $\hat{\boldsymbol{n}}$ and area $\mathrm{d}S$. Parallelepiped A gives a contribution $\hat{\boldsymbol{n}} \cdot \boldsymbol{v}(F) dS$, say, to the sum (10.1), while parallelepiped B must give a contribution $-\hat{\boldsymbol{n}} \cdot \boldsymbol{v}(F)\mathrm{d}S$. The only place where such a cancellation cannot occur is on exterior faces. Thus,

$$\sum_{\text{all parallelepipeds}} \nabla \cdot \boldsymbol{v}\, \mathrm{d}V = \sum_{\text{all exterior faces}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}.$$

But the parallelepiped volumes are infinitesimally small, so this sum converts into an integral:

$$\int_V \nabla \cdot \boldsymbol{v}\, \mathrm{d}V = \int_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}.$$

This completes the proof.

10.1.1 Green's theorem

A frequently used corollary of Gauss's theorem is a relation called **Green's theorem**. If ϕ and ψ are two scalar fields, then we have the identities

$$\nabla \cdot (\phi \nabla \psi) = \phi \nabla \cdot \nabla \psi + \nabla \phi \cdot \nabla \psi,$$
$$\nabla \cdot (\psi \nabla \phi) = \psi \nabla \cdot \nabla \phi + \nabla \psi \cdot \nabla \phi.$$

Subtracting these equations gives

$$\begin{aligned} \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) &= \phi \nabla \cdot \nabla \psi - \psi \nabla \cdot \nabla \phi, \\ &= \phi \nabla^2 \psi - \psi \nabla^2 \phi. \end{aligned}$$

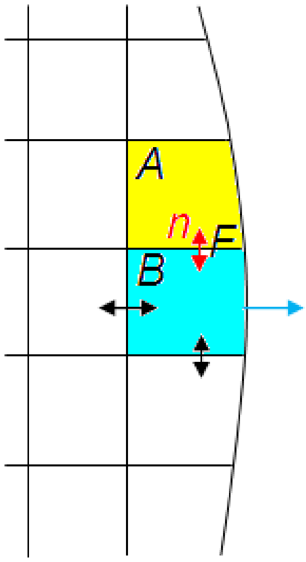

Figure 10.2: Cancellations in Gauss's theorem.

We integrate over a volume V whose boundary is a closed set S. Applying Gauss's theorem gives

$$\begin{aligned}\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) \mathrm{d}V &= \int_V \left[\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi)\right] \mathrm{d}V, \\ &= \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot \mathrm{d}\boldsymbol{S}.\end{aligned}$$

Thus, we have Green's theorem:

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) \mathrm{d}V = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot \mathrm{d}\boldsymbol{S},$$

where V is a region of $\mathbb{R}^3$ whose boundary is the closed set S.

10.1.2 Other forms of Gauss's theorem

Although the form $\int_V \nabla\cdot\boldsymbol{v}\mathrm{d}V = \int_S \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$ is the most common statement of Gauss's theorem, there are other forms. For example, let

$$\boldsymbol{v}(\boldsymbol{x}) = v(\boldsymbol{x})\boldsymbol{a},$$

where $\boldsymbol{a}$ is a *constant vector*. We have

$$\int_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V = \int_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V = \boldsymbol{a}\cdot\int_V (\nabla v)\mathrm{d}V.$$

However, applying Gauss's theorem gives

$$\int_V \nabla \cdot \boldsymbol{v}\, \mathrm{d}V = \int_S v\boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} = \boldsymbol{a} \cdot \int_S v\, \mathrm{d}\boldsymbol{S}.$$

Equating both sides,

$$\boldsymbol{a} \cdot \int_V \nabla v\, \mathrm{d}V = \boldsymbol{a} \cdot \int_S v\, \mathrm{d}\boldsymbol{S},$$

or

$$\boldsymbol{a} \cdot \left[\int_V \nabla v\, \mathrm{d}V - \int_S v\, \mathrm{d}\boldsymbol{S} \right] = 0.$$

Since this holds for arbitrary vector fields of the form $\boldsymbol{v} = v(\boldsymbol{x})\boldsymbol{a}$, it must be true that $[\cdots] = 0$, or

$$\int_V \nabla v\, \mathrm{d}V = \int_S v\, \mathrm{d}\boldsymbol{S}.$$

Similarly, letting $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} \times \boldsymbol{u}(\boldsymbol{x})$, where $\boldsymbol{a}$ is a constant vector, gives

$$\int_V \nabla \times \boldsymbol{u}\, \mathrm{d}V = \int_S \mathrm{d}\boldsymbol{S} \times \boldsymbol{u}.$$

Worked examples

1. Evaluate by using Gauss's theorem $\int_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$, where

$$\boldsymbol{v} = 8xz\hat{\boldsymbol{x}} + 2y^2\hat{\boldsymbol{y}} + 3yz\hat{\boldsymbol{z}}$$

and S is the surface of the unit cube in the positive octant, one of whose vertices lies at $(0,0,0)$.

We compute:

$$\begin{aligned} \int_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} &= \int_V \mathrm{d}V\, \nabla \cdot \boldsymbol{v}, \\ &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, \left(8z + 4y + 3y\right), \\ &= 1 \cdot 1 \cdot \int_0^1 8z\, \mathrm{d}z + 1 \cdot 1 \cdot \int_0^1 7y\, \mathrm{d}y, \\ &= 4 + \tfrac{7}{2} = \tfrac{15}{2}. \end{aligned}$$

2. A fluid is confined in a container of volume V with closed boundary S. The velocity of the fluid is $\boldsymbol{v}(\boldsymbol{x},t)$. The velocity satisfies the so-called no-throughflow condition

$$\boldsymbol{v}\cdot\hat{\boldsymbol{n}} = 0,\ \text{on } S,$$

where $\hat{\boldsymbol{n}}$ is the outward-pointing normal to the surface. Now suppose that a pollutant is introduced to the fluid, of concentration $C(\boldsymbol{x},t)$. The pollutant must satisfy the equation

$$\frac{\partial C}{\partial t} + \nabla\cdot(\boldsymbol{v}C) = 0.$$

Prove that the total amount of pollutant,

$$P(t) = \int_V C(\boldsymbol{x},t)\,\mathrm{d}V,$$

stays the same over time (hence P is in fact independent of time).

Proof: We have

$$\begin{aligned}
\frac{dP}{dt} &= \frac{d}{dt}\int_V C(\boldsymbol{x},t)\,\mathrm{d}V,\\
&= \int_V \frac{\partial C(\boldsymbol{x},t)}{\partial t}\,\mathrm{d}V,\\
&= -\int_V \nabla\cdot(\boldsymbol{v}C)\,\mathrm{d}V,\\
&= -\int_S C(\boldsymbol{x}\in S,t)\boldsymbol{v}(\boldsymbol{x}\in S,t)\cdot\mathrm{d}\boldsymbol{S}.
\end{aligned}$$

But

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{v}|_{\boldsymbol{x}\in S} = 0,$$

hence

$$\frac{dP}{dt} = 0,$$

and the amount of pollutant P is constant ('conserved').

10.2 Stokes's Theorem

Theorem 10.2 *Let S be an open, two-sided surface bounded by a closed, non-intersecting*

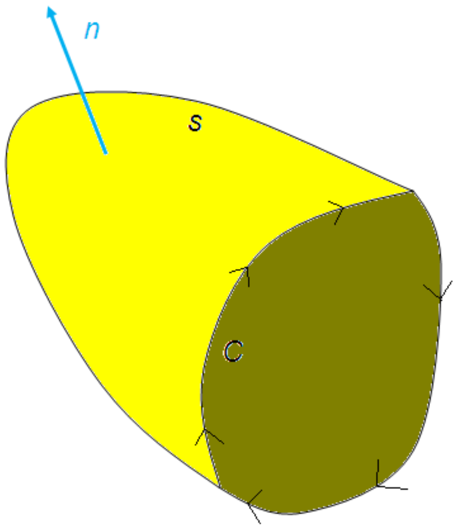

Figure 10.3: Stokes theorem: S is a surface; C is its boundary. The boundary can be given a definite orientation so the curve is called **two-sided**.

curve C, and let $\boldsymbol{v}(\boldsymbol{x})$ be a vector field with continuous derivatives. Then,

$$\oint_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} = \int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S},$$

where C is treated in the positive direction: an observer walking along the boundary of S, with his head pointing in the direction of the positive normal to S, has the surface on his left.

For the $S - C$ curve to which the theorem refers, see Figure 10.3.

Proof: First, consider a rectangle in the x-y plane of sides of length Δx and Δy, with one vertex positioned at (x, y) (Figure 10.4). Label the edges Exp, Exm, Eyp, and Eym. We compute

$$\sum_{\text{all edges}} \boldsymbol{v} \cdot \Delta\boldsymbol{x},$$

where $\Delta\boldsymbol{x}$ is the line element on each edge, and we compute in an anticlockwise sense. For example, in the x-direction, along Exp we have $\mathrm{d}\boldsymbol{x} = \hat{\boldsymbol{x}}\mathrm{d}x$ and along Exm we have $\mathrm{d}\boldsymbol{x} = -\hat{\boldsymbol{x}}\mathrm{d}x$. Adding up these contributions to $\boldsymbol{v} \cdot \Delta\boldsymbol{x}$ gives

$$[v_1(x, y, z)\Delta x - v_1(x, y + \Delta y, z)]\, \Delta x.$$

Similarly, the contributions along Eyp and Eym give

$$[v_2(x + \Delta x, y) - v_2(x, y)]\, \Delta y.$$

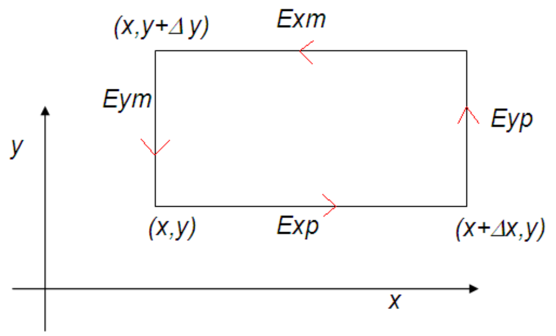

Figure 10.4: Line integration over a rectangle.

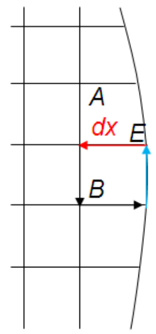

Figure 10.5: Cancellations in Stokes's theorem.

Summing over these four contributions (i.e. summing over the four edges), we have

$$\sum_{\text{all edges}} \boldsymbol{v}\cdot\Delta\boldsymbol{x} = \left[v_1(x,y) - v_1(x,y+\Delta y)\right]\Delta x + \left[v_2(x+\Delta x,y) - v_2(x,y)\right]\Delta y$$

We apply Taylor's theorem to these increments and omit terms that are $O(\Delta x^2, \Delta y^2)$. This procedure is rigorous in the limit as the parallelogram area goes to zero. We obtain

$$\begin{aligned}\sum_{\text{all edges}} \boldsymbol{v}\cdot\Delta\boldsymbol{x} &= \left[v_1(x,y) - v_1(x,y+\Delta y)\right]\Delta x + \left[v_2(x+\Delta x,y) - v_2(x,y)\right]\Delta y \\ &= \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)_{(x,y)} \Delta x\Delta y.\end{aligned}$$

However, $\mathrm{d}\boldsymbol{S} = \hat{\boldsymbol{z}}\Delta x\Delta y$ pointing out of the page, hence

$$\sum_{\text{all edges}} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} = (\nabla\times\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{S}.$$

For the second and final step, consider a surface S with boundary C. We break this surface up into many infinitesimally small parallelograms. By the previous result, we have

$$\sum_{\text{all parallelograms}} (\nabla\times\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{S} = \sum_{\text{all parallelograms}} \left(\sum_{\text{all edges}} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x}\right). \tag{10.2}$$

Consider, however, two neighbouring parallelograms (Figure 10.5). Call them A and B These will share a common edge, E, with line element $\mathrm{d}\boldsymbol{x}$. Parallelogram A gives a contribution a, say, to the sum (10.1), while parallelepiped B must give a contribution $-a$. The only place where such a cancellation cannot occur is on exterior edges. Thus,

$$\sum_{\text{all parallelograms}} (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} = \sum_{\text{all exterior edges}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}.$$

But the parallelogram areas are infinitesimally small, so this sum converts into an integral:

$$\int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}.$$

This completes the proof.

Example: Given a vector $\boldsymbol{v} = -\hat{\boldsymbol{x}}y + \hat{\boldsymbol{y}}x$, using Stokes's theorem, show that the integral around a continuous closed curve in the xy plane

$$\tfrac{1}{2} \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} = \tfrac{1}{2} \oint (x\,\mathrm{d}y - y\,\mathrm{d}x) = S,$$

the area enclosed by the curve.

Proof:

$$\begin{aligned}
\tfrac{1}{2} \oint_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} &= \tfrac{1}{2} \int_S [\nabla \times (-\hat{\boldsymbol{x}}y + \hat{\boldsymbol{y}}x)] \cdot \mathrm{d}\boldsymbol{S}, \\
&= \tfrac{1}{2} \int_S (2\hat{\boldsymbol{z}}) \cdot \mathrm{d}\boldsymbol{S}, \\
&= \tfrac{1}{2} \int_S (2\hat{\boldsymbol{z}}) \cdot (\mathrm{d}x\,\mathrm{d}y\,\hat{\boldsymbol{z}}), \\
&= \int_S \mathrm{d}x\,\mathrm{d}y = S.
\end{aligned}$$

Green's theorem in the plane

The last example hints at the following result: let S be a patch of area entirely contained in the xy plane, with boundary C, and let $\boldsymbol{v} = (v_1(x,y), v_2(x,y), 0)$ be a smooth vector field. Then,

$$\begin{aligned}
\int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} &= \int_S (\nabla \times \boldsymbol{v}) \cdot (\mathrm{d}x\,\mathrm{d}y\,\hat{\boldsymbol{z}}), \\
&= \int_S \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

But by Stokes's theorem,

$$\begin{aligned}
\int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} &= \int_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}, \\
&= \int_C (v_1 \mathrm{d}x + v_2 \mathrm{d}y).
\end{aligned}$$

Putting these equations together, we have Green's theorem in the plane:

$$\int_S \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \int_C (v_1 \mathrm{d}x + v_2 \mathrm{d}y) \, .$$

10.3 Potential theory

A vector field $\boldsymbol{v}$ is irrotational if and only if

- $\nabla \times \boldsymbol{v} = 0$ if and only if
- $\boldsymbol{v} = -\nabla \mathcal{U}$ if and only if
- The line integral $\int_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}$ depends only on the initial and final points of the path C and is independent of the details of the path between these terminal points.

Proving that $\boldsymbol{v} = -\nabla \mathcal{U} \implies \nabla \times \boldsymbol{v} = 0$ was trivial and we have done this already. Until now, we have been unable to prove the converse, namely that $\nabla \times \boldsymbol{v} \implies \boldsymbol{v} = -\nabla \mathcal{U}$. Let us do so now.

Consider an open subset $\Omega \in \mathbb{R}^3$ that is **simply connected**, i.e. contains no 'holes'. Let us take an arbitrary closed, smooth curve C in Ω. Because Ω is simply connected, it is possible to find a surface S that lies entirely in Ω, such that (S, C) have the properties mentioned in Stokes's theorem. Suppose now that $\nabla \times \boldsymbol{v} = 0$ for **all points** $\boldsymbol{x} \in \Omega$. Now, by Stokes's theorem,

$$\begin{aligned} 0 &= \int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S}, \\ &= \oint_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}. \end{aligned}$$

This last result is true for **all closed, piecewise smooth contours** in the domain Ω. The only way for this relationship to be satisfied for **all contours** is if $\boldsymbol{v} = -\nabla \mathcal{U}$, for some function $\mathcal{U}(\boldsymbol{x})$, since then,

$$\begin{aligned} \oint_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} &= -\oint_C (\nabla \mathcal{U}) \cdot \mathrm{d}\boldsymbol{x}, \\ &= -\left[\mathcal{U}(a) - \mathcal{U}(a) \right], \\ &= 0, \end{aligned}$$

for some reference point a on the contour C. Thus, we have proved that a vector field $\boldsymbol{v}$ is irrotational **if and only if** $\boldsymbol{v} = -\nabla \mathcal{U}$.

Simple-connectedness

Simple-connectedness will not be an issue in this module, as we usually work with vector fields defined on the whole of $\mathbb{R}^3$. On the other hand, it is not hard to find a domain Ω that is not simply connected. For example, consider a portion of the xy plane with a hole (Figure 10.6). The closed

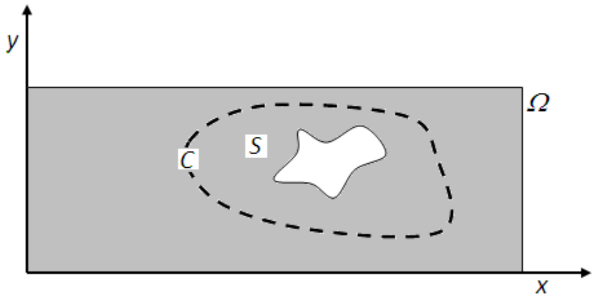

Figure 10.6: The set Ω is not simply connected.

curve C surrounds a region S; however, S is not contained entirely in Ω. We have knowledge of $\nabla \times \boldsymbol{v}$ only in Ω; we are unable to say anything about $\nabla \times \boldsymbol{v}$ in certain parts of the region S, and are therefore unable to apply the arguments of Stokes's theorem to this particular (S, C) pair. Again, it is not hard to find examples of such domains: imagine the domain of the vector field for flow over an aerofoil: such a domain is obviously not simply connected.

A more precise definition of simple-connectedness than the vague condition that 'the set should contain no holes' is the following: for any two closed paths $C_0 : [0,1] \to \Omega$, $C_1 : [0,1] \to \Omega$ based at $\boldsymbol{x}_0$, i.e.

$$\boldsymbol{x}_{C_0}(0) = \boldsymbol{x}_{C_1}(0) = \boldsymbol{x}_0,$$

there exists a continuous map

$$H : [0,1] \times [0,1] \to \Omega,$$

such that

$$\begin{aligned} H(t,0) &= \boldsymbol{x}_{C_0}(t), \qquad 0 \le t \le 1, \\ H(t,1) &= \boldsymbol{x}_{C_1}(t), \qquad 0 \le t \le 1, \\ H(0,s) &= H(1,s) = \boldsymbol{x}_0, \qquad 0 \le s \le 0. \end{aligned}$$

Such a map is called a **homotopy** and C_0 and C_1 are called homotopy equivalent. One can think of this map as a 'continuous deformation of one loop into another'. Because a point is, trivially, a loop, in a simply-connected set, a loop can be continuously deformed into a point. Note in the example Figure 10.6, the loop C cannot be continuously deformed into a point without leaving the set Ω. This is a more relational - or topological way - of describing the 'hole' in the set in Figure 10.6.

Worked examples

1. In thermodynamics, the energy of a system of gas particles is expressed in differential form:

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y,$$

where

- A is the temperature;
- B is minus the pressure;
- x has the interpretation of entropy;
- y has the interpretation of container volume.

The temperature and the pressure are known to satisfy the following relation:

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}.$$

Prove that for any closed path C in xy-space (i.e. in entropy/volume-space),

$$\oint_C [A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y] = 0.$$

Proof: We may regard

$$\boldsymbol{v}(x,y) = (A(x,y), B(x,y))$$

as a vector field, and we may take

$$\mathrm{d}\boldsymbol{S} = \mathrm{d}x\,\mathrm{d}y\hat{\boldsymbol{z}}$$

as an area element, pointing out of the xy-plane. Now let S be the patch of area in xy space enclosed by the curve C. We have

$$\begin{aligned}
\int_S (\nabla\times\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{S} &= \int_S \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y,\\
&= \int_S \left[\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right]\mathrm{d}x\,\mathrm{d}y,\\
&= \int_S \left(\frac{\partial A}{\partial y} - \frac{\partial A}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y,\\
&= 0.
\end{aligned}$$

But by Stokes's theorem,

$$\begin{aligned} 0 &= \int_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S}, \\ &= \int_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}, \\ &= \int_C [A\mathrm{d}x + B\mathrm{d}y], \end{aligned}$$

as required. Because $A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y$ integrates to zero when the integral is a closed contour, there exists a potential $E(x,y)$, such that

$$\mathrm{d}E = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y.$$

The function E is called the **thermodynamic energy**. The integral of $\mathrm{d}E$ around a closed path is identically zero, and **the energy is path-independent**.

In general, the **differential form**

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y$$

is **exact** if and only if

- There is a function $\phi(x,y)$, such that

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y := \mathrm{d}\phi,$$

  if and only if

- The following relation holds:

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$$

2. In mechanics, particles experience a **force field** $\boldsymbol{F}(\boldsymbol{x})$. The force is called conservative if a potential function exists:

$$\boldsymbol{F} = -\nabla \mathcal{U}.$$

Thus, a force is conservative if and only if $\nabla \times \boldsymbol{F} = 0$.

3. Show that the three-dimensional gravitational force

$$\boldsymbol{F} = -\frac{\alpha \boldsymbol{r}}{|\boldsymbol{r}|^3}$$

is a conservative force, where α is a positive constant. We compute $\nabla \times \boldsymbol{F}$ by application of the following chain rule:

$$\nabla \times (\phi \boldsymbol{u}) = \phi \nabla \times \boldsymbol{u} + \nabla \phi \times \boldsymbol{u},$$

and we take $\phi = r^{-3}$ and $\boldsymbol{u} = \boldsymbol{r}$:

$$\nabla \times \boldsymbol{F} = -\alpha \left\{ \frac{1}{r^3} (\nabla \times \boldsymbol{r}) + \left[\nabla \left(r^{-3} \right) \right] \times \boldsymbol{r} \right\},$$

Now

$$\nabla \times \boldsymbol{r} = \nabla \times \left(\tfrac{1}{2} \nabla r^2 \right) = 0.$$

Also,

$$\nabla r^{-3} = -\frac{3\boldsymbol{r}}{r^5},$$

Hence,

$$\begin{aligned} \nabla \times \boldsymbol{F} &= -\alpha \left[\frac{1}{r^3} \nabla \times \boldsymbol{r} - \left(\nabla r^{-3} \right) \times \boldsymbol{r} \right], \\ &= -\alpha \left[0 - \left(\frac{3\boldsymbol{r}}{r^5} \right) \times \boldsymbol{r} \right], \\ &= 0. \end{aligned}$$

Thus, both contributions to $\nabla \times \boldsymbol{F}$ are zero, so $\nabla \times \boldsymbol{F} = 0$, and gravity is conservative.

See if you can show that

$$\mathcal{U} = -\frac{\alpha}{r}$$

is a suitable potential, $\boldsymbol{F} = -\nabla \left(-\alpha r^{-1} \right)$.

4. Show that the force

$$\boldsymbol{F} = \alpha (x^2 \hat{\boldsymbol{x}} + y \hat{\boldsymbol{y}})$$

is a conservative force and construct its potential.

We have

$$\nabla \times \boldsymbol{F} = \alpha \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ x^2 & y & 0 \end{vmatrix} = \alpha \hat{\boldsymbol{z}} \left(\partial_x y - \partial_y x^2 \right) = 0.$$

Next, we take

$$F_x = \alpha x^2 = -\partial_x \mathcal{U}.$$

Ordinary integration gives

$$\mathcal{U}(x,y) = -\tfrac{1}{3}\alpha x^3 + f(y),$$

where $f(y)$ is a function to be determined. But we also have

$$F_y = \alpha y = -\partial_y \mathcal{U},$$

which gives

$$\mathcal{U}(x,y) = -\tfrac{1}{2}\alpha y^2 + g(x).$$

Putting these results together, we have

$$\mathcal{U}(x,y) = -\alpha\left(\tfrac{1}{3}x^3 + \tfrac{1}{2}y^2\right) + \text{Const.},$$

and the constant is immaterial because only *gradients* of the potential are important.

5. Recall that the vorticity $\boldsymbol{\omega}(\boldsymbol{x})$ measures the amount of swirl in a fluid velocity field $\boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$. Show that all irrotational flows

$$\boldsymbol{\omega} = 0,$$

are potential flows,

$$\boldsymbol{v} = \nabla\phi.$$

Show that the potential for an incompressible irrotational flow satisfies Laplace's equation:

$$\nabla \cdot \boldsymbol{v} = 0 \text{ and } \boldsymbol{\omega} = 0 \implies \nabla^2\phi = 0.$$

The study of the equation $\nabla^2\phi = 0$ is called **harmonic analysis**.

If the flow is irrotational, then $\nabla \times \boldsymbol{v} = 0$, which implies, by Stokes's theorem,

$$\boldsymbol{v} = \nabla\phi,$$

(note the sign), for some *velocity potential* ϕ. We are to assume that the flow is incompressible:

$$0 = \nabla \cdot \boldsymbol{v} = \nabla \cdot \nabla\phi = \nabla^2\phi.$$

Thus, an incompressible, irrotational flow satisfies

$$\nabla^2\phi = 0.$$

Chapter 11

Curvilinear coordinate systems

Overview and introduction

So far we have restricted ourselves to Cartesian coordinate systems. A Cartesian coordinate system offers a unique advantage in that the distinguished directions $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ all point in constant directions. However, many physical problems are not well suited to solution in Cartesian coordinates. For instance, in the atmosphere, fluid flow takes place on a sphere, and latitude and longitude are more appropriate labels for position in space. Such a problem naturally leads to the use of spherical polar coordinates. In fact, the coordinate system we use should be chosen to fit the problem in hand, and to exploit any type of symmetry or constraint therein. Then, hopefully, the problem will be more amenable to solution than if we had stubbornly persisted with the Cartesian framework.

Unfortunately, there is a high price to pay for this freedom of choice (for coordinate systems). In an arbitrary coordinate system, the distinguished directions are no longer constant, and the operators div, grad, and curl become very cumbersome. Nevertheless, we must be willing to pay the ultimate price for this freedom, and derive expressions for div, grad, and curl in **orthogonal curvilinear coordinate systems**.

11.1 Coordinate transformations

In three dimensions, three variables are necessary and sufficient to specify the location of a particle. We have used the Cartesian triple (x, y, z), where the equations $x = \text{Const.}$, $y = \text{Const.}$, and $z = \text{Const.}$ describe three mutually perpendicular families of planes. Suppose now we superimpose on these planes a second family of surfaces. These surfaces need not be planes; nor need they be parallel. In the Cartesian framework, a point is specified by the intersection of the three planes; in the new framework, the same point is specified by the intersection of three surfaces. In the new

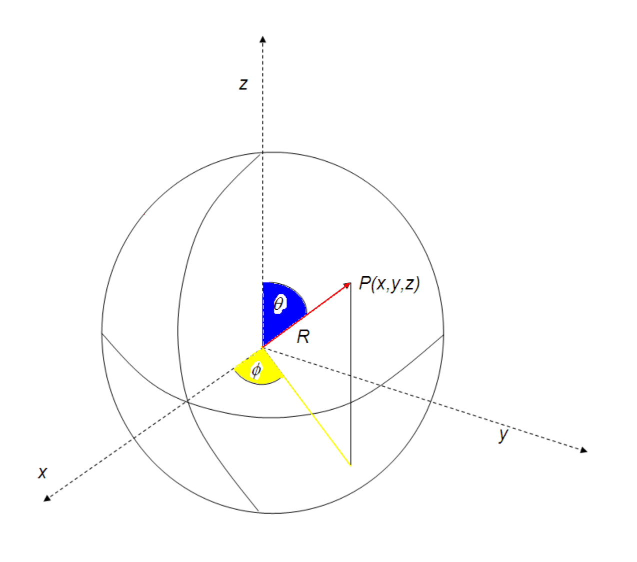

Figure 11.1: Spherical polar coordinates

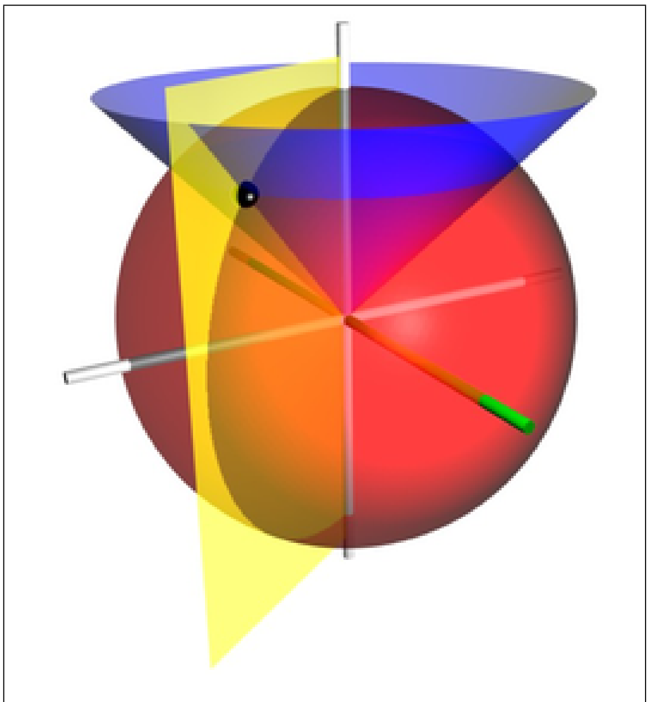
Figure 11.2: Planes generated by pherical polar coordinates. From http://en.wikipedia.org/wiki/Spherical_coordinates, 16th Aug. 2010.

framework, let the new surfaces be described by

$$q_1 = \text{Const.}, \qquad q_2 = \text{Const.}, \qquad q_3 = \text{Const.}.$$

Because the point in question can be described adequately in both frameworks, as the point of intersection of three surfaces, we may write

$$x = x(q_1, q_2, q_3), \qquad y = y(q_1, q_2, q_3), \qquad z = z(q_1, q_2, q_3),$$

and

$$q_1 = q_1(x, y, z), \qquad q_2 = q_2(x, y, z), \qquad q_3 = q_3(x, y, z),$$

where each function written here is assumed smooth. That is, there is a smooth, invertible map connecting the two coordinate systems. This map is called a **coordinate transformation**.

Example: Consider **spherical polar coordinates** as shown in Figure 11.1. The point P can either be labelled by the Cartesian triple (x, y, z), or by its radial distance R from the origin, together with two angles: the **azimuthal angle** and the **polar angle**. The azimuthal angle φ is the angle between the x-axis and the projection of the radius vector $\boldsymbol{x} \equiv \boldsymbol{r} \equiv \overrightarrow{OP}$ on to the x-y plane. The polar angle θ is the angle between the z-direction and the radius vector. Here are the surfaces generated by these new coordinates:

- The surface $R = \mathrm{Const.}$ is a **sphere** of radius R centred at O (q_1),
- The surface $\theta = \mathrm{Const.}$ is a **cone** whose tip lies at the origin O (q_2),
- The surface $\varphi = \mathrm{Const.}$ is a plane parallel to the z-axis, given by $y = x\tan\varphi$ (q_3).

The point P is given by the intersection of these surfaces, or by the intersection of the planes $x = \mathrm{Const.}$, $y = \mathrm{Const.}$, and $z = \mathrm{Const.}$ (See Figure 11.2). These two coordinate systems are related through

$$\begin{aligned} x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\sin\varphi, \\ z &= r\cos\theta, \end{aligned}$$

with inverse transformation

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2}, \\ \theta &= \cos^{-1}\left(z/r\right), \\ \varphi &= \tan^{-1}\left(y/x\right). \end{aligned}$$

Note: Particular care must be taken with the inverse $\tan^{-1}(y/x)$. Where necessary, we must add or subtract 2π to the answer to obtain an angle $\varphi \in [0, 2\pi)$.

11.2 The line element, tangent vectors, scale factors

Recall, in a Cartesian frame, that a small increment of length $\mathrm{d}s$ is given by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

The quantity $\mathrm{d}s$ is called the **line element**. Let us take a coordinate transformation

$$x = x(q_1, q_2, q_3), \qquad y = y(q_1, q_2, q_3), \qquad z = z(q_1, q_2, q_3),$$

and

$$q_1 = q_1(x, y, z), \qquad q_2 = q_2(x, y, z), \qquad q_3 = q_3(x, y, z),$$

and compute the line element i.t.o. the q's. This is possible because the line element exists independent of its description in Cartesian coordinates. We have,

$$\mathrm{d}x = \frac{\partial x}{\partial q_1}\mathrm{d}q_1 + \frac{\partial x}{\partial q_2}\mathrm{d}q_2 + \frac{\partial x}{\partial q_3}\mathrm{d}q_3,$$

and similarly for $\mathrm{d}y$ and $\mathrm{d}z$. Thus, in vector notation,

$$\mathrm{d}\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial q_1}\mathrm{d}q_1 + \frac{\partial \boldsymbol{x}}{\partial q_2}\mathrm{d}q_2 + \frac{\partial \boldsymbol{x}}{\partial q_3}\mathrm{d}q_3$$

Substitution of these **differentials** into the definition of the line element gives

$$\begin{aligned}
\mathrm{d}s^2 = \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} &= \left(\frac{\partial \boldsymbol{x}}{\partial q_1}\mathrm{d}q_1 + \frac{\partial \boldsymbol{x}}{\partial q_2}\mathrm{d}q_2 + \frac{\partial \boldsymbol{x}}{\partial q_3}\mathrm{d}q_3\right)\cdot\left(\frac{\partial \boldsymbol{x}}{\partial q_1}\mathrm{d}q_1 + \frac{\partial \boldsymbol{x}}{\partial q_2}\mathrm{d}q_2 + \frac{\partial \boldsymbol{x}}{\partial q_3}\mathrm{d}q_3\right)\\
&= \left(\frac{\partial \boldsymbol{x}}{\partial q_1}\cdot\frac{\partial \boldsymbol{x}}{\partial q_1}\right)\mathrm{d}q_1^2 + \left(\frac{\partial \boldsymbol{x}}{\partial q_2}\cdot\frac{\partial \boldsymbol{x}}{\partial q_2}\right)\mathrm{d}q_2^2 + \left(\frac{\partial \boldsymbol{x}}{\partial q_3}\cdot\frac{\partial \boldsymbol{x}}{\partial q_3}\right)\mathrm{d}q_3^2\\
&\quad + \left(\frac{\partial \boldsymbol{x}}{\partial q_1}\cdot\frac{\partial \boldsymbol{x}}{\partial q_2}\right)\mathrm{d}q_1\,\mathrm{d}q_2 + \left(\frac{\partial \boldsymbol{x}}{\partial q_1}\cdot\frac{\partial \boldsymbol{x}}{\partial q_3}\right)\mathrm{d}q_1\,\mathrm{d}q_3 + \left(\frac{\partial \boldsymbol{x}}{\partial q_2}\cdot\frac{\partial \boldsymbol{x}}{\partial q_3}\right)\mathrm{d}q_2\,\mathrm{d}q_3\\
&\quad + \left(\frac{\partial \boldsymbol{x}}{\partial q_2}\cdot\frac{\partial \boldsymbol{x}}{\partial q_1}\right)\mathrm{d}q_2\,\mathrm{d}q_1 + \left(\frac{\partial \boldsymbol{x}}{\partial q_3}\cdot\frac{\partial \boldsymbol{x}}{\partial q_1}\right)\mathrm{d}q_3\,\mathrm{d}q_1 + \left(\frac{\partial \boldsymbol{x}}{\partial q_3}\cdot\frac{\partial \boldsymbol{x}}{\partial q_2}\right)\mathrm{d}q_3\,\mathrm{d}q_2.
\end{aligned}$$

In more compact form, this is written as

$$\begin{aligned}
\mathrm{d}s^2 = g_{11}\mathrm{d}q_1^2 + g_{22}\mathrm{d}q_2^2 + g_{33}\mathrm{d}q_3^2&\\
+ g_{12}\mathrm{d}q_1\,\mathrm{d}q_2 + g_{13}\mathrm{d}q_1\,\mathrm{d}q_3 + g_{23}\mathrm{d}q_2\,\mathrm{d}q_3&\\
+ g_{21}\mathrm{d}q_2\,\mathrm{d}q_1 + g_{31}\mathrm{d}q_3\,\mathrm{d}q_1 + g_{32}\mathrm{d}q_3\,\mathrm{d}q_2.&
\end{aligned}$$

and

$$g_{ij} = \frac{\partial \boldsymbol{x}}{\partial q_i}\cdot\frac{\partial \boldsymbol{x}}{\partial q_j} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j}$$

is called the **metric tensor**.

The expression we have derived for the line element is clearly very complicated. Therefore, we restrict ourselves to **orthogonal coordinate systems**:

> A coordinate system is orthogonal if g_{ij} is a diagonal matrix.

The reason for this nomenclature is clear: the vector

$$\frac{\partial \boldsymbol{x}}{\partial q_i} \tag{11.1}$$

is **normal** to the surface $q_i = \mathrm{Const.}$. Thus, the coordinate surfaces are mutually perpendicular if

$$\left(\frac{\partial \boldsymbol{x}}{\partial q_i}\right)\cdot\left(\frac{\partial \boldsymbol{x}}{\partial q_j}\right) = 0, \qquad i \neq j,$$

in which case the metric tensor is diagonal. In this context, we actually call the vectors (11.1) the **tangent vectors** of the coordinate system, because $\partial\boldsymbol{x}/\partial q_1$ is tangent to the surfaces $q_2 = \mathrm{Const.}$

and $q_3 =$ Const. &c. Restricting to such coordinate systems, the line element becomes

$$\mathrm{d}s^2 = g_{11}\mathrm{d}q_1^2 + g_{22}\mathrm{d}q_2^2 + g_{33}\mathrm{d}q_3^2,$$

or

$$\mathrm{d}s^2 = h_1^2\mathrm{d}q_1^2 + h_2^2\mathrm{d}q_2^2 + h_3^2\mathrm{d}q_3^2,$$

where

$$h_i = \sqrt{g_{ii}}, \qquad \text{no sum over } i$$

are the **scale factors of the orthogonal coordinate system**. Moreover, we have three mutually orthogonal vectors $\partial\boldsymbol{x}/\partial q_i$, which we may take to form a basis. Indeed, we take unit vectors

$$\hat{\boldsymbol{q}}_i = \frac{\partial\boldsymbol{x}}{\partial q_i}\Big/\left|\frac{\partial\boldsymbol{x}}{\partial q_i}\right| = \frac{1}{h_i}\frac{\partial\boldsymbol{x}}{\partial q_i}.$$

and thus any vector $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \hat{\boldsymbol{q}}_1 A_1 + \hat{\boldsymbol{q}}_2 A_2 + \hat{\boldsymbol{q}}_3 A_3,$$

where

$$A_i = \boldsymbol{A}\cdot\hat{\boldsymbol{q}}_i$$

is the component of the vector $\boldsymbol{A}$ in the $\hat{\boldsymbol{q}}_i$ direction (and NOT in any particular Cartesian direction).

Example: Consider spherical polar coordinates again, where

$$\begin{aligned} x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\sin\varphi, \\ z &= r\cos\theta, \end{aligned}$$

with inverse transformation

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2}, \\ \theta &= \cos^{-1}(z/r), \\ \varphi &= \tan^{-1}(y/x). \end{aligned}$$

Let take the position vector

$$\boldsymbol{x} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z,$$

and compute the tangent vectors:

$$\begin{aligned}\frac{\partial \boldsymbol{x}}{\partial r} &= \hat{\boldsymbol{x}}\frac{\partial x}{\partial r} + \hat{\boldsymbol{y}}\frac{\partial x}{\partial r} + \hat{\boldsymbol{z}}\frac{\partial x}{\partial r},\\ &= \hat{\boldsymbol{x}}\frac{\partial}{\partial r}\left(r\sin\theta\cos\varphi\right) + \hat{\boldsymbol{x}}\frac{\partial}{\partial r}\left(r\sin\theta\sin\varphi\right) + \hat{\boldsymbol{x}}\frac{\partial}{\partial r}\left(r\cos\theta\right),\\ &= \hat{\boldsymbol{x}}\sin\theta\cos\varphi + \hat{\boldsymbol{y}}\sin\theta\sin\varphi + \hat{\boldsymbol{z}}\cos\theta.\end{aligned}$$

$$\begin{aligned}\frac{\partial \boldsymbol{x}}{\partial \theta} &= \hat{\boldsymbol{x}}\frac{\partial x}{\partial \theta} + \hat{\boldsymbol{y}}\frac{\partial x}{\partial \theta} + \hat{\boldsymbol{z}}\frac{\partial x}{\partial \theta},\\ &= \hat{\boldsymbol{x}}\frac{\partial}{\partial \theta}\left(r\sin\theta\cos\varphi\right) + \hat{\boldsymbol{x}}\frac{\partial}{\partial \theta}\left(r\sin\theta\sin\varphi\right) + \hat{\boldsymbol{x}}\frac{\partial}{\partial \theta}\left(r\cos\theta\right),\\ &= r\left[\hat{\boldsymbol{x}}\cos\theta\cos\varphi + \hat{\boldsymbol{y}}\cos\theta\sin\varphi - \hat{\boldsymbol{z}}\sin\theta\right].\end{aligned}$$

$$\begin{aligned}\frac{\partial \boldsymbol{x}}{\partial \varphi} &= \hat{\boldsymbol{x}}\frac{\partial x}{\partial \varphi} + \hat{\boldsymbol{y}}\frac{\partial y}{\partial \varphi} + \hat{\boldsymbol{z}}\frac{\partial z}{\partial \varphi},\\ &= \hat{\boldsymbol{x}}\frac{\partial}{\partial \varphi}\left(r\sin\theta\cos\varphi\right) + \hat{\boldsymbol{x}}\frac{\partial}{\partial \varphi}\left(r\sin\theta\sin\varphi\right) + \hat{\boldsymbol{x}}\frac{\partial}{\partial \varphi}\left(r\cos\theta\right),\\ &= r\left[-\hat{\boldsymbol{x}}\sin\theta\sin\varphi + \hat{\boldsymbol{y}}\sin\theta\cos\varphi\right].\end{aligned}$$

Now compute

$$\begin{aligned}\left(\frac{\partial \boldsymbol{x}}{\partial r}\right)\cdot\left(\frac{\partial \boldsymbol{x}}{\partial \theta}\right) &= r\left(\hat{\boldsymbol{x}}\sin\theta\cos\varphi + \hat{\boldsymbol{y}}\sin\theta\sin\varphi + \hat{\boldsymbol{z}}\cos\theta\right)\cdot\left(\hat{\boldsymbol{x}}\cos\theta\cos\varphi + \hat{\boldsymbol{y}}\cos\theta\sin\varphi - \hat{\boldsymbol{z}}\sin\theta\right),\\ &= r\left[\sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi - \sin\theta\cos\theta\right] = 0.\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial \boldsymbol{x}}{\partial r}\right)\cdot\left(\frac{\partial \boldsymbol{x}}{\partial \varphi}\right) &= r\left(\hat{\boldsymbol{x}}\sin\theta\cos\varphi + \hat{\boldsymbol{y}}\sin\theta\sin\varphi + \hat{\boldsymbol{z}}\cos\theta\right)\cdot\left(-\hat{\boldsymbol{x}}\sin\theta\sin\varphi + \hat{\boldsymbol{y}}\sin\theta\cos\varphi\right),\\ &= r\left[-\sin^2\theta\sin\varphi\cos\varphi + \sin^2\theta\sin\varphi\cos\varphi\right] = 0.\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial \boldsymbol{x}}{\partial \varphi}\right)\cdot\left(\frac{\partial \boldsymbol{x}}{\partial \theta}\right) &= r^2\left(-\hat{\boldsymbol{x}}\sin\theta\sin\varphi + \hat{\boldsymbol{y}}\sin\theta\cos\varphi\right)\cdot\left(\hat{\boldsymbol{x}}\cos\theta\cos\varphi + \hat{\boldsymbol{y}}\cos\theta\sin\varphi - \hat{\boldsymbol{z}}\sin\theta\right),\\ &= r^2\left[-\sin\theta\cos\theta\sin\varphi\cos\varphi + \sin\theta\cos\theta\sin\varphi\cos\varphi\right] = 0,\end{aligned}$$

and the coordinate system is orthogonal. Now we compute the scale factors:

$$\begin{aligned}h_r^2 &= \left(\frac{\partial \boldsymbol{x}}{\partial r}\right)\cdot\left(\frac{\partial \boldsymbol{x}}{\partial r}\right),\\ &= \left(\hat{\boldsymbol{x}}\sin\theta\cos\varphi + \hat{\boldsymbol{y}}\sin\theta\sin\varphi + \hat{\boldsymbol{z}}\cos\theta\right)\cdot\left(\hat{\boldsymbol{x}}\sin\theta\cos\varphi + \hat{\boldsymbol{y}}\sin\theta\sin\varphi + \hat{\boldsymbol{z}}\cos\theta\right),\\ &= 1.\end{aligned}$$

$$
\begin{aligned}
h_\theta^2 &= \left(\frac{\partial \boldsymbol{x}}{\partial \theta}\right) \cdot \left(\frac{\partial \boldsymbol{x}}{\partial \theta}\right), \\
&= r^2 \left(\hat{\boldsymbol{x}} \cos\theta \cos\varphi + \hat{\boldsymbol{y}} \cos\theta \sin\varphi - \hat{\boldsymbol{z}} \sin\theta\right) \cdot \left(\hat{\boldsymbol{x}} \cos\theta \cos\varphi + \hat{\boldsymbol{y}} \cos\theta \sin\varphi - \hat{\boldsymbol{z}} \sin\theta\right), \\
&= r^2
\end{aligned}
$$

$$
\begin{aligned}
h_\varphi^2 &= \left(\frac{\partial \boldsymbol{x}}{\partial \varphi}\right) \cdot \left(\frac{\partial \boldsymbol{x}}{\partial \varphi}\right), \\
&= r^2 \left(-\hat{\boldsymbol{x}} \sin\theta \sin\varphi + \hat{\boldsymbol{y}} \sin\theta \cos\varphi\right) \cdot \left(-\hat{\boldsymbol{x}} \sin\theta \sin\varphi + \hat{\boldsymbol{y}} \sin\theta \cos\varphi\right), \\
&= r^2 \sin^2\theta.
\end{aligned}
$$

Thus, spherical polar coordinates are orthogonal, the line element is

$$
\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2,
$$

and the unit vectors are

$$
\begin{aligned}
\hat{\boldsymbol{r}} &= \hat{\boldsymbol{x}} \sin\theta \cos\varphi + \hat{\boldsymbol{y}} \sin\theta \sin\varphi + \hat{\boldsymbol{z}} \cos\theta, \\
\hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{x}} \cos\theta \cos\varphi + \hat{\boldsymbol{y}} \cos\theta \sin\varphi - \hat{\boldsymbol{z}} \sin\theta. \\
\hat{\boldsymbol{\varphi}} &= -\hat{\boldsymbol{x}} \sin\varphi + \hat{\boldsymbol{y}} \cos\varphi.
\end{aligned}
$$

These unit vectors point in the directions of increasing r, φ, and θ, respectively (Figure 11.3). Note that the unit vectors, although of constant magnitude, vary in direction as the point P is varied. They are not constant vectors, and do not go to zero when differentiated. It is for this reason that developing expressions for div, grad, and curl in curvilinear coordinates is complicated. It is to this issue that we now turn.

11.3 Grad, div, and curl in curvilinear coordinate systems

To avoid confusion, in this section we use the notation ψ for scalar fields. The use of φ to label a function is avoided because it is conventional to use this symbol for the azimuthal coordinate in the spherical polar system.

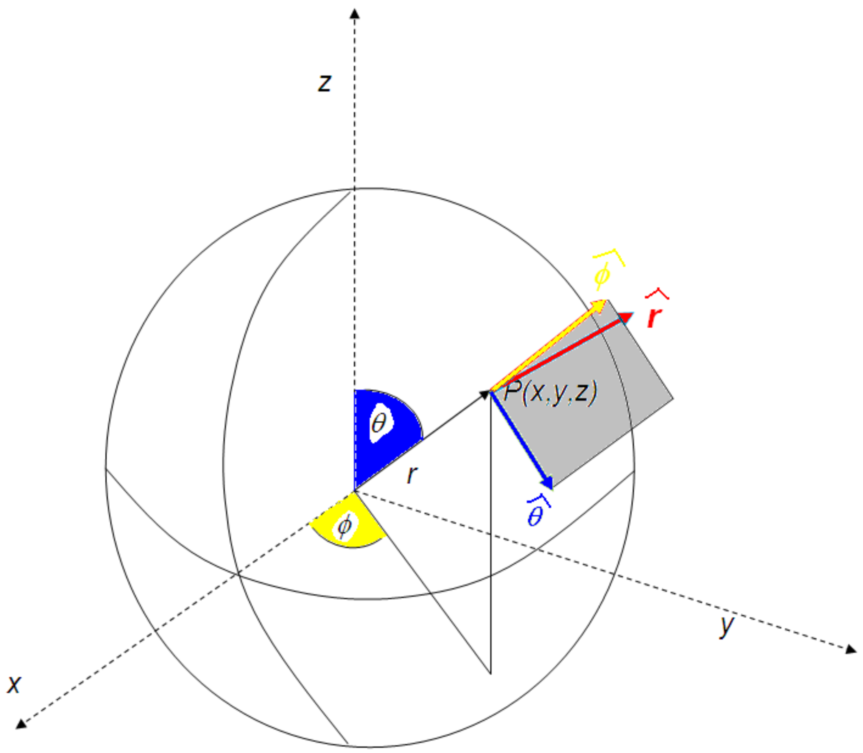

Figure 11.3: The unit vectors for spherical polar coordinates

11.3.1 The gradient

Because $\hat{\boldsymbol{q}}_i$ form an orthogonal basis, any vector (such as $\nabla\psi$) can be written as

$$\nabla\psi = \sum_{i=1}^{3} \hat{\boldsymbol{q}}_i \left[(\nabla\psi)\cdot\hat{\boldsymbol{q}}_i\right].$$

Now consider $(\nabla\psi)\cdot\hat{\boldsymbol{q}}_i$. This is nothing other than the directional derivative of ψ in the q_i-direction:

$$(\nabla\psi)\cdot\hat{\boldsymbol{q}}_i = \lim_{\delta q_i\to 0}\frac{\psi\left(q_i + h_i\delta q_i\right) - \psi\left(q_i\right)}{h_i\delta q_i},$$

where $h_i\delta q_i$ is a small increment of **length** in the q_i-direction (δq_i is not, by itself, an increment of length). Thus,

$$(\nabla\psi)\cdot\hat{\boldsymbol{q}}_i = \frac{1}{h_i}\frac{\partial\psi}{\partial q_i},$$

and hence,

$$\nabla\psi = \sum_{i=1}^{3}\frac{\hat{\boldsymbol{q}}_i}{h_i}\frac{\partial\psi}{\partial q_i},$$

or

$$\nabla\psi(q_1,q_2,q_3)=\frac{\hat{\boldsymbol{q}}_1}{h_1}\frac{\partial\psi}{\partial q_1}+\frac{\hat{\boldsymbol{q}}_2}{h_2}\frac{\partial\psi}{\partial q_2}+\frac{\hat{\boldsymbol{q}}_3}{h_3}\frac{\partial\psi}{\partial q_3}. \tag{11.2}$$

11.3.2 The divergence

Recall Gauss's theorem: In three dimensions, given a vector field $\boldsymbol{v}(\boldsymbol{x})$ and a volume V with bounding surface S,

$$\int_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V=\int_S \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}.$$

Here, we view Gauss's theorem as a *definition* of divergence:

$$\nabla\cdot\boldsymbol{v}=\lim_{\int_V \mathrm{d}V\to 0}\frac{\int_V \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}}{\int_V \mathrm{d}V}. \tag{11.3}$$

Thus,

$$\nabla\cdot\boldsymbol{v}(q_1,q_2,q_3)=\lim_{\int_V \mathrm{d}V\to 0}\frac{\int_V \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}}{\int_V \mathrm{d}V},\qquad \mathrm{d}V=h_1h_2h_3\,\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3.$$

Refer to Figure 11.4: we compute the area integrals associated with a small parallelepiped formed by the intersection of 6 surfaces,

$$q_1=\mathrm{Const.},\qquad q_1+\mathrm{d}q_1=\mathrm{Const.},\qquad \text{\&c.}$$

On the face labelled $Fq1p$ in Figure 11.4, we have

$$\begin{aligned}
\mathrm{d}\boldsymbol{S} &= \left[\frac{\partial\boldsymbol{x}}{\partial q_2}\times\frac{\partial\boldsymbol{x}}{\partial q_3}\right]_{(q_1+\mathrm{d}q_1,q_2,q_3)}\mathrm{d}q_2\mathrm{d}q_3,\\
&= \left[h_2h_3\left(\hat{\boldsymbol{q}}_2\times\hat{\boldsymbol{q}}_3\right)\mathrm{d}q_2\mathrm{d}q_3\right]_{(q_1+\mathrm{d}q_1,q_2,q_3)},\\
&= \hat{\boldsymbol{q}}_1h_2h_3\mathrm{d}q_2\mathrm{d}q_3\big|_{(q_1+\mathrm{d}q_1,q_2,q_3)}.
\end{aligned}$$

Hence,

$$\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}=(v_1h_2h_3)(q_1+\mathrm{d}q_1,q_2,q_3)dq_2dq_3.$$

Similarly, on the face labelled $Fq1m$, we have

$$\mathrm{d}\boldsymbol{S}=-\hat{\boldsymbol{q}}_1h_2h_3\mathrm{d}q_2\mathrm{d}q_3\big|_{(q_1,q_2,q_3)}.$$

Hence,

$$\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}=-(v_1h_2h_3)(q_1,q_2,q_3)dq_2dq_3.$$

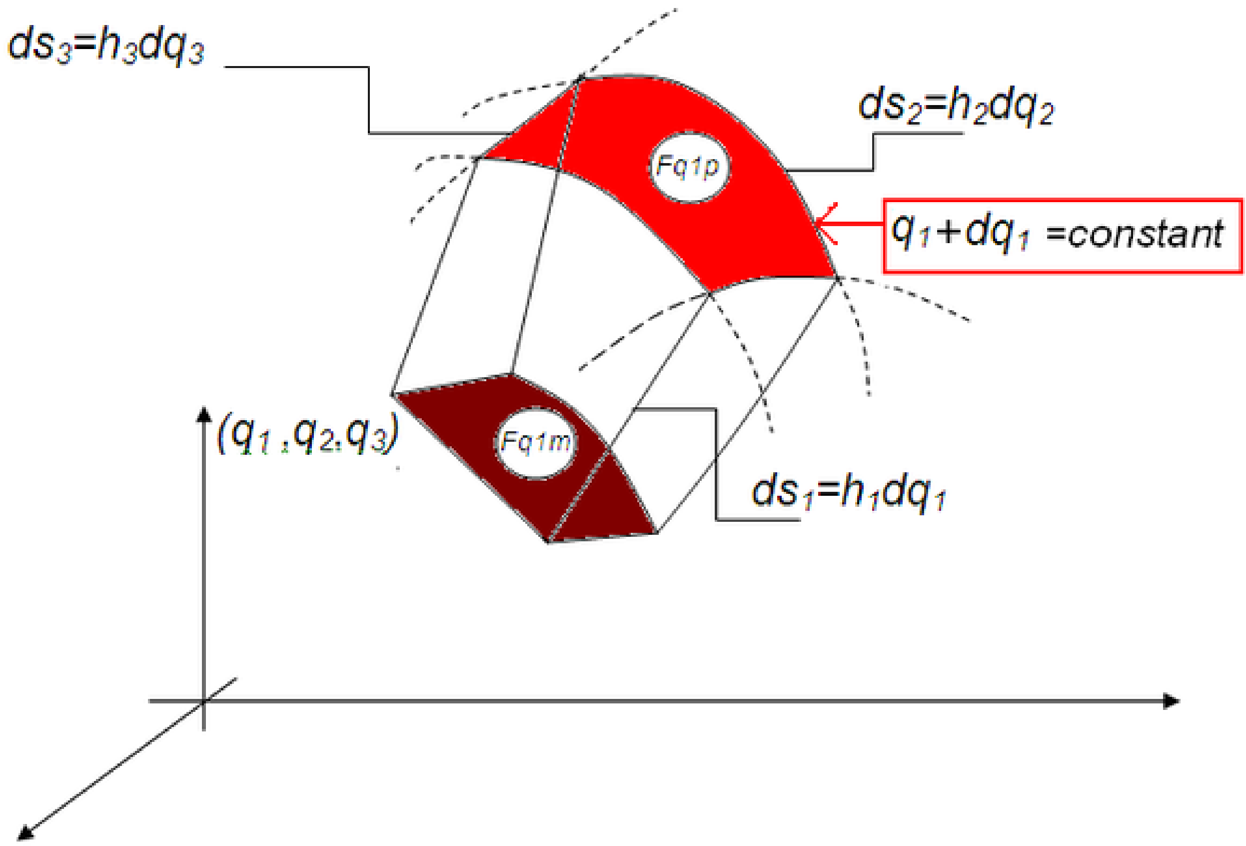

Figure 11.4: The volume element in curvilinear coordinates: this sketch forms a basis for deriving div and grad in curvilinear coordinates.

Adding these contributions gives

$$[(v_1h_2h_3)(q_1+\mathrm{d}q_1,q_2,q_3)-(v_1h_2h_3)(q_1,q_2,q_3)]\,dq_2dq_3=\frac{\partial}{\partial q_1}(v_1h_2h_3)\,\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3.$$

Adding up the other contributions gives

$$\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}=\left[\frac{\partial}{\partial q_1}(v_1h_2h_3)+\frac{\partial}{\partial q_2}(v_2h_3h_1)+\frac{\partial}{\partial q_3}(v_3h_1h_2)\right]\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3$$

Applying the definition of the divergence (11.3) gives

$$\begin{aligned}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S} &= \left[\frac{\partial}{\partial q_1}(v_1h_2h_3)+\frac{\partial}{\partial q_2}(v_2h_3h_1)+\frac{\partial}{\partial q_3}(v_3h_1h_2)\right]\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3,\\ &\overset{\text{divergence}}{=} \nabla\cdot\boldsymbol{v}\,\mathrm{d}V,\\ &= \nabla\cdot\boldsymbol{v}\,h_1h_2h_3\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3.\end{aligned}$$

Hence,

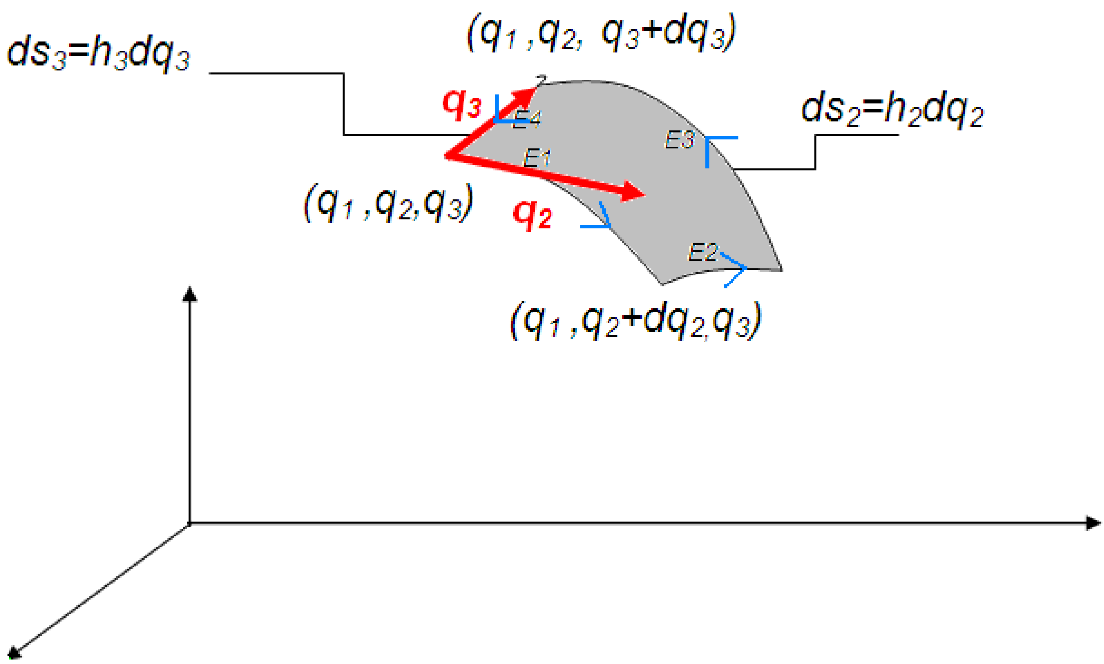

Figure 11.5: The area element in curvilinear coordinates: this sketch forms a basis for deriving the curl operator in curvilinear coordinates.

$$\nabla\cdot\boldsymbol{v}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(v_1h_2h_3\right)+\frac{\partial}{\partial q_2}\left(v_2h_3h_1\right)+\frac{\partial}{\partial q_3}\left(v_3h_1h_2\right)\right]. \tag{11.4}$$

11.3.3 The curl

This is the last operator to compute. Let us take Stokes's theorem for an areal patch S with boundary C, and integrate a vector field $\boldsymbol{v}$ in the usual manner:

$$\int_S \nabla\times\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}=\oint_C \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x}.$$

In particular, let S be a patch of area on the surface

$$q_1=\mathrm{Const.},$$

as shown in Figure 11.5. Thus,

$$\mathrm{d}\boldsymbol{S}=\hat{\boldsymbol{q}}_1h_2h_3\mathrm{d}q_2\mathrm{d}q_3.$$

We compute the line integral around boundary of this areal patch in a sense given by the right-hand rule:

$$\oint_C \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} = \underbrace{\left[\boldsymbol{v}\cdot(\hat{\boldsymbol{q}}_2 h_2 \mathrm{d}q_2)\right](q_1,q_2,q_3)}_{\text{Contribution from edge } E1} + \underbrace{\left[\boldsymbol{v}\cdot(\hat{\boldsymbol{q}}_3 h_3 \mathrm{d}q_3)\right](q_1,q_2+\mathrm{d}q_2,q_3)}_{E2}$$
$$- \underbrace{\left[\boldsymbol{v}\cdot(\hat{\boldsymbol{q}}_2 h_2 \mathrm{d}q_2)\right](q_1,q_2+\mathrm{d}q_2,q_3+\mathrm{d}q_3)}_{E3} - \underbrace{\left[\boldsymbol{v}\cdot(\hat{\boldsymbol{q}}_3 h_3 \mathrm{d}q_3)\right](q_1,q_2,q_3+\mathrm{d}q_3)}_{E4}$$

Tidy up a little bit:

$$\oint_C \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} = [v_2 h_2](q_1,q_2,q_3)\mathrm{d}q_2 + [v_3 h_3](q_1,q_2+\mathrm{d}q_2,q_3)\mathrm{d}q_3$$
$$- [v_2 h_2](q_1,q_2+\mathrm{d}q_2,q_3+\mathrm{d}q_3)\mathrm{d}q_2 - [v_3 h_3](q_1,q_2,q_3+\mathrm{d}q_3)\mathrm{d}q_3.$$

Pair up terms ready for a Taylor expansion:

$$\oint_C \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} = [v_2 h_2](q_1,q_2,q_3)\mathrm{d}q_2 - [v_2 h_2](q_1,q_2+\mathrm{d}q_2,q_3+\mathrm{d}q_3)\mathrm{d}q_2$$
$$+ [v_3 h_3](q_1,q_2+\mathrm{d}q_2,q_3)\mathrm{d}q_3 - [v_3 h_3](q_1,q_2,q_3+\mathrm{d}q_3)\mathrm{d}q_3.$$

Expand:

$$\oint_C \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} = -\left[\frac{\partial}{\partial q_3}(v_2 h_2)\right]\mathrm{d}q_2\mathrm{d}q_3 + \left[\frac{\partial}{\partial q_2}(v_3 h_3)\right]\mathrm{d}q_2\mathrm{d}q_3.$$

Note that we have neglected the term

$$-\left[\frac{\partial}{\partial q_2}(v_2 h_2)\right]\mathrm{d}q_2\mathrm{d}q_2$$

from the first Taylor expansion because it is second order in the small quantity $\mathrm{d}q_2$. Next, we consider

$$\begin{aligned}\int_S (\nabla\times\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{S} &= \int_S (\nabla\times\boldsymbol{v})\cdot\hat{\boldsymbol{q}}_1 h_2 h_3 \mathrm{d}q_2\mathrm{d}q_3,\\ &= (\nabla\times\boldsymbol{v})\cdot\hat{\boldsymbol{q}}_1 h_2 h_3 \mathrm{d}q_2\mathrm{d}q_3, \qquad (*)\end{aligned}$$

since the areal patch is infinitesimal. But by Stokes's theorem,

$$\begin{aligned}\int_S (\nabla\times\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{S} &= \oint_C \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x},\\ &= -\left[\frac{\partial}{\partial q_3}(v_2 h_2)\right]\mathrm{d}q_2\mathrm{d}q_3 + \left[\frac{\partial}{\partial q_2}(v_3 h_3)\right]\mathrm{d}q_2\mathrm{d}q_3. \qquad (**)\end{aligned}$$

Equate $(*)$ and $(**)$:

$$(\nabla\times\boldsymbol{v})\cdot\hat{\boldsymbol{q}}_1 h_2h_3\mathrm{d}q_2\mathrm{d}q_3 = -\left[\frac{\partial}{\partial q_3}(v_2h_2)\right]\mathrm{d}q_2\mathrm{d}q_3+\left[\frac{\partial}{\partial q_2}(v_3h_3)\right]\mathrm{d}q_2\mathrm{d}q_3$$

But

$$(\nabla\times\boldsymbol{v})\cdot\hat{\boldsymbol{q}}_1=(\nabla\times\boldsymbol{v})_1,$$

the component of the curl in the first ($\hat{\boldsymbol{q}}_1$) direction. Hence,

$$(\nabla\times\boldsymbol{v})_1=\frac{1}{h_2h_3}\left[\frac{\partial}{\partial q_2}(v_3h_3)-\frac{\partial}{\partial q_3}(v_2h_2)\right]$$

By construction, $(\hat{\boldsymbol{q}}_1,\hat{\boldsymbol{q}}_2,\hat{\boldsymbol{q}}_3)$ form a right-handed system. Thus, we may obtain the other components of the curl through cyclic permutations:

$$\begin{aligned}(\nabla\times\boldsymbol{v})_2 &= \frac{1}{h_3h_1}\left[\frac{\partial}{\partial q_3}(v_1h_1)-\frac{\partial}{\partial q_1}(v_3h_3)\right],\\ (\nabla\times\boldsymbol{v})_3 &= \frac{1}{h_1h_2}\left[\frac{\partial}{\partial q_1}(v_2h_2)-\frac{\partial}{\partial q_2}(v_1h_1)\right].\end{aligned}$$

This result may be summarized succinctly in determinant form:

$$\nabla\times\boldsymbol{v}=\frac{1}{h_1h_2h_3}\begin{vmatrix}\hat{\boldsymbol{q}}_1h_1 & \hat{\boldsymbol{q}}_2h_2 & \hat{\boldsymbol{q}}_3h_3\\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3}\\ h_1v_1 & h_2v_2 & h_3v_3\end{vmatrix},$$

where $\boldsymbol{v}=v_1\hat{\boldsymbol{q}}_1+v_2\hat{\boldsymbol{q}}_2+v_3\hat{\boldsymbol{q}}_3$ is the vector field $\boldsymbol{v}(q_1,q_2,q_3)$ written in curvilinear coordinates.

11.4 Some applications of our results

1. Compute the Laplacian $\nabla^2\psi$ of a scalar field $\psi(\boldsymbol{x})$ (a) in general, curvilinear coordinates (q_1,q_2,q_3); (b) in spherical polar coordinates.

   Let $\boldsymbol{v}=\nabla\psi$. Using the definition of the gradient,

$$v_i=\frac{1}{h_i}\frac{\partial\psi}{\partial q_i},$$

   (no sum), and

$$\boldsymbol{v}=\sum_{i=1}^{3}\hat{\boldsymbol{q}}_iv_i.$$

Next, using the definition of divergence,

$$\nabla \cdot \boldsymbol{v} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(v_1 h_2 h_3 \right) + \frac{\partial}{\partial q_2} \left(v_2 h_3 h_1 \right) + \frac{\partial}{\partial q_3} \left(v_3 h_1 h_2 \right) \right],$$

we obtain

$$\nabla \cdot (\nabla \psi) = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left[\left(\frac{1}{h_1} \frac{\partial \psi}{\partial q_1} \right) h_2 h_3 \right] + \text{Cyclic permutations} \right\}$$

But $\nabla^2 \psi = \nabla \cdot (\nabla \psi)$, hence

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial q_3} \right) \right]. \qquad (11.5)$$

For the spherical-polar case, let $q_1 \to r$, $q_2 \to \theta$, and $q_3 \to \varphi$. Then,

$$\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{r}}, \qquad \hat{\boldsymbol{q}}_2 = \hat{\boldsymbol{\theta}}, \qquad \hat{\boldsymbol{q}}_3 = \hat{\boldsymbol{\varphi}},$$

and

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = r \sin\theta.$$

Substituting these relations into Eq. (11.5), we obtain

$$\nabla^2 \psi \left(r, \theta, \varphi \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \varphi^2}.$$

2. Using spherical polar coordinates, show that the central force

$$\boldsymbol{F} = \alpha |\boldsymbol{r}|^n \boldsymbol{r}, \qquad n \neq -2.$$

is conservative. Show also that

$$\mathcal{U} = -\frac{\alpha}{n+2} |\boldsymbol{r}|^{n+2}$$

is a potential, $\boldsymbol{F} = -\nabla \mathcal{U}$.

We use the assignments made in the last exercise and compute

$$\begin{aligned}
\nabla \times \boldsymbol{F} &= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{\boldsymbol{q}}_1 h_1 & \hat{\boldsymbol{q}}_2 h_2 & \hat{\boldsymbol{q}}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}, \\
&= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} r & \hat{\boldsymbol{\varphi}} r \sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_r & r F_\theta & r \sin\theta F_\varphi, \end{vmatrix},
\end{aligned}$$

Note that the force can be written as

$$\boldsymbol{F} = \alpha |\boldsymbol{r}|^n |\boldsymbol{r}| \hat{\boldsymbol{r}},$$

so

$$F_r = \alpha r^{n+1}, \qquad F_\varphi = 0, \qquad F_\vartheta = 0.$$

Thus, a central force only has a radial component, when expressed in spherical polar coordinates. The curl is then

$$\begin{aligned}
\nabla \times \boldsymbol{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} r & \hat{\boldsymbol{\varphi}} r \sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_r(r) & 0 & 0 \end{vmatrix}, \\
&= \frac{1}{r^2 \sin\theta} \left[\hat{\boldsymbol{r}} \left(\partial_\theta 0 - \partial_\varphi 0 \right) - \hat{\boldsymbol{\theta}} r \left(\partial_r 0 - \partial_\varphi F_r(r) \right) + \hat{\boldsymbol{\varphi}} r \sin\theta \left(\partial_r 0 - \partial_\theta F_r(r) \right) \right], \\
&= 0.
\end{aligned}$$

Next, we compute $-\nabla \mathcal{U}$. For spherical polar coordinates,

$$\begin{aligned}
\nabla \psi &= \frac{\hat{\boldsymbol{q}}_1}{h_1} \frac{\partial \psi}{\partial q_1} + \frac{\hat{\boldsymbol{q}}_2}{h_2} \frac{\partial \psi}{\partial q_2} + \frac{\hat{\boldsymbol{q}}_3}{h_3} \frac{\partial \psi}{\partial q_3}, \\
&= \hat{\boldsymbol{r}} \frac{\partial \psi}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r} \frac{\partial \psi}{\partial \theta} + \frac{\hat{\boldsymbol{\varphi}}}{r \sin\theta} \frac{\partial \psi}{\partial \varphi}.
\end{aligned}$$

But $\mathcal{U}$ is radially symmetric, $\mathcal{U} = \mathcal{U}(r)$, so

$$\begin{aligned} -\nabla\mathcal{U} &= -\hat{\boldsymbol{r}}\frac{\partial\mathcal{U}}{\partial r} = -\hat{\boldsymbol{r}}\mathcal{U}'(r), \\ &= -\hat{\boldsymbol{r}}\frac{d}{dr}\left(-\frac{\alpha}{n+2}r^{n+2}\right), \\ &= +\hat{\boldsymbol{r}}\alpha r^{n+1}, \\ &= \alpha r^n \boldsymbol{r}, \qquad \boldsymbol{r} = r\hat{\boldsymbol{r}}, \\ &= \boldsymbol{F}. \end{aligned}$$

Note in particular, that if

$$\boldsymbol{F} = \frac{\alpha}{|\boldsymbol{r}|^3}\boldsymbol{r},$$

then $n = -3$ and

$$\mathcal{U} = \frac{\alpha}{r}.$$

When $\alpha = -Gm_1m_2$ this is the gravitational force.

Chapter 12

Special Curvinlinear coordinate systems

Overview

In this section we study two special coordinate systems that commonly occur in fluid flow, electromagnetism, and quantum mechanics. Because of their importance to your later courses, this is a particularly important chapter.

12.1 Spherical polar coordinates

We have already encountered this system, but let us recall it briefly. The coordinate system is shown in Fig. 12.1. The point P is labelled by its radial distance r from the origin, together with two angles: the azimuthal angle φ, and the angle θ between the z-direction and the radius vector $\boldsymbol{r}$ extending from the origin O to the point P. The Cartesian coordinate system (x, y, z) and the spherical polar coordinate system are related through

$$\begin{aligned} x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\sin\varphi, \\ z &= r\cos\theta, \end{aligned}$$

with inverse transformation

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2}, \\ \theta &= \cos^{-1}(z/r), \\ \varphi &= \tan^{-1}(y/x). \end{aligned}$$

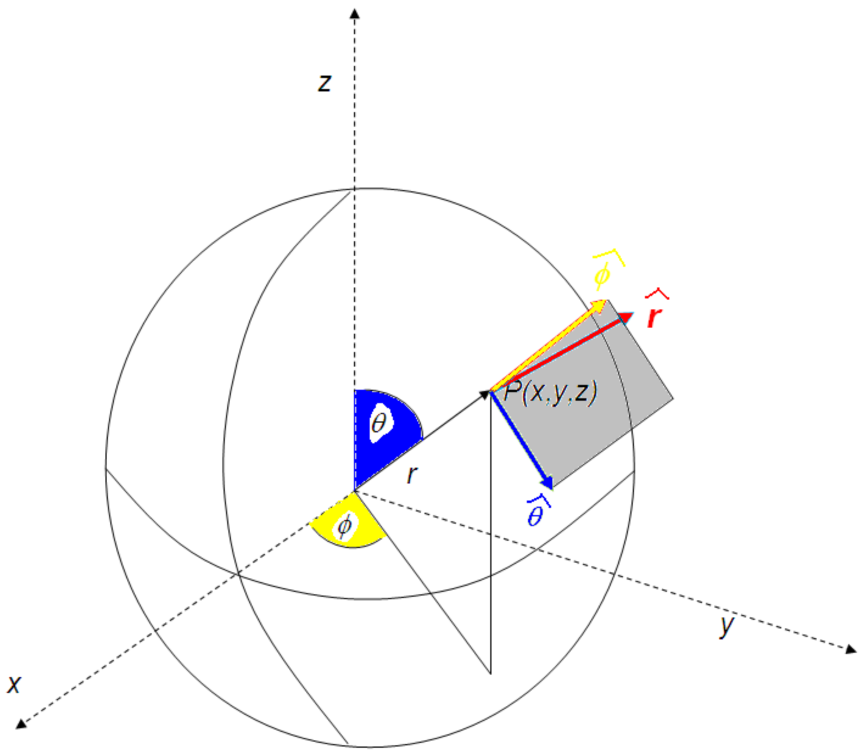

Figure 12.1: Spherical polar coordinates

Recall, we made the identification $q_1 \to r$, $q_2 \to \theta$, and $q_3 \to \varphi$, and we wrote

$$\begin{aligned} \boldsymbol{r} \equiv \boldsymbol{x} &= \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z, \\ &= \hat{\boldsymbol{x}}r\sin\theta\cos\varphi + \hat{\boldsymbol{y}}r\sin\theta\sin\varphi + \hat{\boldsymbol{z}}r\cos\theta. \end{aligned}$$

From this we computed the vectors

$$\hat{\boldsymbol{r}} = \frac{\partial \boldsymbol{x}}{\partial r}\Big/\left|\frac{\partial \boldsymbol{x}}{\partial r}\right|, \qquad \hat{\boldsymbol{\theta}} = \frac{\partial \boldsymbol{x}}{\partial \theta}\Big/\left|\frac{\partial \boldsymbol{x}}{\partial \theta}\right|, \qquad \hat{\boldsymbol{\varphi}} = \frac{\partial \boldsymbol{x}}{\partial \varphi}\Big/\left|\frac{\partial \boldsymbol{x}}{\partial \varphi}\right|.$$

These were found to be

$$\begin{aligned} \hat{\boldsymbol{r}} &= \hat{\boldsymbol{x}}\sin\theta\cos\varphi + \hat{\boldsymbol{y}}\sin\theta\sin\varphi + \hat{\boldsymbol{z}}\cos\theta, \\ \hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{x}}\cos\theta\cos\varphi + \hat{\boldsymbol{y}}\cos\theta\sin\varphi - \hat{\boldsymbol{z}}\sin\theta, \\ \hat{\boldsymbol{\varphi}} &= -\hat{\boldsymbol{x}}\sin\varphi + \hat{\boldsymbol{y}}\cos\varphi. \end{aligned}$$

and are mutually orthogonal:

$$\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\theta}} = 0;$$

Also,

$$\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}} + \text{ Cyclic permutations.}$$

Note that it also follows that

$$\boldsymbol{x} \equiv \boldsymbol{r} = r\hat{\boldsymbol{r}}.$$

We also computed the scale factors

$$h_r = \left|\frac{\partial \boldsymbol{x}}{\partial r}\right|, \qquad h_\theta = \left|\frac{\partial \boldsymbol{x}}{\partial \theta}\right|, \qquad h_\varphi = \left|\frac{\partial \boldsymbol{x}}{\partial \varphi}\right|,$$

which we found to be equal to

$$h_r = 1, \qquad h_\theta = r, \qquad h_\varphi = r\sin\theta.$$

Once we know the scale factors and the unit vectors, we may compute grad, div, curl, and the Laplacian in the spherical polar system. We recall again the identifications $\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{r}}$, $\hat{\boldsymbol{q}}_2 = \hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{q}}_3 = \hat{\boldsymbol{\varphi}}$, along with

$$q_1 = r, \qquad q_2 = \theta, \qquad q_3 = \varphi.$$

Thus,

1. The gradient: General case:

$$\nabla\psi(q_1, q_2, q_3) = \frac{\hat{\boldsymbol{q}}_1}{h_1}\frac{\partial\psi}{\partial q_1} + \frac{\hat{\boldsymbol{q}}_2}{h_2}\frac{\partial\psi}{\partial q_2} + \frac{\hat{\boldsymbol{q}}_3}{h_3}\frac{\partial\psi}{\partial q_3};$$

   Spherical polar coordinates:

$$\nabla\psi(r, \theta, \varphi) = \hat{\boldsymbol{r}}\frac{\partial\psi}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial\psi}{\partial\theta} + \frac{\hat{\boldsymbol{\varphi}}}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}.$$

2. The divergence: General case:

$$\nabla\cdot\boldsymbol{v} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(v_1h_2h_3) + \frac{\partial}{\partial q_2}(v_2h_3h_1) + \frac{\partial}{\partial q_3}(v_3h_1h_2)\right];$$

   Spherical polar coordinates:

$$\nabla\cdot\boldsymbol{v}(r, \theta, \varphi) = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2v_r\right) + r\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + r\frac{\partial v_\varphi}{\partial\varphi}\right]$$

3. The curl: General case:

$$(\nabla \times \boldsymbol{v})(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{\boldsymbol{q}}_1 h_1 & \hat{\boldsymbol{q}}_2 h_2 & \hat{\boldsymbol{q}}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix};$$

Spherical polar coordinates:

$$(\nabla \times \boldsymbol{v})(r, \theta, \varphi) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} r & \hat{\boldsymbol{\varphi}} r \sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ v_r & r v_\theta & r \sin\theta v_\varphi, \end{vmatrix}$$

4. The Laplacian: General case:

$$\nabla^2 \psi(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial q_3} \right) \right].$$

Spherical polar coordinates:

$$\nabla^2 \psi \left(r, \theta, \varphi \right) = \frac{1}{r^2 \sin\theta} \left[\sin\theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin\theta} \frac{\partial^2 \psi}{\partial \varphi^2} \right].$$

12.2 Cylindrical coordinates

Consider **cylindrical polar coordinates** as shown in Fig. 12.2. The point P can either be labelled by the Cartesian triple (x, y, z), or by the following quantities:

- The distance z between the point P and its projection on to the xy-plane;
- The distance ρ from the origin O to the projection of P on to the xy-plane;
- The angle φ that the projection makes with the x-axis;

Note that φ is the same as the azimuthal angle in the spherical polar system; otherwise these two systems are different. The surfaces generated by these new coordinates are two planes and a cylinder:

- The plane $z = \text{Const.}$;
- The plane $y = x \tan\varphi$ (i.e. the plane $\varphi = \text{Const.}$);

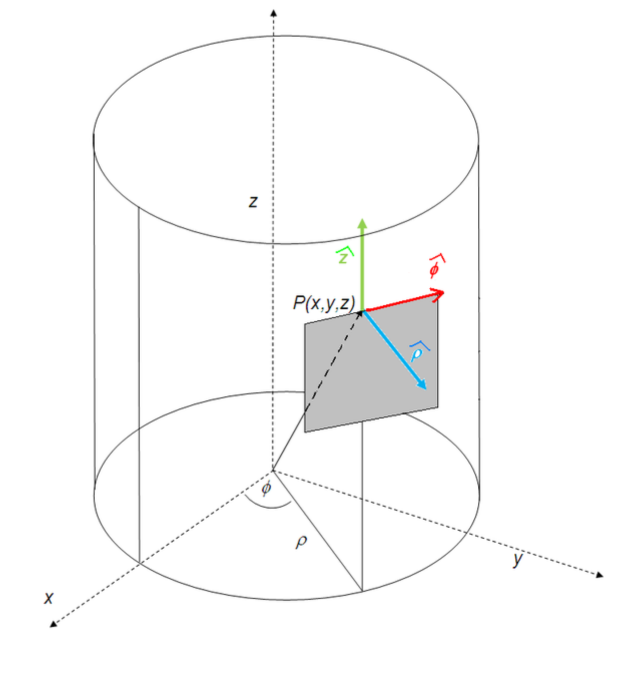

Figure 12.2: Cylindrical polar coordinates

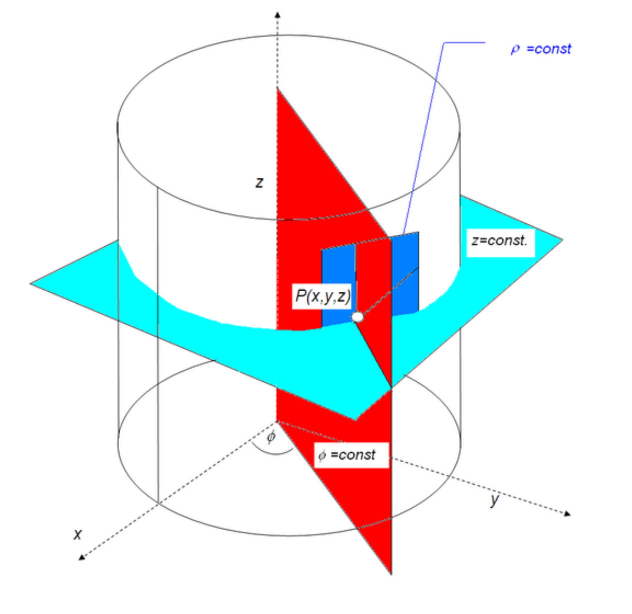

Figure 12.3: Planes generated by cylindrical polar coordinates

- The cylinder $\rho^2 = x^2 + y^2 = \text{Const.}$;

see Fig. 11.2.

The Cartesian and the cylindrical coordinate systems are related through

$$\begin{aligned} x &= \rho \cos \varphi, \\ y &= \rho \sin \varphi, \\ z &= z, \end{aligned}$$

with inverse transformation

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2}, \\ \varphi &= \tan^{-1}(y/x), \\ z &= z. \end{aligned}$$

Thus, we have

$$\begin{aligned} \boldsymbol{r} \equiv \boldsymbol{x} &= \hat{\boldsymbol{x}} x + \hat{\boldsymbol{y}} y + \hat{\boldsymbol{z}} z, \\ &= \hat{\boldsymbol{x}} \rho \cos \varphi + \hat{\boldsymbol{y}} \rho \sin \varphi + \hat{\boldsymbol{z}} z. \end{aligned}$$

Let's compute the tangent vectors:

$$\hat{\boldsymbol{\rho}} = \frac{\partial \boldsymbol{x}}{\partial \rho} \Big/ \left|\frac{\partial \boldsymbol{x}}{\partial \rho}\right|, \qquad \hat{\boldsymbol{\varphi}} = \frac{\partial \boldsymbol{x}}{\partial \varphi} \Big/ \left|\frac{\partial \boldsymbol{x}}{\partial \varphi}\right|, \qquad \hat{\boldsymbol{z}} = \frac{\partial \boldsymbol{x}}{\partial z} \Big/ \left|\frac{\partial \boldsymbol{x}}{\partial z}\right|.$$

First,

$$\frac{\partial \boldsymbol{x}}{\partial \rho} = \hat{\boldsymbol{x}} \cos\varphi + \hat{\boldsymbol{y}} \sin\varphi,$$

and this has unit length, hence

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}} \cos\varphi + \hat{\boldsymbol{y}} \sin\varphi.$$

Next,

$$\frac{\partial \boldsymbol{x}}{\partial \varphi} = -\hat{\boldsymbol{x}}\rho \sin\varphi + \hat{\boldsymbol{y}}\rho \cos\varphi,$$

which has length ρ, hence

$$\hat{\boldsymbol{\varphi}} = -\hat{\boldsymbol{x}} \sin\varphi + \hat{\boldsymbol{y}} \cos\varphi.$$

Finally, the third tangent vector must simply be $\hat{\boldsymbol{z}}$. We assemble these results:

$$\begin{aligned}
\hat{\boldsymbol{\rho}} &= \hat{\boldsymbol{x}} \cos\varphi + \hat{\boldsymbol{y}} \sin\varphi, \\
\hat{\boldsymbol{\varphi}} &= -\hat{\boldsymbol{x}} \sin\varphi + \hat{\boldsymbol{y}} \cos\varphi, \\
\hat{\boldsymbol{z}} &= \hat{\boldsymbol{z}}.
\end{aligned}$$

These are quite clearly mutually orthogonal:

$$\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\theta}} = 0.$$

We must also compute the scale factors:

$$h_\rho = \left|\frac{\partial \boldsymbol{x}}{\partial \rho}\right| = |\hat{\boldsymbol{x}} \cos\varphi + \hat{\boldsymbol{y}} \sin\varphi| = 1,$$

$$h_\varphi = \left|\frac{\partial \boldsymbol{x}}{\partial \varphi}\right| = |-\hat{\boldsymbol{x}}\rho \sin\varphi + \hat{\boldsymbol{y}}\rho \cos\varphi| = \rho,$$

and

$$h_z = 1.$$

For convenience, let us assemble these results also:

$$
\begin{aligned}
h_\rho &= 1, \\
h_\varphi &= \rho, \\
h_z &= 1.
\end{aligned}
$$

The line element in this coordinate system is thus

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2.$$

Now, we make the identifications $\hat{\boldsymbol{q}}_1 = \hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{q}}_2 = \hat{\boldsymbol{\varphi}}$, and $\hat{\boldsymbol{q}}_3 = \hat{\boldsymbol{z}}$, along with

$$q_1 = \rho, \qquad q_2 = \varphi, \qquad q_3 = z.$$

Thus,

1. The gradient: General case:

$$\nabla\psi(q_1, q_2, q_3) = \frac{\hat{\boldsymbol{q}}_1}{h_1}\frac{\partial\psi}{\partial q_1} + \frac{\hat{\boldsymbol{q}}_2}{h_2}\frac{\partial\psi}{\partial q_2} + \frac{\hat{\boldsymbol{q}}_3}{h_3}\frac{\partial\psi}{\partial q_3};$$

   Cylindrical polar coordinates:

$$\nabla\psi(\rho, \varphi, z) = \hat{\boldsymbol{\rho}}\frac{\partial\psi}{\partial\rho} + \frac{\hat{\boldsymbol{\varphi}}}{\rho}\frac{\partial\psi}{\partial\varphi} + \hat{\boldsymbol{z}}\frac{\partial\psi}{\partial z}.$$

2. The divergence: General case:

$$\nabla\cdot\boldsymbol{v} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(v_1h_2h_3) + \frac{\partial}{\partial q_2}(v_2h_3h_1) + \frac{\partial}{\partial q_3}(v_3h_1h_2)\right];$$

   Cylindrical polar coordinates:

$$\nabla\cdot\boldsymbol{v}(\rho, \varphi, z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v_\rho) + \frac{1}{\rho}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_z}{\partial z}.$$

3. The curl: General case:

$$(\nabla\times\boldsymbol{v})(q_1, q_2, q_3) = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{\boldsymbol{q}}_1h_1 & \hat{\boldsymbol{q}}_2h_2 & \hat{\boldsymbol{q}}_3h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1v_1 & h_2v_2 & h_3v_3 \end{vmatrix};$$

Cylindrical polar coordinates:

$$(\nabla \times \boldsymbol{v})(\rho, \varphi, z) = \frac{1}{\rho} \begin{vmatrix} \hat{\boldsymbol{\rho}} & \hat{\boldsymbol{\varphi}}\rho & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ v_\rho & \rho v_\varphi & v_z, \end{vmatrix}$$

4. The Laplacian: General case:

$$\nabla^2 \psi(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial q_3} \right) \right].$$

Cylindrical polar coordinates:

$$\nabla^2 \psi \left(\rho, \varphi, z\right) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2},$$

and NOTE THE EXPONENT in the ∂_ρ derivative: it is 1 (in the spherical polar coordinate case the corresponding radial exponent is 2).

12.3 Physical application

1. Solve Laplace's equation $\nabla^2 \psi = 0$, in cylindrical coordinates, for $\psi = \psi(\rho)$.

   For this particular function,

   $$\partial_\varphi \psi = \partial_z \psi = 0,$$

   hence,

   $$0 = \nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) = 0.$$

   This means that the $(\cdot) = \mathrm{Const.} := k$, hence,

   $$\rho \frac{\partial \psi}{\partial \rho} = k,$$

   or

   $$\frac{d\psi}{d\rho} = \frac{k}{\rho}.$$

   Separating the variables gives

   $$\mathrm{d}\psi = k \frac{\mathrm{d}\rho}{\rho};$$

integration gives the final answer:

$$\psi(\rho) = \psi_0 + k\log(\rho), \qquad \rho \neq 0.$$

and there are two constants of integration because the equation is second-order.

2. For the flow of an incompressible viscous fluid, the Navier–Stokes equations lead to

$$-\nabla \times (\boldsymbol{v} \times (\nabla \times \boldsymbol{v})) = \nu \nabla^2 (\nabla \times \boldsymbol{v}), \tag{12.1}$$

where $\boldsymbol{v}(\boldsymbol{x})$ is the fluid velocity and ν is the constant kinematic viscosity. For axial flow in a cylindrical pipe we take the velocity $\boldsymbol{v}$ to be

$$\boldsymbol{v} = \hat{\boldsymbol{z}} v(\rho). \tag{12.2}$$

Show that the left-hand side of Eq. (12.1) is identically zero when the velocity has the form (12.2). Hence, $\boldsymbol{v} = \hat{\boldsymbol{z}} v(\rho)$ must satisfy

$$\nabla^2 (\nabla \times \boldsymbol{v}) = 0.$$

Show that this leads to the ordinary differential equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d^2 v}{d\rho^2}\right) - \frac{1}{\rho^2}\frac{dv}{d\rho} = 0,$$

with solution

$$v = v_0 + a_2 \rho^2,$$

where v_0 and a_2 are constants. Show that the **boundary condition**

$$v = 0, \qquad \text{on } \rho = R, \text{ the pipe wall}$$

leads to the final form

$$v(\rho) = v_0 \left(1 - \frac{\rho^2}{R^2}\right).$$

Let's focus on the LHS first. With $\boldsymbol{v} = \hat{\boldsymbol{z}} v(\rho)$, we compute

$$\nabla \times \boldsymbol{v} = \frac{1}{\rho} \begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\varphi}} & \hat{\boldsymbol{z}} \\ \partial_\rho & \partial_\varphi & \partial_z \\ 0 & 0 & v(\rho) \end{vmatrix} = -\hat{\boldsymbol{\varphi}} \frac{\partial v}{\partial \rho}.$$

Now we take

$$\boldsymbol{v}\times(\nabla\times\boldsymbol{v})=\begin{vmatrix}\hat{\boldsymbol{\rho}} & \hat{\boldsymbol{\varphi}} & \hat{\boldsymbol{z}}\\ 0 & 0 & v(\rho)\\ 0 & -\frac{\partial v}{\partial\rho} & 0\end{vmatrix}=+\hat{\boldsymbol{\rho}}v(\rho)\frac{\partial v}{\partial\rho},$$

and this determinant expansion is legitimate because $(\hat{\boldsymbol{\rho}},\hat{\boldsymbol{\varphi}},\hat{\boldsymbol{z}})$ form a right-handed orthonormal triad. Finally, we take the curl of this expression:

$$\nabla\times(\boldsymbol{v}\times(\nabla\times\boldsymbol{v}))=\frac{1}{\rho}\begin{vmatrix}\hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\varphi}} & \hat{\boldsymbol{z}}\\ \partial_\rho & \partial_\varphi & \partial_z\\ v(\rho)\frac{\partial v}{\partial\rho} & 0 & 0\end{vmatrix}=0.$$

Thus, for $\boldsymbol{v}=\hat{\boldsymbol{z}}v(\rho)$, the LHS of the fluid equation is identically zero, and we are forced to consider

$$\nabla^2(\nabla\times\boldsymbol{v})=0,$$

or

$$\nabla^2(-\hat{\boldsymbol{\varphi}}v'(\rho))=0.$$

Some care is required here because $\hat{\boldsymbol{\varphi}}$ is non-constant and cannot be taken outside the differential operator. However, we can cross both sizes with $\hat{\boldsymbol{z}}$ and take the constant vector $\hat{\boldsymbol{z}}$ inside the operator:

$$0=\nabla^2(\hat{\boldsymbol{\varphi}}\times\hat{\boldsymbol{z}}v'(\rho)).$$

But $(\hat{\boldsymbol{\rho}},\hat{\boldsymbol{\varphi}},\hat{\boldsymbol{z}})$ are a right-handed orthonormal triad, so $\hat{\boldsymbol{\varphi}}\times\hat{\boldsymbol{z}}=\hat{\boldsymbol{\rho}}$, and we solve

$$0=\nabla^2(\hat{\boldsymbol{\rho}}v'(\rho))=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}(\hat{\boldsymbol{\rho}}v'(\rho))\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}(\hat{\boldsymbol{\rho}}v'(\rho))+\frac{\partial^2}{\partial z^2}(\hat{\boldsymbol{\rho}}v'(\rho)).\tag{12.3}$$

Let's consider

$$\hat{\boldsymbol{\rho}}=\hat{\boldsymbol{x}}\cos\varphi+\hat{\boldsymbol{y}}\sin\varphi.$$

Evidently,

$$\frac{\partial\hat{\boldsymbol{\rho}}}{\partial\rho}=\frac{\partial\hat{\boldsymbol{\rho}}}{\partial z}=0,$$

and

$$\frac{\partial\hat{\boldsymbol{\rho}}}{\partial\varphi}=-\hat{\boldsymbol{x}}\sin\varphi+\hat{\boldsymbol{y}}\cos\varphi,\qquad\frac{\partial^2\hat{\boldsymbol{\rho}}}{\partial\varphi^2}=-\hat{\boldsymbol{x}}\cos\varphi-\hat{\boldsymbol{y}}\sin\varphi=-\hat{\boldsymbol{\rho}}.$$

Substitute these expressions back into Eq. (12.3):

$$\begin{aligned}0&=\frac{\hat{\boldsymbol{\rho}}}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial v'(\rho)}{\partial\rho}\right)+\frac{v'(\rho)}{\rho^2}\frac{\partial^2\hat{\boldsymbol{\rho}}}{\partial\varphi^2}+0,\\ &=\frac{\hat{\boldsymbol{\rho}}}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial v'(\rho)}{\partial\rho}\right)-\hat{\boldsymbol{\rho}}\frac{v'(\rho)}{\rho^2}.\end{aligned}$$

Hence,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial v'(\rho)}{\partial\rho}\right) = \frac{v'(\rho)}{\rho^2},$$

as required. Substitution of the trial solution $v = v_0 + a_2\rho^2$ into the LHS gives

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho 2a_2) = 2a_2.$$

Substitution into the RHS gives

$$\frac{v'(\rho)}{\rho^2} = \frac{2a_2\rho^2}{\rho^2} = 2a_2.$$

Hence, LHS = RHS, and $v = v_0 + a_2\rho^2$ is a solution. Note that if $v(R) = 0$, then

$$v_0 + a_2R^2 = 0 \implies a_2 = -\frac{v_0}{R^2},$$

hence

$$v = v_0\left(1 - \frac{\rho^2}{R^2}\right).$$

This is the celebrated **Poiseuille flow**, observed in flows in blood vessels.

Chapter 13

Special integrals involving curvilinear coordinate systems

A mathematician is someone to whom

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x = \sqrt{\pi}$$

is as obvious as $1 + 1 = 2$.

William Thomson, 1st Baron Kelvin of Largs.[1]

Overview

In this section we carry out some special integrations in various spatial dimensions. These require clever substitutions involving curvilinear coordinates.

13.1 The gamma integral

Consider the integral

$$\Gamma(n+1) = \int_0^{\infty} t^n \mathrm{e}^{-t} \mathrm{d}t, \qquad n \in \{0, 1, 2, \cdots\}.$$

If $n = 0$, the integration is easy:

$$\Gamma(1) = \int_0^{\infty} \mathrm{e}^{-t} \mathrm{d}t = -\mathrm{e}^{-t}\big|_0^{\infty} = 1.$$

[1] William Thomson, b. 1824 Belfast, d. 1907 Largs, Scotland. Kelvin was born in Belfast but moved to Scotland as a child. There is a very impressive statue of Kelvin in the Belfast botanical gardens.

Otherwise, we do integration by parts:

$$\begin{aligned}
\Gamma(n+1) &= \int_0^\infty \underbrace{t^n}_{u}\,\underbrace{\mathrm{e}^{-t}\mathrm{d}t}_{dv}, \\
&= -t^n\mathrm{e}^{-t}\Big|_0^\infty - n\int_0^\infty \underbrace{\left(-\mathrm{e}^{-t}\right)}_{v}\underbrace{t^{n-1}\mathrm{d}t}_{du}, \\
&= n\int_0^\infty t^{n-1}\mathrm{e}^{-t}\mathrm{d}t, \\
&= n\Gamma(n-1).
\end{aligned}$$

Now, we repeat this integration by parts until we are left with one integral evaluation, $\Gamma(1)$:

$$\Gamma(n+1) = n(n-1)\ldots 2.\Gamma(1) = n(n-1)\ldots 2.1 = n!$$

Thus, for $n \in \{0, 1, 2, \cdots\}$,

$$n! = \Gamma(n+1) := \int_0^\infty t^n\mathrm{e}^{-t}\mathrm{d}t.$$

Note, however, that the integral

$$\Gamma(x+1) = \int_0^\infty t^x\mathrm{e}^{-t}\mathrm{d}t$$

is valid for all $x \geq 0$, and that

$$\Gamma(x+1) = x\Gamma(x) \qquad x > 0.$$

This gives a generalization of the factorial function to positive real numbers:

$$x! := \Gamma(x+1).$$

Note: Let $t = u^2$ in $\Gamma(x+1) = \int_0^\infty t^x\mathrm{e}^{-t}\mathrm{d}t$. Then,

$$\mathrm{d}t = 2u\mathrm{d}u.$$

The integral, re-expressed in the u-variable, also ranges from 0 to $+\infty$:

$$\Gamma(x+1) = 2\int_0^\infty u^{2x+1}\mathrm{e}^{-u^2}\mathrm{d}u,$$

which is an alternative expression for the Gamma function. Setting $x = 0$ gives

$$1 = \Gamma(1) = 2\int_0^\infty u\mathrm{e}^{-u^2}\mathrm{d}u$$

13.2 The exponential integral

In this section we compute the integral

$$I := \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x.$$

First, let us derive the area element in two-dimensional spherical polar coordinates.

In two dimensions, the spherical polar coordinates are as follows:

$$x = r\cos\varphi, \qquad y = r\sin\varphi,$$

where $r = \sqrt{x^2+y^2}$ is the distance from the origin to the point $P(x,y)$ and $\varphi = \tan^{-1}(y/x)$ is the angle between the x-axis and the radius vector $\boldsymbol{r} = \overrightarrow{OP}$. Based on the identity

$$\boldsymbol{r} \equiv \boldsymbol{x} = r\cos\varphi\hat{\boldsymbol{x}} + r\sin\varphi\hat{\boldsymbol{y}},$$

we compute the tangent vectors:

$$\frac{\partial \boldsymbol{x}}{\partial r} = \cos\varphi\hat{\boldsymbol{x}} + \sin\varphi\hat{\boldsymbol{y}} = \hat{\boldsymbol{r}},$$

since $|\cos\varphi\hat{\boldsymbol{x}} + \sin\varphi\hat{\boldsymbol{y}}|^2 = \cos^2\varphi + \sin^2\varphi = 1$,

$$\frac{\partial \boldsymbol{x}}{\partial \varphi} = -r\sin\varphi\hat{\boldsymbol{x}} + r\cos\varphi\hat{\boldsymbol{y}}, \qquad \hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\boldsymbol{x}} + \cos\varphi\hat{\boldsymbol{y}},$$

since $|-r\sin\varphi\hat{\boldsymbol{x}} + r\cos\varphi\hat{\boldsymbol{y}}|^2 = r^2$. The scale factors are thus

$$h_r = \left|\frac{\partial \boldsymbol{x}}{\partial r}\right| = 1, \qquad h_\varphi = \left|\frac{\partial \boldsymbol{x}}{\partial \varphi}\right| = r.$$

Hence, the line element is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2.$$

and an infinitesimal patch of area is

$$\mathrm{d}S = h_r h_\varphi \mathrm{d}r\mathrm{d}\varphi = r\mathrm{d}r\mathrm{d}\varphi.$$

But $\mathrm{d}S = \mathrm{d}x\mathrm{d}y$, hence

$$\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\varphi.$$

Now we compute I. First, take

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x\right), \\ &= \left(\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\mathrm{d}y\right), \end{aligned}$$

since x is a 'dummy variable' of integration. Re-write this as

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\, \mathrm{e}^{-(x^2+y^2)}.$$

Now introduce polar coordinates. To enumerate all points (x, y) in the plane, the angle φ must go between 0 and 2π, and the radius vector r must go from 0 to ∞. Thus,

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\, \mathrm{e}^{-(x^2+y^2)}, \\ &= \int_0^{2\pi}\int_0^{\infty} r\mathrm{d}r\mathrm{d}\varphi\, \mathrm{e}^{-r^2}, \\ &= \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\infty} r\mathrm{d}r\, \mathrm{e}^{-r^2}, \\ &= 2\pi\int_0^{\infty} r\mathrm{d}r\, \mathrm{e}^{-r^2}, \\ &= 2\pi\int_0^{\infty} \left(-\tfrac{1}{2}\right)\frac{d}{dr}\mathrm{e}^{-r^2}\mathrm{d}r, \\ &= \pi\left[-\mathrm{e}^{-r^2}\right]_0^{\infty}, \\ &= \pi. \end{aligned}$$

Hence $I^2 = \pi$, and

$$I = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}.$$

One final note: Recall

$$\begin{aligned} \Gamma(x+1) &= \int_0^{\infty} t^x\mathrm{e}^{-t}\mathrm{d}t, \\ &= 2\int_0^{\infty} u^{2x+1}\mathrm{e}^{-u^2}\mathrm{d}u. \end{aligned}$$

Take the second form with $x = -1/2$:

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^{\infty} u^0\mathrm{e}^{-u^2}\mathrm{d}u = \int_{-\infty}^{\infty} \mathrm{e}^{-u^2}\mathrm{d}u = \sqrt{\pi}.$$

Thus, for half-integers $n+\frac{1}{2}$, where $n \in \{0,1,2,\cdots\}$,

$$\left(n+\tfrac{1}{2}\right)! = \left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\cdots\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right),$$

or

$$\left(n+\tfrac{1}{2}\right)! = \left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\cdots\tfrac{1}{2}\sqrt{\pi}.$$

13.3 The volume of an n-ball

In n dimensions, the ball centred at 0 of radius r is a subset of $\mathbb{R}^n$ such that

$$x_1^2 + x_2^2 + \cdots x_n^2 \leq r^2.$$

We would like to find the volume of this ball:

$$V_n(r) = \int\cdots\int_{x_1^2+\cdots+x_n^2\leq r^2} \mathrm{d}x_1\cdots\mathrm{d}x_n.$$

In analogy with polar coordinates in two-dimensional space, let us write the volume element as

$$\mathrm{d}x_1\cdots\mathrm{d}x_n = r^{n-1}\mathrm{d}r\mathrm{d}\Omega_n,$$

where $\mathrm{d}\Omega_n$ is a differential involving angles $\varphi_1,\cdots,\varphi_{n-1}$ that are unspecified polar coordinates on the sphere in $\mathbb{R}^n$. It is not necessary to know what these angles are, suffice to say that

$$\mathrm{d}\Omega_n = f\left(\varphi_1,\cdots,\varphi_{n-1}\right)\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_{n-1},$$

where $f(\cdots)$ is some function. The differential $\mathrm{d}\Omega_n$ is the element of **solid angle** in n dimensions, and its integral over all possible values of $\varphi_1,\cdots,\varphi_{n-1}$ gives the **surface area** of the unit sphere in n dimensions, $S_n(1)$. Thus,

$$\begin{aligned} V_n(r) &= \int\cdots\int_{x_1^2+\cdots+x_n^2\leq r^2} \mathrm{d}x_1\cdots\mathrm{d}x_n, \\ &= \int \mathrm{d}\Omega_n \int_0^r r^{n-1}\mathrm{d}r, \\ &= \frac{S_n(1)r^n}{n}. \end{aligned}$$

This gives a relationship between surface area and volume in n dimensions.

Now, consider the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{-\boldsymbol{x}^2} \mathrm{d}^n x &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{e}^{-x_1^2 - \cdots x_n^2} \mathrm{d}x_1 \cdots \mathrm{d}x_n, \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{-x_1^2} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{e}^{-x_n^2} \mathrm{d}x_n, \\ &= I^n, \\ &= \pi^{n/2}. \qquad (*)\end{aligned}$$

But we can write $\int_{-\infty}^{\infty} \mathrm{e}^{-\boldsymbol{x}^2} \mathrm{d}^n x$ in general spherical polar coordinates as

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{-\boldsymbol{x}^2} \mathrm{d}^n x &= \int \mathrm{d}\Omega_n \int_0^{\infty} r^{n-1} \mathrm{e}^{-r^2}\, \mathrm{d}r, \\ &= \tfrac{1}{2} S_n(1) \Gamma(n/2). \qquad (**)\end{aligned}$$

Equating (*) and (**) gives

$$S_n(1) = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

hence,

$$V_n(r) = \frac{2\pi^{n/2}}{n\Gamma(n/2)} r^n.$$

Check: $n = 2$ gives $V_2(r) = \pi r^2$, $n = 3$ gives

$$\frac{2\pi\sqrt{\pi}}{3\frac{1}{2}\sqrt{\pi}} r^3 = \tfrac{4}{3}\pi r^3.$$

13.4 The Jacobian

Recall, in two dimensions, in spherical polar coordinates,

$$\mathrm{d}S = \mathrm{d}x\, \mathrm{d}y = r\, \mathrm{d}r\, \mathrm{d}\varphi = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} \mathrm{d}r\, \mathrm{d}\varphi.$$

The determinant

$$J := \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix}$$

is called the **Jacobian** of the coordinate transformation $(x, y) \to (r, \varphi)$. In general, in n dimensions, given a coordinate transformation

$$q_1 = q_1(x_1, \cdots x_n), \cdots, q_n = q_n(x_1, \cdots x_n),$$

the volume element has the form

$$\mathrm{d}V_n = \mathrm{d}x_1 \cdots \mathrm{d}x_n = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \cdots & \frac{\partial x_1}{\partial q_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial q_1} & \cdots & \frac{\partial x_n}{\partial q_n} \end{vmatrix} \mathrm{d}q_1 \cdots \mathrm{d}q_n,$$

and

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \cdots & \frac{\partial x_1}{\partial q_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial q_1} & \cdots & \frac{\partial x_n}{\partial q_n} \end{vmatrix}$$

is the **Jacobian**. For orthogonal coordinate systems, this always reduces to

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \cdots & \frac{\partial x_1}{\partial q_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial q_1} & \cdots & \frac{\partial x_n}{\partial q_n} \end{vmatrix} = h_{q_1} \cdots h_{q_n}.$$

We prove these facts now.

Proof: Form the tangent vectors

$$\boldsymbol{t}_1 = \frac{\partial \boldsymbol{x}}{\partial q_1}, \cdots, \boldsymbol{t}_n = \frac{\partial \boldsymbol{x}}{\partial q_n},$$

where

$$\hat{\boldsymbol{q}}_i = \boldsymbol{t}_i / |\boldsymbol{t}_i|.$$

Recall, in two dimensions,

$$\mathrm{d}V_2 = |\boldsymbol{t}_1 \times \boldsymbol{t}_2| \, \mathrm{d}q_1 \, \mathrm{d}q_2,$$

in three dimensions,

$$\mathrm{d}V_3 = |\boldsymbol{t}_1 \cdot (\boldsymbol{t}_2 \times \boldsymbol{t}_3) \,| \, \mathrm{d}q_1 \, \mathrm{d}q_2 \, \mathrm{d}q_3.$$

In both cases, we have the formula

$$\mathrm{d}V_n = \begin{vmatrix} | & \cdots & | \\ \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_n \\ | & \cdots & | \end{vmatrix} \mathrm{d}q_1 \cdots \mathrm{d}q_n, \qquad n = 2, 3,$$

where the i^{th} column of this determinant is the column vector $\boldsymbol{t}_i$. Now there is nothing special about dimensions $n = 2$ or $n = 3$, so this formula must hold in an arbitrary spatial dimension:

$$\mathrm{d}V_n = \begin{vmatrix} | & | & | \\ \boldsymbol{t}_1 & | & \boldsymbol{t}_n \\ | & | & | \end{vmatrix} \mathrm{d}q_1 \cdots \mathrm{d}q_n, \qquad n \in \{1, 2, \cdots\}.$$

In other words,

$$\mathrm{d}V_n = \mathrm{d}x_1 \cdots \mathrm{d}x_n = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \cdots & \frac{\partial x_1}{\partial q_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial x_n}{\partial q_1} & \cdots & \frac{\partial x_n}{\partial q_n} \end{vmatrix} \mathrm{d}q_1 \cdots \mathrm{d}q_n.$$

For orthogonal curvilinear coordinates,

$$\begin{aligned} \mathrm{d}V_n &= \begin{vmatrix} | & | & | \\ \boldsymbol{t}_1 & | & \boldsymbol{t}_n \\ | & | & | \end{vmatrix} \mathrm{d}q_1 \cdots \mathrm{d}q_n, \\ &= \begin{vmatrix} | & | & | \\ \hat{\boldsymbol{q}}_1 & | & \hat{\boldsymbol{q}}_n \\ | & | & | \end{vmatrix} h_1 \cdots h_n \, \mathrm{d}q_1 \cdots \mathrm{d}q_n, \\ &= \left| P \begin{pmatrix} | & | & | \\ \hat{\boldsymbol{q}}_1 & | & \hat{\boldsymbol{q}}_n \\ | & | & | \end{pmatrix} P^T \right| h_1 \cdots h_n \, \mathrm{d}q_1 \cdots \mathrm{d}q_n, \end{aligned}$$

where P is an orthogonal matrix $|PP^T| = 1$ that rotates the matrix

$$\begin{pmatrix} | & | & | \\ \hat{\boldsymbol{q}}_1 & | & \hat{\boldsymbol{q}}_n \\ | & | & | \end{pmatrix}$$

into the identity matrix,

$$P \begin{pmatrix} | & | & | \\ \hat{\boldsymbol{q}}_1 & | & \hat{\boldsymbol{q}}_n \\ | & | & | \end{pmatrix} P^T = \mathbb{I}_n.$$

Thus,

$$\mathrm{d}V_n = h_1 \cdots h_n \, \mathrm{d}q_1 \cdots \mathrm{d}q_n,$$

as required.

In conclusion, for orthogonal coordinates,

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \cdots & \frac{\partial x_1}{\partial q_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial q_1} & \cdots & \frac{\partial x_n}{\partial q_n} \end{vmatrix} = h_1 \cdots h_n.$$

13.5 The ball in $\mathbb{R}^4$

In this section we construct coordinates for the ball in $\mathbb{R}^4$ and compute its volume from this construction. Our approach for developing coordinates is based on an analogy with three dimensional space.

Recall the construction of a ball in $\mathbb{R}^3$. Topologically, we take two identical discs (balls in $\mathbb{R}^2$) and let them sit one on top of the other. We glue the boundary edges of these two balls together. We then 'inflate' the glued-together object so that the two discs are pushed in opposite directions into the third dimension. In terms of coordinates, this construction is summarized by the augmentation of the two-dimensional coordinate system

$$\begin{aligned} x &= r\cos\varphi, \\ y &= r\sin\varphi, \qquad 0 \le \varphi < 2\pi, \qquad r = \sqrt{x^2+y^2}, \end{aligned}$$

to the following form:

$$\begin{aligned} z &= r\cos\theta, \\ x &= \sin\theta\, r\cos\varphi, \\ y &= \sin\theta\, r\sin\varphi, \end{aligned}$$

where

$$0 \le \varphi < 2\pi, \qquad 0 \le \theta < \pi, \qquad r = \sqrt{x^2+y^2+z^2}.$$

We now repeat the same steps: A ball in four dimensions is constructed from two identical three-balls. We sit these balls one on top of the other and glue their boundaries together (these boundaries are actually spheres). We then 'inflate' this object so that the two ball-interiors are pushed in opposite directions into the fourth dimension. In coordinate terms, we have

$$\begin{aligned} w &= r\cos\psi, \\ z &= \sin\psi \; r\cos\theta, \\ y &= \sin\psi \; r\sin\theta\sin\varphi, \\ x &= \sin\psi \; r\sin\theta\cos\varphi, \end{aligned}$$

where

$$0 \le \varphi < 2\pi, \qquad 0 \le \theta < \pi, \qquad 0 \le \psi < \pi, \qquad r = \sqrt{x^2 + y^2 + z^2 + w^2}.$$

For notational convenience, we re-write this system as

$$\begin{aligned} x_1 &= r\cos\psi, \\ x_2 &= r\sin\psi\cos\theta, \\ x_3 &= r\sin\psi\sin\theta\sin\varphi, \\ x_4 &= r\sin\psi\sin\theta\cos\varphi. \end{aligned}$$

Now a general vector $\boldsymbol{x}$ in $\mathbb{R}^4$ is written as

$$\boldsymbol{x} = \hat{\boldsymbol{e}}_1 x_1 + \hat{\boldsymbol{e}}_2 x_2 + \hat{\boldsymbol{e}}_3 x_3 + \hat{\boldsymbol{e}}_4 x_4,$$

where

$$\begin{aligned} \hat{\boldsymbol{e}}_1 &= (1, 0, 0, 0)\,, \\ \hat{\boldsymbol{e}}_2 &= (0, 1, 0, 0)\,, \\ \hat{\boldsymbol{e}}_3 &= (0, 0, 1, 0)\,, \\ \hat{\boldsymbol{e}}_4 &= (0, 0, 0, 1)\,. \end{aligned}$$

Hence,

$$\boldsymbol{x} = \hat{\boldsymbol{e}}_1 r\cos\psi + \hat{\boldsymbol{e}}_2 r\sin\psi\cos\theta + \hat{\boldsymbol{e}}_3 r\sin\psi\sin\theta\cos\varphi + \hat{\boldsymbol{e}}_4 r\sin\psi\sin\theta\sin\varphi.$$

Now, we can compute tangent vectors.

Clearly,

$$\hat{\boldsymbol{r}} = \frac{\partial \boldsymbol{x}}{\partial r} = \hat{\boldsymbol{e}}_1 \cos\psi + \hat{\boldsymbol{e}}_2 \sin\psi\cos\theta + \hat{\boldsymbol{e}}_3 \sin\psi\sin\theta\cos\varphi + \hat{\boldsymbol{e}}_4 \sin\psi\sin\theta\sin\varphi.$$

is the radial tangent vector with unit norm. Next,

$$\frac{\partial \boldsymbol{x}}{\partial \psi} = -\hat{\boldsymbol{e}}_1 r \sin\psi + \hat{\boldsymbol{e}}_2 r \cos\psi \cos\theta + \hat{\boldsymbol{e}}_3 r \cos\psi \sin\theta \cos\varphi + \hat{\boldsymbol{e}}_4 r \cos\psi \sin\theta \sin\varphi.$$

with norm r, hence

$$\hat{\boldsymbol{\psi}} = -\hat{\boldsymbol{e}}_1 \sin\psi + \hat{\boldsymbol{e}}_2 \cos\psi \cos\theta + \hat{\boldsymbol{e}}_3 \cos\psi \sin\theta \cos\varphi + \hat{\boldsymbol{e}}_4 \cos\psi \sin\theta \sin\varphi.$$

Again,

$$\frac{\partial \boldsymbol{x}}{\partial \theta} = \hat{\boldsymbol{e}}_1 0 + r \sin\psi \left[-\hat{\boldsymbol{e}}_2 \sin\theta + \hat{\boldsymbol{e}}_3 \cos\theta \cos\varphi + \hat{\boldsymbol{e}}_4 \cos\theta \sin\varphi\right],$$

with norm $r\sin\psi$, hence

$$\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{e}}_2 \sin\theta + \hat{\boldsymbol{e}}_3 \cos\theta \cos\varphi + \hat{\boldsymbol{e}}_4 \cos\theta \sin\varphi.$$

Finally,

$$\frac{\partial \boldsymbol{x}}{\partial \varphi} = \hat{\boldsymbol{e}}_1 0 + \hat{\boldsymbol{e}}_2 0 + r \sin\psi \sin\theta \left[-\hat{\boldsymbol{e}}_3 \sin\varphi + \hat{\boldsymbol{e}}_4 \cos\varphi\right],$$

with norm $r \sin\psi \sin\theta$, hence

$$\hat{\boldsymbol{\varphi}} = -\hat{\boldsymbol{e}}_3 \sin\varphi + \hat{\boldsymbol{e}}_4 \cos\varphi.$$

Let's assemble these results.

Tangent vectors:

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \hat{\boldsymbol{e}}_1 \cos\psi + \hat{\boldsymbol{e}}_2 \sin\psi \cos\theta + \hat{\boldsymbol{e}}_3 \sin\psi \sin\theta \cos\varphi + \hat{\boldsymbol{e}}_4 \sin\psi \sin\theta \sin\varphi, \\
\hat{\boldsymbol{\psi}} &= -\hat{\boldsymbol{e}}_1 \sin\psi + \hat{\boldsymbol{e}}_2 \cos\psi \cos\theta + \hat{\boldsymbol{e}}_3 \cos\psi \sin\theta \cos\varphi + \hat{\boldsymbol{e}}_4 \cos\psi \sin\theta \sin\varphi, \\
\hat{\boldsymbol{\theta}} &= -\hat{\boldsymbol{e}}_2 \sin\theta + \hat{\boldsymbol{e}}_3 \cos\theta \cos\varphi + \hat{\boldsymbol{e}}_4 \cos\theta \sin\varphi, \\
\hat{\boldsymbol{\varphi}} &= -\hat{\boldsymbol{e}}_3 \sin\varphi + \hat{\boldsymbol{e}}_4 \cos\varphi.
\end{aligned}$$

Scale factors:

$$\begin{aligned}
h_r &= 1, \\
h_\psi &= r, \\
h_\theta &= r \sin\psi, \\
h_\varphi &= r \sin\psi \sin\theta.
\end{aligned}$$

It is straightforward to check that these vectors are orthogonal: there are $(4-1)! = 6$ relations to check. For example,

$$\begin{aligned}
\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\psi}} &= \left[\hat{\boldsymbol{e}}_1\cos\psi+\hat{\boldsymbol{e}}_2\sin\psi\cos\theta+\hat{\boldsymbol{e}}_3\sin\psi\sin\theta\cos\varphi+\hat{\boldsymbol{e}}_4\sin\psi\sin\theta\sin\varphi\right]\\
&\quad\cdot\left[-\hat{\boldsymbol{e}}_1\sin\psi+\hat{\boldsymbol{e}}_2\cos\psi\cos\theta+\hat{\boldsymbol{e}}_3\cos\psi\sin\theta\cos\varphi+\hat{\boldsymbol{e}}_4\cos\psi\sin\theta\sin\varphi\right],\\
&= -\cos\psi\sin\psi+\sin\psi\cos\psi\left[\cos^2\theta+\sin^2\theta\left(\cos^2\varphi+\sin^2\varphi\right)\right],\\
&= -\cos\psi\sin\psi+\sin\psi\cos\psi=0.
\end{aligned}$$

Now let's compute the volume of the four-ball:

$$\begin{aligned}
V_4 &= \int_0^R \mathrm{d}r\int_0^\pi \mathrm{d}\psi\int_0^\pi \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\, h_r h_\psi h_\theta h_\varphi,\\
&= \int_0^R \mathrm{d}r\int_0^\pi \mathrm{d}\psi\int_0^\pi \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\, r^3\sin\psi\sin^2\theta,\\
&= \left(\int_0^R r^3\mathrm{d}r\right)\left(\int_0^\pi \mathrm{d}\psi\,\sin^2\psi\right)\left(\int_0^\pi \mathrm{d}\theta\,\sin\theta\right)\left(\int_0^{2\pi}\mathrm{d}\varphi\right),\\
&= \left(\frac{1}{4}r^4\right)\left[\tfrac{1}{2}\left(\psi-\sin\psi\cos\psi\right)_0^\pi\right]\left(-\cos\pi+\cos 0\right)2\pi,\\
&= \tfrac{1}{2}\pi^2 r^4.
\end{aligned}$$

Check against the general formula:

$$\begin{aligned}
V_n &= \frac{2\pi^{n/2}}{n\Gamma(n/2)}r^4,\\
&= \frac{2\pi^2}{4\Gamma(2)}r^4, \qquad n=4,\\
&= \frac{2\pi^2}{4\cdot 1!}r^4,\\
&= \tfrac{1}{2}\pi^2 r^4.
\end{aligned}$$

13.6 One more integral

The last integral in this chapter is the following one:

$$I(\boldsymbol{x}) = \int_{-\infty}^{\infty}\mathrm{d}k_x\int_{-\infty}^{\infty}\mathrm{d}k_y\int_{-\infty}^{\infty}\mathrm{d}k_z\,\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{1+\boldsymbol{k}^2}, \qquad \boldsymbol{k}=(k_x,k_y,k_z).$$

First, let us re-write this in a more suggestive form:

$$I(\boldsymbol{x}) = \int \mathrm{d}^3k\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{1+k^2},$$

where the range is implicit and is equal to the whole of $\mathbb{R}^3$.

To do this integral, we go over to polar coordinates in $\boldsymbol{k}$:

$$\begin{aligned} k_z &= k\cos\theta, \\ k_y &= k\sin\theta\sin\varphi, \\ k_x &= k\sin\theta\cos\varphi, \qquad k=\sqrt{k_x^2+k_y^2+k_z^2}. \end{aligned}$$

As usual,

$$\mathrm{d}^3k = k^2\sin\theta\,\mathrm{d}k\mathrm{d}\theta\mathrm{d}\varphi.$$

Hence,

$$I=\int_0^\infty k^2\mathrm{d}k\int_0^\pi\sin\theta\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\,\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{1+k^2}.$$

We choose a coordinate system in $\boldsymbol{x}$-space such that $\boldsymbol{x}$ aligns with the k_z-axis. Then,

$$\boldsymbol{k}\cdot\boldsymbol{x}=k|\boldsymbol{x}|\cos\theta,$$

and

$$\begin{aligned} I(\boldsymbol{x}) &= \int_0^\infty k^2\mathrm{d}k\int_0^\pi\sin\theta\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\frac{\mathrm{e}^{\mathrm{i}k|\boldsymbol{x}|\cos\theta}}{1+k^2}, \\ &= 2\pi\int_0^\infty\frac{k^2}{1+k^2}\mathrm{d}k\int_0^\pi\sin\theta\mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}k|\boldsymbol{x}|\cos\theta} \end{aligned}$$

Now we use a neat trick:

$$\sin\theta\mathrm{e}^{\mathrm{i}kx\cos\theta}=-\frac{1}{\mathrm{i}kx}\frac{d}{d\theta}\mathrm{e}^{\mathrm{i}kx\cos\theta}.$$

Hence,

$$\begin{aligned} I(\boldsymbol{x}) &= 2\pi\int_0^\infty\frac{k^2}{1+k^2}\mathrm{d}k\int_0^\pi\sin\theta\mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}k|\boldsymbol{x}|\cos\theta}, \\ &= 2\pi\int_0^\infty\mathrm{d}k\frac{k^2}{1+k^2}\frac{\mathrm{i}}{kx}\int_0^\pi\mathrm{d}\theta\frac{d}{d\theta}\mathrm{e}^{\mathrm{i}kx\cos\theta}, \\ &= 2\pi\int_0^\infty\mathrm{d}k\,\frac{k^2}{1+k^2}\frac{\mathrm{i}}{kx}\left[\mathrm{e}^{-\mathrm{i}kx}-\mathrm{e}^{\mathrm{i}kx}\right], \\ &= \frac{4\pi}{x}\int_0^\infty\mathrm{d}k\frac{k\sin(kx)}{1+k^2}, \\ &= \frac{2\pi}{x}\int_{-\infty}^\infty\mathrm{d}k\frac{k\sin(kx)}{1+k^2}. \end{aligned}$$

In another course, you will hopefully be exposed to complex-variable theory, which determines this integral through Cauchy's residue theorem: $\int_{-\infty}^\infty\mathrm{d}k\cdots=\pi\mathrm{e}^{-x}$, hence

$$I(\boldsymbol{x})=\frac{2\pi}{x}\left(2\pi\frac{\mathrm{e}^{-x}}{2}\right)=2\pi^2\frac{\mathrm{e}^{-x}}{x},$$

and the final answer is a function of the *scalar* $x = |\boldsymbol{x}|$.

This completes the chapter about special integrals.

Chapter 14

The calculus of variations I

14.1 Overview

Recall the technique of extremization in ordinary calculus. For a **real-valued function**

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R}, \\ x &\rightarrow f(x), \end{aligned}$$

the extreme points are given by

$$f'(x) = 0,$$

and the minima satisfy

$$f'(x) = 0, \qquad f''(x) > 0.$$

In this chapter we extremize **functionals**. A functional is a map from a set of functions to the real line. First, consider

$$\Omega = \{f | f \text{ is a differentiable real-valued function}\}.$$

Then a functional S is a map

$$\begin{aligned} S : \Omega &\rightarrow \mathbb{R}, \\ f &\rightarrow S[f]. \end{aligned}$$

Extremising such maps is a tricky business, although we tackle it now.

14.2 Functionals involving functions of a single real variable

In this section we consider the set

$$\Omega = \{f | f \text{ is a differentiable real-valued function}\},$$

and examine functionals of the form

$$S[f] = \int_{x_1}^{x_2} \ell\left(f(x), f'(x), x\right) \mathrm{d}x.$$

We wish to find a function $f_0(x) \in \Omega$ that extremizes S. In this section **we assume that such a function exists**. Let

$$S[f_0] = \min_{f\in\Omega} S[f] \text{ or } \max_{f\in\Omega} S[f],$$

since we do not specify whether $f_0(x)$ is a minimum or a maximum. We introduce the deformation

$$f\left(x, \alpha\right) = f_0(x) + \alpha\eta(x),$$

where $\eta(x)$ **is a differentiable function that vanishes at** $x = x_1$ **and** $x = x_2$ **but is otherwise arbitrary**. Now, we introduce a function of the α-variable:

$$S(\alpha) = \int_{x_1}^{x_2} \ell\left(f\left(x, \alpha\right), \partial_x f\left(x, \alpha\right), x\right) \mathrm{d}x.$$

If $f_0(x)$ extremizes the functional $S[f]$, then the difference between $S[f_0]$ and neighbouring functions (slightly deformed functions) is very small. Thus, we have a condition for f_0 to be an extreme value:

$$\left.\frac{dS(\alpha)}{d\alpha}\right|_{\alpha=0} = 0.$$

Now we compute $dS(\alpha)/d\alpha$:

$$\begin{aligned}
\frac{dS(\alpha)}{d\alpha} &= \frac{d}{d\alpha}\int_{x_1}^{x_2} \ell\left(f\left(x, \alpha\right), \partial_x f\left(x, \alpha\right), x\right) \mathrm{d}x, \\
&= \int_{x_1}^{x_2} \frac{\partial}{\partial\alpha}\ell\left(f\left(x, \alpha\right), \partial_x f\left(x, \alpha\right), x\right) \mathrm{d}x \\
&= \int_{x_1}^{x_2} \frac{\partial}{\partial\alpha}\ell\left(f_0\left(x\right) + \alpha\eta(x), \partial_x f_0\left(x\right) + \alpha\partial_x\eta(x), x\right) \mathrm{d}x, \\
&= \int_{x_1}^{x_2} \left[\frac{\partial\ell}{\partial f}\frac{\partial}{\partial\alpha}\left[f_0\left(x\right) + \alpha\eta(x)\right] + \frac{\partial\ell}{\partial\left(\partial_x f\right)}\frac{\partial}{\partial\alpha}\left[\partial_x f_0\left(x\right) + \alpha\partial_x\eta(x)\right]\right] \mathrm{d}x, \\
&= \int_{x_1}^{x_2} \left[\frac{\partial\ell}{\partial f}\eta(x) + \frac{\partial\ell}{\partial\left(\partial_x f\right)}\frac{d\eta}{dx}\right] \mathrm{d}x.
\end{aligned}$$

Do some integration by parts:

$$\begin{aligned}\frac{dS(\alpha)}{d\alpha} &= \int_{x_1}^{x_2}\left[\frac{\partial\ell}{\partial f}\eta(x)+\frac{\partial\ell}{\partial\left(\partial_x f\right)}\frac{d\eta}{dx}\right]\mathrm{d}x,\\ &= \int_{x_1}^{x_2}\frac{\partial\ell}{\partial f}\eta(x)\mathrm{d}x+\int_{x_1}^{x_2}\left[\frac{d}{dx}\left(\frac{\partial\ell}{\partial\left(\partial_x f\right)}\eta(x)\right)-\left(\frac{d}{dx}\frac{\partial\ell}{\partial\left(\partial_x f\right)}\right)\eta(x)\right]\mathrm{d}x,\\ &= \int_{x_1}^{x_2}\left[\frac{\partial\ell}{\partial f}\eta(x)-\left(\frac{d}{dx}\frac{\partial\ell}{\partial\left(\partial_x f\right)}\right)\eta(x)\right]\mathrm{d}x+\left(\frac{\partial\ell}{\partial\left(\partial_x f\right)}\eta(x)\right)\Bigg|_{x_1}^{x_2}.\end{aligned}$$

But by construction, $\eta(x_1)=\eta(x_2)=0$, hence

$$\frac{dS(\alpha)}{d\alpha}=\int_{x_1}^{x_2}\left[\frac{\partial\ell}{\partial f}-\left(\frac{d}{dx}\frac{\partial\ell}{\partial\left(\partial_x f\right)}\right)\right]\eta(x)\mathrm{d}x.$$

Now let's evaluate at $\alpha=0$, where $dS(\alpha)/d\alpha=0$. This means that the function-evaluation

$$\ell(f_0+\alpha\eta(x),\partial_x f_0+\alpha\partial_x\eta(x),x)$$

in the last string of equations is converted into the function-evaluation

$$\ell(f_0,\partial_x f_0,x).$$

Hence,

$$0=\frac{dS(\alpha)}{d\alpha}\bigg|_{\alpha=0}=\int_{x_1}^{x_2}\left[\frac{\partial\ell}{\partial f}-\left(\frac{d}{dx}\frac{\partial\ell}{\partial\left(\partial_x f\right)}\right)\right]_{f_0}\eta(x)\mathrm{d}x$$

Now recall that the function $\eta(x)$ is arbitrary (except at the endpoints, and except for the differentiability criterion). In particular, we may choose it such that it always has the same sign as the square brackets $[\cdots]$. Thus, we have the integral of a non-negative quantity over a finite interval being zero: the only way for such a relation to be satisfied is for the quantity itself to be everywhere zero, or

$$\left[\frac{\partial\ell}{\partial f}-\left(\frac{d}{dx}\frac{\partial\ell}{\partial\left(\partial_x f\right)}\right)\right]_{f_0}=0.$$

This is the celebrated **Euler–Lagrange equation** (EL). Note that $\partial\ell/\partial f$ DOES NOT MEAN 'the derivative of the function ℓ w.r.t. the function f; instead it means '**the derivative of the function ℓ w.r.t. its first slot**'; similarly $\partial\ell/\partial(\partial_x f)$ simply means '**the derivative of the function ℓ w.r.t. its second slot**'.

In future, we shall write $y(x) \equiv f(x)$, and write the EL equation as

$$\frac{d}{dx}\frac{\partial \ell}{\partial y_x} - \frac{\partial \ell}{\partial y} = 0,$$

the solution of which is $y(x)$, the extremized trajectory of the functional $S[y]$. Again, $\ell = \ell(y(x), y_x(x), x)$, and $\partial \ell / \partial y_x$ means 'the derivative of the function ℓ w.r.t. its second slot, subsequently evaluated at $y_x(x) \equiv y'(x)$.

Example:

Theorem 14.1 *The shortest distance between two points in a plane is a line.*

Proof: Form the line element

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2.$$

Along curves $y = y(x)$, this is

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \left(\frac{dy}{dx}\right)^2 \mathrm{d}x^2,$$

hence

$$\mathrm{d}s = \sqrt{1 + y_x^2}\, \mathrm{d}x.$$

We wish to minimize the functional

$$S[y] = \int_{x_1}^{x_2} \mathrm{d}s = \int_{x_1}^{x_2} \sqrt{1 + y_x^2} \mathrm{d}x.$$

Here

$$\ell(y, y_x, x) = \sqrt{1 + y_x^2},$$

and

$$\partial_y \ell = 0, \qquad \partial_{y_x} \ell = \frac{y_x}{\sqrt{1 + y_x^2}}, \qquad \partial_x \ell = 0.$$

The EL equation

$$\frac{d}{dx}\frac{\partial \ell}{\partial y_x} - \frac{\partial \ell}{\partial y} = 0,$$

reduces to

$$\frac{d}{dx}\frac{y_x}{\sqrt{1 + y_x^2}} = 0,$$

or

$$\frac{y_x}{\sqrt{1 + y_x^2}} = \mathrm{Const.} := k.$$

Tidy up:

$$y_x^2 = k^2\left(1 + y_x^2\right),$$

or

$$y_x^2(1-k^2) = k^2 \implies y_x = \sqrt{k^2/(1-k^2)} := m.$$

Thus, we solve

$$y_x(x) = m,$$

or

$$y(x) = mx + c,$$

which is the equation of a straight line. The constants m and c can be determined with reference to the fixed endpoints (x_1, y_1) and (x_2, y_2).

Example:

> Fermat's principle of least time states that the path taken by a beam of light is such that the time of travel is minimum.

Here we show that Fermat's principle implies Snell's law of refraction. For a beam of light in a plane,

$$\mathrm{d}t = \frac{\mathrm{d}s}{c(x,y)} = \frac{n(x,y)}{c_0}\mathrm{d}s,$$

where $n(x,y)$ is the index of refraction and c_0 is the speed of light in a vacuum. Hence, over a path $(x, y(x))$, we have

$$\mathrm{d}t = \frac{n(x,y(x))}{c_0}\sqrt{1+y_x^2(x)}\mathrm{d}x,$$

and we seek to minimize the functional

$$S = \int_{x_1}^{x_2} \mathrm{d}t = \int_{x_1}^{x_2} \frac{n(x,y(x))}{c_0}\sqrt{1+y_x^2(x)}\mathrm{d}x.$$

Setting $c_0 = 1$, we have

$$\ell(y, y_x, x) = n(x,y)\sqrt{1+y_x^2},$$

and

$$\partial_y \ell = n_y(x,y)\sqrt{1+y_x^2}, \qquad \partial_{y_x}\ell = \frac{n(x,y)y_x}{\sqrt{1+y_x^2}}, \qquad \partial_x \ell = n_x(x)\sqrt{1+y_x^2}$$

The EL equation

$$\frac{d}{dx}\frac{\partial \ell}{\partial y_x} - \frac{\partial \ell}{\partial y} = 0,$$

reduces to

$$\frac{d}{dx}\frac{n(x,y(x))y_x(x)}{\sqrt{1+y_x(x)^2}} = n_y(x,y(x))\sqrt{1+y_x(x)^2}.$$

This is the final result and does not simplify any further without specification of $n(x,y)$. Note that d/dx is a TOTAL DERIVATIVE:

$$\frac{d}{dx}\ell(y(x),y_x(x),x) = \frac{\partial \ell}{\partial y}y_x + \frac{\partial \ell}{\partial y_x}y_{xx} + \frac{\partial \ell}{\partial x},$$

hence

$$\frac{d}{dx}\frac{n(x,y)y_x}{\sqrt{1+y_x^2}} = \left[n_x(x,y)+n_y(x,y)y_x\right]\frac{y_x}{\sqrt{1+y_x^2}} + n(x,y)\frac{d}{dx}\left(\frac{y_x}{\sqrt{1+y_x^2}}\right).$$

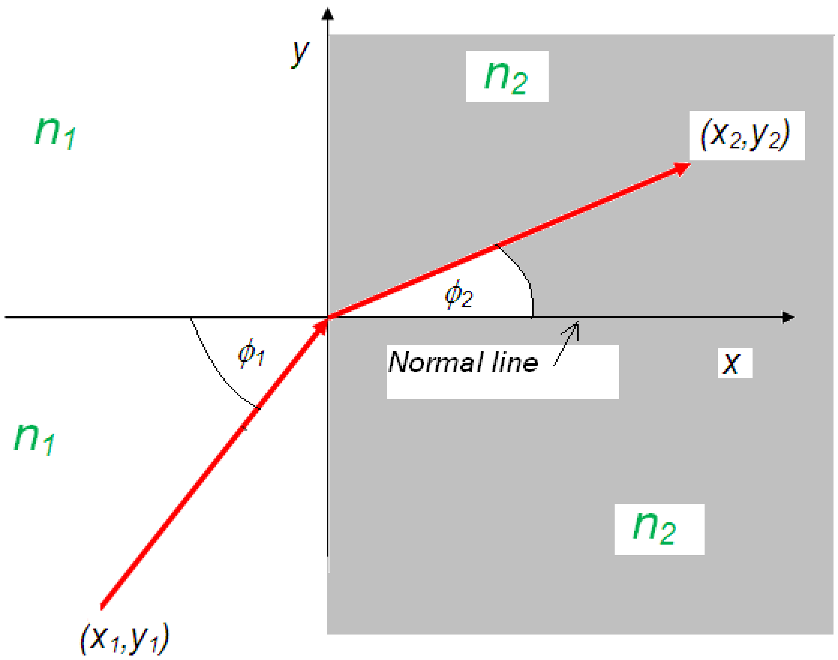

Figure 14.1: Snell's law of refraction

Suppose now we take

$$n(x,y) = \begin{cases} n_m, & x<0 \\ n_p, & x>0. \end{cases}$$

(See Fig. 14.1). Unfortunately, now $n(x,y)$ is discontinuous. However, it is still piecewise differentiable, on the half-planes $x<0$ and $x>0$. Let us take separate variations in these two spaces:

$$\begin{aligned}
\frac{dS_m}{d\alpha} &= \frac{d}{d\alpha}\int_{(x_1<0,y_1)}^{(0,0)} n_m\sqrt{1+y_x^2}\big|_{y(x,\alpha)}\mathrm{d}x,\\
&= \int_{(x_1<0,y_1)}^{(0,0)} n_m\left(\frac{y_x}{\sqrt{1+y_x^2}}\right)_{y(x,\alpha)}\eta_x(x)\mathrm{d}x,\\
&= n_m\left(\frac{y_x}{\sqrt{1+y_x^2}}\eta(x)\right)_{(x_1,y_1)}^{(0,0)} - \int_{(x_1<0,y_1)}^{(0,0)} n_m\frac{d}{dx}\left(\frac{y_x}{\sqrt{1+y_x^2}}\right)_{y(x,\alpha)}\eta(x)\mathrm{d}x,\\
\frac{dS_m}{d\alpha}\bigg|_{\alpha=0} &= n_m\left(\frac{y_x(0_-)}{\sqrt{1+y_x(0_-)^2}}\eta(0_-)\right) - \int_{(x_1<0,y_1)}^{(0,0)} n_m\frac{d}{dx}\left(\frac{y_x}{\sqrt{1+y_x^2}}\right)_{y(x)}\eta(x)\mathrm{d}x.
\end{aligned}$$

Here, we have used the notation

$$\eta(0_-) = \lim_{\varepsilon\to 0,\varepsilon>0}\eta(-\varepsilon), \qquad \text{\&c.}$$

and have chosen a path that penetrates the interface $x = 0$ at $y = 0$. By continuity, the light ray must pass through this point as it enters into the upper half-plane. Thus, the second component of the variation is

$$\frac{dS_p}{d\alpha} = -n_p\left(\frac{y_x(0_+)}{\sqrt{1+y_x(0_+)^2}}\eta(0_+)\right) - \int_{(0,0)}^{(x_2>0,y_2)} n_p\frac{d}{dx}\left(\frac{y_x}{\sqrt{1+y_x^2}}\right)_{y(x)}\eta(x)\mathrm{d}x.$$

Putting these two components together, the stationarity condition

$$\begin{aligned}
0 = \eta\,(0)&\left[\frac{n_m y_x(0_-)}{\sqrt{1+y_x(0_-)^2}} - \frac{n_p y_x(0_+)}{\sqrt{1+y_x(0_+)^2}}\right]\\
&- \int_{(x_1<0,y_1)}^{(0,0)} n_p\frac{d}{dx}\left(\frac{y_x}{\sqrt{1+y_x^2}}\right)_{y(x)}\eta(x)\mathrm{d}x - \int_{(0,0)}^{(x_2>0,y_2)} n_p\frac{d}{dx}\left(\frac{y_x}{\sqrt{1+y_x^2}}\right)_{y(x)}\eta(x)\mathrm{d}x,
\end{aligned}$$

The two integrals are identically zero if $y(x)$ is piecewise linear:

$$y_{p,m} = M_{p,m}x$$

(Moreover, this solution satisfies the interfacial condition at $y = 0$). In order for the boundary term to vanish, we need

$$\frac{n_m M_m}{\sqrt{1+M_m^2}} = \frac{n_p M_p}{\sqrt{1+M_p^2}}, \qquad (*)$$

Note that the slope of the line $L_m : y_m(x) = M_m x$ is $\tan\varphi_m = M_m/1$. Hence,

$$\sin\varphi_m = \frac{M_m}{\sqrt{1+M_m^2}}$$

Similarly, the slope of the line $L_p : y_p(x) = M_p x$ is $\tan\varphi_p = M_p$, and

$$\sin\varphi_p = \frac{M_p}{\sqrt{1+M_p^2}}$$

Substituting these angles in to Eq. (*),

$$n_p \sin\varphi_p = n_m \sin\varphi_m.$$

Re-arranging gives

$$\frac{\sin\varphi_m}{\sin\varphi_p} = \frac{n_p}{n_m},$$

which is precisely Snell's law.

14.3 Surfaces of minimal area

Before considering the problem of finding surfaces of minimal area, we prove the following theorem:

Theorem 14.2 *Given a function $\ell = \ell(y, y_x)$, $\partial_x \ell = 0$, where $y(x)$ satisfies Euler's equation,*

$$\frac{d}{dx}\frac{\partial\ell}{\partial y_x} = \frac{\partial\ell}{\partial y},$$

then

$$y - y_x\frac{\partial\ell}{\partial y_x} = \text{Const.}$$

Proof: First, consider in general (i.e. $\partial_x\ell$ not necessarily zero)

$$D := \frac{\partial\ell}{\partial x} - \frac{d}{dx}\left(\ell - y_x\frac{\partial\ell}{\partial y_x}\right).$$

We operate on the second term with the total derivative:

$$D = \frac{\partial\ell}{\partial x} - \left(\frac{\partial\ell}{\partial y}y_x + \frac{\partial\ell}{\partial y_x}y_x x + \frac{\partial\ell}{\partial x}\right) + \left(y_{xx}\frac{\partial\ell}{\partial y_x} + y_x\frac{d}{dx}\frac{\partial\ell}{\partial y_x}\right).$$

Effecting cancellations gives

$$D = y_x\left(\frac{\partial\ell}{\partial y} - \frac{d}{dx}\frac{\partial\ell}{\partial y_x}\right),$$

which is zero, by EL. Hence,

$$\text{EL holds iff } \frac{\partial \ell}{\partial x} - \frac{d}{dx}\left(\ell - y_x \frac{\partial \ell}{\partial y_x}\right) = 0.$$

Therefore, in the special case where $\partial_x \ell = 0$, we have

$$0 = \frac{d}{dx}\left(\ell - y_x \frac{\partial \ell}{\partial y_x}\right),$$

or

$$\ell - y_x \frac{\partial \ell}{\partial y_x} = \text{Const.} \tag{14.1}$$

as required.

Now we move onto the real subject of this section: consider two parallel coaxial wire circles to be connected by a surface of minimum area that is generated by revolving a curve $y(x)$ around the x-axis (Fig. 14.2). The curve is required to pass through fixed end points (x_1, y_1) and (x_2, y_2). The

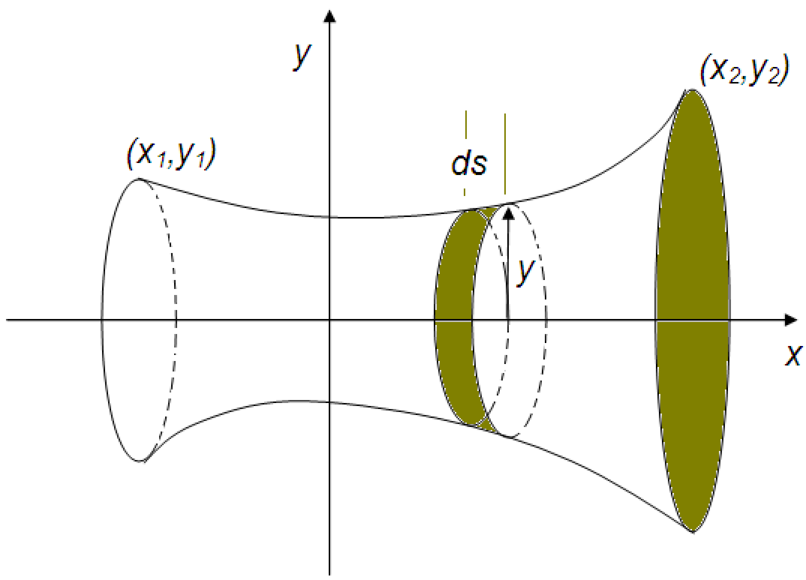

Figure 14.2: Surface of revolution: It is desired to find the surface of minimum area.

variational problem is to choose the curve $y(x)$ so that the area of the resulting surface will be a **minimum**.

From the figure, the area element is

$$\mathrm{d}A = 2\pi y \mathrm{d}s = 2\pi y\sqrt{1+y_x^2}\mathrm{d}x.$$

The functional to minimize is therefore

$$S[y] = \int_{x_1}^{x_2} 2\pi y \sqrt{1+y_x^2}\mathrm{d}x.$$

Neglecting the 2π, we obtain

$$\ell(y, y_x, x) = y\left(1+y_x^2\right)^{1/2}.$$

We have $\partial_x \ell = 0$, so the simplified version of EL (Eq. (14.1)) gives

$$y\sqrt{1+y_x^2} - yy_x^2 \frac{1}{\sqrt{1+y_x^2}} = \mathrm{Const.} = c_1.$$

Tidying up gives

$$\frac{y}{\sqrt{1+y_x^2}} = c_1.$$

Squaring gives

$$\frac{y^2}{1+y_x^2} = c_1^2.$$

Solve for y_x:

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{c_1^2} - 1}.$$

Separate variables:

$$\mathrm{d}x = \frac{\mathrm{d}y}{\sqrt{\frac{y^2}{c_1^2} - 1}}.$$

Integrating gives

$$x = c_1 \cosh^{-1} \frac{y}{c_1} + c_2.$$

Inverting gives

$$\boxed{y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right).}$$

This is the final answer. However, the answer requires further study, and this investigation highlights some of the pitfalls of variational calculus.

14.3.1 The minimum area

Consider again the solution

$$y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right).$$

to the extremal problem. The constants of integration c_1 and c_2 are fixed with reference to the end points of the wire (x_1, y_1) and (x_2, y_2). For simplicity, we take

$$(x_1, y_1) = (-x_0, 1)\,, \qquad (x_2, y_2) = (x_0, 1)\,.$$

The wire frame is symmetric about $x = 0$, so the surface of minimal area ought to have this symmetry too: $c_2 = 0$. Hence,

$$y = c_1 \cosh\left(\frac{x}{c_1}\right),$$

and

$$y = 1 \text{ at } x = x_0 \implies 1 = c_1 \cosh\left(\frac{x_0}{c_1}\right). \qquad (**)$$

We substitute this relation into the area integral:

$$\begin{aligned}
A &= 2\pi \int_{-x_0}^{x_0} y(x)\sqrt{1 + y_x(x)^2}\mathrm{d}x, \\
&= 2\pi c_1 \int_{-x_0}^{x_0} \cosh\left(\frac{x}{c_1}\right)\sqrt{1 + \sinh^2\left(\frac{x}{c_1}\right)}\mathrm{d}x, \\
&= 2\pi c_1 \int_{-x_0}^{x_0} \cosh\left(\frac{x}{c_1}\right)\cosh\left(\frac{x}{c_1}\right)\mathrm{d}x, \\
&= 4\pi c_1 \int_{0}^{x_0} \cosh^2\left(\frac{x}{c_1}\right)\mathrm{d}x, \\
&= \pi c_1^2 \left[\sinh\left(\frac{2x_0}{c_1}\right) + \frac{2x_0}{c_1}\right].
\end{aligned}$$

Finally, we are left with an area equation

$$A = \pi c_1^2 \left[\sinh\left(\frac{2x_0}{c_1}\right) + \frac{2x_0}{c_1}\right],$$

where (see Eq. (**))

$$1 = c_1 \cosh(x_0/c_1)$$

We can solve this last equation to obtain $c_1 = c_1(x_0)$. Unfortunately, only a numerical solution exists. This is shown in Fig. 14.3. Below a critical value $x_{0c} = 0.662$ two solutions to this equation exist. We plug the two solutions into the area formula. We see that the upper branch $c_1 \geq 0.5$ produces the curve with smaller area. This corresponds to the minimum of the functional. As x_0 is increased (corresponding to increasing the gap between the two wire rings), the two solution branches move closer together until they collide and annihilate each other at $x_0 = x_{0c} \approx 0.662$. Thereafter, no solution exists. At this critical value, the area of the curve equals

$$A(x_{0c}) = 2\pi.$$

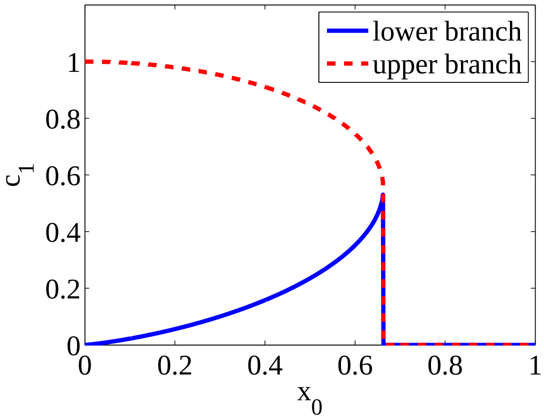

Figure 14.3: The solution of the equation $1 = c_1 \cosh(x_0/c_1)$ for various values of x_0. Below a critical value $x_0 = 0.662$ two solutions exits, called the *upper branch* and the *lower branch*. Above this value, no solution exists.

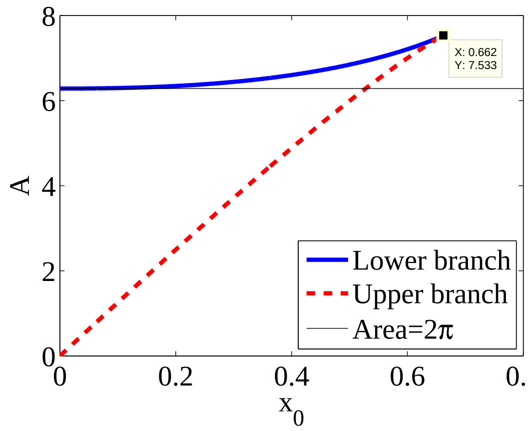

Figure 14.4: Area of surface of revolution associated with the two solutions of $1 = c_1 \cosh(x_0/c_1)$.

Physically, you can think of this situation $x_0 \to x_{0c}$ as corresponding to a soap film. The film forms the surface of revolution so as to minimize its area and hence its energy. As the gap between the two wire rings is increased, the soap film is stretched. At the critical value, the film ruptures. However, the soap film does not go away: instead it forms two disc-like surfaces around the two wire rings of unit radius, to give a total area 2π. This area is less than the two surfaces obtained by the surface of revolution and is therefore the preferred state.

This exercise contains an important lesson: A solution that satisfies the EL equations does not necessarily minimize the functional. Careful study of the different solutions is required to establish minimality. In other words, the EL equations are a necessary condition for minimality, but they are not sufficient. Two typical solutions from the two branches ('catenary curves') for $x_0 = 0.5$ are shown in Fig. 14.5.

14.3.2 Mechanics

In classical mechanics, Newton's equations can be derived from the condition that the functional

$$S[x(t)] = \int_{t_1}^{t_2} [K(x_t(t) - \mathcal{U}(x)] \,\mathrm{d}t$$

be stationary. In this context, the function S to be extremized is called the **action**. Here

$$K = \tfrac{1}{2} m x_t^2 = \tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2$$

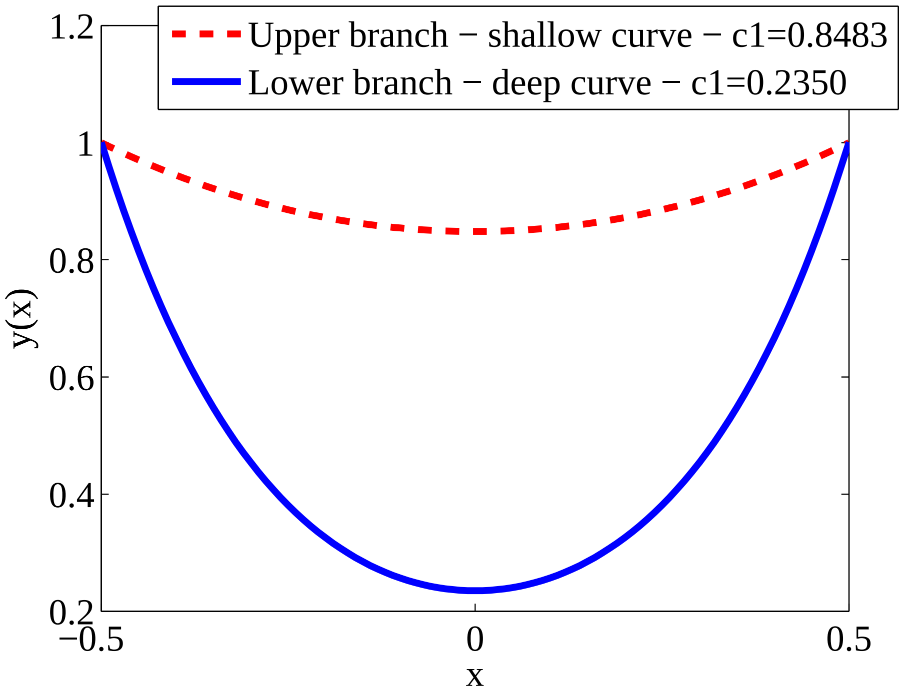

Figure 14.5: Solutions of the EL equation for the soap-film problem are 'catenary curves'. Shwon are the two solutions at $x_0 = 0.5$.

is the so-called kinetic energy and $\mathcal{U}(x)$ is the potential energy. Thus, we have $\ell = K - \mathcal{U}$, and the EL equation

$$\frac{d}{dt}\frac{\partial \ell}{\partial x_t} = \frac{\partial \ell}{\partial x}$$

becomes

$$m\frac{d}{dt}x_t = -\frac{\partial}{\partial x}\mathcal{U}(x),$$

or

$$m\frac{d^2x}{dt^2} = -\frac{\partial \mathcal{U}}{\partial x} = F(x),$$

which is Newton's law.

It should be straightforward to see that this generalizes to n particles interacting via a potential-energy function $\mathcal{U}(x_1, \cdots x_n)$: the action

$$S[x_1, \cdots x_n] = \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} \tfrac{1}{2} m_i \left(\frac{dx_i}{dt}\right)^2 - \mathcal{U}(x_1, \cdots x_n) \right] \mathrm{d}t$$

is stationary iff

$$m_i \frac{d^2 x_i}{dt^2} = -\frac{\partial \mathcal{U}}{\partial x_i}\left(x_1, \cdots x_n\right), \qquad i \in \{1, \cdots n\}.$$

It might seem quixotic to introduce this new formalism simply to recover Newton's laws. However, the action principle is independent of our choice of coordinates. So we may express it in terms of suitable curvilinear coordinates:

$$\delta \int_{t_1}^{t_2} \left[K\left(q_1, \cdots, q_n, (q_1)_t, \cdots (q_n)_t\right) - \mathcal{U}(q_1 \cdots q_n)\right] \mathrm{d}t \Longleftrightarrow \frac{d}{dt}\frac{\partial K}{\partial (q_i)_t} = \frac{\partial}{\partial q_i}\left[K - \mathcal{U}(q_1 \cdots q_n)\right].$$

A simple example should suffice: Consider a single particle experiencing a central potetnial $\mathcal{U} = \mathcal{U}(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. In spherical polar coordinates, the line element is

$$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2,$$

hence

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2.$$

In more compact notation,

$$\left(\frac{ds}{dt}\right)^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2.$$

But

$$K = \tfrac{1}{2}\left(\frac{ds}{dt}\right)^2 = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right).$$

The action is thus

$$S = \int_{t_1}^{t_2}\left[\tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) - \mathcal{U}(r)\right]\mathrm{d}t.$$

The EL equations are

$$\begin{aligned}
\frac{d}{dt}(m\dot{r}) &= \frac{\partial}{\partial r}\left[\tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}mr^2\sin^2\theta\dot{\varphi}^2 - \mathcal{U}(r)\right], \\
\frac{d}{dt}\left(mr^2\dot{\theta}\right) &= mr^2\dot{\varphi}^2\cos\theta\sin\theta, \\
\frac{d}{dt}\left(mr^2\sin^2\theta\dot{\varphi}\right) &= 0.
\end{aligned}$$

The last equation clearly gives

$$mr^2\sin^2\theta\dot{\varphi} = L = \text{Const.} \implies \dot{\varphi} = \frac{L}{mr^2\sin^2\theta}.$$

Plug this into the second equation

$$\begin{aligned}
\frac{d}{dt}\left(mr^2\dot{\theta}\right) &= mr^2\dot{\varphi}^2\cos\theta\sin\theta, \\
&= mr^2\frac{L^2}{mr^2\sin^2\theta mr^2\sin^2\theta}\sin\theta\cos\theta, \\
&= \frac{L^2}{mr^2}\frac{\cos\theta}{sin^3\theta}, \\
&= -\frac{L^2}{2mr^2}\frac{\partial}{\partial\theta}\frac{1}{\sin^2\theta}.
\end{aligned}$$

Multiply both equations by $r^2\dot\theta$:

$$\begin{aligned}
mr^2\dot\theta\frac{d}{dt}(r^2\dot\theta) &= -\frac{L^2}{2m}\frac{d\theta}{dt}\frac{d}{d\theta}\frac{1}{\sin^2\theta},\\
\tfrac{1}{2}m\frac{d}{dt}\left(r^2\dot\theta\right)^2 &= -\frac{L^2}{2m}\frac{d}{dt}\frac{1}{\sin^2\theta},\\
\tfrac{1}{2}m(r^2\dot\theta)^2+\tfrac{1}{2}\frac{L^2}{m}\frac{1}{\sin^2\theta} &= \text{Const.} = J^2.
\end{aligned}$$

But $L = mr^2\sin^2\theta\dot\varphi$, hence

$$J^2 = \tfrac{1}{2}mr^4\dot\theta^2+\tfrac{1}{2}mr^4\sin^2\theta\dot\varphi^2,$$

and

$$\tfrac{1}{2}mr^2\dot\theta^2+\tfrac{1}{2}mr^2\sin^2\theta\dot\varphi^2 = \frac{J^2}{r^2}$$

Finally, note the first equation of the EL set (radial equation):

$$\begin{aligned}
\frac{d}{dt}(m\dot r) &= \frac{\partial}{\partial r}\left[\tfrac{1}{2}mr^2\dot\theta^2+\tfrac{1}{2}mr^2\sin^2\theta\dot\varphi^2-\mathcal{U}(r)\right],\\
&= \frac{\partial}{\partial r}\left[\frac{J^2}{r^2}-\mathcal{U}(r)\right].
\end{aligned}$$

Thus, three-dimensional central-force motion reduces to a quasi-one-dimensional equation:

$$m\ddot r = \frac{\partial}{\partial r}\left[\frac{J^2}{r^2}-\mathcal{U}(r)\right].$$

Chapter 15

The calculus of variations II: Constraints

15.1 Overview

In this section we find the extreme points of functionals subject to various constraints. We first of all recall the theory of constrained optimization for calculus.

15.2 Functions

Consider a function $f(x, y)$. We are to find the extreme points of this function **subject to the constraint** that

$$\psi(x, y) = 0.$$

We call the function to be extremized the **objective function**. You might recall that the correct way to do the extremization is to form a new function

$$f_\lambda(x, y) := f(x, y) - \lambda\psi(x, y).$$

We extremize this new ('auxiliary') function:

$$\nabla f_\lambda(x, y) = 0 \implies \begin{cases} \frac{\partial f}{\partial x} = \lambda\frac{\partial \psi}{\partial x}, \\ \frac{\partial f}{\partial y} = \lambda\frac{\partial \psi}{\partial y} \end{cases} .$$

Here λ is a constant, which can be obtained by solving the second of these equations:

$$\lambda = \frac{\partial f}{\partial y}\bigg/\frac{\partial \psi}{\partial y}.$$

Now, substitute this into the first equation:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}\left(\frac{\partial\psi/\partial x}{\partial\psi/\partial y}\right). \qquad (*)$$

We need to solve for an extreme point (x_0, y_0), and this requires two equations. We have precisely this number of equations: Eq. (*) and the constraint:

$$\begin{aligned} \psi(x,y) &= 0, \\ \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial y}\left(\frac{\partial\psi/\partial x}{\partial\psi/\partial y}\right), \end{aligned}$$

with solution(s) (x_0, y_0).

To see why this method works, consider a simple constraint of the form

$$\psi(x,y) = y - \psi_0(x) = 0. \tag{15.1}$$

The equation $\psi(x,y) = 0$ can always be **locally inverted** to yield $y =$ some function(x), however, a global inverse of the form (15.1) is rather special. Nevertheless, let's proceed with the analysis. Consider now the function $f(x,y)$ to be minimized, subject to the constraint (15.1). Without knowledge of constraint theory, the natural thing to do is to solve

$$0 = \frac{d}{dx}f(x, y = \psi_0(x)) = f_x(x, \psi_0(x)) + f_y(x, \psi_0(x))\frac{d\psi_0}{dx}.$$

In other words,

$$\begin{aligned} y &= \psi_0(x), \\ f_x(x,y) &= -f_y(x,y)\frac{d\psi_0}{dx}, \end{aligned}$$

Note, however, $\partial_y\psi = 1$ and $\partial_x\psi = -d\psi_0/dx$. Hence, we have solved nothing other than

$$\begin{aligned} 0 &= \psi(x,y), \\ f_x(x,y) &= f_y(x,y)\left(\frac{\partial\psi/\partial x}{\partial\psi/\partial y}\right), \end{aligned}$$

or $\nabla f_\lambda = 0$, with $f_\lambda(x,y) = f(x,y) - \lambda(y - \psi_0(x))$!!

The constant λ is called the **Lagrange multiplier** and this method of constrained variation. This example shows that the **method of Lagrange multipliers** is noting other than a simple mnemonic for inverting the constraint function and substituting the result into the objective function.

Example: Minimize the function

$$f(x,y,z) = s_1^2 x^2 + s_2^2 y^2 + s_3^2 z^2,$$

subject to the constraint that

$$r_1 x + r_2 y + r_3 z = \mu.$$

Here $(s_1, s_2, s_3, r_1, r_2, r_3)$ and μ are positive constants. Form the auxiliary function

$$f_\lambda(x,y,z) = \left(s_1^2 x^2 + s_2^2 y^2 + s_3^2 z^2\right) - \lambda\left(r_1 x + r_2 y + r_3 z - \mu\right).$$

and set $\nabla f_\lambda = 0$. We obtain,

$$\begin{aligned} 2s_1^2 x &= \lambda r_1, \\ 2s_2^2 y &= \lambda r_2, \\ 2s_3^2 z &= \lambda r_3. \end{aligned}$$

Focussing on the third equation gives

$$\lambda = \frac{2s_3^2 z}{r_3}.$$

Substitution into the other two equations gives

$$2s_1^2 x = 2s_3^2 z \frac{r_1}{r_3}, \qquad 2s_2^2 y = 2s_3^2 z \frac{r_2}{r_3},$$

Hence,

$$x = z\frac{s_3^2}{s_1^2}\frac{r_1}{r_3}, \qquad y = z\frac{s_3^2}{s_2^2}\frac{r_2}{r_3}.$$

But $r_1 x + r_2 y + r_3 z = \mu$. So we have a triple of linear equations:

$$\begin{aligned} x &= z\frac{s_3^2}{s_1^2}\frac{r_1}{r_3}, \\ y &= z\frac{s_3^2}{s_2^2}\frac{r_2}{r_3}, \\ r_1 x + r_2 y + r_3 z &= \mu. \end{aligned}$$

Substitution of the first two equations into the third yields

$$z\left(\frac{s_3^2}{s_1^2}\frac{r_1^2}{r_3} + \frac{s_3^2}{s_2^2}\frac{r_2^2}{r_3} + r_3\right) = \mu,$$

hence

$$
\begin{aligned}
z &= z_0 := \frac{\mu}{\left(\frac{s_3^2}{s_1^2}\frac{r_1^2}{r_3} + \frac{s_3^2}{s_2^2}\frac{r_2^2}{r_3} + r_3\right)}, \\
x &= \frac{s_3^2}{s_1^2}\frac{r_1}{r_3} z_0, \\
y &= z\frac{s_3^2}{s_2^2}\frac{r_2}{r_3} z_0.
\end{aligned}
$$

Finally, the minimum value of the objective function is

$$
\begin{aligned}
f_0 &= s_1^2 \frac{s_3^4}{s_1^4}\frac{r_1^2}{r_3^2} z_0^2 + s_2^2 \frac{s_3^4}{s_2^4}\frac{r_2^2}{r_3^2} z_0^2 + z_0^2, \\
&= \frac{s_3^4 z_0^2}{r_3^2}\left(\frac{r_1^2}{s_1^2} + \frac{r_2^2}{s_2^2} + \frac{r_3^2}{s_3^2}\right), \\
&= \frac{s_3^4}{r_3^2}\frac{\mu^2}{\left(\frac{s_3^2}{s_1^2}\frac{r_1^2}{r_3} + \frac{s_3^2}{s_2^2}\frac{r_2^2}{r_3} + r_3\right)^2}\left(\frac{r_1^2}{s_1^2} + \frac{r_2^2}{s_2^2} + \frac{r_3^2}{s_3^2}\right), \\
&= \frac{\mu^2}{\left(\frac{r_1^2}{s_1^2} + \frac{r_2^2}{s_2^2} + \frac{r_3^2}{s_3^2}\right)^2}\left(\frac{r_1^2}{s_1^2} + \frac{r_2^2}{s_2^2} + \frac{r_3^2}{s_3^2}\right), \\
&= \frac{\mu^2}{\left(\frac{r_1^2}{s_1^2} + \frac{r_2^2}{s_2^2} + \frac{r_3^2}{s_3^2}\right)}.
\end{aligned}
$$

Interpretation: (x, y, z) are weights in a portfolio of stocks labelled 1, 2, and 3. r_i is the return generated by the i^{th} stock, and

$$\mu = r_1 x + r_2 y + r_3 z$$

is the desired return on the portfolio. The quantity s_i is the standard deviation of the return on the i^{th} stock and represents the riskiness of investing in this stock. The quantity

$$f_0 = s_1^2 x + s_2^2 y + s_3^2 z$$

is the square of the standard deviation of the portfolio, and the minimum level of risk is

$$\text{MIN RISK} = \frac{\mu}{\left(\frac{r_1^2}{s_1^2} + \frac{r_2^2}{s_2^2} + \frac{r_3^2}{s_3^2}\right)^{1/2}}$$

which is realised when the fraction of the portfolio in each stock is given by the Lagrange-multiplier procedure just derived. If we want a return μ on an investment, a portfolio is less risky than investing in one stock ($\mu = r_1 x$, $y = z = 0$), since

$$\frac{\mu}{\left(\frac{r_1^2}{s_1^2}\right)^{1/2}} \geq \frac{\mu}{\left(\frac{r_1^2}{s_1^2} + \frac{r_2^2}{s_2^2} + \frac{r_3^2}{s_3^2}\right)^{1/2}}.$$

This is the mathematical statement that "you should not put all your eggs in the one basket".

You should note that the list of assumptions in this calculations is as long as your arm: failure to understand the limitations of these assumptions results in financial crises such as the 2007 subprime mortgage crisis (seriously!).

15.3 Functionals: Holonomic constraints

Now we pass over to functionals. Suppose we are to minimize the functional

$$S[f,g] = \int_{x_1}^{x_2} \ell(f, g, f_x, g_x, x)\mathrm{d}x,$$

subject to the constraint

$$\psi(f(x), g(x), x) = 0$$

We DO NOT consider constraints involving the derivatives of f and g. The pointwise constraint $\psi(f(x), g(x), x)$ is called a **holonomic** constraint. In reality there is an infinite number of constraints, one at each point x. Thus, any Lagrange multiplier in the constant must be labelled by the point x: $\lambda \to \lambda(x)$. We therefore minimize the auxiliary functional

$$S_\lambda[f,g] = \int_{x_1}^{x_2} \left[\ell(f, g, f_x, g_x, x) - \lambda(x)\psi(f, g, x)\right] \mathrm{d}x.$$

To do this, we introduce the deformed trajectories

$$f_\alpha = f_0(x) + \alpha\eta(x), \qquad g_\alpha = g_0(x) + \beta\zeta(x),$$

where (f_0, g_0) is the solution (assumed to exist) and η and ζ are differentiable functions that vanish at the end points x_1 and x_2. We solve for

$$\nabla_{\alpha,\beta} S(\alpha,\beta) = 0, \qquad S(\alpha,\beta) = \int_{x_1}^{x_2} \left[\ell(f_\alpha, g_\alpha, f_{\alpha,x}, g_{\alpha,x}, x) - \lambda(x)\psi(f_\alpha, g_\alpha, x)\right].$$

For example, let's do the α-variation:

$$\begin{aligned}
\frac{\partial S}{\partial \alpha} &= \int_{x_1}^{x_2} \left[\frac{\partial \ell}{\partial f_\alpha} \frac{\partial f_\alpha}{\partial \alpha} + \frac{\partial \ell}{\partial (f_{\alpha,x})} \frac{\partial f_{\alpha,x}}{\partial \alpha} - \lambda(x) \frac{\partial \psi}{\partial f_\alpha} \frac{\partial f_\alpha}{\partial x} \right] \mathrm{d}x, \\
&= \int_{x_1}^{x_2} \left[\frac{\partial \ell}{\partial f_\alpha} \eta(x) + \frac{\partial \ell}{\partial (f_{\alpha,x})} \frac{d\eta}{dx} - \lambda(x) \frac{\partial \psi}{\partial f_\alpha} \eta(x) \right] \mathrm{d}x, \\
&= \int_{x_1}^{x_2} \left[\frac{\partial \ell}{\partial f_\alpha} \eta(x) - \left(\frac{d}{dx} \frac{\partial \ell}{\partial (f_{\alpha,x})} \right) \eta(x) - \lambda(x) \frac{\partial \psi}{\partial f_\alpha} \eta(x) \right] \mathrm{d}x + \left(\frac{\partial \ell}{\partial (f_{\alpha,x})} \eta(x) \right)_{x_1}^{x_2}, \\
&= \int_{x_1}^{x_2} \left[\frac{\partial \ell}{\partial f_\alpha} - \left(\frac{d}{dx} \frac{\partial \ell}{\partial (f_{\alpha,x})} \right) - \lambda(x) \frac{\partial \psi}{\partial f_\alpha} \right] \eta(x) \, \mathrm{d}x
\end{aligned}$$

Stationarity means that $[\cdots] = 0$ at $\alpha = 0$. In other words,

$$\left[\frac{d}{dx} \frac{\partial \ell}{\partial f_x} \right]_{f_0} = \left[\frac{\partial \ell}{\partial f} - \lambda(x) \frac{\partial \psi}{\partial f} \right]_{f_0}$$

Similarly,

$$\left[\frac{d}{dx} \frac{\partial \ell}{\partial g_x} \right]_{g_0} = \left[\frac{\partial \ell}{\partial g} - \lambda(x) \frac{\partial \psi}{\partial g} \right]_{g_0}$$

We now have three equations in the unknowns $(f_0(x), g_0(x), \lambda(x))$:

$$\begin{aligned}
\left[\frac{d}{dx} \frac{\partial \ell}{\partial f_x} \right]_{f_0, g_0} &= \left[\frac{\partial \ell}{\partial f} - \lambda(x) \frac{\partial \psi}{\partial f} \right]_{f_0, g_0}, \\
\left[\frac{d}{dx} \frac{\partial \ell}{\partial g_x} \right]_{f_0, g_0} &= \left[\frac{\partial \ell}{\partial g} - \lambda(x) \frac{\partial \psi}{\partial g} \right]_{f_0, g_0}, \\
\psi\left(f_0(x), g_0(x) \right) = 0.
\end{aligned}$$

These are the **constrained Euler–Lagrange equations**. Usually we will just write them as

$$\begin{aligned}
\frac{d}{dx} \frac{\partial \ell}{\partial f_x} &= \frac{\partial \ell}{\partial f} - \lambda(x) \frac{\partial \psi}{\partial f}, \\
\frac{d}{dx} \frac{\partial \ell}{\partial g_x} &= \frac{\partial \ell}{\partial g} - \lambda(x) \frac{\partial \psi}{\partial g}, \\
\psi\left(f(x), g(x) \right) &= 0.
\end{aligned}$$

Example: Consider a single particle in two dimensions experiencing the potential

$$\mathcal{U}(x, y) = mgy.$$

However, the coordinates (x, y) are constrained such that $x^2 + y^2 = R^2 = \text{Const.}$. In other words,

$$\psi(x, y) = x^2 + y^2 - R^2, \qquad \psi(x, y) = 0.$$

Find the equations of motion.

We have the constrained action

$$S = \int_{t_1}^{t_2} \left[\tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy - \lambda(t)\left(x^2 + y^2 - R^2\right)\right] \mathrm{d}t.$$

The first EL equation is

$$\frac{d}{dt}(m\dot{x}) = \frac{\partial}{\partial x}(-mgy) + 2\lambda x \implies m\ddot{x} = 2\lambda x.$$

The second one is

$$\frac{d}{dt}(m\dot{y}) = \frac{\partial}{\partial x}(-mgy) + 2\lambda y \implies m\ddot{y} = -mg + 2\lambda y.$$

From the first EL equation, $\lambda = m\ddot{x}/(2x)$. Substitute this into the second EL equation to obtain

$$m\ddot{y} = -mg + my\frac{\ddot{x}}{x},$$

or

$$\ddot{y} - \frac{y\ddot{x}}{x} = -g.$$

Because the constraint function gives

$$x^2 + y^2 = R^2.$$

it is natural to introduce the parametrization

$$x = R\cos\varphi, \qquad y = R\sin\varphi.$$

where

$$\tan\varphi = \frac{y}{x}.$$

Differentiate x and y:

$$\dot{y} = R\cos\varphi\dot{\varphi}, \qquad \ddot{y} = R\cos\varphi\ddot{\varphi} - R\sin\varphi\dot{\varphi}^2.$$

$$\dot{x} = -R\sin\varphi\dot{\varphi}, \qquad \ddot{x} = -R\sin\varphi\ddot{\varphi} - R\cos\varphi\dot{\varphi}^2.$$

Put them together:

$$\begin{aligned}\ddot{y} - \frac{y}{x}\ddot{x} &= R\cos\varphi\ddot{\varphi} - R\sin\varphi\dot{\varphi}^2 - \frac{\sin\varphi}{\cos\varphi}\left(-R\sin\varphi\ddot{\varphi} - R\cos\varphi\dot{\varphi}^2\right), \\ &= R\ddot{\varphi}\left(\cos\varphi + \frac{\sin^2\varphi}{\cos\varphi}\right) = 0, \\ &= R\ddot{\varphi}\frac{1}{\cos\varphi}.\end{aligned}$$

But the EOM is

$$\ddot{y} - \frac{y}{x}\ddot{x} = -g.$$

Hence,

$$R\ddot{\varphi} = -g\cos\varphi,$$

or

$$\ddot{\varphi} = -\frac{g}{R}\cos\varphi.$$

Introducing the angle

$$\theta := \varphi - \tfrac{3}{2}\pi,$$

this is

$$\ddot{\theta} = -\frac{g}{R}\sin\theta,$$

which is the equation of motion for a pendulum.

15.4 Global constraints

In the previous section we dealt with holonomic constraints, where the constraint was pointwise, and therefore really represented an infinite number of constraints, parametrized by a non-constant Lagrangian multiplier. Now we look at a global constraints.

Example: A wire cable hangs between two supports. The points of support are located at $(\pm x_0, 1)$. Find the curve that minimizes the gravitational energy of the chain.

The energy is given by

$$\mathrm{d}E = \rho \mathrm{d}s g y(x),$$

where ρ is the mass per unit length, $\mathrm{d}s$ is an element of length along the chain, g is gravity, and $y(x)$ is the height above zero of the chain. The total energy is thus

$$E = \rho g \int_{-x_0}^{x_0} \mathrm{d}s y(x) = \rho g \int_{-x_0}^{x_0} \sqrt{1 + y_x^2}\, y(x)\mathrm{d}x.$$

However, the total length of the chain is constant. This represents a constraint:

$$L = \int_{-x_0}^{x_0} \sqrt{1+y_x^2}\mathrm{d}x.$$

The functional to extremize is thus

$$S = \int_{-x_0}^{x_0} \left[\sqrt{1+y_x^2}y(x) - \lambda\sqrt{1+y_x^2}\right]\mathrm{d}x,$$

where we take λ to be a constant because there is only one, global constraint (previously the constraint was a pointwise one). The EL equation is

$$\frac{d}{dx}\left[\frac{\partial}{\partial y_x}\left(\sqrt{1+y_x^2}y(x) - \lambda\sqrt{1+y_x^2}\right)\right] = \frac{\partial}{\partial y}\left(\sqrt{1+y_x^2}y(x) - \lambda\sqrt{1+y_x^2}\right)$$

Or,

$$\frac{d}{dx}\left[\frac{y_x}{\sqrt{1+y_x^2}}(y-\lambda)\right] = \sqrt{1+y_x^2}.$$

Calling

$$\ell = \sqrt{1+y_x^2}y(x) - \lambda\sqrt{1+y_x^2}, \qquad \lambda = \text{Const.},$$

we have, from the EL equation,

$$\frac{\partial \ell}{\partial x} - \frac{d}{dx}\left(\ell - y_x\frac{\partial \ell}{\partial y_x}\right) = 0.$$

But $\partial_x \ell = 0$ because we have taken λ to be constant. Thus,

$$\ell - y_x\frac{\partial \ell}{\partial y_x} = \text{Const.} = c_1,$$

or

$$\sqrt{1+y_x^2}y(x) - \lambda\sqrt{1+y_x^2} - y_x\frac{y_x(y-\lambda)}{\sqrt{1+y_x^2}}$$

Re-arranging gives

$$\sqrt{1+y_x^2}\left[y(x)-\lambda\right] - \frac{y_x^2\left[y(x)-\lambda\right]}{\sqrt{1+y_x^2}} = c_1$$

Hence,

$$\begin{aligned} \left[y(x)-\lambda\right] &= c_1\sqrt{1+y_x^2}, \\ y_x^2 &= \frac{\left[y(x)-\lambda\right]^2}{c_1^2} - 1. \end{aligned}$$

Introduce the substitution

$$y(x) - \lambda = c_1 \cosh z.$$

Then

$$\frac{dy}{dx} = c_1 \sinh(z) \frac{dz}{dx}.$$

Hence,

$$c_1^2 \sinh^2(z) z_x^2 = \cosh^2(z) - 1 = \sinh^2(z),$$

$$z_x = \frac{1}{c_1} \implies z = \frac{x + c_2}{c_1}.$$

The final solution is thus

$$y(x) = \lambda + c_1 \cosh\left(\frac{x + c_2}{c_1}\right).$$

The constants λ, c_1, and c_2 can be obtained from the two initial conditions and the arc-length constraint.

15.5 Geodesics

A **geodesic** is the shortest path between two points on a curved surface. Recall, in ordinary (Euclidean) space, the shortest distance between two points is a line. In curved spaces (e.g. on the sphere), the shortest distance between two points is along a special curve, determined by an extremization procedure.

Consider a curve $\boldsymbol{x}(t) = (x(t), y(t), z(t))$ in space, subject to the constraint that

$$\psi(x, y, z) = 0.$$

The constraint forces the path to 'live' on a certain surface. This is a standard holonomic constraint. For example, if

$$\psi = x^2 + y^2 + z^2 - R^2,$$

then the constraint functional forces the curve $\boldsymbol{x}(t)$ on to the sphere. To minimize the distance between two points, we solve the extremization problem for the objective functional

$$S = \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \mathrm{d}s - \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \lambda(t) \psi\left(x(t), y(t), z(t)\right) \mathrm{d}t.$$

where $\boldsymbol{x}_1 = \boldsymbol{x}(t_1)$ and $\boldsymbol{x}_2 = \boldsymbol{x}(t_2)$ are the fixed end points. But

$$\mathrm{d}s = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \mathrm{d}t := \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \mathrm{d}t.$$

Thus, we extremize

$$S = \int_{t_1}^{t_2} \left[\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} - \lambda(t)\psi\left(x(t), y(t), z(t)\right) \right] \mathrm{d}t.$$

The EL equation in the x-variable is

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = -\lambda(t)\frac{\partial \psi}{\partial x}.$$

Thus, the four equations to solve are

$$\begin{aligned}
\frac{d}{dt}\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} &= -\lambda(t)\frac{\partial \psi}{\partial x}, \\
\frac{d}{dt}\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} &= -\lambda(t)\frac{\partial \psi}{\partial y}, \\
\frac{d}{dt}\frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} &= -\lambda(t)\frac{\partial \psi}{\partial z}, \\
\psi(x, y, z) &= 0.
\end{aligned}$$

Let's focus on the sphere again. The EL equations to solve are

$$\begin{aligned}
\frac{d}{dt}\frac{\dot{x}_i}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}} &= -2\lambda(t)x_i, \qquad i = 1, 2, 3, \\
x_1^2 + x_2^2 + x_3^2 &= R^2.
\end{aligned}$$

Calling $D := \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}$, we have

$$\frac{\frac{d}{dt}\frac{\dot{x}_1}{D}}{2x_1} = \frac{\frac{d}{dt}\frac{\dot{x}_2}{D}}{2x_2} = \frac{\frac{d}{dt}\frac{\dot{x}_3}{D}}{2x_3} = -\lambda.$$

Expand derivatives in the first two terms:

$$\frac{\ddot{x}_1 D - \dot{x}_1 \dot{D}}{2x_1 D^2} = \frac{\ddot{x}_2 D - \dot{x}_2 \dot{D}}{2x_2 D^2}$$

Re-arranging gives

$$\frac{x_2\ddot{x}_1 - \ddot{x}_2 x_1}{x_2\dot{x}_1 - \dot{x}_2 x_1} = \frac{\dot{D}}{D}.$$

Similarly,

$$\frac{x_3\ddot{x}_2 - \ddot{x}_3 x_2}{x_3\dot{x}_2 - \dot{x}_3 x_2} = \frac{\dot{D}}{D}.$$

Equate these expressions:

$$\frac{x_2\ddot{x}_1 - \ddot{x}_2 x_1}{x_2\dot{x}_1 - \dot{x}_2 x_1} = \frac{x_3\ddot{x}_2 - \ddot{x}_3 x_2}{x_3\dot{x}_2 - \dot{x}_3 x_2}.$$

Re-write this equation again:

$$\begin{aligned}
\frac{x_2\ddot{x}_1 - \ddot{x}_2 x_1}{x_2\dot{x}_1 - \dot{x}_2 x_1} &= = \frac{x_3\ddot{x}_2 - \ddot{x}_3 x_2}{x_3\dot{x}_2 - \dot{x}_3 x_2}, \\
\frac{\frac{d}{dt}\left(x_2\dot{x}_1 - \dot{x}_2 x_1\right)}{x_2\dot{x}_1 - \dot{x}_2 x_1} &= \frac{\frac{d}{dt}\left(x_3\dot{x}_2 - \dot{x}_3 x_2\right)}{x_3\dot{x}_2 - \dot{x}_3 x_2}, \\
\frac{d}{dt}\log\left(x_2\dot{x}_1 - \dot{x}_2 x_1\right) &= \frac{d}{dt}\log\left(x_3\dot{x}_2 - \dot{x}_3 x_2\right), \\
x_2\dot{x}_1 - \dot{x}_2 x_1 &= c_1\left(x_3\dot{x}_2 - \dot{x}_3 x_2\right).
\end{aligned}$$

Solve for x_2 alone:

$$\begin{aligned}
\frac{\dot{x}_1 + c_1\dot{x}_3}{x_1 + c_1 x_3} &= \frac{\dot{x}_2}{x_2}, \\
\frac{d}{dt}\log\left(x_1 + c_1 x_3\right) &= \frac{d}{dt}\log x_2, \\
x_1 + c_1 x_3 = c_2 x_2, &
\end{aligned}$$

and restoring the usual notation, this is

$$x + c_1 z = c_2 y.$$

This is **the equation of a plane** that passes through $(0,0,0)$. Thus, the shortest distance between two points on a sphere is a curve that is given by the intersection of the sphere with a plane passing through the origin, i.e. **a great circle**.

Chapter 16

Fin

Vector calculus was invented by mathematical physicists to formulate Electromagnetism.[1] It is thus the mathematical basis of Electromagnetism, and it also provides the mathematical key to understanding fluid mechanics, quantum mechanics, heat and mass transfer, and partial differential equations. When combined with geometry, such that differential laws can be formulated in non-flat spaces, one has the mathemtical tools at hand to study Relativity and Quantum Field Theory. It is thus indispensable in mathematical physics. I hope this module has succeeded in creating a foundation for you to study these topics in more detail in later years.

[1] *Vector analysis, a text-book for the use of students of mathematics and physics, founded upon the lectures of J. Willard Gibbs*, E. B. Wilson and J. W. Gibbs (1902)

Appendix A

Taylor's theorem in multivariate calculus

We consider here an expression for the first-order terms in Taylor's expansion in multivariate calculus. This result is a simple consequence of single-variable version of Taylor's theorem, together with the standard rules of partial derivatives. We shall show, for $f(x, y)$ sufficiently smooth,

$$f(x+\delta x, y+\delta y) = f(x,y) + f_x(x,y)\delta x + f_y(x,y)\delta y + O(\delta x^2, \delta y^2, \delta x \delta y),$$

where $f_x = \partial f/\partial x$, and $f_y = \partial f/\partial y$.

Proof: Call

$$F(x) := f(x, y+\delta y), \qquad \text{fixed } y.$$

From the single-variable version of Taylor's theorem,

$$F(x+\delta x) = F(x) + F'(x)\delta x + O(\delta x^2);$$

in other words,

$$f(x+\delta x, y+\delta y) = f(x, y+\delta y) + f_x(x, y+\delta y)\delta x + O(\delta x^2). \tag{A.1}$$

Now introduce

$$G_0(y) = f(x,y), \qquad \text{fixed } x,$$

and

$$G_1(y) = f_x(x,y), \qquad \text{fixed } x.$$

Hence,

$$G_0(y+\delta y) = G_0(y) + G_0'(y)\delta y + O(\delta y^2) \implies f(x, y+\delta y) = f(x,y) + f_y(x,y)\delta y + O(\delta y^2);$$

similarly,

$$G_1(y+\delta y) = G_1(y) + G_1'(y)\delta y + O(\delta y^2) \implies f_x(x, y+\delta y) = f_x(x,y) + \partial_y f_x(x,y) + O(\delta y^2).$$

Consider again Eq. (A.1):

$$f(x+\delta x, y+\delta y) = \underbrace{f(x, y+\delta y)}_{=f(x,y)+f_y(x,y)\delta y+O(\delta y^2)} + \delta x \underbrace{[f_x(x, y+\delta y)]}_{f_x(x,y)+\partial_y f_x(x,y)+O(\delta y^2)} + O(\delta x^2).$$

Hence,

$$f(x+\delta x, y+\delta y) = f(x,y) + f_x(x,y)\delta x + f_y(x,y)\delta y + f_{xy}(x,y)\delta x\delta y + O(\delta x^2, \delta y^2).$$

Appendix B

Fubini's theorem and multivariate integration

Consider the problem of finding the area of a right-angled triangle (Fig. B.1). The first goal of

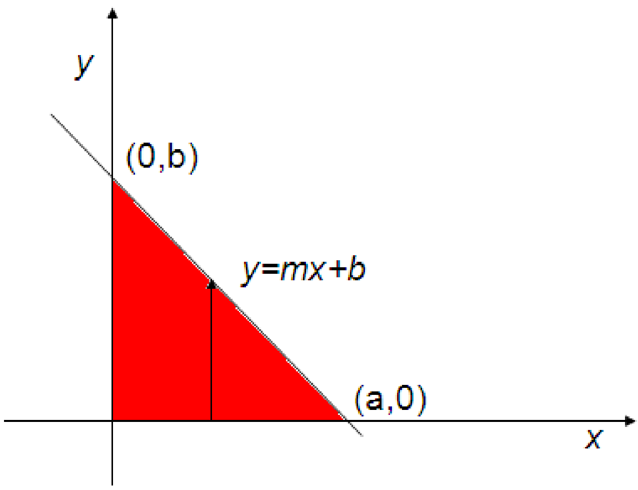

Figure B.1: Right-angled triangle with vertices at $(0, 0)$, $(a, 0)$, and $(0, b)$.

this Appendix is to compute from first-principles the area of the triangle using Riemann integration, using two separate approaches. We shall show that these approaches give the same answer. This is an illustration of a general principle called **Fubini's theorem**, which we state at towards the end of this Appendix.

B.1 Riemann integration and Fubini's theorem – Heuristics

There are at least two ways of calculting the area of the triangle in Figure B.1 using Riemann integration. The first involves **a sum over boxes**, whereby we break up the triangle into small boxes and sum over all such boxes. To do this, we fit rows of boxes into the triangle, where each row is parallel to the x-axis. Each box has sides of length Δx (See Fig. B.2).

- First row of boxes: N_1 boxes fit into the first row, with $\Delta y = m(N_1 \Delta x) + b$, hence

$$N_1 = \frac{\Delta y - b}{m\Delta x}.$$

- N_2 boxes are placed into the second row, with

$$N_1 = \frac{2\Delta y - b}{m\Delta x}.$$

- One continues thus until the last row is reached, in which precisely one box fits. This is the N_y^{th} row, and $N_y \Delta y = m\Delta x + b$, hence

$$N_y = \frac{m\Delta x + b}{\Delta y}.$$

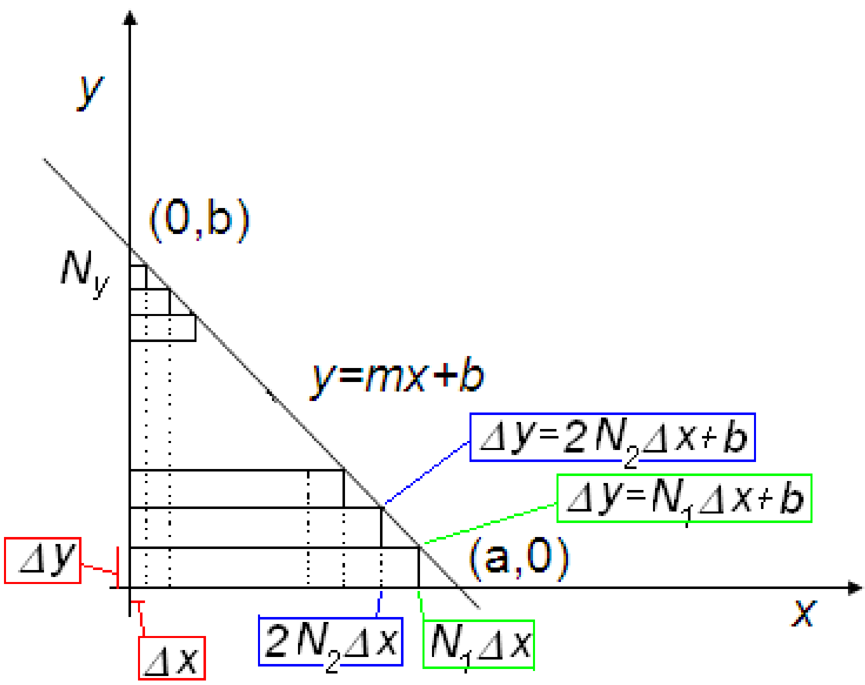

Figure B.2: The area of a triangle computed as a Riemann sum, where each summand is a small box.

We sum of the total area of the boxes:

$$
\begin{aligned}
\text{Area} &= \Delta x \Delta y N_1 + \Delta x \Delta y N_2 + \cdots + \Delta x \Delta y, \\
&= \Delta x \Delta y \left(\frac{\Delta y - b}{m\Delta x}\right) + \Delta x \Delta y \left(\frac{2\Delta y - b}{m\Delta x}\right) + \cdots + \Delta x \Delta y, \\
&= \Delta y \left(\frac{\Delta y - b}{m}\right) + \Delta y \left(\frac{2\Delta y - b}{m}\right) + \cdots + \Delta x \Delta y, \\
&= \frac{\Delta y^2}{m} \sum_{j=1}^{N_y} j - \frac{\Delta y}{m} \sum_{j=1}^{n} b, \\
&= \frac{\Delta y^2}{m} \tfrac{1}{2} N_y (N_y + 1) - \frac{b\Delta y}{m} N_y,
\end{aligned}
$$

Now use the formula for N_y:

$$
\begin{aligned}
\text{Area} &= \tfrac{1}{2} \frac{\Delta y^2}{m} \left(\frac{m\Delta x + b}{\Delta y}\right) \left(\frac{m\Delta x + b}{\Delta y} + 1\right) - \frac{b\Delta y}{m} \left(\frac{m\Delta x + b}{\Delta y}\right), \\
&= \frac{1}{2m} (m\Delta x + b)(m\Delta x + b + \Delta y) - \frac{b}{m}(m\Delta x + b), \\
&\overset{\Delta x, \Delta y \to 0}{=} \frac{b^2}{2m} - \frac{b^2}{m}, \\
&= -\frac{b^2}{2m}.
\end{aligned}
$$

Using $m = -b/a$, this is

$$
\text{Area} = \tfrac{1}{2} ab.
$$

Another way of calculating the area of the traingle involves **a sum over strips**: we break up the triangle into small vertical strips and sum over all such strips. There are N strips of width Δx, hence $(N+1)\Delta x = 1$, and $N = (a/\Delta x) - 1$ (Fig. B.3). The height of the j^{th} strip is

$$
y_j = m x_j + b = m(j\Delta x) + b.
$$

We sum over the area of each strip as follows:

$$\begin{aligned}
\text{Area} &= \sum_{j=1}^{N} y_j \Delta x, \\
&= \sum_{j=1}^{N} (m\Delta x j + b)\Delta x, \\
&= m\Delta x^2 \sum_{j=1}^{N} j + b\Delta x \sum_{j=1}^{N} (1), \\
&= \tfrac{1}{2} m\Delta x^2 N(N+1) + b\Delta x N, \\
&= \tfrac{1}{2} m\Delta x^2 \left(\frac{a}{\Delta x}\right)\left(\frac{a}{\Delta x} - 1\right) + b\Delta x \left(\frac{a}{\Delta x} - 1\right), \\
&= \tfrac{1}{2} ma\,(a - \Delta x) + (a - \Delta x)b, \\
&\overset{\Delta x \to 0}{=} \tfrac{1}{2} ma^2 + ab, \\
&= \tfrac{1}{2}\left(-\frac{b}{a}\right) a^2 + ab, \\
&= \tfrac{1}{2} ab.
\end{aligned}$$

Both methods give the same answer. Indeed, we could have computed the area strip-wise by using

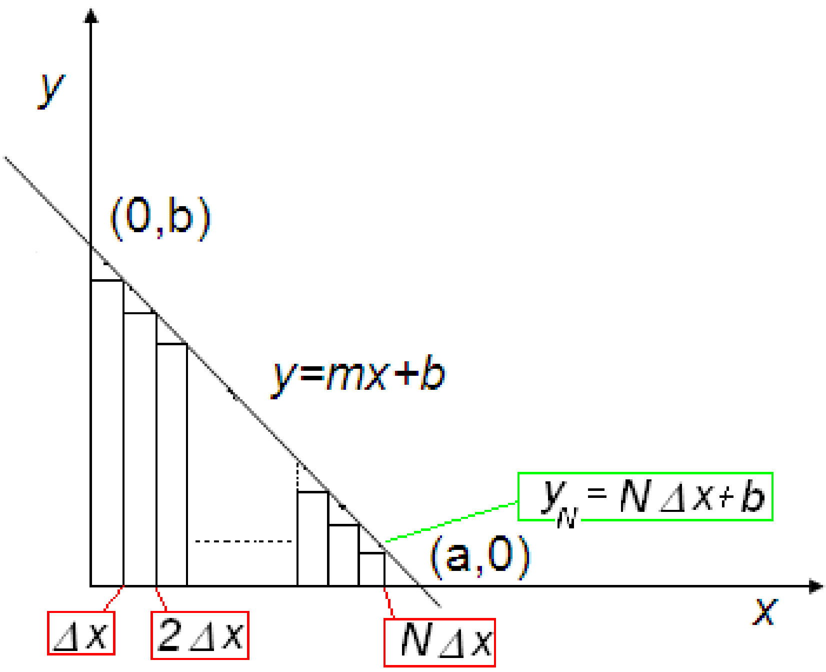

Figure B.3: The area of a triangle computed as a Riemann sum, where each summand is a small strip.

strips *parallel* to the x-axis, rather than perpendicular, and we would still get the same answer. One can think of this result as being the equivalence of the 'limit over strips' and 'the limit over squares'.

This fact fits into a much more general result called **Fubini's theorem**

B.2 Fubini's theorem for areas

We now consider the problem of Riemann integration in two dimensions for a more general domain. Let Ω be a region in $\mathbb{R}^2$ whose boundary is a piecewise continuous closed curve (Figure B.4). The

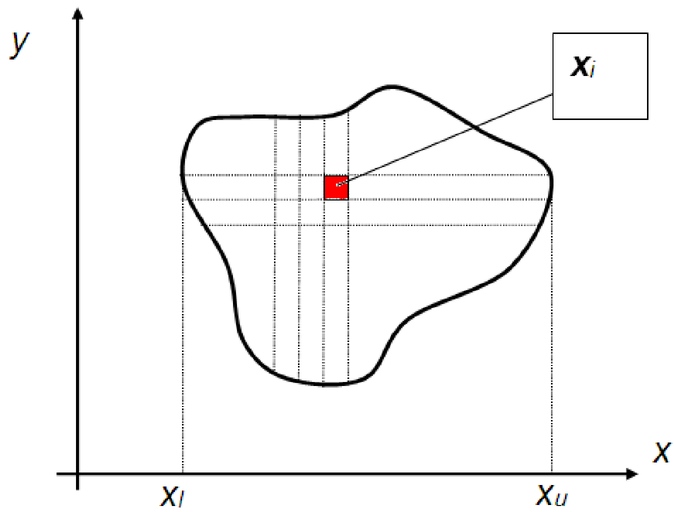

Figure B.4: Riemann integration in $\mathbb{R}^2$ as a 'limit over boxes'. Each small box has sides of length Δx and Δy.

goal here is to compute the area of the region Ω. We start with a **sum over boxes**: we fit a grid of small boxes into the region Ω, of sides of length Δx and Δy. The i^{th} box has its centre at $\boldsymbol{x}_i$. Those sections of Ω not covered by the grid become vanishingly small as Δx and Δy are reduced indefinitely. We therefore have

$$\text{area}(\Omega) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{\boldsymbol{x}_i \in \Omega} \Delta x \Delta y.$$

We interpret this integral as the double integral over Ω:

$$\int\int_{\Omega} \mathrm{d}x\,\mathrm{d}y \stackrel{\text{def}}{=} \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{\boldsymbol{x}_i \in \Omega} \Delta x \Delta y;$$

for sets Ω with piecewise continuous boundaries, the limit can be shown to exist and from Figure B.4, it follows that the double integral is equal to the area of the region Ω.

However, the area of Ω can also be computed as a 'limit over strips'. Consider Figure B.5. In this figure, the region Ω is filled in with narrow strips of width Δx. The strips are centred at x_i, with

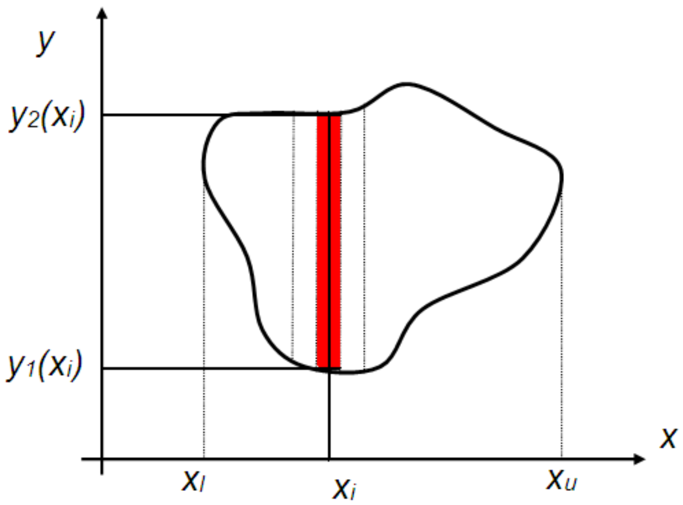

Figure B.5: Riemann integration in $\mathbb{R}^2$ as a 'limit over strips'. Each small strip has width Δx.

$i = 1, 2, \cdots$. Also, for each x-value, the there are two y-boundary points, denoted by $y_1(x)$ and $y_2(x)$. Thus,

$$\text{area}(\Omega) = \lim_{\Delta x \to 0} \Delta x \sum_{x_i} \left[y_2(x_i) - y_1(x_i) \right]. \tag{B.1}$$

But

$$y_2(x_i) - y_1(x_i) = \int_{y_1(x_i)}^{y_2(x_i)} \mathrm{d}y.$$

Define

$$F(x) := \int_{y_1(x)}^{y_2(x)} \mathrm{d}y.$$

Thus, Equation (B.1) becomes

$$\text{area}(\Omega) = \lim_{\Delta x \to 0} \sum_{x_i} \Delta x F(x_i).$$

But this is an ordinary (one-dimensional) Riemann integral:

$$\text{area}(\Omega) = \int_{x_\mathrm{l}}^{x_\mathrm{u}} \mathrm{d}x \, F(x).$$

One restores the definition of $F(x)$ to obtain

$$\text{area}(\Omega) = \int_{x_\mathrm{l}}^{x_\mathrm{u}} \mathrm{d}x \left[\int_{y_1(x)}^{y_2(x)} \mathrm{d}y \right].$$

But

$$\text{area}(\Omega) = \int\int_{\Omega} \mathrm{d}x\,\mathrm{d}y,$$

and we therefore have Fubini's theorem for areas, in the general case:

$$\int\int_{\Omega} \mathrm{d}x\,\mathrm{d}y = \int_{x_\mathrm{l}}^{x_\mathrm{u}} \mathrm{d}x \left[\int_{y_1(x)}^{y_2(x)} \mathrm{d}y\right].$$

In other words, the difficult and slightly weird double integral on the left-hand side can be converted into a repeated application of single integrals on the right-hand side, albeit that the limits of integration on the innermost integral are not necessarily constants.

Finally, there is nothing special about strips parallel to the y-axis, so we have, quite generally,

$$\int\int_{\Omega} \mathrm{d}x\,\mathrm{d}y = \int_{x_\mathrm{l}}^{x_\mathrm{u}} \mathrm{d}x \left[\int_{y_1(x)}^{y_2(x)} \mathrm{d}y\right] = \int_{y_\mathrm{l}}^{y_\mathrm{u}} \mathrm{d}y \left[\int_{x_1(y)}^{x_2(y)} \mathrm{d}x\right]. \tag{B.2}$$

Here, y_l and y_u refer to the (constant) extent of the domain in the y-direction, and $x_1(y)$ and $x_2(y)$ refer to the boundary curves, where the x-coordinates are viewed as parametric functions of y. Thus, Fubini's theorem is sometimes restated as the fact that 'the order in which the integration is performed does not matter'.

In Fubini's theorem, the parametric boundary curves $y_1(x)$ and $y_2(x)$ (and their inverses) refer to the upper and lower segments of the boundary in Figure B.4. Of course, itt is possible for certain ranges of x, that three or more parametric paths are required to describe the boundary (draw this!), but this does not affect any of the arguments given in this Appendix.

B.3 Integration of functions of several variables

Of course, not only do we wish to compute areas, but also n-fold integrals of functions. Focusing on the two-dimensional case, we wish to define and compute integrals such as

$$\mathcal{I} = \int\int_{\Omega} \mathrm{d}x\,\mathrm{d}y\, f(x,y),$$

where $f(x,y)$ is a continuous function and Ω is a set whose boundary is a closed, piecewise continuous curve. Following on from the previous section, this is naturally defined to be

$$\int\int_{\Omega} \mathrm{d}x\,\mathrm{d}y\, f(x,y) \stackrel{\text{def}}{=} \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sum_{\boldsymbol{x}_i\in\Omega} f(\boldsymbol{x}_i)\Delta x \Delta y; \tag{B.3}$$

the main purpose of the analytic theory of Riemann integration is to show that this limit exists for continuous functions and for sets whose boundaries are closed and piecewise continuous; the existence of this limit is taken as given in this module (this is not an analysis course).

As in the previous section, we note that the sum in Equation (B.3) can be computed in many ways. We threfore choose to compute the sum in the following way. We identify **columns** of cells as those at a fixed x-value (see Figure B.6). Then,

- Run over all columns, $I = 1, 2, \cdots$.
- What at the I^{th} column, run over all cells in that column. The cells in this one given column are given a label J. So, for a fixed I, there are many J-values, with $J = 1, 2, \cdots, N_I$. The number N_I depends on the column number: near x_{l} and x_{u} the number N_I is relatively small while closer to the middle of the allowed range of x-values there are relatively many cells in the columns, and N_I is relatively large.

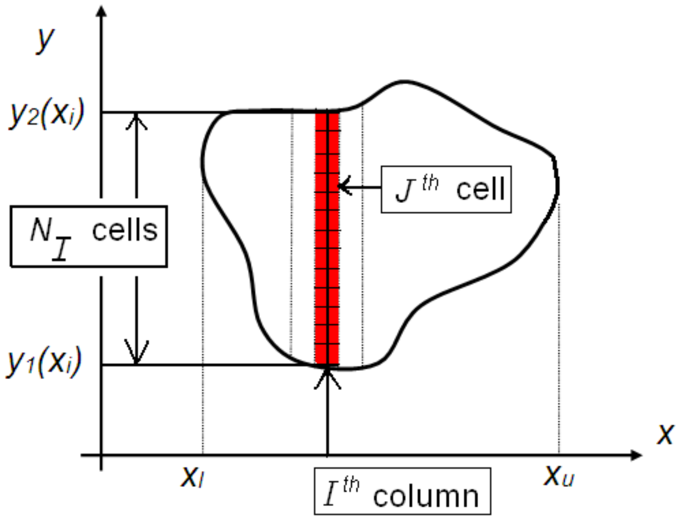

Figure B.6: Riemann integration for a function $f(x,y)$ $\mathbb{R}^2$ as a 'limit over strips'. As before, each small strip has width Δx.

Thus, we have

$$\begin{aligned}
\mathcal{I} &= \int\int_{\Omega} \mathrm{d}x\,\mathrm{d}y\, f(x,y), \\
&= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sum_{\boldsymbol{x}_i\in\Omega} f(\boldsymbol{x}_i)\Delta x\Delta y, \\
&= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sum_{I} \Delta x \sum_{J=1}^{N_I} \Delta y\, f(x_I, y_J).
\end{aligned} \tag{B.4}$$

If the double limit can be performed sequentially (and it can, but this is beyond the scope of this module, and is the crux of Fubini's theorem), then we can re-write Equation (B.4) as

$$\mathcal{I} = \lim_{\Delta x \to 0} \sum_I \Delta x \left[\lim_{\Delta y \to 0} \sum_{J=1}^{N_I} \Delta y \, f(x_I, y_J) \right], \tag{B.5}$$

and we identify the limit inside the square brackets as

$$\lim_{\Delta y \to 0} \sum_{J=1}^{N_I} \Delta y \, f(x_I, y_J) = \int_{y_1(x_I)}^{y_2(x_I)} f(x_I, y) \, \mathrm{d}y,$$

which is an ordinary single integral with respect to y carried out at fixed $x = x_I$. Thus, Equation (B.5) becomes

$$\mathcal{I} = \lim_{\Delta x \to 0} \sum_I \Delta x \left[\int_{y_1(x_I)}^{y_2(x_I)} f(x_I, y) \, \mathrm{d}y \right],$$

and the outermost limit (as $\Delta x \to 0$) converts into an ordinary single integral with respect to x:

$$\mathcal{I} = \int_{x_\mathrm{l}}^{x_\mathrm{u}} \mathrm{d}x \left[\int_{y_1(x)}^{y_2(x)} f(x, y) \, \mathrm{d}y \right],$$

and Fubini's theorem for the double integral of functions (i.e. not just areas) is recoverd: multiple integration amounts to repeated applications of ordinary (single-variable) integration. For the sake of simplifiying notation, we sometimes omit the parametric dependence of the y-limits on x, and write

$$\int_{x_\mathrm{l}}^{x_\mathrm{u}} \mathrm{d}x \left[\int_{y_1(x)}^{y_2(x)} f(x, y) \, \mathrm{d}y \right] \equiv \int_\Omega \mathrm{d}x \left[\int \mathrm{d}y \, f(x, y) \right]$$

Again, there is nothing special about the column treatment in Figure B.6: one could just as easily sum over rows of cells, leading to the following result:

$$\mathcal{I} = \int_{y_\mathrm{l}}^{y_\mathrm{u}} \mathrm{d}y \left[\int_{x_1(y)}^{x_2(y)} f(x, y) \, \mathrm{d}x \right] \equiv \int_\Omega \mathrm{d}y \left[\int \mathrm{d}x \, f(x, y) \right],$$

where y_l, y_u, $x_1(y)$ and $x_2(y)$ are the same as defined previously after Equation (B.2). Thus, we have the following general statement of Fubini's theorem:

Let Ω be a region of $\mathbb{R}^2$ with boundary C, where C is a closed, piecewise differentiable curve. Let $f(x, y)$ be a continuous function on Ω and C. Then

$$\int \int_\Omega f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_\Omega \left[\mathrm{d}x \int \mathrm{d}y \, f(x, y) \right] = \int_\Omega \mathrm{d}y \left[\int \mathrm{d}x \, f(x, y) \right]. \tag{B.6}$$

The result extends to n-fold integrals over finite domains in $\mathbb{R}^n$.

Finally, it would be bold (and indeed erroneous) to claim that we have proved Equation (B.6). In particular, the claim after Equation (B.4) about the order in which the limits are taken would need to be shown rigorously lots epsilons and deltas, or in the much more general context of the more powerful theory of Lebesgue integration and measure theory.