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Critical points, cusps and induced expansion in dimension one

THÈSE

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par

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Mis en page avec la classe thloria.

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Introduction

1 Introduction générale

Ce travail de thèse se situe dans les domaines de la *dynamique de l'intervalle* et de la *dynamique holmorphe*. On s'intéresse particulièrement aux notions de *dilatation induite* pour des transformations lisses par morceaux de l'intervalle, que l'on appelle des *applications à singularités*. Ces transformations, peu étudiées auparavant, peuvent avoir des singularités où la dérivée tend vers $\pm\infty$. La dilatation induite pour une telle transformation f est une collection de sous-intervalles avec des itérées de f associées, telles que l'itérée de f, restreinte au sous-intervalle, soit dilatante. Construire de la dilatation induite est une technique féconde pour résoudre des problèmes en l'absence d'hyperbolicité.

C'est Robert May en 1976 [27] qui a attiré l'attention du grand public scientifique vers la dynamique de l'intervalle. Depuis, des propriétés de systèmes écologiques, biologiques, météorologiques, financières et physiques, entre autres, ont été modélisées par des applications de l'intervalle. La dynamique définie par ces applications, bien que déterministe, peut admettre un comportement dit "chaotique". On peut s'attendre à un comportement tout aussi compliqué au sein des systèmes globaux.

Depuis les années soixante-dix, les connaissances dans le domaine de la dynamique de l'intervalle se sont beaucoup approfondies [6]. Mais ces systèmes dynamiques conservent toujours de nombreux secrets. Edward Lorenz, en 1963, a introduit un modèle en trois dimensions pour étudier la météorologie (pas pour prévoir, mais pour comprendre !). Quand ce modèle est réduit à un modèle unidimensionnel, on retrouve souvent des applications à singularités [23]. Or, tandis que la dynamique des applications lisses avec des points critiques est assez bien comprise, la dynamique des applications à singularités est encore très peu connue.

C'est dans ce sens que débute cette thèse : on développe la théorie ergodique pour une classe générale d'applications à singularités. Nous concentrons nos efforts sur des sujets liés à l'existence de dilatation induite qui sera due à un exposant de Lyapunov positif. La plupart des résultats sont nouveaux même pour des applications lisses ($C^{1+\epsilon}$), et même si certains sont déjà connus sous une condition supplémentaire (points critiques non-plats).

Les mêmes idées marchent tout aussi bien dans le cadre des applications rationnelles (complexes) de la sphère de Riemann qui font le sujet du deuxième chapitre. En y adaptant nos techniques, nous démontrons de nouveaux résultats. Une mesure invariante absolument continue par rapport à une mesure conforme, et avec exposant de Lyapunov positif, a sa densité minorée par une constante positive sur le support de la mesure conforme. En plus, cette mesure est engendrée par une application de Markov induite avec temps de retour intégrable.

L'un des buts des mathématiciens est de classifier les objets étudiés. En reconnaissant quand deux objets sont "équivalents", on évite de dupliquer son travail inutilement. Une façon de le faire, c'est par *conjugaison* ou *changement de coordonnées*. Si on ne s'intéresse qu'à certaines propriétés d'une application, et si ces propriétés sont préservées par une certaine conjugaison, alors on peut regarder l'application initiale et l'application résultante comme étant la même.

Dans le troisième chapitre, on obtient ainsi une classification des applications lisses de l'intervalle admettant une mesure d'entropie maximale qui est absolument continue par rapport à la mesure de Lebesgue. On explore aussi le monde des applications à singularités, en les étudiant grâce aux conjugaisons.

Les notions de dimensions fractales sont à la fois utiles et belles. Elles permettent de décrire certaines caractéristiques de systèmes qui sont souvent préservées par conjugaisons lisses. Dans le quatrième chapitre on étudie la coïncidence de la dimension hyperbolique avec la dimension de Hausdorff. On peut regarder la dimension hyperbolique comme une mesure du potentiel pour dilatation induite dans un système.

Nous terminons avec des estimations fines pour des applications de premier retour au voisinage des points critiques, qui nous permettent de démontrer un résultat d'expansivité pour des cocycles bien adaptés.

2 Un tour des résultats

Les applications à singularités

Une *application à singularités* est une application f de l'intervalle I, lisse par morceaux, telle que $|\log|Df||$ tend vers ∞ aux bords de chaque composant connexe du domaine de définition $\mathcal{D}_f$ de f. Ici, lisse veut dire : l'application f est C^1 sur $\mathcal{D}_f$ et il existe des constantes $C > \epsilon > 0$ telles que

$$|Df(x) - Df(x')| < C|x - x'|^\epsilon$$

là où $|Df| < 2$, et telles que

$$\left|\frac{1}{Df(x)} - \frac{1}{Df(x')}\right| < C|x - x'|^\epsilon$$

sur $\{|Df| > \frac{1}{2}\}$. Les constantes $2, \frac{1}{2}$ ne sont pas importantes ; il faut tout simplement un chevauchement. Cette définition est une adaptation de la notion de classe $C^{1+\epsilon}$.

Si on se donne une mesure de probabilité invariante μ pour f, alors son *exposant de Lyapunov* χ_μ est

$$\chi_\mu := \int \log|Df|\, d\mu$$

quand cette quantité existe. Pour des applications à singularités, l'intégrale n'est pas toujours définie, même si μ est ergodique et absolument continue (section 3.4). Si l'intégrale est définie, positive et finie, on dit que μ a son exposant de Lyapunov positif et fini. Supposons dans la suite que f est une application à singularités avec une mesure de probabilité ergodique et invariante avec exposant de Lyapunov positif et fini.

On construit l'extension naturelle $\Pi : (Y, F, \overline{\mu}) \to (I, f, \overline{\mu})$. L'ensemble Y est la collection d'éléments y constitués d'un point de I et une suite de branches inverses de f, tels que le point est contenu dans le domaine de définition de la composition des branches inverses. On peut regarder Y comme l'ensemble d'orbites en arrière (branche par branche) des points de I. La projection Π est canonique, et $F : Y \to Y$ est inversible et satisfait $\Pi \circ F = f \circ \Pi$. La mesure $\overline{\mu}$ sur Y est l'unique mesure ergodique et invariante qui projette sur μ.

L'intégrabilité de $\log|Df|$ implique que $\lim \frac{1}{n} \log|Df(\Pi F^{-n} y)| = 0$ pour $\overline{\mu}$-presque tout y, c'est-à-dire que, le long de presque toute orbite en arrière, $|Df|$ est bornée entre des suites sous-exponentiellement croissante et décroissante. La propriété $C^{1+\epsilon}$ borne localement la distorsion avec une borne qui dépend de façon polynomiale de la dérivée (une borne sur la distorsion veut dire qu'une application est "presque" affine). Grâce à cette borne, la taille des boules sur lesquelles on peut contrôler la distorsion, le long de presque toute orbite en arrière, décroît sous-exponentiellement. Avec la décroissance exponentielle due à l'exposant de Lyapunov positif, on conclut, dans théorème 1.4.1, que presque partout, pour presque toute orbite en arrière, on peut tirer en arrière un intervalle avec distorsion bornée le long de l'orbite. Il s'agit d'une généralisation du théorème de variété instable de Ledrappier (théorème 8, [21]).

Supposons désormais, en plus, que I admet une partition finie telle que f restreinte à chaque élément de la partition soit monotone.

Le Lemme de Volume établit une belle relation entre l'exposant de Lyapunov, l'entropie h_μ et la dimension de Hausdorff d'une mesure μ (de probabilité, invariante et ergodique avec $0 < \chi_\mu < \infty$), de la forme

$$\mathrm{HD}(\mu) = \frac{h_\mu}{\chi_\mu}.$$

Ce lemme a été démontré par Franz Hofbauer et Peter Raith [16] sous la condition que la dérivée soit de "p-variation bornée". Cette condition n'est pas vérifiée par des applications à singularités puisque la dérivée n'est pas bornée. Nous nous servons des bonnes propriétés "Markov" des "regularly returning intervals"[1] (appelés "nice" par Marco Martens [25]) pour donner une démonstration efficace du Lemme de Volume pour des applications à singularités.

Le premier chapitre s'achève avec ce résultat : les conditions suivantes sont équivalentes (théorème 1.9.12).

1. $h_\mu = \chi_\mu$;
2. μ est absolument continue par rapport à la mesure de Lebesgue ;
3. la densité de μ par rapport à Lebesgue est définie et minorée par une constante positive sur un intervalle ;
4. μ est engendrée par une application de Markov induite dilatante avec temps de retour intégrable.

Même dans le cadre d'applications lisses, ces deux dernières conditions n'étaient connues que dans des cas particuliers. Ce résultat nous permet d'améliorer une réponse de Michael Benedicks et Michał Misiurewicz [1] à une question de Pierre Collet et Jean-Pierre Eckmann : si une application $C^{1+\epsilon}$ admet une mesure de probabilité μ ergodique et invariante, absolument continue par rapport à la mesure de Lebesgue et avec exposant de Lyapunov (ou entropie) positif, alors

$$\int \log |Df| dx > -\infty$$

sur son support. En particulier, il n'y a pas de point critique de type $e^{-\frac{1}{|x|^a}}$ dans le support de μ si $a \geq 1$. Hans Thunberg [39] a aussi posé cette question.

Le fait qu'une application de Markov induite dilatante avec temps de retour intégrable engendre une mesure de probabilité absolument continue est connu depuis longtemps ([6], lemme V.3.1, voir aussi [46]). La construction des applications de Markov induites dilatantes avec temps de retour intégrable est une façon productive de construire des mesures continues.

Pour démontrer que le temps de retour de la condition 4 soit intégrable, l'étape cruciale est de construire un sous-ensemble W_A^* de Y de mesure positive tel que $\Pi_* \overline{\mu}_{|W_A^*}$ soit de densité minorée

[1] des intervalles à retour régulier

sur l'intervalle qui constitue son support. Le résultat en découle puisque le temps de retour pour l'application de premier retour à W_A^* est $\overline{\mu}$-intégrable.

Le fait que la densité soit minorée vient de la construction de W_A^*. Pour chaque $y \in W_A^*$, il existe un $W_y \ni y$ tel que $\Pi W_y = \Pi W_A^*$ est un intervalle W, et tel que W_y peut être tiré en arrière par les mêmes branches inverses que pour y avec une borne uniforme sur la distorsion. Cette borne sur la distorsion impliquera que la densité de la mesure conditionnelle sur W_y ne varie pas beaucoup. En projettant on trouvera que la densité sera minorée sur W.

Les applications rationnelles

Nos techniques se transfèrent à la dynamique holomorphe. Pour une application rationnelle f, l'absolue continuité de la mesure μ par rapport à une mesure t-conforme implique que la densité de μ est minorée par une constante positive m-presque partout, et qu'il existe une application de Markov induite avec temps de retour intégrable (théorèmes 2.1.2, 2.1.4). Juan Rivera-Letelier et Felix Przytycki [34] ont déjà trouvé une application pour ces résultats.

L'existence de l'application induite est bien plus délicate à démontrer que dans le cas réel. En effet, l'absence des ensembles "regularly returning" fait que les bonnes propriétés "Markov" de disjonction ne sont plus automatiques.

Mesures d'entropie maximale

Dans la section 3.2, on classifie les applications de classe $C^{1+\epsilon}$ avec une mesure d'entropie maximale qui est absolument continue par rapport à la mesure de Lebesgue. Dans le cadre des applications unimodales, la classification est complète. Une application tente $T_s : [0,1] \to [0,1]$ est une application linéaire par morceaux donnée par

$$\begin{aligned} T_s(x) &= sx, & 0 \leq x \leq \frac{1}{2}, \\ T_s(x) &= s(1-x), & \frac{1}{2} < x \leq 1. \end{aligned}$$

Soit f une application unimodale de classe $C^{1+\epsilon}$ ou C^r avec entropie topologique positive, et soit μ sa mesure d'entropie maximale. Alors, soit $\mathrm{HD}(\mu) < 1$, soit il y a une conjugaison topologique entre f et une application tente T_s avec $s = 2^{2^{-k}}$ pour un certain $k > 0$, et sur un intervalle qui contient le point critique la conjugaison est aussi lisse que f. En particulier, si f n'est pas renormalisable, alors la mesure d'entropie maximale est absolument continue si et seulement si f est $C^{1+\epsilon}$ (ou C^r si f l'est) conjuguée à T_2, et donc $C^{1+\epsilon}$ conjuguée au polynôme de Chebyshev $4x(1-x)$.

On donne l'idée de la démonstration : si la mesure d'entropie maximale est de dimension un, elle est absolument continue et donc il y a une application induite dilatante. La conjugaison avec le modèle linéaire définie une application induite dilatante pour l'application linéaire par morceaux. Ces deux applications induites admettent des mesures absolument continues avec densités lisses. Comme la conjugaison préserve les mesures, elle aussi est lisse sur un intervalle. On peut étendre la régularité de la conjugaison en se servant de sa définition. Elle va être lisse "partout" sauf sur l'ensemble post-critique PC, qui devra alors être un ensemble fini. On déduit aussi que $f^{-1}(PC)$ doit être contenu dans la réunion de PC avec l'ensemble des points critiques. Pour des applications unimodales non renormalisables, ceci est le cas si et seulement si f est topologiquement conjuguée à T_2.

Hyperbolicité

Un ensemble X, compact et invariant en avant pour une application différentiable f, est dit *hyperbolique* si une itérée de f est dilatante sur X, c'est-à-dire, s'il existe un $n > 0$ tel que $|Df^n_{|X}| > 1$. La *dimension hyperbolique* d'un ensemble K est le supremum des dimensions de Hausdorff des sous-ensembles hyperboliques de K. Une question naturelle est : pour quels f et K la dimension hyperbolique est-elle égale à la dimension de Hausdorff ? On étudie cette question dans le cadre des applications C^2 de l'intervalle avec un nombre fini de points critiques non-plats[2].

Une condition topologique, que les points critiques soient non récurrents, suffit pour démontrer que les deux notions de dimension coïncident pour l'ensemble de Julia d'une telle application (corollaire 4.3.4).

La démonstration se fait par récurrence sur le nombre de points critiques. Il y a au moins un voisinage d'un point critique qui est disjoint des orbites en avant de l'ensemble critique. L'application de premier retour à ce voisinage est une application induite dilatante. Les mesures conformes servent ici pour majorer la dimension des points qui passent infiniment souvent par ce voisinage. Puis on peut enlever le voisinage et toutes ses préimages et considérer un autre point critique. La présence de points paraboliques ajoute une complication, mais, en utilisant encore des applications induites, on peut majorer la dimension des points qui passent infiniment souvent près des points paraboliques mais qui ne passent pas près des points critiques. Les seuls points qui restent ne passent ni près des points paraboliques, ni près des points critiques, et on les traite avec le théorème d'hyperbolicité de Mañé.

3 Une note au lecteur

On suppose une familiarité de la part du lecteur avec l'analyse de base, avec la dynamique de l'intervalle, avec la dynamique holomorphe et avec la théorie ergodique. Les livres de De Melo et Van Strien [6], de Carleson et Gamelin [5] et de Walters [44] sont des références excellentes. Pour le lecteur qui cherche une vulgarisation du domaine, on ne peut pas faire mieux que le livre de Ian Stewart [37], *Does God play dice?*, avec traduction en français *Dieu, joue-t-il aux dés ?*.

[2]non-plat veut dire comme x^a, pour un $a > 1$, à difféomorphisme près

1

Cusp maps

Contents

1.1 Definitions of cusp maps

Smooth maps of the interval have generated a lot of interest and a huge body of work [6]. Maps which are not smooth due to the presence of singularities, or poles, are also worth studying due, for example, to their links with the Lorenz map [23]. However, the presence of poles has prevented much progress from being made. At critical points one has negative Schwarzian derivative which allows one to control the accumulation of distortion of iterates. At poles, conversely, the Schwarzian derivative is unboundedly positive. With poles, the fact that an iterate maps an interval diffeomorphically onto nested intervals does not imply that the pullback of the inner interval is relatively far from singular points.

Our goal is to develop the ergodic theory for interval maps which are smooth except at singularities where the derivative, on at least one side, may tend to infinity.

For continuous maps with two smooth monotone branches, if the norm of the derivative tends to infinity as one approaches the turning point, the turning point is called a *cusp*. This leads us to introduce the following definitions.

Definition 1.1.1 *A map $f : \bigcup_j I_j \to I$ is a* basic-cusp map *if $\{I_j\}_j$ is a finite or countable collection of disjoint open subintervals of I such that f is a C^1 diffeomorphism on each (open) interval $I_j =: (p_j, q_j)$ and such that the following limits exist and equal either 0 or $\pm\infty$:*

$$\lim_{x \searrow p_j^+} Df(x), \lim_{x \nearrow q_j^-} Df(x).$$

The set $X := I \setminus \bigcup_j I_j$ *is called the* singular set *or the* set of singularities of f. *Let the* invariant set K_f *be the set of all* x *such that* $f^n(x) \in I \setminus X$ *for all* $n \geq 0$. *It is forward-invariant but generally is not compact.*

Let $A \subset \mathbb{R}$. Recall that a map $g : A \to \mathbb{R}$ is $C^{1+\epsilon}$ with constants C and ϵ if, for all x, x' in A,

$$|Df(x) - Df(x')| < C|x - x'|^\epsilon.$$

If $f : \bigcup I_j \to I$ is a basic-cusp map, then each restriction $f_{|I_j} : I_j \to I$ extends continuously to a map $f_j : \overline{I_j} \to I$ and, if one compactifies $\mathbb{R}$ to get $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, then $Df_j : \overline{I_j} \to \overline{\mathbb{R}}$ is continuous.

Definition 1.1.2 *A basic-cusp map* $f : \bigcup_{j \geq 1} I_j \to I$ *is a* cusp map *(*with constants C, ϵ*) if there exist constants* $C > \epsilon > 0$ *such that on every* $\overline{I_j}$,

- *for all* x, x' *such that* $|Df_j(x)|, |Df_j(x')| \leq 2$,

$$|Df_j(x) - Df_j(x')| < C|x - x'|^\epsilon; \tag{1.1}$$

- *for all* x, x' *such that* $|Df_j(x)|, |Df_j(x')| \geq 2^{-1}$,

$$\left| \frac{1}{Df_j(x)} - \frac{1}{Df_j(x')} \right| < C|x - x'|^\epsilon.$$

One can check that poles of the form x^α where $0 < \alpha < 1$ (these are *poles of root type*) satisfy this latter relation. In figure 1.1 we present (the graphs of) some cusp maps.

Definition 1.1.3 *A cusp map* $f : \bigcup_j I_j \to I$ *is a* continuous cusp map *if* $\overline{K_f} = I$ *and there exists a continuous map* $g : I \to I$ *which coincides with* f *on* $\bigcup_j I_j$. *We make no distinction between* f *and* g.

We are also interested in some analogue to C^r maps. We want a condition which is satisfied by higher derivatives of f and which holds for all poles of root type. What appeared naturally and suffices to prove nice C^r distortion properties for induced Markov maps is the following.

Definition 1.1.4 *Let* $r \in \{2, 3, \ldots\}$. *A cusp map* $f : \bigcup_j I_j$ *is a* C^r cusp map *(*with constants p, C*) if there exist constants* $C, p > 1$ *such that, for each* j *one has:*

- f_j *is* C^r *on* I_j;
- *for all* $x \in \overline{I_j}$ *such that* $0 < |Df_j(x)| \leq 2$, *and for all* i *such that* $2 \leq i \leq r$, $|D^i f(x)| < C$;
- *for all* $x \in \overline{I_j}$ *with* $|Df_j(x)| \geq 2$, *and for all* i *such that* $2 \leq i \leq r$,

$$\frac{|D^i f(x)|}{|Df(x)|^p} < C; \tag{1.2}$$

If f *is a* C^r *cusp map for all integers* $r > 1$ *then* f *is a* C^∞ *cusp map.*

Definition 1.1.5 *Let* $f : I \to I$. *A* pole (at p) *is an interval* $(p, p + \epsilon) \subset I$ *such that* $|\varepsilon| > 0$, f *is* C^1 *on* $(p, p + \epsilon)$ *and*

$$\lim_{h \searrow 0^+} |Df(p + h\epsilon)| = \infty.$$

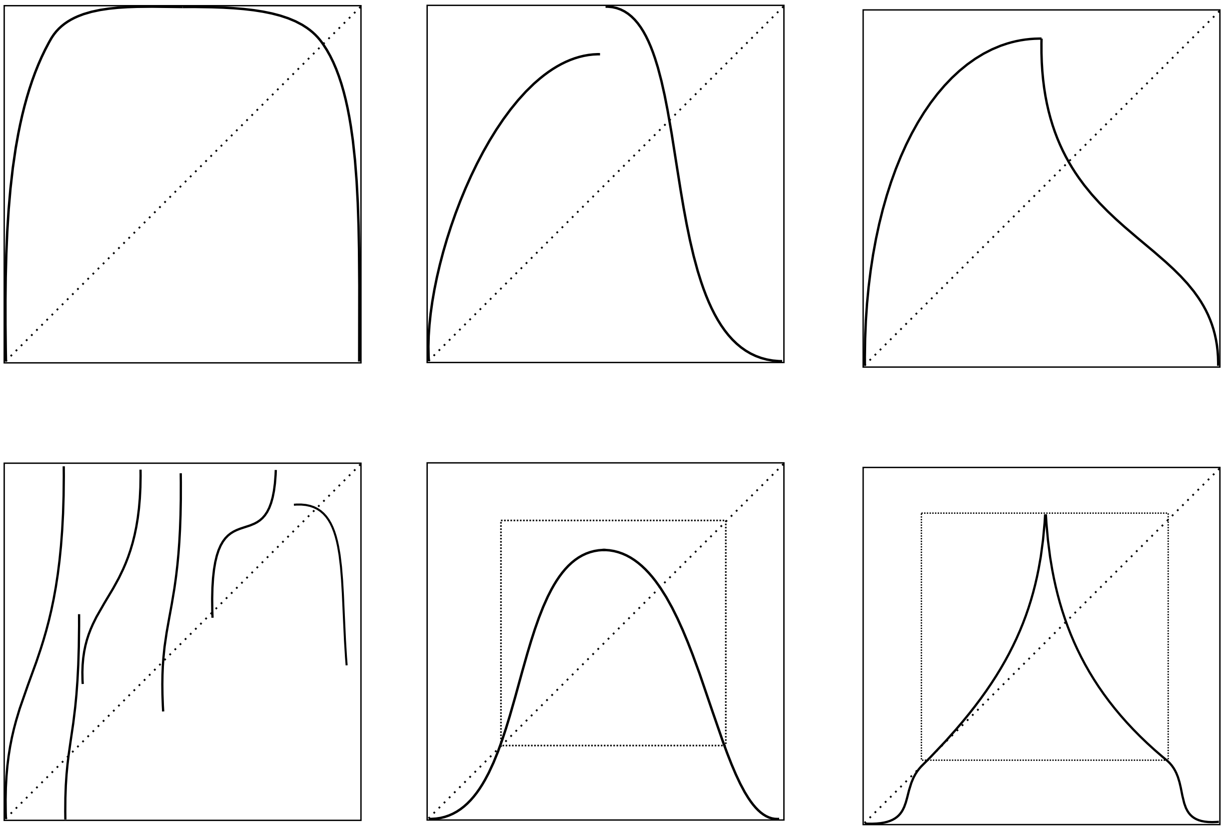

Figure 1.1: Some cusp maps.

For cusp maps f, we shall be interested in ergodic, invariant, probability measures concentrated on the invariant set K_f. Such a measure μ has *positive finite Lyapunov exponent*, often denoted χ_μ, if and only if

$$0 < \chi_\mu := \int \log |Df(x)| d\mu < \infty.$$

In particular the integral must converge. We shall see later that even for fairly simple unimodal cusp maps $\log |Df|$ need not be integrable. When it is, one has $\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)| = \chi_\mu$ almost everywhere.

1.2 Distortion estimates

In the following two lemmas, one can think of ϕ as Df if Df is small, or as $\frac{1}{Df}$ if Df is big.

Lemma 1.2.1 *Let c be a constant such that $0 < c < 1$. Let $\phi : A \to \mathbb{R}$ satisfy $|\phi(x) - \phi(x')| < C|x - x'|^\epsilon$ for some constants $C > \epsilon > 0$ for all $x, x' \in A$.*

Then for all $x, x' \in A$ such that $|x - x'|^\epsilon < c^3$ and $|\phi(x)| > c$,

$$1 - Cc^2 \frac{|x - x'|^\epsilon}{c^3} < \frac{\phi(x')}{\phi(x)} < 1 + Cc^2 \frac{|x - x'|^\epsilon}{c^3}. \tag{1.3}$$

Proof: We have $|\phi(x) - \phi(x')| < C|x - x'|^\epsilon < Cc^3$ and $|\phi(x)| > c$. Thus

$$\frac{|\phi(x) - \phi(x')|}{|\phi(x)|} < \frac{Cc^3}{c} \frac{|x - x'|^\epsilon}{c^3} = Cc^2 \frac{|x - x'|^\epsilon}{c^3}.$$

One recovers (1.3) upon rewriting the lefthand side as $\left|1 - \frac{\phi(x')}{\phi(x)}\right|$ and using the triangle inequalities. □

Lemma 1.2.2 *Let $C > \epsilon > 0$ be constants. There exists a constant c_0, depending on C, ϵ and satisfying $0 < c_0 < \log 2$, with the following property:*

If $\phi : A \to \mathbb{R}$ satisfies $|\phi(x) - \phi(x')| < C|x - x'|^\epsilon$ for all $x, x' \in A$, and if c is a constant satisfying $0 < c \leq c_0$, then for all $x, x' \in A$ such that, $|x - x'|^\epsilon < c^3$ and $|\phi(x)| > c$,

$$\big|\log|\phi(x)| - \log|\phi(x')|\big| < c\frac{|x - x'|^\epsilon}{c^3} < c. \tag{1.4}$$

Proof: There exists a constant a_0, $0 < a_0 < \log 2$ such that $\log\left(1 - \frac{a}{2}\right) > -a$ for all a such that $0 < a < a_0$. Choose $c_0 > 0$ such that $Cc_0 < \frac{1}{4}$ and so that $c_0 < a_0$. Then by Lemma 1.2.1, for x, c satisfying the conditions of this lemma,

$$1 - \frac{c}{2}\frac{|x - x'|^\epsilon}{c^3} < 1 - Cc^2\frac{|x - x'|^\epsilon}{c^3} < \frac{\phi(x + t)}{\phi(x)} < 1 + Cc^2\frac{|x - x'|^\epsilon}{c^3} < 1 + c\frac{|x - x'|^\epsilon}{c^3}.$$

Taking logs and using $c \leq c_0 < a_0$,

$$-c\frac{|x - x'|^\epsilon}{c^3} < \log\left(\frac{\phi(x + t)}{\phi(x)}\right) < c\frac{|x - x'|^\epsilon}{c^3},$$

as required. □

Lemma 1.2.3 *Let $f : \bigcup_j I_j \to I$ be a cusp map with constants C, ϵ. Let $x_1 \in \bigcup_j I_j$ satisfy $|Df(x_1)| = 1$, $x_2 \in \bigcup_j I_j$ satisfy $|Df(x_2)| = 2$ and $x_3 \in \bigcup_j I_j$ satisfy $|Df(x_3)| = 2^{-1}$.*

Let c_0, which depends only on C, ϵ, be given by Lemma 1.2.2. Then $|x_1 - x_l|^\epsilon \geq c_0^3$ for $l = 2, 3$.

Proof: Suppose $x_1 \in I_j$ and let A denote the connected component of $\{x' \in \overline{I_j} : 2^{-1} < |Df_j(x')| < 2\}$ containing x_1. Apply Lemma 1.4 with $\phi := Df$ and $x := x_1$ and $x' \in \partial A$. It gives that if $|x_1 - x'|^\epsilon < c_0^3$ then

$$|\log|Df(x')|| < c_0\frac{|x_1 - x'|^\epsilon}{c_0^3} \leq \log 2,$$

contradicting $x' \in \partial A$ (since $|\log|Df(x')|| = \log 2$). □

Lemma 1.2.4 *Let $f : \bigcup_j I_j \to I$ be a cusp map with constants C, ϵ.*

There exists a constant c_0, $0 < c_0 < 1$ such that if c is a constant satisfying $0 < c < c_0$, then for all $x, x' \in \bigcup_j I_j$ such that $|x - x'|^\epsilon < c^3$ and $c < |Df(x)| < c^{-1}$,

$$\big|\log|Df(x)| - \log|Df(x')|\big| < c\frac{|x - x'|^\epsilon}{c^3} < c. \tag{1.5}$$

Proof: Let c_0 be given by Lemma 1.2.2. Let $x \in I_j$ satisfy $0 < |Df(x)| \leq 1$. Let A denote the connected component of $\{x' \in \overline{I_j} : 0 < |Df_j(x)| < 2\}$ containing x. If $y \in \partial A$ then $|Df_j(y)| \in \{0, 2\}$. By Lemma 1.2.3, if $|Df_j(y)| = 2$ then $\mathrm{dist}(x, y)^\epsilon > c_0^3$. Just as in the proof of Lemma 1.2.3, if $Df_v(y) = 0$, $0 < c < c_0$ and $c \leq |Df(x)| \leq 1$, one arrives at a contradiction if $\mathrm{dist}(x, y)^\epsilon \leq c^3$. Applying Lemma 1.2.2 for $\phi := Df$, we get (1.5) for x such that $|Df(x)| \leq 1$.

Now let $x \in I_j$ satisfy $0 < |Df(x)|^{-1} \leq 1$. Let A' denote the connected component of $\{x' \in \overline{I_j} : 0 < |Df_j(x)|^{-1} < 2\}$ containing x. Continuing as before, one deduces (1.5) for x such that $|Df(x)|^{-1} \leq 1$. □

1.3 The natural extension

Definition 1.3.1 *Let $f : \bigcup_j I_j \to I$ be a cusp map with singular set X. An open subinterval P of I is called an* interval of monotonicity *if f is monotone on P, $\partial P \subset \partial X$ and if f is not monotone on any open interval strictly containing P.*

Remark: By *monotone on P*, we understand of course monotone on $P \cap \bigcup_j I_j$ — we only consider points in the domain of definition of f. We shall treat notions like *injectivity* similarly.

Definition 1.3.2 *A* partition into intervals of monotonicity *for f is a collection $\mathcal{P}$ of intervals of monotonicity of f which form a cover of $I \setminus X$. $\mathcal{P}$ is a* finite *partition if the collection is finite.*

Properties of a partition $\mathcal{P}$ into intervals of monotonicity: Each point of $I \setminus X$ is contained in exactly one element of $\mathcal{P}$. A partition into intervals of monotonicity always exists and is a countable set. The absence of critical points in $I \setminus X$ implies that f is injective on P for each $P \in \mathcal{P}$. On each $P \in \mathcal{P}$ the following limit exists and equals either zero or infinity:

$$\lim_{x \to \partial P_i} |Df(x)|.$$

Suppose f has an invariant probability measure μ on the invariant set K_f with positive finite Lyapunov exponent. Let $\mathcal{P} =: \{P_i\}_i$ be a partition into intervals of monotonicity for f. Let $Z = \{0, 1, 2, \ldots\}^\infty$. Define Y as the set of couples $(x; z)$ where $x \in I, z = (z_0 z_1 z_2 \ldots) \in Z$ are such that there exists an infinite sequence x_i satisfying

- $f(x_{i+1}) = x_i$,
- $x_i \in P_{z_i}$ and
- $x_0 = x$.

Define $F(x; (z_0 z_1 z_2 \ldots)) := (f(x); (i z_0 z_1 \ldots))$ if $f(x) \in P_i$. Then $F : Y \to Y$ is invertible, since if $x' \in P_{z_0}$ and $f(x') = f(x)$ then $x = x'$ by injectivity on each partition element. We shall denote by Π the projection from Y into I (in general it is not onto since some points of I may not have preimages), and by Π' the projection from Y into Z. If $V \subset I$ and $z \in Z$ are such that for all $x \in V$, $(x; z) \in Y$ then

$$(V; z) := \bigcup_{x \in V} (x; z).$$

There is a unique probability measure $\overline{\mu}$ on Y, invariant by F, such that $\Pi_* \overline{\mu} = \mu$. If μ is ergodic then so is $\overline{\mu}$, and importantly $\overline{\mu}$ is also an ergodic invariant measure for the inverse map F^{-1}. We call $(Y, F, \overline{\mu})$ the natural extension of (I, f, μ).

1.4 Ledrappier Theory

We generalise the *unstable manifold* results of Ledrappier [21] to cusp maps. Even for smooth (thus without poles) interval maps, the results of Ledrappier seem badly known. The idea of *Markov extensions* (e.g. [18]), has been popular, but has only been productive for maps with a finite number non-flat critical points.

The generalised results below appear more powerful and easy to use. We shall later use our techniques to prove new results and extend known ones to wider classes of interval maps.

We recently discovered that related unstable manifold results using the "Lyapunov metric" were developed by Newhouse in [28] for $C^{1+\epsilon}$ endomorphisms of $\mathbb{R}^n$. Later we shall treat absolutely continuous measures; in what follows we merely require positive finite Lyapunov exponent.

Theorem 1.4.1 *Let $f : I \to I$ be an cusp map with invariant set K_f. Suppose f has an invariant, ergodic probability measure μ on K_f with positive finite Lyapunov exponent $\chi = \int \log |Df|\, d\mu > 0$. Let $(Y, F, \overline{\mu})$ be the natural extension for (I, f, μ). Let η satisfy $0 < 3\eta < \chi$.*

Then there exists a measurable function α on Y, positive almost everywhere, suth that for $\overline{\mu}$ almost every $y \in Y$ there exists an infinite sequence of open intervals $V_n \subset I$, $n \geq 0$, with the following properties:

- *$V := V_0 := B(\Pi y, \alpha(y)) \ni \Pi y$ and $V_n \ni \Pi F^{-n} y$;*
- *for all $x, x' \in V_n$*
$$\left|\log |Df^n(x)| - \log |Df^n(x')|\right| < \log 4 \left(\frac{|f^n(x) - f^n(x')|}{|V|}\right)^{\epsilon}; \tag{1.6}$$
- *moreover,*
$$\lim_{j\to\infty} \lim_{n\to\infty} \max_{x,x' \in V_{n+j}} \left|\log |Df^n(x)| - \log |Df^n(x')|\right| = 0; \tag{1.7}$$
- *for all $x \in V$,*
$$4^{-1} \leq \prod_{i=1}^{\infty} \frac{Df(\Pi F^{-i} y)}{Df(\Pi F^{-i}(x; \Pi' y))} \leq 4; \tag{1.8}$$
- *for all $x, x' \in V_n$*
$$\left|\log |Df(x)| - \log |Df(x')|\right| < \log 2 \left(\frac{|x - x'|}{|V_n|}\right)^{\epsilon}; \tag{1.9}$$
- *there exists a measurable function ρ on Y, $0 < \rho(y) < \infty$ almost everywhere, such that for all $x \in V_n$*
$$|I|\rho(y)^{-1} e^{n(\chi-\eta)} < |Df^n(x)| < \rho(y) e^{n(\chi+\eta)}, \tag{1.10}$$
so in particular, $|V_n| \leq \rho(y) e^{-n(\chi-\eta)}$;
- *there exists a measurable function γ on Y, $0 < \gamma(y) < \infty$ almost everywhere, such that for all $x \in V_n$,*
  - *if f is not a C^2 cusp map then $\gamma(y) = 1$ everywhere;*
  - *if f is a C^r cusp map, $2 \leq r < \infty$, then*
$$\frac{1}{|Df^n(x)|} \left|D^i \log |Df^n(x)|\right| \leq \gamma(y) \tag{1.11}$$
for $i = 1, 2, \ldots, r-1$, for all $x \in V_n$, for all $n \geq 0$.

Idea of proof: *The results of Section 1.2 essentially imply that there is a good distortion bound on a ball around each point of radius equal to some polynomial of* $\min(|Df|, |Df|^{-1})$. *Integrability of the Lyapunov exponent means that the derivative (of f, not f^n) is bounded by subexponential sequences along almost all backwards orbits (to give a meaning to* almost all *we need the concept of the natural extension). Thus, along backwards orbits the size of the ball on which one has good distortion bounds decreases subexponentially. Positive Lyapunov exponent means that $|Df^n|$ eventually increases exponentially along backwards orbits, so pullbacks of a sufficiently small neighbourhood will inductively always land in the ball with good distortion bounds.*

Proof: Fix $\delta > 0$ such that $\delta < \eta$, $(\chi - 3\delta) > \frac{3}{\epsilon}\delta$ and, if f is C^r with constants C, p, such that $\delta pr < \chi - 2\eta$. Subsequently fix $N > 0$ large enough that, for all $n \geq N$, the following inequalities hold

$$\begin{aligned} \frac{1}{2}\log 2 + \sum_{m \geq N} e^{-m\delta} &< \log 2; && (1.12)\\ e^{-n\delta} &< c_0; && (1.13)\\ 2e^{-n(\chi-\delta)} &< \frac{1}{2}e^{-(n+1)(\delta+\frac{3}{\epsilon}\delta)}; && (1.14)\\ & && (1.15) \end{aligned}$$

where c_0 comes from Lemma 1.2.4.

Let $n \geq N$, so we have $e^{-n\delta} < c_0$. By Lemma 1.2.4, if $e^{-n\delta} < |Df(x)| < e^{n\delta}$ and $|x - x'|^\epsilon < e^{-n3\delta}$, then

$$\big|\log|Df(x)| - \log|Df(x')|\big| < e^{-n\delta}\left(\frac{|x-x'|^\epsilon}{e^{-n3\delta}}\right) < e^{-n\delta}. \tag{1.16}$$

Since $\lim_{n\to\infty}\frac{1}{n}\log|Df^n(\Pi F^{-n}y)| = \chi$ almost everywhere, almost everywhere there exists $n(y) > N$ such that for all $n \geq n(y)$,

$$e^{n(\chi-\delta)} < |Df^n(\Pi F^{-n}y)| < e^{n(\chi+\delta)} \tag{1.17}$$

and also

$$2e^{-n\delta} < |Df(\Pi F^{-n}y)| < \frac{1}{2}e^{n\delta}. \tag{1.18}$$

Denote by B_n the ball $B(\Pi F^{-n}y, \frac{1}{2}e^{-n\frac{3}{\epsilon}\delta})$. By (1.18) and (1.16), if $n \geq n(y)$ then for all $x, x' \in B_n$

$$\big|\log|Df(x)| - \log|Df(x')|\big| < e^{-n\delta}\left(\frac{|x-x'|^\epsilon}{e^{-n3\delta}}\right) < e^{-n\delta}. \tag{1.19}$$

Therefore B_n is disjoint from the singular set and so is contained in the interior of the domain of f.

Almost everywhere, by Hölder continuity of the derivative (one can use Lemma 1.2.4 for example), there exist an open balls V centered on Πy and intervals $V_{n(y)} \ni \Pi F^{-n(y)}y$ such that $f^{n(y)} : V_{n(y)} \to V$ is a diffeomorphism and, for all $x, x' \in V_{n(y)}$ and all $0 \leq i \leq n(y)$,

$$\begin{aligned} &\Big|\log|Df^i(f^{n(y)-i}x)| - \log|Df^i(f^{n(y)-i}x')|\Big| < \\ &\qquad \frac{1}{2}\log 2\left(\frac{|f^{n(y)}(x) - f^{n(y)}(x')|}{|V|}\right)^\epsilon < \frac{1}{2}\log 2, \end{aligned} \tag{1.20}$$

Together with (1.17), this implies $|V_{n(y)}| < 2e^{-n(y)(\chi-\delta)}$.

We now proceed with a simple induction argument to show the existence of a sequence V_n of intervals such that $f^n : V_n \to V$ is a diffeomorphism and for all $x, x' \in V_n$

$$\big|\log|Df^n(x)| - \log|Df^n(x')|\big| < \log 2. \tag{1.21}$$

We take V and $V_{n(y)}$ as before. Suppose that for all n such that $n_0 \geq n \geq n(y)$ there exists $V_n \ni \Pi F^{-n}y$ such that $f^n : V_n \to V$ is a diffeomorphism and, for all $x, x' \in V_n$,

$$\big|\log|Df^n(x)| - \log|Df^n(x')|\big| < \frac{1}{2}\log 2 + \sum_{i=n(y)+1}^{n} e^{-i\delta} < \log 2 - \sum_{i=n+1}^{\infty} e^{-i\delta}.$$

We want to show this statement for n_0+1. By (1.17), one has $|V_{n_0}| < 2e^{-n_0(\chi-\delta)}$. For x, x' in B_{n_0+1} one has $\frac{Df(x')}{Df(x)} < e^{\log 2} = 2$, and $|Df(\Pi F^{-n_0+1}y)| > 2e^{-(n_0+1)\delta}$, so everywhere on B_{n_0+1} one has $|DF| > e^{-(n_0+1)\delta}$. Thus

$$\begin{aligned} f(B_{n_0+1}) &\supset B(\Pi F^{-n_0}y, \frac{1}{2}e^{-(n_0+1)(\delta+\frac{3}{\epsilon}\delta)}) \\ &\supset B(\Pi F^{-n_0}y, 2e^{-n_0(\chi-\delta)}) \supset V_{n_0}. \end{aligned}$$

Thus $V_{n_0} \subset f(B_{n_0+1})$ and there exists $V_{n_0+1} \subset B_{n_0+1}$ such that $f : V_{n_0+1} \to V_{n_0}$ is a diffeomorphism and for all $x, x' \in V_{n_0+1}$,

$$\big|\log|Df(x)| - \log|Df(x')|\big| < e^{-(n_0+1)\delta}. \tag{1.22}$$

Since $\log|Df^{n+1}| = \log|Df^n| + \log|Df|$, this completes the induction argument, which implies (1.21). Statement (1.7) follows from (1.22).

Now, for $n \geq n(y)$, if $x, x' \in V_n$ then (1.21) implies that

$$\frac{|f^n(x) - f^n(x')|}{|V|} \leq 2\frac{|x-x'|}{|V_n|}.$$

By (1.19), if $x, x' \in V_n$ then

$$\big|\log|Df(x)| - \log|Df(x')|\big| < e^{-n\delta}\left(\frac{|x-x'|}{|V_n|}\right)^{\epsilon} < 2e^{-n\delta}\left(\frac{|f^n(x)-f^n(x')|}{|V|}\right)^{\epsilon}.$$

Thus,

$$\big|\log|Df^n(x)| - \log|Df^n(x')|\big| < \sum_{i=0}^{n-1}\big|\log|Df(f^ix)| - \log|Df(f^ix')|\big|$$

$$\begin{aligned} &< \left(2\frac{|f^n(x)-f^n(x')|}{|V|}\right)^{\epsilon}\left(\frac{1}{2}\log 2 + \sum_{j>n(y)} e^{-j\delta}\right) \\ &< 2\log 2\left(\frac{|f^n(x)-f^n(x')|}{|V|}\right)^{\epsilon} \end{aligned}$$

which is (1.6). Set $\alpha(y) := \frac{1}{2}|V|$. The convergence in (1.8) follows from (1.7) and the bounds come from (1.6). It remains to show (1.11).

Suppose now f is a C^r cusp map with constants C, p. For $i \geq 1$ one can expand $D^i \log|Df^n|$ as a polynomial

$$D^i \log|Df^n(x)| = \sum_{j=0}^{n-1} P\left(\frac{D^2f(f^jx)}{Df(f^jx)}, \frac{D^3f(f^jx)}{Df(f^jx)}, \ldots, \frac{D^{i+1}f(f^jx)}{Df(f^jx)}\right)$$

where P is some (calculable) polynomial of order i.

Set $\theta(x) := \min(1, |Df(x)|^p)$ and $\theta_n(x) := \min_{j=0}^{n-1}\theta(f^jx)$. Using the definition of C^r cusp maps, for each $i = 2, \ldots, r$, one has

$$\left|\frac{D^if(f^jx)}{Df(f^jx)}\right| \leq \frac{C}{\theta_n(x)}$$

for all $j = 0, \dots, n-1$. If K_i is the product of the number of terms of the polynomial P_i with the maximum of the moduli of the coefficients of P_i, then

$$\begin{aligned}
\left|D^i \log |Df^n(x)|\right| &\leq \sum_{j=0}^{n-1} \left| P\left(\frac{C}{\theta_n(x)}, \frac{C}{\theta_n(x)}, \dots, \frac{C}{\theta_n(x)}\right)\right| \\
&\leq \sum_{j=0}^{n-1} K_i \left(\frac{C}{\theta_n(x)}\right)^i \\
&\leq nC_0\theta_n(x)^{-i} \leq nC_0\theta_n(x)^{-r}
\end{aligned}$$

for some constant C_0 for all $i = 1, \dots, r-1$. We have already seen that on V_n one has $|Df| > e^{-n\delta}$ for all $n \geq n(y)$. Hence there exists a $b := b(y) > 0$ such that on V_n one has $|Df| > be^{-n\delta}$. Then for $x \in V_n$, $\theta_n(x) > be^{-n\delta p}$ so

$$\begin{aligned}
\left|D^i \log |Df^n(x)|\right| &\leq nC_0 b^{-r} e^{n\delta pr} \\
&\leq C_0 b^{-r} n e^{n(\chi - 2\eta)}.
\end{aligned}$$

By (1.10),

$$\begin{aligned}
\frac{1}{|Df^n(x)|}\left|D^i \log |Df^n(x)|\right| &\leq \rho(y) e^{-n(\chi-\eta)} C_0 b^{-r} n e^{n(\chi-2\eta)} \\
&\leq \rho(y) C_0 b^{-r} n e^{-n\eta}.
\end{aligned}$$

The righthand side is exponentially decreasing so (1.11) follows. □

1.5 Periodic points are dense

The following result is unsurprising and an easy application of Ledrappier theory.

Theorem 1.5.1 *Let f be a cusp map with an ergodic invariant probability measure μ with positive finite Lyapunov exponent χ_μ.*

The closure of the set of periodic points contains the support of μ. The closure of the set of inverse images (i.e. the closure of the backward orbit) of some periodic point contains the support of μ.

Proof: Firstly, if μ has atoms then it is concentrated on a periodic orbit and the result is trivial, so we can assume that μ is non-atomic.

Let U be an open interval of positive measure. For the first part it suffices to show that there are periodic points arbitrarily close (or in) to U. Let $(Y, F, \overline{\mu})$ be the natural extension for (f, I, μ). Modulo a set of $\overline{\mu}$-measure zero one can write

$$\Pi^{-1}U = \bigcup_{k>0} \{y \in \Pi^{-1}U : \alpha(y) > \frac{1}{k}, \rho(y) < k\},$$

where the functions come from Theorem 1.4.1, for η a small positive constant. There is therefore a $k_0 > 2$ such that the set

$$B := \{y \in \Pi^{-1}U : \alpha(y) > \frac{1}{k_0}, \rho(y) < k_0\}$$

is of positive $\overline{\mu}$ measure, i.e., $\overline{\mu}(B) > 0$, and of course $\Pi(B) \subset U$. Then there is an interval $V \subset U$ such that $|V| < k_0$ and $\overline{\mu}(B \cap \Pi^{-1}V) > 0$. Set $V_B := B \cap \Pi^{-1}V$. By ergodicity, almost every point $y \in Y$ enters V_B infinitely often. In particular, μ almost every point x in I is the projection of a point y which enters V_B at infinitely often at times n_j say. But then, by Theorem 1.4.1, x is contained in an interval of size less than $k_0 e^{-n_j(\chi-\eta)}$ which is mapped by f^{n_j} onto V. Thus almost every point is contained in arbitrarily small intervals mapped by iterates of f onto V. Taking x in the interior of V, by the Intermediate Value Theorem, there is a periodic point p in $V \subset U$.

Again using that almost every point is contained in arbitrarily small intervals which get mapped onto V, we conclude that inverse images of p are dense in the support of μ. □

As a corollary we get the following result.

Lemma 1.5.2 *Let f be a cusp map with a partition into a finite number of intervals of monotonicity and a finite number of discontinuities. Suppose f has an ergodic, invariant, probability measure μ with positive finite Lyapunov exponent χ_μ.*

Then given any interval W of positive μ-measure, there is a $j > 0$ such that $\bigcup_{k=0}^{j} f^k(W)$ contains the support of μ.

Proof: By Theorem 1.5.1, W contains a periodic point of period m say. Let $g := f^m_{|W}$. For $k \geq 0$, one has $g^k(W) \subset g^{k+1}(W)$. Let $S_n := \overline{\bigcup_{k\geq 0} g^k(U)} \setminus \bigcup_{k=0}^{n} g^k(U)$.

Since g has only a finite number of discontinuities and intervals of monotonicity, for n sufficiently large S_n has zero measure or else contains a finite number of connected components which are sent monotonically into each other by g. But this latter possibility would imply the existence of a periodic attractor which would attract almost every point in S_n. Since μ is ergodic and has positive Lyapunov exponent, this implies that in this case too $\mu(S_n) = 0$. Thus for n large enough S_n is disjoint from the support of μ. Repeating the argument for each $f^i(W)$ for $1 \leq i < m$, the result follows. □

1.6 Nice intervals

Let f be a cusp map with an ergodic invariant probability measure μ with positive finite Lyapunov exponent χ_μ. Let $(Y, F, \overline{\mu})$ be the natural extension. Modulo a set of $\overline{\mu}$-measure zero one can write

$$Y = \bigcup_{k>0} \{y \in Y : \alpha(y) > \frac{1}{k}, \rho(y) < k, \gamma(y) \leq k\}$$

where the functions come from Theorem 1.4.1, for η a small positive constant. There is therefore a $k_0 > 2$ such that the set

$$A := \{y \in Y : \alpha(y) > \frac{1}{k_0}, \rho(y) < k_0, \gamma(y) < k_0\}$$

is of positive $\overline{\mu}$ measure, i.e., $\overline{\mu}(A) > 0$.

Define the measure m_A on I by $m_A(B) = \overline{\mu}(A \cap \Pi^{-1}B)$. In other words m_A is the push-forward of $\overline{\mu}$ restricted to A. Then $m_A(I) = \overline{\mu}(A) > 0$ and m_A is non-atomic if μ is — if $m_A(x) = \overline{\mu}(A \cap \Pi^{-1}x) \leq \mu(x)$. We summarize in the following lemma:

Lemma 1.6.1 *Let f be a cusp map with an ergodic invariant probability measure μ with positive finite Lyapunov exponent χ_μ. Let $(Y, F, \overline{\mu})$ be the natural extension.*

Given any η satisfying $0 < 3\eta < \chi_\mu$ there exist a subset A of Y, a number $k_0 > 2$ and a non-atomic measure m_A on I with the following properties:

- $m_A(I) = \overline{\mu}(A) > 0$;
- $\mu(B) \geq m_A(B) = \overline{\mu}(A \cap \Pi^{-1}B)$ *for all* $B \subset I$;
- *for all* $y \in Y$, *there exists a sequence of intervals* V_n *such that*
  - $V_0 = B(\Pi y, k_0^{-1})$;
  - *for all* $n > 0$, *for all* $x, x' \in V_n$, *equations (1.6–1.9) hold;*
  - *for all* $n > 0$, *for all* $x \in V_n$,
$$|I|k_0^{-1}e^{n(\chi_\mu - \eta)} \leq |Df^n(x)| \leq k_0 e^{n(\chi_\mu+\eta)};$$
  - $|V_n| \leq k_0 e^{-n(\chi_\mu - \eta)}$;
  - *if* f *is a* C^r *cusp map,*
$$\frac{1}{|Df^n(x)|}\left|D^i \log|Df^n(x)|\right| \leq k_0$$
for $i = 1, 2, \ldots, r-1$, *for all* $x \in V_n$, *for all* $n \geq 0$.

Lemma 1.6.2 *There exists an interval* U *with* $m_A(U) > 0$ *and* $|U| < k_0^{-1}$.

Proof: Cover I by disjoint intervals of size less than k_0^{-1}. At least one of them is of positive measure with respect to m_A. □

Lemma 1.6.3 *Let* m *be a non-atomic measure on* I. *Suppose* J *is an interval such that* $m(J) > 0$. *Then for every* $\varepsilon > 0$ *there exists an interval* $J_\varepsilon \subset J$ *such that* $m(J) > m(J_\varepsilon) > 0$ *and*

$$0 < \frac{|J_\varepsilon|}{dist(J_\varepsilon, \partial J)} < \varepsilon.$$

Proof: Consider increasing open sets $U_i \subset J$ such that $\overline{U_i} \subset U_{i+1}$. Since m has no atoms in ∂J, $m(J) = \sup_i m(U_i)$ and there is an n such that $m(U_n) > 0$.

Let $\delta := \operatorname{dist}(U_n, \partial J) > 0$ and partition U_n into a finite number of (non-degenerate) intervals of size less than $\delta\varepsilon$. At least one of these intervals, call it J', must have positive measure since U_n does. Since m is non-atomic, J' has a subinterval of positive measure less than $m_A(J)$. This subinterval satisfies the claims of the lemma. □

Lemma 1.6.4 *Let* m *be a non-atomic measure on* I. *Let* Z *be the set of all* $x \in I$ *such that there exists an open interval* L *with* $m(L) = 0$ *and* $x \in \partial L$. *Then* $m(Z) = 0$.

Proof: Each open interval L such that $m(L) = 0$ is contained in a maximal such interval V. All such maximal intervals are pairwise disjoint. Thus there are at most a countable number of such maximal intervals V. But m is non-atomic so $m(\partial V) = 0$ and Z is the countable union of sets of measure zero and thus of measure zero itself. □

An open interval U is *regularly returning* if $f^n(\partial U) \cap U = \emptyset$ for all $n > 0$. If A is a connected component of $f^{-n}(U)$ and B is a connected component of $f^{-m}(U)$ with $m \geq n$, it is easy to check that either $A \cap B = \emptyset$ or $B \subset A$, so inverse images of regularly returning intervals are either nested or disjoint.

Proposition 1.6.5 *Let f be a cusp map with a non-atomic ergodic invariant probability measure μ with positive finite Lyapunov exponent χ_μ.*

Then μ almost every point is contained in arbitrarily small regularly returning open intervals, the boundaries of which are preperiodic but not periodic.

Proof: By Lemma 1.6.4 and Theorem 1.5.1, almost every point x is accumulated on both sides by inverse images of some periodic point p. Take one arbitrarily close preperiodic point not in the orbit of x on each side of x and consider the partition defined by the orbits of these two points. The interior of any partition element is regularly returning. The result follows, since the measure of the periodic orbit of p is zero, the measure being non-atomic. □

Lemma 1.6.6 *Let f be a cusp map with a non-atomic ergodic invariant probability measure μ with positive finite Lyapunov exponent χ_μ and let A, k_0 and m_A be given by Lemma 1.6.1. Let U be an interval with $m_A(U) > 0$ and $|U| < k_0^{-1}$ (such an interval exists by Lemma 1.6.2).*

Then for every $\varepsilon > 0$ there exists a regularly returning interval $W \subset U$ such that $m_A(W) > 0$ and

$$0 < \frac{|W|}{dist(W, \partial U)} < \varepsilon.$$

The boundary of W can be taken non-periodic.

Proof: By Lemma 1.6.3, there is an interval $J \subset U$ such that $m_A(J) > 0$ and

$$0 < \frac{|J|}{\mathrm{dist}(J, \partial U)} < \frac{1}{2}\varepsilon.$$

There then exists a $\delta > 0$ such that if $J_\delta = B(J, \delta)$ then

$$0 < \frac{\delta}{\mathrm{dist}(J_\delta, \partial U)} < \varepsilon. \tag{1.23}$$

By Proposition 1.6.5 there is a set X of full μ measure such that for all $x \in X$, x is contained in an open regularly returning interval of size less than δ with non-periodic boundary. Since $m_A \ll \mu$, X is also of full m_A measure.

Let Z be the set of all $x \in I$ such that there exists an open interval V with $m_A(V) = 0$ and $x \in \partial V$. Then $m_A(Z) = 0$ by Lemma 1.6.4. Then $X \setminus Z$ is of full measure with respect to m_A, so $J \cap (X \setminus Z)$ has positive measure.

Let $x \in J \cap (X \setminus Z)$. Then there is a regularly returning open interval $W \ni x$ of size less than δ. Since $x \notin Z$, $m(W) > 0$. The lemma follows by (1.23), as $W \subset J_\delta$. □

1.7 Exponent, Entropy and the Dynamical Volume Lemma

In this section we generalise the results of Hofbauer and Raith [16] to cusp maps. The Hausdorff dimension of a measure, denoted $\mathrm{HD}(\mu)$, is defined in Section 4.1, and the entropy h_μ is defined at the end of this section. In dimension one, a theorem proving that

$$\mathrm{HD}(\mu) = \frac{h_\mu}{\chi_\mu}$$

for an ergodic invariant probability measure with positive Lyapuonv exponent for some class of maps is called a *Dynamical Volume Lemma.* Hofbauer and Raith proved the Dynamical Volume

Lemma for maps for which the derivative has *bounded p-variation*, a condition not satisfied by cusp maps due to the presence of poles.

The results in this section are only proven for measures with positive *finite* Lyapunov exponent. Maps with measures of maximal entropy with infinite Lyapunov exponent and strictly positive finite entropy are simple to construct. Is the Hausdorff dimension of such a measure equal to zero?

One direction of the Dynamical Volume Lemma does not need the existence of a finite partition into intervals of monotonicity:

Theorem 1.7.1 *Let f be a cusp map with an ergodic invariant probability measure μ with positive finite Lyapunov exponent χ_μ and with entropy h_μ. Then for almost all x with respect to μ,*

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq \frac{h_\mu}{\chi_\mu}.$$

Also,

$$HD(\mu) \geq \frac{h_\mu}{\chi_\mu}$$

and so $0 \leq h_\mu \leq \chi_\mu < \infty$.

Proof: We write χ for χ_μ. Let $(Y, F, \overline{\mu})$ be the natural extension. Let η be a small positive constant and let k_0, A, m_A be given by Lemma 1.6.1. Let U be given by Lemma 1.6.2, so $|U| < k_0^{-1}$ and $m_A(U) > 0$. Let $U' \subset U$ be a regularly returning open interval with $m_A(U') > 0$ given by Lemma 1.6.6.

Set $\mathcal{U} := \{U', I \setminus U'\}$; this is a partition of I. Let $\mathcal{P} =: \{P_i\}$ be a partition into intervals of monotonicity. Then $\mathcal{P}_n := \{P_1, P_2, \dots, P_n, (I \setminus \bigcup_{i\leq n} P_i)\}$ is a partition of $I \setminus X$ for each n. Let $\mathcal{Q}_n := \bigvee_{i=0}^n f^{-i}(\mathcal{U} \vee \mathcal{P}_n)$. Then each $\mathcal{Q}_n$ is a finite partition and $\mathcal{Q}_n \leq \mathcal{Q}_{n+1}$. To apply Fact 1.7.4 we need to show that the partitions tend to the partition into points on a set of full measure. Let $U_A := A \cap \Pi^{-1}U'$. Then $\overline{\mu}(U_A) = m_A(U_A) > 0$. By ergodicity, there is a set X of full measure such that every $x \in X$ is the projection of a point y which visits U_A infinitely often. Suppose $x, x' \in X$ are in the same element of the partition $\bigvee_{i\geq 0} \mathcal{Q}_i$. Then $f^n(x)$ is always in the same element of the partition $\mathcal{P}$ as $f^n(x')$, and if $f^n(x)$ is in U then so is $f^n(x')$. It follows by injectivity of f on each P_i that if $f^{n_j} : V_j \to U$ is a diffeomorphism and $V_j \ni x$ then $V_j \ni x'$. But taking n_j such that $F^{n_j}y \in U_A$, one has $|V_j| \leq k_0 e^{-n_j(\chi-\delta)}$, so $|V_j| \to 0$ and $x = x'$.

Applying Fact 1.7.4, we have that $h_\mu = \lim_{n\to\infty} h(f, \mathcal{Q}_n)$. Set $h_n := h(f, \mathcal{Q}_n)$. By definition of entropy, $h_\mu \geq h_n \geq 0$.

Let Z be the set of all $x \in I$ such that there exists an open interval L with $m_A(L) = 0$ and $x \in \partial L$. By Lemma 1.6.4, $m_A(Z) = 0$. Let $X' = X \setminus Z$. Let $p \in X' \cap U'$. Then there are arbitrarily large n and intervals W with $p \in W \subset U'$ such that $f^n : W \to U'$ is a diffeomorphism. For such n, W, for all $j, 0 < j \leq n$, and all $i > 0$, $f^j(W) \cap \partial U' = f^j(W) \cap \partial P_i = \emptyset$, so W is contained in a partition element of $\mathcal{Q}_n$. Also, W is regularly returning since U' is, and $m_A(W) > 0$ since $p \notin Z$.

By Lemma 1.6.3, there is an interval $V \subset W$ such that $m_A(V) > 0$ and

$$0 < \frac{|V|}{\mathrm{dist}(V, \partial W)} < k_0^{-2}.$$

Then for all $x' \in V$, one has $V \subset B(x', k_0|V|) \subset W \subset U$. Let $V_A := A \cap \Pi^{-1}V$. Then by definition of m_A, $\overline{\mu}(V_A) = m_A(V) > 0$.

Then by the Birkhoff Ergodic theorem, almost every $x \in I$ is the projection of a point y which visits V_A infinitely often at times $n_j(y)$ with asymptotic frequency $\lim_{j\to\infty} \frac{j}{n_j(y)} = \overline{\mu}(V_A)$. Examining the ratio of successive terms, it follows that

$$\lim_{j\to\infty} \frac{n_{j+1}(y)}{n_j(y)} = 1. \tag{1.24}$$

Fix such an x. Then there exists an increasing sequence n_j and intervals W_j containing x such that $f^{n_j}(x)$ is in V and $f^{n_j} : W_j \to B(f^{n_j}(x), k_0^{-1})$ is a diffeomorphism with distortion bounded by k_0 such that

$$k_0^{-1} e^{n(\chi-\eta)} < |Df^{n_j}(x')| < k_0 e^{n(\chi+\eta)} \tag{1.25}$$

for all $x' \in W_j$.

Let $s_j > 0$ be the maximal s such that $f^{n_j}(B(x,s)) \subset B(f^{n_j}x, k_0|V|)$. Then $f^{n_j}(B(x,s_j)) \subset W$ so $B(x,s_j) \subset W_j$. Then the distortion bound and (1.25) imply that

$$|V|e^{-n_j(\chi+\eta)} < s_j. \tag{1.26}$$

This, together with (1.24), implies that

$$\frac{-n_j}{\log s_{j+1}} = \frac{-n_{j+1}}{\log s_{j+1}} \frac{n_j}{n_{j+1}} \geq \frac{1}{\chi - 2\eta} \tag{1.27}$$

for all j larger than some j_0.

Let L and $i \geq 0$ be such that $f^i : L \to W$ is a diffeomorphism. Then we claim that L is contained in an element of the partition $\bigvee_{k=0}^{i} f^{-k}\mathcal{Q}_n$. Indeed, $\bigvee_{k=0}^{i} f^{-k}\mathcal{Q}_n \leq \mathcal{Q}_{n+i}$. The same argument works as that which showed W was contained in an element of $\mathcal{Q}_n$.

Consider x, n_j and W_j from before. By the preceding paragraph, W_j is contained in an element, call it Q_j, of the partition $\bigvee_{k=0}^{n_j-n} f^{-k}\mathcal{Q}_n$. By the Shannon-McMillan-Breiman theorem (Fact 1.7.5), almost everywhere

$$\lim_{j\to\infty} \frac{-1}{n_j - n} \log \mu(Q_j) = \lim_{j\to\infty} \frac{-1}{n_j} \log \mu(Q_j) = h_n. \tag{1.28}$$

But

$$B(x, s_j) \subset W_j \subset Q_j,$$

so

$$-\log \mu(Q_j) < -\log \mu(B(x,s_j)).$$

Setting $K_n := \min(n, h_n)$, for all j larger than some $j_1 \geq j_0$, one then has

$$\frac{-1}{n_j} \log \mu(B(x,s_j)) > K_n - \eta.$$

For $j \geq j_1$ and t such that $s_{j+1} < t < s_j$,

$$\frac{\log \mu(B(x,t))}{\log t} \geq \frac{\log \mu(B(x,s_j))}{\log s_{j+1}} \geq \frac{-n_j}{\log s_{j+1}}(K_n - \eta).$$

Using (1.27),

$$\frac{\log \mu(B(x,t))}{\log t} \geq \frac{1}{\chi - 2\eta}(K_n - \eta)$$

so

$$\liminf_{t\searrow 0} \frac{\log \mu(B(x,t))}{\log t} \geq \frac{K_n - \eta}{\chi - 2\eta}.$$

This holds on a set of full measure for all small $\eta > 0$, so, on a set of full measure,

$$\liminf_{t\searrow 0} \frac{\log \mu(B(x,t))}{\log t} \geq \frac{K_n}{\chi}. \tag{1.29}$$

Taking increasing (by n) pairs n, W, K_n increases to h_μ so

$$\liminf_{t\searrow 0} \frac{\log \mu(B(x,t))}{\log t} \geq \frac{h_\mu}{\chi}, \tag{1.30}$$

and this holds for all x from a set of full measure.

This implies by Fact 4.1.1 that $\mathrm{HD}(\mu) \geq \frac{h_\mu}{\chi}$. In particular, $h_\mu \leq \chi$ since the dimension of μ cannot be greater than one. □

The following theorem is the **Dynamical Volume Lemma for cusp maps:**

Theorem 1.7.2 *Let f be a cusp map with a partition into a finite number of intervals of monotonicity. Suppose f has an ergodic invariant probability measure μ with positive finite Lyapunov exponent χ_μ and entropy h_μ. Then for almost all x with respect to μ,*

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \frac{h_\mu}{\chi_\mu},$$

In particular,

$$HD(\mu) = \frac{h_\mu}{\chi_\mu}.$$

Proof: We write χ for χ_μ. Let $(Y, F, \overline{\mu})$ be the natural extension. Let η be a small positive constant and let k_0, A, m_A be given by Lemma 1.6.1. Let U be given by Lemma 1.6.2, so $|U| < k_0^{-1}$ and $m_A(U) > 0$. Let $W \subset U$ be a regularly returning open interval with $m_A(W) > 0$ given by Lemma 1.6.6, such that

$$0 < \frac{|W|}{\mathrm{dist}(W, \partial U)} < k_0^{-2}.$$

Let $W_A = A \cap \Pi^{-1} W$. Let $\{Q_1, \ldots, Q_p\}$ be a finite partition of I into intervals of monotonicity of f. Let X be a set of full μ measure such that for all $x \in X$, $\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)| = \chi$, and such that each x in X is the projection of a point y which enters W_A infinitely often at times n_j, and that the n_j satisfy

$$\lim_{j\to\infty} \frac{n_j}{n_{j+1}} = 1. \tag{1.31}$$

Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\} \vee \{W, I \setminus W\}$. Then $\mathcal{Q}$ is generating with respect to μ. Indeed, suppose $x, x' \in X$ are in the same element of $\bigvee_{i\geq 0} f^{-i}\mathcal{Q}$. But $x \in X$ implies that there are intervals V_j and an increasing sequence n_j such that $f^{n_j} : V_j \to W$ is a diffeomorphism and $|V_j| \leq k_0 e^{-n_j(\chi-\eta)}$ for arbitrarily large n_j. But $V_j \in \bigvee_{i=0}^{n_j} \mathcal{Q}$ by monotonicity on Q_i, so $|x - x'| \leq k_0 e^{-n_j(\chi-\eta)}$ which implies that $x = x'$.

Denote by $Q_n(x)$ the element of $\bigvee_{i=0}^{n} f^{-i}\mathcal{Q}$ containing x. By the Shannon-McMillan-Breiman Theorem (Fact 1.7.5), there is a set $X' \subset X$ of full measure such that for all $x \in X'$

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu(Q_n(x)) = h_\mu.$$

For $x \in X' \subset X$, consider the V_j such that $f^{n_j}(x) \in W$, $f^{n_j} : V_j \to U$ is a diffeomorphism with distortion bounded by k_0 and that

$$k_0^{-1} e^{n_j(\chi-\delta)} < |Df^n_{|V_j}| < k_0 e^{n_j(\chi+\delta)}.$$

Let $r_j > 0$ be the maximal r such that $B(x,r) \subset V_j$. Suppose W_j is mapped by f^{n_j} diffeomorphically onto W. The distortion bound and (1.31) imply that

$$\frac{|W_j|}{\operatorname{dist}(W_j, \partial V_j)} < k_0^{-1} < \frac{1}{2},$$

so $B(x,r_j) \supset W_j$. Note that $W_j = Q_{n_j}(x)$, and for all j large, $\frac{-1}{n_j} \log \mu(Q_{n_j}(x)) < h_\mu + \eta$, so for all j greater than some j_0,

$$\frac{-1}{n_j} \log \mu(B(x,r_j)) < h_\mu + \eta,$$

Also note that $r_j \le k_0 e^{-n_j(\chi-\eta)}$ so $\frac{-1}{n_j} \log r_j \ge \chi - 2\eta$ for all j greater than some $j_1 \ge j_0$, and that r_j is decreasing. Let $j \ge j_1$ and r satisfy $r_j > r \ge r_{j+1}$. Then

$$\frac{\log \mu(B(x,r))}{\log r} \le \frac{\log \mu(B(x,r_{j+1}))}{\log r_j} \le \frac{n_{j+1}}{n_j} \frac{h_\mu + \eta}{\chi - 2\eta}.$$

But $\lim_{j\to\infty} \frac{n_{j+1}}{n_j} = 1$ so, on a set of full measure,

$$\limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \le \frac{h_\mu + \eta}{\chi - 2\eta},$$

and η is arbitrarily small, so

$$\limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \le \frac{h_\mu}{\chi}.$$

Together with Theorem 1.7.1, one has for almost all x,

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \frac{h_\mu}{\chi}.$$

By Fact 4.1.1, $\mathrm{HD}(\mu) = \frac{h_\mu}{\chi}$. This completes the proof of the theorem. $\square$

Entropy

In this note we present some definitions and results concerning entropy, assuming knowledge of measures and algebras, following [44], which we cite as a general reference.

In what follows, $X := (X, \mathcal{B}, m)$ shall be a probability space and $T : X \to X$ a measure-preserving transformation (that is, m is invariant under T, or equivalently $m(T^{-1}A) = m(A)$). A partition of X is a collection of disjoint elements of $\mathcal{B}$ whose union is X.

If ξ and η are finite partitions then $\xi \le \eta$ shall mean that each element of ξ is a union of elements of η. If $\xi = \{A_1, \ldots, A_n\}, \eta = \{C_1, \ldots, C_k\}$ then their join is the partition

$$\xi \vee \eta = \{A_i \cap C_j : 1 \le i \le n, 1 \le j \le k\}.$$

For each $k \ge 0$, $T^{-n}\xi = \{T^{-k}A_1, \ldots, T^{-k}A_n\}$ is a finite partition, as is $\bigvee_{i=0}^k T^{-i}\xi$.

The entropy $H(\xi)$ of a finite partition $\xi = \{A_1, \ldots, A_k\}$ is defined as

$$H(\xi) = -\sum_{i=1}^{k} m(A_i) \log m(A_i),$$

and the entropy of T with respect to ξ is

$$h(T, \xi) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{k} T^{-n}\xi\right).$$

This limit always exists and is finite and non-negative.

Definition 1.7.3 *If $T : X \to X$ is a measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$ then $h(T) = \sup h(T, \mathcal{A})$, where the supremum is taken over all finite sub-algebras $\mathcal{A}$ of $\mathcal{B}$, is called the* entropy *of T. Equivalently, $h(T) = \sup h(T, \xi)$, where the supremum is taken over all finite partitions of $(X, \mathcal{B}, m)$.*

Fact 1.7.4 *Let $(X, \mathcal{B}, m)$ be a probability space and let $\{\mathcal{A}_n\}_1^\infty$ be finite sub-algebras of $\mathcal{B}$ such that $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ and $\bigvee_{n=1}^{\infty} \mathcal{A}_n \doteq \mathcal{B}$. If $T : X \to X$ is measure-preserving then $h(T) = \lim_{n\to\infty} h(T, \mathcal{A}_n)$.*

Fact 1.7.5 ([29], p.39) Shannon-McMillan-Breiman: *Let ξ be a finite partition of the probability space $(X, \mathcal{B}, m)$ and suppose $T : X \to X$ is ergodic. Let $\xi_n = \bigvee_{i=0}^{n-1} T^{-}\xi$ and $\xi(x)$ the partition element containing the point x. Then almost everywhere, $\lim_{n\to\infty} -\frac{1}{n} \log m(\xi_n(x)) = h_\mu$.*

1.8 Ruelle's Inequality stymied

In [35], Ruelle proved an inequality relating entropy and Lyapunov exponents for C^1 maps of compact manifolds. This of course holds for C^1 maps of the interval.

Fact 1.8.1 *Let $f : I \to I$ be a C^1 map of the interal I with an ergodic invariant probability measure μ. Then*

$$h_\mu \leq \max\left(0, \int \log |Df| d\mu\right).$$

As we have seen in the preceding sections, if a measure has positive finite Lyapunov exponent this gives a lot of power. Ruelle's Inequality is fundamental: whenever a measure has positive entropy it has positive Lyapunov exponent, in the C^1 setting.

Unfortunately, for cusp maps $\log |Df|$ is **not necesarily integrable**, even for absolutely continuous invariant measures for cusp maps smoothly conjugate to the Chebyshev map $4x(1-x)$ on the interior of $[0, 1]$. This is the first major difference we notice with ordinary $C^{1+\epsilon}$ maps.

Is it possible to come up with an analogue for cusp maps? We suspect it is possible when the singular set is non-recurrent, or maybe even just slow-recurrent. In the non-recurrent case one can show that there exists an $M < \infty$ such that the lower Lyapunov exponent of *every* point in the invariant set is $< M$. In this situation $\log |Df|$ should be integrable...

1.9 Absolutely continuous measures

An analogous result to the following was proven in [21] in a more restrictive setting. We provide a short proof of the existence of Pesin's partition, taking advantage of the existence of regularly returning intervals.

Theorem 1.9.1 *Let f be a cusp map with a partition into a finite number of intervals of monotonicity. Suppose f has an ergodic, invariant, probability measure μ with positive Lyapunov exponent and with natural extension $(Y, F, \overline{\mu})$. There exists a measurable partition η of Y with the following properties:*

1. *the partition η is increasing by F, $F^{-1}\eta > \eta$, and generates;*
2. *entropy of μ is given by $h(\mu) = H(F^{-1}\eta/\eta)$;*
3. *for $\overline{\mu}$ almost every point y, for all $k \geq 0$, $\Pi F^{-k}(\eta(y))$ is contained in an interval of monotonicity of f, where $\eta(y)$ denotes the element of η containing y;*
4. *for $\overline{\mu}$ almost every point y, $0 < \int_{\eta(y)} \Delta(y, y')dy' < \infty$, where the integration is with respect to the natural Lebesgue measure (i.e. the pullback of Lebesgue measure by $\Pi_{|\eta(y)}$) on each element of η and*
$$\Delta(y, y') = \lim_{n\to\infty} \frac{Df^n(\Pi F^{-n}y)}{Df^n(\Pi F^{-n}y')};$$
5. *let k_0, A, m_A be given by Lemma 1.6.1; there exists a regularly returning interval W with non-periodic boundary such that, on a subset of A of positive measure, $\eta(y) = (W; \Pi' y)$.*

Proof: Let k_0, A, m_A be given by Lemma 1.6.1, and W a regularly returning open interval with non-periodic boundary such that $|W| < k_0^{-1}$ and $m_A(W) > 0$. This exists by Lemma 1.6.6. Let $\mathcal{Q}$ be the partition given by the join of $\{W, I \setminus W\}$ with the finite partition into intervals of monotonicity. We saw in the proof of Theorem 1.7.2 that $\mathcal{Q}$ is a generating partition.

Since $\mathcal{Q}$ is finite and generating, $h(f, \mathcal{Q}) = h_\mu$. Let ξ be the partition of Y given by $\xi := \{\Pi^{-1}Q : Q \in \mathcal{Q}\}$ and let $\eta := \bigvee_{i=0}^{\infty} F^i\xi$. Then ξ is a finite, generating partition of Y, and

$$h_{\overline{\mu}} = h(F, \xi) = H(\xi | \bigvee_{i>0} F^i\xi) = H(F^{-1}\eta|\eta) = h(f, \mathcal{Q}) = h_\mu.$$

Let $W_A := A \cap \Pi^{-1}W$. For almost all y there exists $k > 0$ such that $F^k(y) \in W_A$, so there exist intervals $V_n \ni \Pi F^{-n}y$ for all $n \geq 0$ such that $f^{n+k} : V_n \to W$ is a diffeomorphism. Since W is regularly returning, $V_n \cap \partial W = \emptyset$. It follows that $\eta(y) \supset (V_0; \Pi' y)$, and that if $y \in W_A$, then $\eta(y) = (W; \Pi' y)$.

It remains to show that the integral is positive and finite. For all $k \geq 0$, one has $\eta(y) = F^k([F^{-k}\eta](F^{-k}y))$, so $[F^{-k}\eta](F^{-k}y)$ contains an open interval for almost every y. For almost every y, $|Df(\Pi F^{-i}y)|$ is positive and finite for all i, and by ergodicity there exists a $k \geq 0$ such that $F^{-k}y \in W_A$. One has

$$\int_{\eta(y)} \Delta(y, y')dy' = \int_{F^k([F^{-k}\eta](F^{-k}y))} \Delta(F^{-k}y, F^{-k}y') \prod_{i=1}^{k} \frac{Df(\Pi F^{-i}y)}{Df(\Pi F^{-i}y')} dy'$$

$$= \prod_{i=1}^{k} |Df(\Pi F^{-i}y)| \int_{[F^{-k}\eta](F^{-k}y)} \Delta(F^{-k}y, y')dy'.$$

By (1.8) the last integrand is bounded inside $(4^{-1}, 4)$ since $[F^{-k}\eta](F^{-k}y) \subset \eta(F^{-k}y)$ and $F^{-k}y \in W_A$, so the integrals are positive and finite, completing the proof. □

The Rohlin decomposition $p(y, \cdot)$ for the measure $\overline{\mu}$ with respect to the partition η is a conditional probability measure on each partition element of Y such that, for any measurable set $B \subset Y$ one has

$$\overline{\mu}(B) = \int_Y p(y, B) d\overline{\mu} = \int_Y p(y, B \cap \eta(y)) d\overline{\mu}.$$

By Theorem 1.9.1, if $n > 0$,

$$nh_\mu = H(F^{-n}\eta/\eta) = -\int \log p(y, [F^{-n}\eta](y)) d\overline{\mu}. \tag{1.32}$$

Theorem 1.9.2 *Let f be a cusp map with a partition into a finite number of intervals of monotonicity. Suppose f has an ergodic, invariant, probability measure μ with positive finite Lyapunov exponent χ_μ and entropy h_μ satisying $\chi_\mu = h_\mu$. Let $(Y, F, \overline{\mu})$ be the natural extension.*

Then the Rohlin decomposition for the measure $\overline{\mu}$ with respect to the partition η of Proposition 1.9.1 is given by $q(y, B)$, for y in Y and B a measurable subset, where

$$q(y, B) := \frac{\int_{B \cap \eta(y)} \Delta(y, y') dy'}{\int_{\eta(y)} \Delta(y, y') dy'}. \tag{1.33}$$

Proof: The proof carries over from [21], proposition 3.6, without modification. We include it for completeness.

It follows from (1.33) that $q(y, \cdot)$ is a probability measure on $\eta(y)$, and that if $\eta(y_1) = \eta(y_2)$ then $q(y_1, \cdot) = q(y_2, \cdot)$ by multiplying above and below by $\Delta(y_1, y_2)$. Set $k(y) := \int_{\eta(y)} \Delta(y, y') dy'$. Then, by definition,

$$\begin{aligned} q(y, [F^{-n}\eta](y)) &= \frac{\int_{[F^{-n}\eta](y)} \Delta(y, y') dy'}{k(y)} \\ &= \frac{1}{k(y)} \int_{F^{-n}(\eta(F^n(y))} \Delta(F^n y, F^n y') \frac{Df^n(\Pi y')}{Df^n(\Pi y)} dy' \\ &= \frac{k(F^n y)}{|Df^n(\Pi y)| k(y)}. \end{aligned}$$

Taking logs,

$$\log q(y, [F^{-n}\eta](y)) = \log k(F^n y) - \log |Df^n(\Pi y)| - \log k(y).$$

Integrating, applying [21] lemma 3.5,

$$-\int \log q(y, [F^{-n}\eta](y)) d\overline{\mu} = n\chi_\mu.$$

Let $p(y, \cdot)$ be the Rohlin decomposition for the measure $\overline{\mu}$ with respect to the partition η. By hypothesis and (1.32),

$$\int \log q(y, [F^{-n}\eta](y)) d\overline{\mu} = \int \log p(y, [F^{-n}\eta](y)) d\overline{\mu},$$

so

$$\int \log \frac{q(y, [F^{-n}\eta](y))}{p(y, [F^{-n}\eta](y))} d\overline{\mu} = 0. \tag{1.34}$$

Consider the measure $\overline{\nu}$ defined on the σ-algebra generated by $[F^{-n}\eta]$ defined by

$$\overline{\nu}(B) = \int q(y, B) d\overline{\mu}.$$

By (1.34),

$$\int \left(\log \frac{d\overline{\nu}}{d\overline{\mu}}\right) d\overline{\mu} = 0.$$

Both measures are probabilities, so by concavity of logarithm, $\overline{\nu} = \overline{\mu}$, so $q(y, B) = p(y, B)$ if B is in the σ-algebra generated by $[F^{-n}\eta]$. As η generates and $\eta < F^{-1}\eta$, one concludes that q and p coincide almost everywhere. □

Corollary 1.9.3 *Let f be a cusp map with a partition into a finite number of intervals of monotonicity. Suppose f has an ergodic, invariant, probability measure μ with positive finite Lyapunov exponent χ_μ.*

If the Hausdorff dimension of μ is equal to 1 then μ is absolutely continuous.

Proof: By the Dynamic Volume Lemma (Theorem 1.7.2) if the measure has Hausdorff dimension 1 then the entropy $h(\mu) = \chi_\mu$. Let B be a subset of zero Lebesgue measure. Then $q(y, B) = 0$ for all y, so $\overline{\mu}(\Pi^{-1}B) = 0 = \mu(B)$. □

The following result is new even for $C^{1+\epsilon}$ maps. We recently discovered that Keller [18] has proven this for unimodal maps with negative Schwarzian derivative and non-flat critical points.

Theorem 1.9.4 *Let f be a cusp map with a partition into a finite number of intervals of monotonicity. Suppose f has an ergodic, invariant, probability measure μ with positive finite Lyapunov exponent χ_μ.*

If μ is absolutely continuous with respect to Lebesgue measure λ then there exists $\nu > 0$ and an open interval such that the density of μ is bounded from below by some constant $\nu > 0$ Lebesgue almost everywhere on the interval.

Proof: Let k_0, A, m_A be given by Lemma 1.6.1 and η, W be given by Theorem 1.9.1. Let W_A be the subset of A of positive measure given by Theorem 1.9.1 such that, for all $y \in W_A$, $\eta(y) = (W; \Pi' y)$.We have $4^{-1} \leq \Delta(y, y') \leq 4$ if $y' \in \eta(y)$ for all y in W_A.

Thus, for $y \in W_A$, the density of the Rohlin decomposition on the partition element $\eta(y) = (W; \Pi' y)$ containing y is bounded inside $[4^{-2}|W|^{-1}, 4^2|W|^{-1}]$. Let $W_A^* := \bigcup_{y \in W_A} (W; \Pi' y) = \bigcup_{y \in W_A} \eta(y)$. If we set $\overline{\mu}_A := \overline{\mu}_{|W_A^*}$, then $\Pi_* \overline{\mu}_A$ has support W and the density of $\Pi_* \overline{\mu}_A$ is bounded inside

$$[4^{-2}|W|^{-1}\overline{\mu}(W_A^*), 4^2|W|^{-1}\overline{\mu}(W_A^*)].$$

Since $\overline{\mu}_A$ is a restriction of $\overline{\mu}$, the density of $\Pi_* \overline{\mu}_A$ is less than that of μ almost everywhere. In particular, the density of μ on W is bounded from below by $\nu := 4^{-2}|W|^{-1}\overline{\mu}(W_A^*) > 0$ as required. □

Lemma 1.9.5 *Let g be a piecewise C^1 map with an absolutely continuous invariant probability measure with density ρ. Then*

$$\rho(x) = \sum_{w \in g^{-1}x} \frac{1}{|Dg(w)|} \rho(w).$$

In particular, $\rho(g(x)) \geq \frac{1}{|Dg(x)|} \rho(x)$.

Proof: This is just the change of variables formula. □

Theorem 1.9.6 *Let $f : I \to I$ be a cusp map with a partition into a finite number of intervals of monotonicity and only a finite number of discontinuities. Suppose f has an ergodic, invariant, probability measure μ with positive finite Lyapunov exponent χ_μ, and that μ is absolutely continuous with respect to Lebesgue measure.*

Then the support of μ is a finite union of intervals X on which μ is equivalent to Lebesgue.

Moreover, if, in addition, f has no poles, then the density of μ on X is bounded from below by a positive constant $\delta > 0$.

Proof: Let U be the interval and ν the constant given by Theorem 1.9.4. By Lemma 1.5.2, there exists a $j > 0$ such that $\bigcup_{k=0}^{j} f^k U$ contains the support of μ, and thus equals the support of μ.

If f has no poles, then $|Df|$ is bounded by a positive constant L say. By Lemma 1.9.5, the density of μ is bounded from below by $\frac{\nu}{L^j}$ on its support. □

Theorem 1.9.7 *Let $f : I \to I$ be a $C^{1+\epsilon}$ map of the interval I. Suppose f has an invariant ergodic probability measure μ with positive entropy or with positive Lyapunov exponent such that μ is absolutely continuous with respect to Lebesgue measure.*

Then the support of μ is a finite union of intervals on which

$$\int \log |Df(x)| dx > -\infty,$$

where integration is with respect to Lebesgue measure.

Proof: By Ruelle's Inequality, positive entropy implies that the Lyapunov exponent is positive (and finite, since $|Df|$ is bounded). We can apply the previous theorem, since f extends to be a continuous cusp map on some larger interval. The lower bound on the density implies that anything integrable with respect to μ is integrable with respect to Lebesgue measure on the support of μ. □

This preceding theorem we like, but when can one apply it? Do all absolutely continuous invariant probability measures have positive entropy? No, but if they are ergodic? Perhaps — we are not aware of any counterexamples, but this is hard to show. In the unimodal setting with a non-flat critical point and negative Schwarzian, Blokh and Lyubich [2] prove that every acip has positive entropy.

Unfortunately, flat critical points kill density estimates: a forward invariant set can have density one at the critical point but lower one-sided density of zero at the corresponding side of the critical value, even for C^∞ symmetric maps with negative Schwarzian derivative. Nevertheless, we have the following:

Corollary 1.9.8 *Let g_b be a map from the unimodal family*

$$-1 + b\left(1 - e^{-1+|x|^{-a}}\right).$$

If $a \geq 1$ then no ergodic absolutely continuous invariant probability measure with positive entropy exists.

Proof: Were such a measure to exist, it would have a positive Lyapunov exponent by Ruelle's Inequality and its support would contain the critical point at 0. Now Theorem 1.9.7 implies that $\log |Dg_b(x)|$ is integrable with respect to Lebesgue measure. But $\log |Dg_b(x)| = h(x) + |x|^{-a}$, where h is integrable with respect to Lebesgue measure. If $a \geq 1$ then $|x|^{-a}$ is not integrable. □

Hans Thunberg [39] proves that if $a < \frac{1}{8}$ then there is a positive measure set of parameters b for which an ergodic absolutely continuous invariant probability measure (with positive entropy) exists, and asks whether for $a \geq 1$ no such measure exists. Misiurewicz and Benedicks [1], giving a partial repsonse to a question of Collet and Eckmann, showed that for Misiurewicz maps with negative Schwarzian derivative an absolutely continuous invariant probability measure exists if and only if $\log |Df|$ is integrable. The preceding theorem and corollary provide a more complete response to these questions.

1.9.1 Absolutely continuous measures are generated by Markov maps

When trying to prove the existence of absolutely continuous invariant probability measures, a standard and fruitful technique is to show the existence of an expanding induced map whose domain has full measure in its range. One can push around the absolutely continuous probability measure for the induced map to get an absolutely continuous invariant measure for the original map. If the return time for such a *Markov map* is integrable with respect to Lebesgue, then the resultant measure is finite and can be normalised to give a probability measure.

A natural question is whether for *all* absolutely continuous invariant probability measures the measure can be produced from such an induced map. Henk Bruin [3] has shown that this is the case for unimodal maps with negative Schwarzian derivative and non-flat critical points. He also shows that it is true when "natural" Markov maps exist, but does not prove that they do outside of the unimodal negative Schwarzian non-flat critical point case. We prove a stronger result, dropping the condition on non-flatness of critical points, admitting poles and weakening the condition on the number of singularities.

This next definition shall be repeated in Section 3.1; we include it here for convenience. Let U_i be a finite or countable collection of disjoint closed subintervals of a closed interval U such that the Lebesgue measure of $U \setminus \bigcup_i U_i$ is zero. We call a map $\phi : \bigcup_i U_i \to U$ a *full expanding* $C^{1+\epsilon}$ *Markov map* if ϕ restricted to each U_i is a diffeomorphism onto U, $\#\{U_i\} \geq 2$, $|D\phi| \geq \lambda > 1$ for some constant λ and there exists a constant $C > 0$ such that

$$|D\phi(x) - D\phi(x')| \leq C|\phi(x) - \phi(x')|^\epsilon$$

for all x, x' in the same connected component U_i of the domain of ϕ. Fact 3.1.17 implies that ϕ has an absolutely continuous invariant probability measure ν say whose density is bounded away from zero and infinity on U.

Definition 1.9.9 *If g is a map defined on some space X containing $\bigcup_i U_i$ then we say ϕ is an* induced Markov map *for g if ϕ is a full expanding $C^{1+\epsilon}$ Markov map which coincides with an iterate g^{n_i} of g on each U_i, and ϕ is a C^r induced Markov map if ϕ is an induced Markov map and ϕ is C^r (Definition 3.1.14). Writing $n(x) = n_i$ if $x \in U_i$, if*

$$\int_U n(x)dx = \sum_i n_i |U_i| < \infty$$

then we say ϕ has integrable return time.

If ϕ is an induced Markov map for g with integrable return time then, as is well known,

$$\sum_i \sum_{j=0}^{n_i - 1} g_*^j \nu_{|U_i}$$

is a finite ergodic absolutely continuous invariant measure for g which, when normalised, is an ergodic absolutely continuous invariant probability measure μ for g. We say *μ is generated by* the induced Markov map ϕ.

Theorem 1.9.10 *Let f be a cusp map with a partition into a finite number of intervals of monotonicity. Suppose f has an ergodic, absolutely continuous, invariant, probability measure μ with positive finite Lyapunov exponent χ_μ.*

Then μ is generated by an induced Markov map for f. If f is C^r then μ is generated by a C^r induced Markov map for f.

Proof: Let k_0, A, m_A be given by Lemma 1.6.1 and η, W be given by Theorem 1.9.1. Let W_A be the subset of A of positive measure given by Theorem 1.9.1 such that, for all $y \in W_A$, $\eta(y) = (W; \Pi' y)$. As per the proof of Theorem 1.9.4, set $W_A^* := \bigcup_{y \in W_A}(W; \Pi' y) = \bigcup_{y \in W_A} \eta(y)$ and set $\overline{\mu}_A := \overline{\mu}_{|W_A^*}$. We saw that $\Pi_* \overline{\mu}_A$ has support W and the density of $\Pi_* \overline{\mu}_A$ is bounded from below by a constant $\nu > 0$.

By Lemma 1.6.1, if x is the projection of a point y mapped by F^n into W_A^*, then, since there is a point of W_A in $(W; \Pi' F^n y)$, x is contained in an interval V_n mapped diffeomorphically onto $\overline{W}$ by f^n and, for all $x \in V_n$, $|Df^n(x)| \geq |I| k_0^{-1} e^{n(\chi_\mu - \gamma)}$ where $\gamma < \chi_\mu$ is a small positive constant. Also by Lemma 1.6.1, if f is C^r then for all x in V_n and for $i = 1, 2, \ldots, r-1$ one has

$$\frac{1}{|Df^n(x)|} \left| D^i \log |Df^n| \right| < k_0.$$

Fix an $N > 0$ such that $|I| k_0^{-1} e^{n(\chi_\mu - \gamma)} \geq \lambda > 1$ for some λ. For $y \in W_A^*$, let $m(y)$ denote the time of the Nth return of y to W_A^*. By ergodicity, $m(y)$ exists and is finite almost everywhere on W_A^*, $m(y) \geq N$ and, by a simple generalisation of Kac' Theorem ([31], theorem 2.4.6),

$$\int_{W_A^*} m(y) d\overline{\mu} = N.$$

Let $n(x) := \min\{m(y) : \Pi y = x, y \in W_A^*\}$. Then $n(x)$ is defined almost everywhere in W and there exists an interval $U_x \ni x$ such that $f^n : U_x \to \overline{W}$ is a diffeomorphism with $|Df_{|U_x}| \geq \lambda$. Moreover, by inequality (1.6), one has $|\log |Df^n(x)| - \log |Df^n(x')|| < C |f^n(x) - f^n(x')|^\epsilon$ for $C := |W|^{-\epsilon} \log 4$. We now show $n(x)$ is integrable:

$$\begin{aligned}
\int_W n(x) dx &\leq \frac{1}{\nu} \int_W n(x) d\Pi_* \overline{\mu}_A \\
&= \frac{1}{\nu} \int_{W_A^*} n(\Pi y) d\overline{\mu}_A \\
&\leq \frac{1}{\nu} \int_{W_A^*} m(y) d\overline{\mu}_A \\
&= \frac{1}{\nu} \int_{W_A^*} m(y) d\overline{\mu} \\
&= \frac{1}{\nu} N.
\end{aligned}$$

Since W is regularly returning, each U_x is contained in W. It is straightforward to verify that $n(x)$ is constant on the interior of U_x. Thus if $U_x \cap U_{x'} \neq \emptyset$ then either $U_x = U_{x'}$ or they only overlap on the boundary. This latter possibility is impossible since it would imply that the boundary of W would be periodic.

The map $\phi(x) := f^{n(x)}(x)$ defined on $\bigcup_{x\in W} U_x$ is therefore an induced Markov map with integrable return time. (For the union over all x in W to have a sense we set $U_x := \emptyset$ where $n(x)$ is not defined.) Thus there is an ergodic invariant absolutely continuous invariant probability measure μ' generated by ϕ. The support of μ' coincides with that of μ and both have positive density so by ergodicity they are equal and μ is generated by ϕ. □

Theorem 1.9.11 *Let f be a $C^{1+\epsilon}$ map with a partition into a finite number of intervals of monotonicity.*

Then f has an ergodic, absolutely continuous, invariant, probability measure with positive finite Lyapunov exponent if and only if there exists an induced Markov map with integrable return time.

Proof: One direction is given by Theorem 1.9.10. On the other hand, if there exists an induced Markov map with integrable return time, the measure generated by it will be an ergodic, absolutely continuous, invariant probability measure. The entropy of the measure is positive because it is non-invertible almost everywhere on the range of the Markov map. Then Ruelle's Inequality implies that the Lyapunov exponent is positive, and it is finite because the derivative is bounded.
□

Remark: There are induced Markov maps with integrable return time for cusp maps such that the generated measure has non-integrable Lyapunov exponent.

Summarising the preceding results, we obtain the following:

Theorem 1.9.12 *Let f be a cusp map with a partition into a finite number of intervals of monotonicity. Suppose μ is an ergodic, invariant, probability measure for f with positive finite Lyapunov exponent χ_μ. The following conditions are equivalent:*

1. *μ is absolutely continuous with respect to Lebesgue measure;*
2. *$h_\mu = \chi_\mu$;*
3. *the density of μ with respect to Lebesgue measure (exists and) is bounded from below by a positive constant on an open interval;*
4. *μ is generated by an induced Markov map for f with integrable return time.*

Ledrappier [21] showed equivalence between 1 and 2 for $C^{1+\epsilon}$ maps with non-flat critical points. Bruin [3] showed equivalence of all four conditions in the case of unimodal maps with non-flat critical points and negative Schwarzian derivative. Theorem 1.9.12 represents a substantial improvement.

2

Rational maps with absolutely continuous measures

Contents

2.1 Introduction

The Julia set of a rational map often is a fractal set of zero Lebesgue measure. When this happens the notion of conformal measure replaces that of Lebesgue measure, and it is natural to ask if there are absolutely continuous invariant probability measures (*acips*) with respect to some conformal measure. Conformal measures always exist [38]. It has been shown ([11] amongst others) that acips with respect to conformal measures often exist.

When acips with respect to a conformal measure can be shown to exist, the construction can be used to show the measures are equivalent and that the acips have density bounded from below away from zero. We show that this is *always* true. Moreover, whenever such an acip exists, there exists an expanding induced Markov map which generates the measure.

As usual when dealing with rational maps, we make use of the spherical metric (and derivatives). We make no notational distinction between it and the Euclidian metric. We use $|\cdot|$ to denote the diameter of a set. As far as we are aware, the lower bound for the density and the existence of the expanding induced Markov map are new; everything else is known.

Juan Rivera-Letelier deserves many thanks for many helpful remarks regarding some early drafts of this section, as well as (and more importantly) for his interest in the results and his encouragement to write them up.

Definition 2.1.1 *A measure m is t-*conformal *for a differentiable map g if*

$$m(g(A)) = \int_A |Dg|^t dm$$

on any set A on which g is injective and A and $g(A)$ are measurable.

Theorem 2.1.2 *Let f be a rational map of the Riemann sphere and m a t-conformal measure for f. Suppose that there exists an ergodic, invariant probability measure μ which is absolutely continuous with respect to m and that the Lyapunov exponent of μ is strictly positive.*

Then m and μ are equivalent, non-atomic and of dimension t, and there exists a constant $\varepsilon > 0$ such that the density of μ with respect to m is greater than ε almost everywhere.

Remark: The ergodicity condition is not wholly necessary. It suffices, as in [22], to require that $|Df^n| \to \infty$ almost everywhere with respect to μ.

Definition 2.1.3 *Let f be a rational map with a conformal measure m. Let U and V be open topological balls such that $\overline{U} \subset V$ and $m(U) > 0$ and let $U_i \subset V_i \subset U$ satisfy:*

- $U_i \cap U_j \neq \emptyset \iff i = j$;
- $m(U \setminus \bigcup_i U_i) = 0$;
- *for each i there exists n_i such that $f^{n_i} : V_i \to V$ is a diffeomorphism which maps U_i to U.*

Define $\phi : \bigcup_i U_i \to U$ by $\phi_{|U_i} := f^{n_i}_{|U,i}$. Then ϕ is called an induced Markov map for (f, m)*. If additionally $\sum_i n_i m(U_i) < \infty$ then we say ϕ has integrable return time.*

Theorem 2.1.4 *Let f be a rational map of the Riemann sphere and m a t-conformal measure for f. Suppose that there exists an ergodic, invariant probability measure μ which is absolutely continuous with respect to m and that the Lyapunov exponent of μ is strictly positive.*

Then there exists an induced Markov map for (f, m) with integrable return time.

Remark: One can always generate an ergodic acip if one has an induced Markov map with integrable return time by pushing around the acip for the Markov map. If one does this for the induced Markov map given by the theorem one recovers the original measure μ.

2.2 Density bounds and Markov maps

2.2.1 Constructing a nice partition

Definition 2.2.1 The Natural Extension: *Let $Y \subset \overline{\mathbb{C}}^{\mathbb{N}}$ be the set of sequences $y = (y_0, y_1, \ldots)$ such that $f(y_{i+1}) = y_i$ and define $F : Y \to Y$ by*

$$F(y_0, y_1, \ldots) = (F(y_0), y_0, y_1, \ldots).$$

Then $F : Y \to Y$ is invertible. We shall denote by Π the projection onto the first element, i.e. $\Pi y = y_0$.

If f has an invariant measure μ then there is a unique measure $\overline{\mu}$ on Y, invariant by F, such that $\Pi_* \overline{\mu} = \mu$. If μ is ergodic then so is $\overline{\mu}$, and importantly $\overline{\mu}$ is also an ergodic invariant measure for the inverse map F^{-1}. We call $(Y, F, \overline{\mu})$ the natural extension of $(\overline{\mathbb{C}}, f, \mu)$.

The following fact is lemma 2 of [22].

Fact 2.2.2 *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with an ergodic invariant probability measure μ with positive Lyapunov exponent. For each $\varepsilon > 0$, there exists a countable partition $\mathcal{P}_\varepsilon$ of $\overline{\mathbb{C}}$ with the following properties:*

- *the partition entropy $-\sum_{P\in\mathcal{P}_\varepsilon} \mu(P)\log\mu(P)$ is finite;*
- *if x, x' are in the same element of $\mathcal{P}_\varepsilon$ then*
$$e^{-\varepsilon} \leq \frac{|Df(x)|}{|Df(x')|} \leq e^{\varepsilon};$$
- *denoting by $\partial\mathcal{P}_\varepsilon$ the union of the boundaries of all partition elements, the Lebesgue measure of $\overline{\partial\mathcal{P}_\varepsilon}$ is zero;*
- *for every positive $a < 1$, $\sum_{i=1}^{\infty} m(B(\overline{\partial\mathcal{P}_\varepsilon}, a^i)) < +\infty$.*

Remark: The following lemma is sufficient to prove Fact 2.2.5. That the above Fact follows fairly straightforwardly from the lemma was noticed by Juan Rivera-Letelier (personal communication) — one uses different exponential rates and balls around the critical set to control convergence of the sum. We include the proof of the following lemma.

Lemma 2.2.3 *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with an ergodic invariant probability measure μ with positive Lyapunov exponent. For each $\varepsilon > 0$, there exists a countable partition $\mathcal{P}_\varepsilon$ of $\overline{\mathbb{C}}$ with the following properties:*

- *the partition entropy $-\sum_{P\in\mathcal{P}_\varepsilon} \mu(P)\log\mu(P)$ is finite;*
- *if x, x' are in the same element of $\mathcal{P}_\varepsilon$ then*
$$e^{-\varepsilon} \leq \frac{|Df(x)|}{|Df(x')|} \leq e^{\varepsilon};$$
- *denoting by $\partial\mathcal{P}_\varepsilon$ the union of the boundaries of all partition elements,*
$$\overline{\partial\mathcal{P}_\varepsilon} \setminus \partial\mathcal{P}_\varepsilon \subset Crit,$$
and the Lebesgue measure of $\partial\mathcal{P}_\varepsilon$ is zero;
- *for every positive $a < 1$,*
$$\sum_{i=1}^{\infty} m(B(\partial\mathcal{P}_\varepsilon, Crit, a^i)) < +\infty, \tag{2.1}$$
where $B(A, S, t) := \bigcup_{x\in A} B(x, dist(x, S)t)$.

Proof: We need the following simple but important classical fact which we borrow from [21].

Fact 2.2.4 *Let m be a finite measure on an interval I. For Lebesgue almost every x in I, for all $\alpha < 1$,*
$$\limsup_n \frac{1}{n} \log m([x - \alpha^n, x + \alpha^n]) < 0.$$

Obviously, one can find a finite partition of any subset of $\overline{\mathbb{C}}$ disjoint from a neighbourhood of the critical points. Thus it suffices to show that we can find a suitable partition of a neighbourhood of one critical point. By a diffeomorphic change of coordinates, it suffices to find an appropriate partition of a small ball B centred on 0 for $z :\mapsto z^d + c$ for any positive integer $d > 1$.

We shall use polar coordinates: $B = \{(r, 2\pi\theta) : 0 \le r < r_0, 0 \le \theta < 1\}$, where we identify (r,θ) and $(r,\theta+1)$. Let $C_r := \{(s,\theta) \in B : s = r\}$ and $R_\theta := \{(r,\phi) \in B : \phi = \theta\}$, so C_r is a circle of radius r and R_θ is a ray.

Now we can define a measure m_θ on $[-1,2]$ by $m_\theta(A) := \mu(\bigcup_{\theta\in A} R_\theta)$. Clearly the measure is finite since μ is.

Let $W_\theta(a) := \bigcup_{|\phi|\le a} B^{\theta+\phi}$, so $W_\theta(a)$ is a wedge whose point is the critical point at 0. For all $\theta \in [0,1]$ and $a < \frac{1}{2}$, one has $m_\theta([\theta - a, \theta + a]) = \mu(W_\theta(a))$. By Fact 2.2.4, for almost every $\theta \in [0,1]$, for all positive $\alpha < 1$

$$\sum_{i\ge 1} \mu(W_\theta(\frac{1}{2}\alpha^i)) < \infty. \tag{2.2}$$

We remark that one can replace $\frac{1}{2}$ by any positive constant without altering the convergence of this sum. Note that, using the notation $B(A,S,t)$ from the statement of the Lemma we are currently proving, if $0 \le \theta \le 1$ and $0 < a < \frac{1}{2}$, then $W_\theta(a) = B(R_\theta, 0, \sin(a2\pi))$, so it follows from (2.2) that, for all positive $\alpha < 1$,

$$\sum_{i\ge 1} \mu(B(R_\theta, 0, \alpha^i)) < \infty \tag{2.3}$$

and this holds for Lebesgue almost every θ.

For $0 < a < 1$, $0 < \delta \le 1$, and $\gamma > 0$, set

$$\delta T_\gamma(a) := \bigcup_{n\ge n_0} \bigcup_{|s|\le a} B_{\delta e^{-n\gamma}(1+s)},$$

where n_0 is large enough that $B_{e^{-n_0\gamma}} \subset B$. For a small enough that $1 + a < e^{\frac{\gamma}{2}}$, $\delta T_\gamma(a)$ is a union of disjoint concentric annuli centred on 0. Now for each $\gamma > 0$ we define a measure m_γ on $(e^{-\gamma}, 1]$ by $m_\gamma(A) := \mu(\bigcup_{\delta\in A} \delta T_\gamma(0))$. Then $m_\gamma((e^{-\gamma}, 1]) \le \mu(B)$.

For all $\delta \in (e^{-\gamma}, 1)$ and a such that $1 + a < e^{\frac{\gamma}{2}}$, one has $m_\gamma([\delta - a, \delta + a]) = \mu(\delta T_\gamma(a))$. By Fact 2.2.4, for almost every $\delta \in (e^{-\gamma}, 1)$, for all positive $\alpha < 1$

$$\sum_{i\ge 1} \mu(\delta T_\gamma(\alpha^i)) < \infty. \tag{2.4}$$

Note that $\delta T_\gamma(a) = B(\delta T_\gamma(0), 0, a)$, so for all positive $\alpha < 1$,

$$\sum_{i\ge 1} \mu(B(\delta T_\gamma(0), 0, \alpha^i)) < \infty \tag{2.5}$$

and this holds for Lebsegue almost every δ in $(e^{-\gamma}, 1)$.

Now we are ready to construct the required partition. Let $\varepsilon > 0$ be given. There exists a $\gamma > 0$ and a finite number of rays R_θ such that, for any positive $\delta < 1$, on a partition of $B \setminus 0$ into simply connected sets defined by the rays R_θ and the concentric circles $\delta T_\gamma(0)$, the distortion of f is bounded by $\frac{\varepsilon}{2}$. We can choose δ and the rays such that (2.3) and (2.5) are both satisfied and such that the distortion on each partition element is bounded by ε. Let $\mathcal{P}_\varepsilon$ be such a partition.

Since the change of coordinates was diffeomorphic, it suffices to show (2.1) in our polar coordinate system. Since there only a finite number of rays and only one $\delta T_\gamma(0)$, by (2.3) and (2.5), we conclude that (2.1) holds.

We now check that the entropy of the partition defined by $\delta T_\gamma(0)$ is finite — for the join with the finite partition defined by the rays would then have finite entropy, but this partition is none other than $\mathcal{P}_\varepsilon$. So, let

$$A_n := \bigcup_{\delta e^{-n\gamma} \leq r < \delta e^{(-n+1)\gamma}} C_r.$$

We have $|Df((r,\theta))| = dr^{d-1}$, so n A_n,

$$d\delta^{d-1} e^{-n\gamma(d-1)} \leq |Df| < d\delta^{d-1} e^{(-n+1)\gamma(d-1)}.$$

Hence, on A_n, $\log|Df| \leq \log(d\delta^{d-1}) + (-n+1)\gamma(d-1)$. Only considering n large enough that $A_n \subset B$, we get

$$-\infty < \int_{\bigcup_n A_n} \log|Df| d\mu \leq \sum_n \mu(A_n) \left(\log(d\delta^{d-1} + (-n+1)\gamma(d-1) \right).$$

Thus $\sum_n n\mu(A_n) < \infty$. It is well-known (e.g. lemma 9.3.1 of [32]) that this implies that $\sum -\mu(A_n) \log \mu(A_n) < \infty$. □

Now let $\varepsilon < \chi_\mu/2$ and let $\mathcal{P}_\varepsilon$ be given by the preceding Fact. Define the partition ξ of Y by:

$$\xi = \bigvee_{j=0}^{\infty} F^j(\Pi^{-1}\mathcal{P}_\varepsilon),$$

and let $\xi(y)$ denote the partition element of ξ containing y.

Fact 2.2.5 *([22], lemma 4). The partition ξ has the following properties:*

- *for almost every $y \in Y$, $\Pi_{|\xi(y)}$ is a bijection between $\xi(y)$ and a neighbourhood of Πy;*
- *if $\xi(y') = \xi(y)$ then the following infinite product converges:*

$$\Delta_t(y, y') = \prod_{i=1}^{\infty} \frac{|Df(\Pi F^{-i} y')|^t}{|Df(\Pi F^{-i} y)|^t};$$

- *the function $\Delta_t(\cdot,\cdot)$ is continuous and bounded from zero and infinity on $\xi(y) \times \xi(y)$.*

Remark: Using Lemma 2.2.3, one gets

$$\mathrm{dist}^{-1}(\Pi F^{-n} y, Crit)\mathrm{dist}(\Pi F^{-n} y, \partial\mathcal{P}_\varepsilon)$$

decreases subexponentially. Subexponential approach of $\Pi F^{-n} y$ to Crit (which follows from integrability of the Lyapunov exponent) will then give subexponential approach to $\partial\mathcal{P}_\varepsilon$.

Remark: This fact was actually only stated in [22] for $t = 2$, but holds for all t. On almost every partition element of ξ, iterating by F^{-1} the exponential decrease due to the Lyapunov exponent and the distortion bound on each element of $\mathcal{P}_\varepsilon$ implies that $|\xi(\Pi F^{-n} y)|$ tends to zero exponentially almost everywhere. Hence ξ is a generating partition. Projecting, it follows that $\mathcal{P}_\varepsilon$ is a generating partition. That $\Pi(\xi(y))$ contains a neighbourhood of $\Pi(y)$ is important and follows from the subexponential approach of $\Pi F^{-n} y$ to $\partial\mathcal{P}_\varepsilon$.

The Rohlin decomposition $p(y,\cdot)$ for the measure $\overline{\mu}$ with respect to the partition ξ is a conditional probability measure (the conditional expectation) on each partition element of Y such that, for any measurable set $B \subset Y$ one has

$$\overline{\mu}(B) = \int_Y p(y,B)d\overline{\mu} = \int_Y p(y, B\cap\xi(y))d\overline{\mu}.$$

For all $n > 0$,

$$nh_\mu = H(F^{-n}\xi|\xi) = -\int \log p(y,[F^{-n}](y))d\overline{\mu}. \tag{2.6}$$

In the following proposition, $q(y,\cdot)$ is an educated guess as to the Rohlin decomposition: it could be derived directly if we knew in advance that $\mu(A) = \lim_{n\to\infty} m(f^{-n}A)$ for all measurable sets A.

Proposition 2.2.6 *Suppose additionally that there is a t-conformal measure m supported on the Julia set and that μ is absolutely continuous with respect to m. Then the Rohlin decomposition of $\overline{\mu}$ with respect to the partition ξ is given by $q(y,\cdot)$, where*

$$q(y,dz) := \frac{\Delta_t(y,z)m_{\xi(y)}(dz)}{\int_{\xi(y)}\Delta_t(y,y')m_{\xi(y)}(dy')}, \tag{2.7}$$

where $m_{\xi(y)}$ is the pullback by $\Pi_{|\xi(y)}$ of the conformal measure m restricted to $\Pi\xi(y)$.

Proof: We postpone proving Proposition 2.3.4 until later; applying it gives that $\mathrm{HD}(\mu) = t$. The Dynamical Volume Lemma (see e.g. [22], theorem 3), states that $\mathrm{HD}(\mu) = \frac{h_\mu}{\chi_\mu}$. Putting these together, we get

$$h_\mu = t\chi_\mu. \tag{2.8}$$

The rest of the proof of this proposition is essentially the same as that of proposition 3.6 of [21]; we include it for the reader's convenience.

It follows from (2.7) that $q(y,\cdot)$ is a probability measure on $\xi(y)$, and that if $\xi(y_1) = \xi(y_2)$ then $q(y_1,\cdot) = q(y_2,\cdot)$ by multiplying above and below by $\Delta_t(y_1,y_2)$. Set $k(y) := \int_{\xi(y)}\Delta_t(y,y')dy'$. Then, by definition,

$$\begin{aligned} q(y,[F^{-n}\xi](y)) &= \frac{\int_{[F^{-n}\xi](y)}\Delta_t(y,y')}{k(y)} \\ &= \frac{1}{k(y)}\int_{F^{-n}(\xi(F^n(y))}\Delta_t(F^ny,F^ny')\frac{Df^n(\Pi y')}{Df^n(\Pi y)} \\ &= \frac{k(F^ny)}{|Df^n(\Pi y)|^t k(y)}. \end{aligned}$$

Taking logs,

$$\log q(y,[F^{-n}\xi](y)) = \log k(F^ny) - t\log|Df^n(\Pi y)| - \log k(y).$$

Integrating, applying the classical lemma 3.5 of [21],

$$-\int \log q(y,[F^{-n}\xi](y))d\overline{\mu} = nt\chi_\mu.$$

Let $p(y,\cdot)$ be the Rohlin decomposition for the measure $\overline{\mu}$ with respect to the partition ξ. Recalling (2.8) and (2.6),

$$\int \log q(y,[F^{-n}\xi](y))d\overline{\mu} = \int \log p(y,[F^{-n}\xi](y))d\overline{\mu},$$

so

$$\int \log \frac{q(y, [F^{-n}\xi](y))}{p(y, [F^{-n}\xi](y))} d\overline{\mu}. \tag{2.9}$$

Consider the measure $\overline{\nu}$ defined on the σ-algebra of $[F^{-n}\xi]$ measurable sets defined by

$$\overline{\nu}(B) = \int q(y, B) d\overline{\mu}.$$

By (2.9),

$$\int \left(\log \frac{d\overline{\nu}}{d\overline{\mu}}\right) d\overline{\mu} = 0.$$

Both measures are probabilities, so by concavity of logarithm, $\overline{\nu} = \overline{\mu}$, so $q(y, B) = p(y, B)$ if B is $[F^{-n}\xi]$ measurable. As ξ generates and $\xi < F^{-1}\xi$, one concludes that q and p coincide almost everywhere. □

2.2.2 Proof of Theorems 2.1.2, 2.1.4

Using the results and notation of Subsection 2.2.1, to each y we can associate numbers $\theta(y)$ and $M(y)$ such that $M(y) > \theta(y) > 0$ almost everywhere and $B(\Pi y, \theta(y)) \subset \Pi\xi(y)$, and such that $\Delta_t(\cdot, \cdot)$ is bounded inside $[M^{-1}(y), M(y)]$ on $\xi(y) \times \xi(y)$. There exist $M > \theta_0 > 0$ such that, setting

$$A := \{y \in Y : \theta(y) > \theta_0, M(y) < M\},$$

A is of positive measure. Hence there is an $x \in \overline{\mathbb{C}}$ such that $A_x := A \cap \Pi^{-1}B(x, \frac{\theta_0}{4})$ has positive measure. We write $B_x := B(x, \frac{\theta_0}{2})$. For each y in A_x we have that $\Pi\xi(y) \supset B(\Pi y, \theta_0) \supset B_x$. On each such $\xi(y)$ the ratio of the conditional density at any two points is bounded by M^2. Projecting, one finds that on B_x the density of μ with respect to m is bounded from below almost everywhere with respect to m by

$$M^{-2}\overline{\mu}(A_x)m(B_x)^{-1} =: \nu > 0.$$

There is an $n > 0$ such that $f^n(B_x) \supset \mathcal{J}$. By invariance of the measure, the density of μ with respect to m is bounded from below almost everywhere with respect to m by $D^{-n}\nu$, where D is the maximal value taken by $|Df|^t$. It follows that μ and m are equivalent and that m is ergodic and non-atomic, completing the proof of Theorem 2.1.2.

Now let γ be such that $\frac{\theta_0}{4} < \gamma < \frac{\theta_0}{2}$ and such that, for almost every $y \in Y$, $\Pi F^{-n}y$ accumulates sub-exponentially on $\partial B(x, \gamma)$ (again, one shows that there exists such a γ using Fact 2.2.4). Set $U' := B(x, \gamma)$ and set $V' := B(x, \frac{\theta_0}{2}) = B_x$, so $\overline{U'} \subset V' \subset \Pi\xi(y)$ for all y in A_x. Let η satisfy $0 < \eta < \chi_\mu$. By convergence of the product in Fact 2.2.5, there exists a function $\rho > 0$ on Y such that, for almost all y in Y, for all $y' \in \xi(y)$,

$$|Df^n(\Pi F^{-n}y')| \geq \rho(y)e^{n(\chi_\mu - \eta)}$$

for all $n \geq 1$. Thus, modulo a set of measure zero, one can write

$$Y = \bigcup_{t>0} Y_t,$$

where

$$Y_t := \{y : \text{ for all } n > 0, \operatorname{dist}(\Pi F^{-n}y, \partial U') \geq te^{-n\frac{\chi_\mu - \eta}{2}}$$

$$\text{and } |Df^n_{|\Pi F^{-n}\xi(y)}| \geq te^{n(\chi_\mu - \eta)}\}.$$

Then there exists a $t_0 > 0$ such that $A_{t_0} := A_x \cap Y_{t_0}$ has positive measure, and there exists an $M > 0$ such that, for all $n \geq M$,

$$|\Pi F^{-n}\xi(y)| < |\overline{\mathbb{C}}|\frac{1}{t_0}e^{-n(\chi_\mu-\eta)} < t_0 e^{-n\frac{\chi_\mu-\eta}{2}}$$

for all $y \in A_{t_0}$. Thus for all $y \in A_{t_0}$, for all $n \geq M$,

$$\Pi F^{-n}\xi(y) \cap \partial U' = \emptyset. \tag{2.10}$$

We need to do some work now in order to get good disjointedness properties.
Remark: In the interval case one can use preperiodic points to define regularly returning intervals, from which good disjointedness properties follow immediately. The rational case is substantially more complicated.

Recall that for all $y \in A_{t_0}$, $\Pi\xi(y) \supset V'$. Define

$$A'_* := \Pi^{-1}U' \cap \bigcup_{y \in A_{t_0}} \xi(y).$$

Then $\overline{\mu}(A'_*) \geq \overline{\mu}(A_{t_0}) > 0$. For each $y \in A'_*$, let $U'_y = \Pi^{-1}U' \cap \xi(y)$, so (2.10) immediately gives

$$\Pi F^{-n}U'_y \cap \partial U' = \emptyset$$

for all $n \geq M$. Pulling back this relation, if $n \geq 0$ and there exists a $k > 0$ such that

$$\Pi F^{-n-k}U'_y \cap \partial \Pi F^{-n}U'_y \neq \emptyset, \tag{2.11}$$

then for each n the minimal such k (if such a k exists) satisfies $k \leq M$.

Each periodic point of period $\leq M$ is a fixed point of $f^{M!}$. The fixed points of $f^{M!}$ we denote by $\mathrm{Fix}(M!)$. There exists a continuous function σ such that $\sigma(0) = 0$ and such that, for any set V, $V \cap f^{-k}V \neq \emptyset$ and $0 < k \leq M$ imply that

$$\mathrm{dist}(V, \mathrm{Fix}(M!)) \leq \sigma(|V|). \tag{2.12}$$

For each y in A'_*, let $\{n_j(y)\}$ be the collection of $n \geq 0$ for which there exists a $k > 0$ satisfying

$$\Pi F^{-n-k}U'_y \cap \partial \Pi F^{-n}U'_y \neq \emptyset,$$

and order $\{n_j(y)\}$ so that $\{n_j(y)\}$ is increasing in j. Note that $\{n_j(y)\}$ may be empty or finite. It is immediate that if $n_i(y), n_{i+1}(y) \in \{n_j(y)\}$, then $n_{i+1}(y) - n_i(y) = 1$, and we showed that for each such n the minimal k is $\leq M$.

Suppose there is a positive measure subset of A'_* such that for each y there the set $\{n_j(y)\}$ is infinite (and thus equal to $\mathbb{N}$). Then (2.12) gives that

$$\mathrm{dist}(\Pi F^{-n}U'_y, \mathrm{Fix}(M!)) \leq \sigma(|\Pi F^{-n}U'_y|).$$

But $|\Pi F^{-n}U'_y|$ is exponentially decreasing, so the accumulation set of $\Pi F^{-n}U'_y$ is contained in $\mathrm{Fix}(M!)$. Since μ is non-atomic, this is impossible. We deduce that for almost every y in A'_*, the set $\{n_j(y)\}$ is finite. For each y in A'_*, let $n(y) = \sup\{n \in \{0\} \cup \{n_j(y)\}\}$. We remark that $n(y)$ is constant on each U'_y.

Then there exists an $N > 0$ such that the set

$$A_* := \{y \in A'_* : n(y) \leq N\}$$

is of positive measure. Summarising, we have shown that, for all y in A_*,

$$\Pi F^{-n-k} U'_y \cap \partial \Pi F^{-n} U'_y = \emptyset, \tag{2.13}$$

for all $n \geq N$ and all $k > 0$.

Recall that $\xi = \bigvee_{j=0}^{\infty} F^{-j}\Pi^{-1}\mathcal{P}_\epsilon$. Let $\xi_N := \bigvee_{j=0}^{N} F^{-j}\Pi^{-1}\mathcal{P}_\epsilon$ (each element of ξ_N corresponds to a branch of f^{-n} on an element of $\mathcal{P}_\epsilon$). There is an element $Q \in \xi_N$ such that $\overline{\mu}(A_* \cap Q) > 0$. Let

$$A_Q := F^{-N}(A_* \cap Q)$$

and set $U := \Pi A_Q$. Then $f^n : U \to U'$ is a diffeomorphism which extends to a diffeomorphism from $V \subset \Pi F^{-N}\xi(F^N y)$ to $V' \subset \Pi\xi(F^N y)$ for $y \in A_Q$ for some $V \supset \overline{U}$. For y in A_Q let $U_y := \Pi^{-1}U \cap \xi(y)$. See figure 2.1.

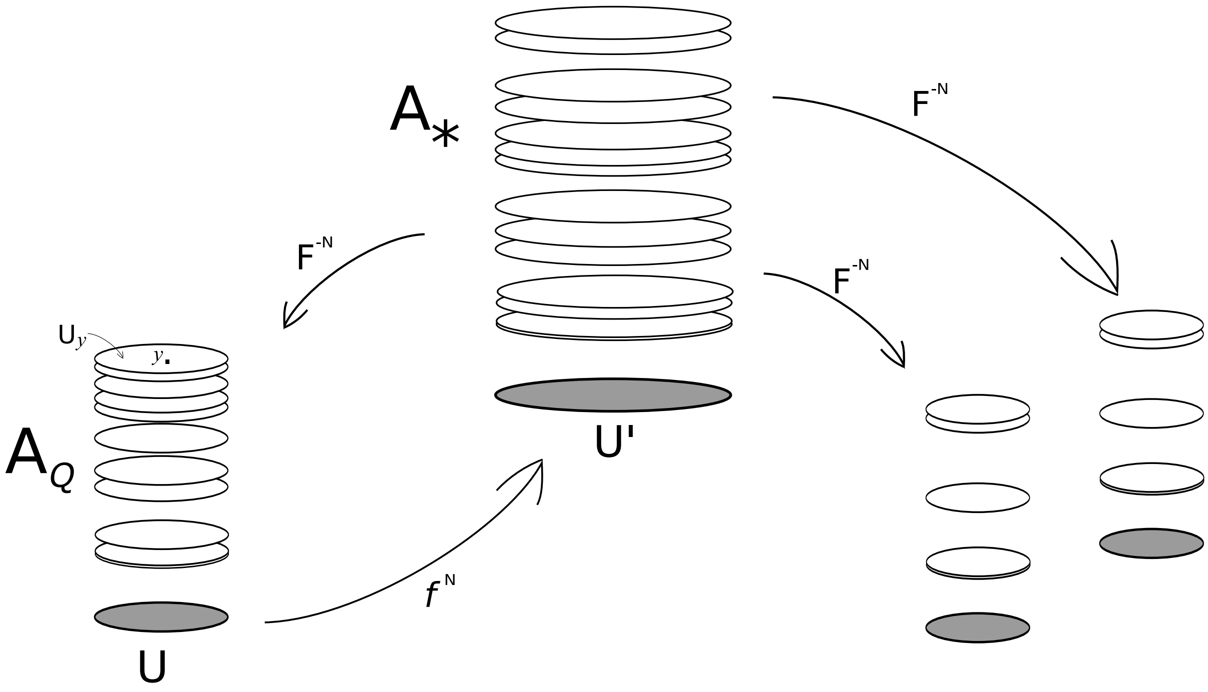

Figure 2.1: A_Q has positive measure.

By (2.13), for all $n \geq 0$, and all $y \in A_Q$,

$$\partial U \cap \Pi F^{-n} U_y = \emptyset.$$

This is nearly a "regularly returning" property: It implies that for any $n, n' \geq 0$ and $y, y' \in A_Q$, $\Pi F^{-n}U_y$ and $\Pi F^{-n'}U_{y'}$ are either nested or disjoint. Also, using the earlier estimate we have

$$|Df^{n+N}_{|F^{-n-N}\xi(F^N y)}| \geq t_0 e^{(n+N)(\chi_\mu - \eta)}$$

for $y \in A_Q$, so, since $|Df^N|$ is bounded, there exists an N_1 such that, for all $n \geq N_1$,

$$|Df^n_{|F^{-n-N}\xi(F^N y)}| \geq 2$$

for all $y \in A_Q$.

Let $\overline{\mu}_Q := \overline{\mu}_{|A_Q}$. As before, the density of $\Pi_* \overline{\mu}_Q$ on U is bounded from below by

$$\nu' := M_0^{-2} \overline{\mu}(A_Q) m(U)^{-1} > 0,$$

where M_0 is a positive constant depending on N and the (bounded) distortion of $f^n_{|U}$. For each y in A_Q, let $m(y)$ denote the N_1th return time of y to A_Q. By ergodicity, $m(y)$ is defined and finite almost everywhere, $m(y) \geq N_1$ and, by a simple generalisation of Kac' Theorem ([31], theorem 2.4.6),

$$\int_{A_*} m(y) d\overline{\mu} = N_1.$$

Let $n(x) := \min\{m(y) : \Pi y = x, y \in A_*\}$. Then $n(x)$ is defined almost everywhere in U and there exist neighbourhoods $V_x \supset U_x \ni x$ such that $V_x \cap \partial U = \emptyset$ and $f^{n(x)} : V_x \to V$ is a diffeomorphism with $|Df^n_{|V_x}| > 2$ and $f^{n(x)}(U_x) = U$. Each U_x is $\Pi F^{-n(x)} U_y$ for some y in A_Q, so the topological balls U_x are either nested or disjoint. It follows that $n(\cdot)$ is constant on each U_x. Let

$$\{U_i\}_i := \{U_x : x \in U\}.$$

Then $m(U \setminus \bigcup_i U_i) = 0$. Defining $\phi(x) := f^{n(x)}(x)$ on $\bigcup_i U_i$ we obtain an expanding induced Markov map for (f, m). It remains to show that ϕ has integrable return time. But

$$\begin{aligned}
\sum_i n_i m(U_i) = \int_U n(x) dm &\leq \frac{1}{\nu'} \int_U n(x) d\Pi_* \overline{\mu}_Q \\
&= \frac{1}{\nu'} \int_{A_Q} n(\Pi y) d\overline{\mu}_Q \\
&\leq \frac{1}{\nu'} \int_{A_Q} m(y) d\overline{\mu}_Q \\
&= \frac{1}{\nu'} \int_{A_Q} m(y) d\overline{\mu} \\
&= \frac{1}{\nu'} N_1 < \infty,
\end{aligned}$$

completing the proof of Theorem 2.1.4. □

2.3 Hausdorff dimension of the measure

This section's main result, which we believe is old and reasonably well known (in [22], however, the author supposes that t is the minimal exponent for which a conformal measure exists), but perhaps without easy-to-find proofs or references, is that if μ is absolutely continuous with respect to a t-conformal measure m then the Hausdorff dimension of μ is equal to t.

Henceforth, f shall be a rational map and μ shall be an ergodic invariant probability measure with positive Lyapunov exponent for f.

2.3.1 Ledrappier theory

The standard proofs of the following Pesin-Ledrappier theory result (see also Fact 2.2.5) use the local non-flatness around critical points. We provide an alternative proof which just uses that rational maps are C^2 (actually $C^{1+H\ddot{o}lder}$ suffices).

Proposition 2.3.1 *Let f be a rational map of $\overline{\mathbb{C}}$. Suppose f has an invariant, ergodic probability measure μ with positive Lyapunov exponent $\chi = \int_I \log|Df|\,d\mu > 0$. Let $(Y, F, \overline{\mu})$ be the natural extension for (f, μ). Let $\eta > 0$.*

Then there exists a measurable function α on Y, positive almost everywhere, such that for $\overline{\mu}$ almost every $y \in Y$ there exists an infinite sequence of simply connected open sets $V_n \subset \overline{\mathbb{C}}$, $n \geq 0$, with the following properties:

- *$V := V_0 := B(\Pi y, \alpha(y)) \ni \Pi y$ and $V_n \ni \Pi F^{-n} y$;*
- *$f^n : V_n \to V$ is a diffeomorphism and for all $x \in V_n$*
$$\big|\log|Df^n(x)| - \log|Df^n(\Pi F^{-n}y)|\big| < \log 4; \tag{2.14}$$
- *for all $x \in V_n$*
$$\big|\log|Df(x)| - \log|Df(\Pi F^{-n}y)|\big| < \log 2;$$
- *there exists a measurable function ρ on Y, $0 < \rho(y) < \infty$ almost everywhere, such that*
$$\rho(y)^{-1} e^{n(\chi-\eta)} < |Df^n(\Pi F^{-n}y)| < \rho(y) e^{n(\chi+\eta)}.$$

In particular, $|V_n| \leq 2\rho(y) e^{-n(\chi-\eta)} |\overline{\mathbb{C}}|$.

Proof: Since f is C^2, there exists a $C > 0$ such that, for all $x, x' \in \overline{\mathbb{C}}$,
$$|Df(x) - Df(x')| < C|x - x'|.$$
We shall fix $\delta > 0$ such that $\chi - \delta > 6\delta$ and $N > 0$ large enough that the following inequalities hold
$$\log 2 + \sum_{n \geq N} e^{-n\delta} \;<\; \log 4; \tag{2.15}$$
$$4e^{-N(\chi-\delta)} \;<\; e^{-(N+1)6\delta}; \tag{2.16}$$
$$|\log(1 \pm Ce^{-2N\delta})| \;<\; e^{-N\delta}. \tag{2.17}$$
Of course, (2.16) and (2.17) also hold for all $n \geq N$. By the C^2 property, if $|x - x'| < e^{-n3\delta}$ then
$$|Df(x) - Df(x')| < Ce^{-n3\delta}.$$
If also $|Df(x)| > e^{-n\delta}$ and $n > N$ then
$$1 - Ce^{-2n\delta} < \frac{|Df(x')|}{|Df(x)|} < 1 + Ce^{-2n\delta}$$
$$\Rightarrow \big|\log|Df(x)| - \log|Df(x')|\big| < e^{-n\delta}. \tag{2.18}$$
Since $\lim_{n\to\infty} \frac{1}{n} \log|Df^n(\Pi F^{-n}y)| = \chi$ almost everywhere, almost everywhere there exists $n(y) > N$ such that for all $n \geq n(y)$,
$$e^{n(\chi-\delta)} < |Df^n(\Pi F^{-n}y)| < e^{n(\chi+\delta)}$$
and also
$$|Df(\Pi F^{-n}y)| \geq e^{-n\delta}. \tag{2.19}$$

Almost everywhere, by continuity of the derivative, there exist an open ball V centred on Πy and neighbourhoods $V_{n(y)} \ni \Pi F^{-n(y)}y$ such that $f^{n(y)} : V_{n(y)} \to V$ is a diffeomorphism and, for all $x, x' \in V_{n(y)}$ and all $0 \le i \le n(y)$,

$$\left|\log|Df^i(f^{n(y)-i}x)| - \log|Df^i(f^{n(y)-i}x')|\right| < \log 2.$$

Choosing $|V| < 1$, one has $|V_{n(y)}| < 2e^{-n(\chi-\delta)}$.

Denote by B_n the ball $B(\Pi F^{-n}y, e^{-n3\delta})$. By (2.19) and (2.18), if $n \ge n(y)$, for all $x \in B_n$

$$\left|\log|Df(x)| - \log|Df(\Pi F^{-n}y)|\right| < e^{-n\delta}. \tag{2.20}$$

We now proceed with a simple induction argument. We take V as before. Suppose $n_0 \ge n(y)$ and for each n such that $n(y) \le n \le n_0$ there exists $V_n \ni \Pi F^{-n}y$ such that $f^n : V_n \to V$ is a diffeomorphism and, for all $x \in V_n$,

$$\left|\log|Df^n(x)| - \log|Df^n(\Pi F^{-n}y)|\right| < \log 2 + \sum_{i=n(y)+1}^{n} e^{-i\delta} < \log 4 - \sum_{i=n+1}^{\infty} e^{-i\delta}.$$

We want to show this remains true with $n_0 + 1$ in place of n_0. One has $|V_{n_0}| < 4e^{-n_0(\chi-\delta)}$. But by (2.20) and (2.16)

$$|f(B_{n_0+1})| \supset B(\Pi F^{-n_0}y, e^{-(n_0+1)3\delta}2e^{-2(n_0+1)\delta}) \supset B(\Pi F^{-n_0}y, 4e^{-n_0(\chi-\delta)}).$$

Thus $V_{n_0} \subset f(B_{n_0+1})$ and there exists $V_{n_0+1} \subset B_{n_0+1}$ such that $f : V_{n_0+1} \to V_{n_0}$ is a diffeomorphism and for all $x \in V_{n_0+1}$,

$$\left|\log|Df(x)| - \log|Df(\Pi F^{-(n_0+1)}y)|\right| < e^{-(n_0+1)\delta}.$$

Since $\log|Df^{n_0+1}| = \log|Df^{n_0}| + \log|Df|$, this completes the induction argument, which implies (2.14). Fixing $\alpha(y) = \frac{1}{2}|V|$, the proposition follows. $\square$

2.3.2 Popular balls

Let $(Y, F, \overline{\mu})$ be the natural extension. Modulo a set of $\overline{\mu}$-measure zero one can write

$$Y = \bigcup_{k>0} \{y \in Y : \alpha(y) > \frac{1}{k}, \rho(y) < k\},$$

where the functions come from Proposition 2.3.1, for η a small positive constant. There is therefore a $k_0 > 2$ such that the set

$$A := \{y \in Y : \alpha(y) > \frac{1}{k_0}, \rho(y) < k_0\}$$

is of positive $\overline{\mu}$ measure, i.e., $\overline{\mu}(A) > 0$.

Define the measure m_A on $\overline{\mathbb{C}}$ by $m_A(B) = \overline{\mu}(A \cap \Pi^{-1}B)$. In other words m_A is the push-forward of $\overline{\mu}$ restricted to A. Then $m_A(\overline{\mathbb{C}}) = \overline{\mu}(A) > 0$ and m_A is non-atomic if μ is. We shall use the following lemmas.

Lemma 2.3.2 *There exists a ball U with $m_A(U) > 0$ and $|U| < k_0^{-1}$.*

Proof: Cover $\overline{\mathbb{C}}$ by balls of diameter less than k_0^{-1}. At least one of them is of positive measure with respect to m_A. □

Lemma 2.3.3 *Let U be a ball given by Lemma 2.3.2. Then for any $V \subset U$ with $m_A(V) > 0$ there is a set X of full μ-measure such that for every x in X there is an increasing sequence n_j and simply connected neighbourhoods U_j of x such that*

- $f^{n_j}(x) \in V$;
- $$f^{n_j} : U_j \to U$$
  is a diffeomorphism with distortion bounded by $\log 4$;
- *for all* $i \leq n_j$, $f^i(U_j) \cap \partial\mathcal{P} = \emptyset$;
- $\lim_{j\to\infty} \frac{j}{n_j} = m(V)$;
- $\lim_{j\to\infty} \frac{n_{j+1}}{n_j} = 1$;
- $$k_0^{-1} e^{n(\chi-\eta)} < |Df^{n_j}(x')| < k_0 e^{n(\chi+\eta)}$$
  for all x' in U_j, where χ is the Lyapunov exponent of μ.

Proof: This lemma follows easily from Lemma 2.3.2 and Proposition 2.3.1 since, by ergodicity of $\overline{\mu}$, almost every point is the projection of a point in Y which enters $A \cap \Pi^{-1}V$ infinitely often at times n_j with asymptotic frequency $\lim_{j\to\infty} \frac{j}{n_j} = \overline{\mu}(V_A)$. Examining the ratio of successive terms, we deduce that

$$\lim_{j\to\infty} \frac{n_{j+1}}{n_j} = 1.$$

□

2.3.3 Hausdorff dimension of the absolutely continuous measure

Proposition 2.3.4 *Let f be a rational map of the Riemann sphere and m a t-conformal measure for f supported on the Julia set. Suppose that there exists an ergodic, invariant measure μ which is absolutely continuous with respect to m and that the Lyapunov exponent of μ is strictly positive.*

Then $HD(\mu) = t$.

To show that $\mathrm{HD}(\mu) \leq t$, it suffices to show that $\mathrm{HD}(X) \leq t$, where X is given by Lemma 2.3.3. But by Lemma 2.3.3, there is a cover of X with sets V_α of arbitrarily small diameter which get mapped with bounded distortion by some iterates f^{n_α} onto U. By Besicovitch' theorem, we can take a subcover $\mathcal{W}$ of intersection multiplicity bounded by some constant M.

Now given any $W \in \mathcal{W}$, the distortion bound and conformality of m imply that

$$\frac{1}{4}\frac{|U|}{m(U)} m(W) \leq |W|^t \leq 4\frac{|U|}{m(U)} m(W),$$

so

$$\sum_{W\in\mathcal{W}} |W|^t \leq 4\frac{|U|}{m(U)} \sum_{W\in\mathcal{W}} m(W) \leq 4M\frac{|U|}{m(U)}.$$

This is independent of the maximal diameter of sets in the cover, so the Hausdorff dimension of X, and thus that of μ, is less than or equal to t.

We shall proceed with a proof by contradiction to prove the other direction. Suppose $\mathrm{HD}(\mu) < s < t$. Let $\eta > 0$ be small enough that $(t-s)(\chi-\eta) > 2\eta$ and let U be given by Lemma 2.3.2. There is a ball $V \subset U$ such that $m_A(V) > 0$ and

$$0 < \mathrm{dist}(V, \partial U).$$

Let X be given by Lemma 2.3.3. Due to the distortion bound, there is a constant $\kappa > 0$ such that, for any $x \in X, V_j, U_j$, one has $\frac{\mathrm{dist}(V_j, \partial U_j)}{|U_j|} > \kappa > 0$ for some constant κ. In particular, since $x \in V_j$, one has

$$\frac{\mathrm{dist}(x, \partial U_j)}{|U_j|} > \kappa > 0.$$

Choose ε such that

$$(t-s)(\chi-\eta) - \chi\varepsilon - 2\eta - \eta\varepsilon > \varepsilon > 0.$$

Let X' be a set of full measure with Hausdorff dimension strictly less than s. There is an $N > 0$ and a subset X_0 of $X \cap X'$ of positive measure with the following property. For each x in X_0 there is a j_x such that $n_{j_x}(x) < N$ and for all $j \geq j_x$

$$\frac{n_{j+1}(x)}{n_j(x)} < 1 + \varepsilon,$$

where the $n_j(x)$ are given by Lemma 2.3.3. For each $x \in X_0$ one has $|U_{j_x}| > k_0^{-1}|U|e^{-N(\chi+\eta)} =: K_0$ say.

Let $\mathcal{W}'$ be a cover of X_0 by sets of diameter less than $\gamma < 2^{-1}K_0\kappa$ and let $\mathcal{W}$ be a cover of X_0 defined as follows:

$$\mathcal{W} = \bigcup_{W \in \mathcal{W}'} \{B(x_W, |W|)\}$$

where x_W is some point in $W \cap X_0$ (this intersection can be assumed non-empty). Thus $\mathcal{W}$ is a cover of X_0 such that each element is a ball centred on a point of x, and

$$\sum_{W \in \mathcal{W}} |W|^t \leq 2^t \sum_{W \in \mathcal{W}'} |W|^t.$$

Now let W be some element of $\mathcal{W}$ centred on $x \in X_0$. The diameter of W is less than $K_0\kappa$ so $W \subset U_{j_x}$. Let j be minimal such that $U_{j+1} \not\supset W$. Then $\mathrm{dist}(x, \partial U_{j+1}) < |W|$ so

$$\kappa < \frac{\mathrm{dist}(x, \partial U_{j+1})}{|U_{j+1}|} < \frac{|W|}{|U|}$$

and thus $|W| > |U_{j+1}|\kappa$. We deduce that

$$m(W) < m(U_j) < C|U_j|^t = C_0\left(\frac{|U_j|}{|U_{j+1}|}\right)^t |U_{j+1}|^t < C_1\left(\frac{|U_j|}{|U_{j+1}|}\right)^t |W|^t$$

for some constants C_0, C_1.

One has $n_j > N \geq n_{j_x}$ so $\frac{n_{j+1}}{n_j} < 1 + \varepsilon$. Using the distortion bound again, one has

$$\frac{|U_j|}{|U_{j+1}|} \leq C_2 e^{-n_j(\chi-\eta)+n_{j+1}(\chi+\eta)} = C_2 e^{\chi(n_{j+1}-n_j)+\eta(n_{j+1}+n_j)} \leq C_2 e^{n_j(\chi\varepsilon+2\eta+\eta\varepsilon)}$$

Thus

$$|W|^s = |W|^{s-t}|W|^t > |W|^{s-t}m(W)C_3e^{-n_j(\chi\varepsilon+2\eta+\eta\varepsilon)}$$

for some constant $C_3 > 0$. Recall that $|W| < |U_j|$, so $|W| < C_4e^{-n_j(\chi-\eta)}$ for some $C_4 > 0$. Thus $|W|^{s-t} > C_5e^{(t-s)n_j(\chi-\eta)}$, so

$$|W|^s > C_6m(W)e^{n_j((t-s)(\chi-\eta)-(\chi\varepsilon+2\eta+\eta\varepsilon))}$$

for some positive constants C_5, C_6. By choice of ε, we have

$$|W|^s > C_6m(W)e^{n_j\varepsilon}.$$

Then

$$\sum_{W\in\mathcal{W}} |W|^s > C_6m(X_0) > 0.$$

Note that the last inequality is where we use that μ is absolutely continuous with respect to m. The constants not depending on the choice of cover $\mathcal{W}'$, we find that the Hausdorff dimension of X_0 is greater than or equal to s, contradicting choice of X'. $\square$

3

Conjugacies

Contents

3.1 Markov maps

Often in one-dimenional dynamics, whether trying to prove the existence of absolutely continuous invariant measures, dynamical volume lemmas, or whatever, one constructs an induced map which is expanding. These induced maps ideally take the form of Markov maps. We present these maps and some of their properties for later use.

We cite again [6] as a general reference, and refer to [26] where it is proven that a conjugacy between full C^r Markov maps which preserves multipliers is actually C^r, and that the density of the absolutely continuous invariant measure for a C^r Markov map is C^{r-1}. We shall give an alternative proof of this latter result using the natural extension.

Definition 3.1.1 *Let U_i be a finite or countable collection of disjoint closed subintervals of a closed interval U. We call a map $\phi : \bigcup_i U_i \to U$ a* Markov map *if ϕ restricted to each U_i is a homeomorphism onto U and $\#\{U_i\} \geq 2$.*

Definition 3.1.2 *Let $\phi : \bigcup_i U_i \to U$ be a Markov map. It has* eternally bounded distortion *if ϕ is a C^1 diffeomorphism on each U_i and if there is a constant $C > 0$ such that for all i, for all $n > 0$, for all x, x' in the same connected component of the domain of ϕ^n,*

$$\left|\log|D\phi^n(x)| - \log|D\phi^n(x')|\right| < C.$$

Lemma 3.1.3 *Let $\phi : \bigcup_i U_i \to U$ be a Markov map with eternally bounded distortion.*
Then all periodic points of ϕ are hyperbolic repelling.

Proof: Let p be a periodic point of period n, let J be the connected component of the domain of ϕ^n containing p and let $g := \phi^n_{|J}$. Then p is a fixed point of $g : J \to U$, and $U \setminus J$ has a non-empty connected component W say. Then $g^{-n}(W)$ are pairwise disjoint so their size goes to zero and $|Dg^n_{|g^{-n}(W)}|$ goes to infinity. The bound on distortion then implies that $|Dg^n(p)|$ goes to infinity. $\square$

Lemma 3.1.4 *Let $\phi : \bigcup_i U_i \to U$ be a Markov map with eternally bounded distortion. Then there exists positive constants $\lambda > 1$, $N \geq 1$ such that for all $n \geq N$, $|D\phi^n| \geq \lambda^n$ everywhere on its domain.*

Proof: Let $\log C > 0$ be the distortion constant. It suffices to show that there exists an n such that $|D\phi^n| > 1$ everywhere on its domain. This will follow due to the distortion bound if, for some n, every component of the domain of ϕ^n is smaller than $C^{-1}|U|$.

We shall proceed with a proof by contradiction. Let V_n be a nested sequence of intervals such that for each n, V_n is a connected component of the domain of ϕ^n, $V_n \supset V_{n+1}$ and $|V_n| > C^{-1}|U|$. Let $c = \inf_n |V_n|$ and suppose that $c > 0$. Let $\varepsilon > 0$ satisfy $\varepsilon < c|U|^{-1}C^{-1}$ and let n be sufficently large that $|V_n| < c + \varepsilon$. Then on V_n,

$$|D\phi^n| < C|U|c^{-1}.$$

Let $g := \phi^n_{|V_n}$ and let $W = g^{-1}(V_n)$. Then $c|U|^{-1}C^{-1} < |W|$. So

$$|L| < c + \varepsilon - c|U|^{-1}C^{-1} < c$$

for any connected component L of $V_n \setminus W$. Thus $V_{2n} \subset W$. Similarly $V_{kn} \subset g^{-k}(V_n)$ for all $k \geq 0$ and so $\bigcap V_n$ contains a fixed point p. By Lemma 3.1.3, p is a hyperbolic repelling fixed point. The distortion bound will then give that $|V_n|$ tends to zero, contradicting our supposition. $\square$

Definition 3.1.5 *Let $\phi : \bigcup_i U_i \to U$ be a Markov map. It is* full *if $\lambda(U \setminus \bigcup_i U_i) = 0$, where λ denotes Lebesgue measure.*

Lemma 3.1.6 *Let $\phi : \bigcup_i U_i \to U$ be a Markov map with eternally bounded distortion. Let $\mathcal{U}_n$ be the collection of connected components of the domain of ϕ^n. For each $J \in \mathcal{U}_n$ let p_J be a fixed point of ϕ^n contained in J and set $u_J := |D\phi^n(p_J)|^{-1}$.*

Then ϕ is full if and only if there exists $\varepsilon > 0$ such that $\sum_{J \in \mathcal{U}_n} u_J > \varepsilon$ for all positive n.

Proof: Let $C > 1$ be such that for all n, for all x, x' in the same component of the domain of ϕ^n,

$$C^{-1} < \frac{D\phi^n(x)}{D\phi^n(x')} < C.$$

Then $C^{-1}|u_J| < |J| < Cu_J$, so

$$C^{-1} \sum_{J \in \mathcal{U}_n} u_J < \sum_{J \in \mathcal{U}_n} |J| < C \sum_{J \in \mathcal{U}_n} u_J.$$

If ϕ is full,

$$\sum_{J \in \mathcal{U}_{n+1}} |J| = |U| = \sum_{J \in \mathcal{U}_n} |J|,$$

so
$$C^{-1}|U| = C^{-1}\sum_{J\in\mathcal{U}_n}|J| < \sum_{J\in\mathcal{U}_n} u_J.$$

If ϕ is not full, set $A := U\setminus\bigcup_i U_i$. Then $\lambda(A) > 0$. Let $J\in\mathcal{U}_n$ and let $\mathcal{W} = \{W\in\mathcal{U}_{n+1} : W\cap J\neq\emptyset\}$. Then $\phi^n : J\to U$ is a diffeomorphism with distortion bounded by $C > 1$ which sends $\bigcup_{W\in\mathcal{W}}$ to $U\setminus A$.

Then
$$\frac{\sum_{W\in\mathcal{W}}|W|}{|J|} \leq 1 - \frac{|A|}{C|U|} =: \delta < 1.$$
The δ is independent of J, so
$$\sum_{W\in\mathcal{U}_{n+1}}|W| \leq \delta\sum_{J\in\mathcal{U}_n}|J|,$$
and $\sum_{J\in\mathcal{U}_n}|J|\leq\delta^n|U|$. But $|J| > C^{-1}u_J$, so $\sum_{J\in\mathcal{U}_n}u_J\leq C\delta^n|U|$. □

Definition 3.1.7 *Let $\phi : \bigcup_i U_i \to U$ and $\psi : \bigcup_i V_i\to V$ be Markov maps. A* conjugacy *h between ϕ and ψ shall be a homeomorhism from U onto $h(U) = V$ such that $h\circ\phi = \psi\circ h$.*

Definition 3.1.8 *A conjugacy h between differentiable Markov maps is said to* preserve multipliers (of period points) *if for each $q > 0$,*
$$|D\phi^q(p)| = |D\psi^q(h(p))|$$
for all periodic points p of period q.

Theorem 3.1.9 *Let $\phi : \bigcup_i U_i\to U$ and $\psi : \bigcup_i V_i \to V$ be Markov maps with eternally bounded distortion. Suppose there is a conjugacy h between ϕ and ψ that preserves multipliers of periodic points.*

Then ϕ is full if and only if ψ is full. Moreover, if they are full then h is Lipschitz and thus absolutely continuous.

Proof: That ϕ is full if and only if ψ is full follows immediately from Lemma 3.1.6. Now suppose ϕ and ψ are full with a common distortion bound $\log C > 0$. By Lemma 3.1.3, all periodic points are hyperbolic repelling so the size of connected components of the domain of ϕ^n tends to zero. To show h is Lipschitz, it suffices to show that there is a constant K such that, for all n, if W is a connected component of the domain of ϕ^n then $|h(W)| < K|W|$.

One can take $K = C^2$. Indeed, h preserves multipliers. Let w be the multiplier of a fixed point of ϕ^n in W. Then $C^2|W| > Cw^{-1} > |h(W)|$, on applying the distortion bound twice, as required. □

Definition 3.1.10 *Let $\phi : \bigcup_i U_i\to U$ be a Markov map. It is a $C^{1+\epsilon}$* Markov map *if ϕ is a C^1 diffeomorphism on each U_i and if there is a constant $C > 1$ such that for all x, x' in the same U_i,*
$$\big|\log|D\phi(x)| - \log|D\phi(x')|\big| < C|\phi(x)-\phi(x')|^\epsilon.$$

Definition 3.1.11 *Let $\phi : \bigcup_i U_i\to U$ be a Markov map. It is an eternally $C^{1+\epsilon}$* Markov map *if ϕ is a C^1 diffeomorphism on each U_i and if there is a constant $C > 1$ such that for all $n > 0$, for all x, x' in the same connected component of the domain of ϕ^n,*
$$\big|\log|D\phi^n(x)| - \log|D\phi^n(x')|\big| < C|\phi^n(x)-\phi^n(x')|^\epsilon.$$

Definition 3.1.12 *A Markov map $\phi : \bigcup_i U_i \to U$ is expanding if $D\phi$ exists and is greater than some constant $\lambda > 1$ everywhere on $\bigcup_i U_i$.*

Lemma 3.1.13 *Any expanding $C^{1+\epsilon}$ Markov map is eternally $C^{1+\epsilon}$.*

Proof: Let ϕ be a $C^{1+\epsilon}$ Markov map with constants C, ϵ such that $|D\phi| \geq \lambda$ where λ is some constant greater than 1. We have

$$\begin{aligned}
\big|\log|D\phi^n(x)| - \log|D\phi^n(x')|\big| &\leq \sum_{j=0}^{n-1} \big|\log|D\phi(\phi^j(x))| - \log|D\phi(\phi^j(x'))|\big| \\
&\leq \sum_{j=0}^{n-1} C|\phi^{j+1}(x) - \phi^{j+1}(x')|^\epsilon \\
&\leq \sum_{j=0}^{n-1} C\lambda^{-\epsilon(n-j-1)}|\phi^n(x) - \phi^n(x')|^\epsilon \\
&\leq \sum_{j=0}^{n-1} C\frac{1}{1-\lambda^\epsilon}|\phi^n(x) - \phi^n(x')|^\epsilon,
\end{aligned}$$

as required. □

Note that if ϕ is eternally $C^{1+\epsilon}$ then it has eternally bounded distortion.

Definition 3.1.14 *If a Markov map $\phi : \bigcup U_i \to U$ is C^r on each U_i, for integer $r \geq 2$, then we say ϕ is a C^r* Markov *map provided the following condition is satisfied:*

$$\left|\frac{D^i\phi(x)}{(D\phi(x))^i}\right| < L, \tag{3.1}$$

for some $L > 0$, for all x in the domain of ϕ and for $i = 2, 3, \ldots, r$.

Remark: By Lemma 3.1.19, one has equivalence between (3.1) and

$$\left|\frac{1}{(D\phi(x))^i} D^i \log|D\phi(x)|\right| \leq K$$

for some $K > 0$, for all x in the domain of ϕ and for $i = 1, 2, 3, \ldots, r-1$.

The construction of absolutely continuous invariant measures for a full eternally $C^{1+\epsilon}$ Markov map ϕ is well-known. We shall use just assume one property of such a measure μ, that $\mu(A) = \lim_{n\to\infty} \lambda(\phi^{-n}(A))$, where λ denotes Lebesgue measure. From this property we shall study the measure $\overline{\mu}$ in the natural extension to deduce further results, notably that if ϕ is C^r then the density is C^{r-1}. Superficially the proof looks quite different to that of Martens and de Melo [26], even if the essence is the same. Hopefully our more elementary proof will be helpful in understanding why the proof of Martens and de Melo works.

Fact 3.1.15 *([6], Theorem V.2.2) Let $\phi : \bigcup_i U_i \to U$ be a full eternally $C^{1+\epsilon}$ Markov map.*

Then ϕ has an invariant probability measure μ which is absolutely continuous with respect to Lebesgue measure λ. The measure ν satisfies

$$\mu(A) = \lim_{n\to\infty} \lambda(\phi^{-n}(A))$$

for all Borel sets $A \subset U$.

We shall again need a natural extension: Let $Z = \{0, 1, 2, \ldots\}^{\infty}$. Define Y as the set of couples $(x; z)$ where $x \in U, z = (z_0 z_1 z_2 \ldots) \in Z$ are such that there exists an infinite sequence x_i satisfying

- $\phi(x_{i+1}) = x_i$,
- $x_i \in U_{z_i}$ and
- $x_0 = x$.

Define $F(x; (z_0 z_1 z_2 \ldots)) := (\phi(x); (iz_0 z_1 \ldots))$ if $\phi(x) \in P_i$. Then $\phi : Y \to Y$ is invertible. We shall denote by Π the projection from Y onto U and by Π' the projection from Y into Z. If $V \subset U$ and $z \in Z$ are such that for all $x \in V$, $(x; z) \in Y$ then

$$(V; z) := \bigcup_{x \in V} (x; z).$$

Let $\overline{\mu}$ be the unique probability measure on Y, invariant by F, such that $\Pi_* \overline{\mu} = \mu$. We call $(Y, F, \overline{\mu})$ the natural extension of (U, ϕ, μ). Now we introduce a function Δ which will enable us to avoid indeterminate limits (which tend to $\frac{0}{0}$).

Lemma 3.1.16 *Let $\phi : \bigcup_i U_i \to U$ be a full eternally $C^{1+\epsilon}$ Markov map with natural extension $(Y, F, \overline{\mu})$. For $x, x' \in U$ and $z \in Z$ define the function $\Delta : U \times U \times Z \to \mathbb{R}^+$ by*

$$\Delta(x, x', z) := \lim_{n \to \infty} \frac{D\phi^n(\Pi F^{-n}(x; z))}{D\phi^n(\Pi F^{-n}(x'; z))} = \prod_{i=1}^{\infty} \frac{D\phi(\Pi F^{-i}(x; z))}{D\phi(\Pi F^{-i}(x'; z))}.$$

Then $\Delta(x, x', z)$ is well-defined (i.e. the limit converges), uniformly bounded away from zero and infinity and there is a constant $K > 1$ such that for all x, x', x_0 in U and all z in Z one has

$$|\Delta(x, x_0, z) - \Delta(x', x_0, z)| \leq K|x - x'|^{\epsilon}$$

and so also

$$|\Delta(x_0, x, z) - \Delta(x_0, x', z)| \leq K|x - x'|^{\epsilon}$$

Proof: That the limit converges is easily deduced from the absolute convergence of the series below. The map ϕ is expanding so there is a $\lambda > 1$ such that $|\Pi F^{-n} x - \Pi F^{-n} x'| \leq \lambda^{-n}|U|$. Then

$$\begin{aligned}
\sum_{i=1}^{\infty} \big|\log|D\phi(\Pi F^{-i}(x; z))| - \log|D\phi(\Pi F^{-i}(x'; z))|\big| &\leq \sum_{i=1}^{\infty} C\lambda^{-i\epsilon}|x - x'|^{\epsilon} \\
&\leq C\frac{1}{1 - \lambda^{-\epsilon}}|x - x'|^{\epsilon} \\
&= K_1|x - x'|^{\epsilon}
\end{aligned}$$

for some constant $K_1 > 0$. Then

$$e^{-K_1|U|^{\epsilon}} \leq \Delta(\cdot, \cdot, \cdot) \leq e^{K_1|U|^{\epsilon}} =: K_2$$

say, which gives the uniform bounds away from zero and infinity. Now

$$\begin{aligned}\left|\Delta(x,x_0,z)-\Delta(x',x_0,z)\right| &= \Delta(x,x_0,z)\left|1-\frac{\Delta(x',x_0,z)}{\Delta(x,x_0,z)}\right| \\ &= \Delta(x,x_0,z)\left|1-\Delta(x',x,z)\right| \\ &\leq K_2\left|1-e^{-K_1|x-x'|^\epsilon}\right| \\ &\leq K_2K_3|x-x'|^\epsilon \\ &= K|x'-x|^\epsilon\end{aligned}$$

for some $K_3 > 0$ and $K := K_2K_3$. □

Theorem 3.1.17 *Let $\phi : \bigcup_i U_i \to U$ be a full eternally $C^{1+\epsilon}$ Markov map. Then ϕ has a unique absolutely continuous invariant probability measure μ. The density of μ is Hölder continuous with Hölder exponent ϵ and the density is bounded away from 0 and ∞. If, moreover, ϕ is also a C^r Markov map, $r \geq 2$, then the density of μ is C^{r-1}.*

Proof: By Fact 3.1.15, there exists a unique absolutley continuous invariant probability measure μ and for all subsets A of U, $\mu(A) = \lim_{n\to\infty} \lambda(\phi^{-n}(A))$. One has

$$\begin{aligned}\frac{d\overline{\mu}(A;z)}{d\overline{\mu}(U;z)} &= \lim_{n\to\infty}\frac{\lambda(\Pi F^{-n}(A;z))}{\lambda(\Pi F^{-n}(U;z))} \\ &= \lim_{n\to\infty}\frac{\int_A \frac{1}{|D\phi^n(\Pi F^{-n}(x;z))|}dx}{\int_U \frac{1}{|D\phi^n(\Pi F^{-n}(x;z))|}dx} \\ &= \lim_{n\to\infty}\frac{\int_A \frac{D\phi^n(\Pi F^{-n}(x_0;z))}{D\phi^n(\Pi F^{-n}(x;z))}dx}{\int_U \frac{D\phi^n(\Pi F^{-n}(x_0;z))}{D\phi^n(\Pi F^{-n}(x;z))}dx} \\ &= \frac{\int_A \Delta(x_0,x,z)dx}{\int_U \Delta(x_0,x,z)dx}\end{aligned}$$

for any fixed x_0 in U. Taking a closer look at the density,

$$\frac{d\overline{\mu}(x;z)}{d\overline{\mu}(U;z)} = \frac{\Delta(x_0,x,z)dx}{\int_U \Delta(x_0,x',z)dx'} = \frac{\Delta(x_0,x,z)dx}{C_z},$$

where $C_z := \int_U \Delta(x_0,x',z)dx'$ is uniformly bounded away from zero and infinity. Thus

$$\frac{d\overline{\mu}(\Pi^{-1}x)}{1} = \frac{d\overline{\mu}(\Pi^{-1}x)}{d\overline{\mu}(\Pi^{-1}U)} = \int_Z C_z^{-1}\Delta(x_0,x,z)dx d\mu'(z)$$

where μ' is the projection by Π' of $\overline{\mu}$ onto Z, so

$$d\mu(x) = \int_Z C_z^{-1}\Delta(x_0,x,z)d\mu'(z)dx.$$

Since C_z is uniformly bounded away from zero, by Lemma 3.1.16 there is a constant $C_1 > 0$ such that, for all $x, x' \in U$, for all $z \in Z$,

$$|C_z^{-1}\Delta(x_0,x,z) - C_z^{-1}\Delta(x_0,x',z)| \leq C_1|x-x'|^\epsilon.$$

Thus
$$|d\mu(x) - d\mu(x')| \le \int_Z C_1|x - x'|^\epsilon d\mu'(z)dx = C_1|x - x'|^\epsilon dx.$$
Thus the density is Hölder continuous with exponent ϵ.

Suppose additionally that ϕ is a C^r Markov map. Regarding $\Delta(x_0, x, z)$ as a function of x alone, for $i \le r - 1$ we have

$$\begin{aligned} D^i \log \Delta(x_0, x, z) &= \sum_{n=1}^{\infty} D^i \log \frac{D\phi(\Pi F^{-n}(x_0; z))}{D\phi(\Pi F^{-n}(x; z)} \\ &= \sum_{n=1}^{\infty} -D^i \log |D\phi(\Pi F^{-n}(x; z))| \\ &= \lim_{n\to\infty} D^i \log |Dg_n(x)| \end{aligned}$$

where g_n is the branch of ϕ^{-n} sending x to $\Pi F^{-n}(x; z)$. In the last paragraph of the proof of Lemma 3.1.19, we show that there is a polynomial, P_i say, such that for all $n \ge 1$,

$$D^i \log |Dg_n(x)| = P_i \left(\frac{D \log |D\phi^n(g_n(x))}{(D\phi^n(g_n(x)))^1}, \ldots, \frac{D^i \log |D\phi^n(g_n(x))}{(D\phi^n(g_n(x)))^i} \right).$$

Let us examine $u_{j,n}(x) := \frac{D^j \log |D\phi^n(g_n(x))}{(D\phi^n(g_n(x)))^j}$ for $1 \le j \le i$. We have

$$\begin{aligned} u_{j,n}(x) &= \frac{1}{(D\phi^n(g_n(x)))^j} \sum_{k=1}^{n} D^j \log |D\phi(g_k(x))| \\ &= \sum_{k=1}^{n} \frac{(D\phi(g_k(x)))^j}{(D\phi^n(g_n(x)))^j} \frac{D^j \log |D\phi(g_k(x))|}{(D\phi(g_k(x)))^j}. \end{aligned}$$

By definition of C^r Markov maps there exists a K independent of j such that

$$\frac{|D^j \log |D\phi(g_k(x))||}{|D\phi(g_k(x))|^j} \le K.$$

Since $|D\phi| \ge \lambda > 1$,
$$\frac{|D\phi(g_k(x))|^j}{|D\phi^n(g_n(x))|^j} \le \lambda^{-j(n-1)}.$$
Thus the above series converges (uniformly w.r.t. x) and $\lim_{n\to\infty} u_{j,n}(x)$ exists, call it $u_j(x)$. Then $|u_j(x)| \le K \frac{\lambda}{\lambda-1}$ and

$$D^i \log \Delta(x_0, x, z) = \lim_{n\to\infty} D^i \log |Dg_n(x)| = P_i\left(u_1(x), u_2(x), \ldots, u_i(x)\right).$$

Thus $D^i \log \Delta(x_0, x, z)$ exists for all x. The $u_j(x)$ being bounded, there exists a constant $C_2 > 0$ such that $|P_i\left(u_1(x), \ldots, u_i(x)\right)| \le C_2$, so $|D^i \log \Delta(x_0, x, z)|$ is bounded by C_2.

We now need to show that $D^i \log \Delta$ is continuous. Let $\gamma > 0$ be given. Since the $u_j(x)$ are bounded and the $u_{j,n}(x)$ converge uniformly, there exists an $N > 0$ such that for all x in U,

$$P_i(u_1(x), \ldots, u_i(x)) - P_i(u_{1,N}(x), \ldots, u_{i,N}(x))| < \frac{1}{2}\gamma.$$

By continuity of $u_{j,N}$, there is a δ such that $|x-x'|<\delta$ implies

$$|P_i(u_{1,N}(x),\ldots,u_{i,N}(x)) - P_i(u_{j,N}(x'),\ldots,u_{i,N}(x'))| < \frac{1}{2}\gamma.$$

Putting together these two inequalities gives

$$|D^i \log\Delta(x_0,x,z) - D^i\log\Delta(x_0,x',z)| = |P_i(u_1(x),\ldots,u_i(x)) - P_i(u_1(x'),\ldots,u_i(x'))| \leq \gamma$$

if $|x-x'|<\delta$. Thus $D^i\log\Delta(x_0,x,z)$ is continuous in x and bounded for $i=1,2,\ldots,r-1$. Recall that

$$d\mu(x) = \int_Z C_z^{-1}\Delta(x_0,x,z)d\mu'(z)dx,$$

so

$$D^{r-1}\log\left(\frac{d\mu(x)}{dx}\right) = \int_Z C_z^{-1}D^{r-1}\log\Delta(x_0,x,z)d\mu'(z)$$

is bounded and continuous, as the average of uniformly bounded continuous functions. Thus $D^{r-1}\left(\frac{d\mu(x)}{dx}\right)$ is continuous, so the density $\frac{d\mu(x)}{dx}$ is C^{r-1}. □

Corollary 3.1.18 *Let $\phi : \bigcup_i U_i \to U$ and $\psi : \bigcup_i V_i \to V$ be eternally $C^{1+\epsilon}$ Markov maps. Suppose there is an absolutely continuous conjugacy h between ϕ and ψ. Then h is a $C^{1+\epsilon}$ diffeomorphism. If moreover ϕ and ψ are C^r Markov maps then h is C^r.*

Proof: Let μ and ν be the unique absolutely continuous invariant probability measures for ϕ and ψ respectively. Then $h_*\mu=\nu$, and $Dh(x) = \frac{d\mu(x)}{d\nu(h(x))}$ so Dh is as smooth as the densities of μ and ν. □

Nonlinearity

When trying to find estimates of smoothness of iterates of a smooth map f, a crucial object of study is the *nonlinearity* $D\log|Df| = \frac{D^2 f}{Df}$. Here we present some useful properties.

Lemma 3.1.19 *Let $\phi : A \to \phi(A)$ be a C^r map of a subset of $\mathbb{R}$. The following properties are equivalent:*

1. *$D^i\log|D\phi|$ is bounded for $i=2,3,\ldots,r-1$;*
2. *$(D\phi)^i D^i \log|D\phi^{-1}|$ is bounded for $i=2,3,\ldots,r-1$;*
3. *$\frac{D^{i+1}\phi}{D\phi}$ is bounded for $i=2,3,\ldots,r-1$;*

Proof: Clearly if any of these bounds hold then $D\phi\neq 0$. For simplicity we suppose that $D\phi>0$. Define the following:

$$\begin{aligned}\eta_i(f) &:= D^i\log Df;\\ \theta_i(f) &:= (Df)^{-i}\eta_i(f);\\ \psi_i(f) &:= \frac{D^{i+1}f}{Df}.\end{aligned}$$

One has

$$D\psi_i = \psi_{i+1} - \psi_i\psi_1.$$

We know that $\eta_1(\phi) = \psi_1(\phi)$, by induction $\eta_i(\phi)$ is a polynomial in $\psi_1(\phi), \psi_2(\phi), \dots, \psi_i(\phi)$ and that (1) is equivalent to (3).

For clarity, let $g := \phi^{-1}$. Then $g \circ \phi(x) = x$ so $Dg(\phi(x)) = (D\phi(x))^{-1}$. Deriving with respect to x we get

$$D^2 g(\phi(x)) D\phi(x) = \frac{-D^2\phi(x)}{(D\phi(x))^{-2}},$$

so replacing $D\phi$ by $(Dg)^{-1}$, we get

$$\eta_1(g)_{|\phi(x)} = \theta_1(f)_{|x}.$$

Deriving $\eta_i(g)_{|\phi(x)}$ with respect to x one obtains $D\phi(x)\eta_{i+1}(g)_{|\phi(x)}$ and deriving θ_i one obtains $(D\phi)(\theta_{i+1} - i\theta_i\theta_1)$, so

$$\eta_2(g)_{|\phi(x)} = \theta_2(\phi)_{|x} - (\theta_1(\phi)_{|x})^2.$$

Again an inductive argument will give that $\eta_i(g)_{|\phi(x)}$ is a polynomial in $\theta_j(\phi)_x$, $j \leq i$, such that the sum of js in any term equals i. Multiplying across by $(D\phi)^i$, one concludes that $(D\phi)^i \eta_i(g)_{|\phi(x)}$ can be written as a polynomial in $\eta_j(\phi)_{|x}$ for $j \leq i$ so one has (1) is equivalent to (2). $\square$

Schwarzian derivative

If f is C^3 and $Df(x) \neq 0$ then the Schwarzian derivative $Sf(x)$ of f at x is defined as

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2}\left(\frac{D^2 f(x)}{Df(x)}\right)^2.$$

Using the notation of nonlinearity, one can also write

$$Sf = D\eta_1(f) - \frac{1}{2}(\eta_1(f))^2.$$

Negative Schwarzian derivative is a powerful tool to control distortion along inverse branches. We refer to [6] for details.

3.2 Visible measures of maximal entropy in dimension one

3.2.1 Introduction and statement of results

Remark: The results of this section extend easily to cusp maps with a finite partition into intervals of monotonicity under the proviso that the Lyapunov exponent of the measures of maximal entropy for such maps have positive finite Lyapunov exponent. A general result (without conditions on the Lyapunov exponent) appears difficult.

There are two special types of measure in one-dimensional dynamics. One is measures of maximal entropy, the other is measures which are absolutely continuous with respect to Lebesgue measure. For the class of maps we consider, measures of maximal entropy always exist and have at most a finite number of ergodic components ([13], [14]).

A natural question is for which maps are measures of maximal entropy also absolutely continuous. In the case of rational maps of the Riemann sphere, the question is slightly different. Anna Zdunik [45] shows the following dichotomy: for every rational map, either the measure of maximal entropy is of Hausdorff dimension strictly smaller than the dimension of the Julia set or the map is *critically finite with parabolic orbifold.*

We provide the real one-dimensional analogue of Zdunik's dichotomy: if f is a $C^{1+\epsilon}$ map of positive topological entropy with a finite number of critical points, an ergodic measure of maximal entropy have dimension less than one or f is conjugate to a piecewise linear map on some restrictive interval, the conjugacy is smooth outside the (finite) post-critical set and the measure is supported on the forward orbit of the restrictive intervals. The conjugacy is C^r if f is a C^r map.

Let f be a $C^{1+\epsilon}$ map with critical set X. The *post-critical set* PC for f is defined by $PC = \bigcup_{i\geq 1} f^i(X)$.

We say f has a (non-trivial) *restrictive* interval $J \subset I$ of period $n > 0$ if there exists an interval $J(\neq I)$ disjoint from periodic attractors such that $J, f(J), \ldots, f^{n-1}(J)$ are disjoint, $f^n(\partial J) \subset \partial J$ and $f^n(J) \subset J$. Also, f is *renormalisable of type* n if f has a non-trivial restrictive interval J of period n containing at least one turning point. The map $f^n_{|J} : J \to J$ is then a *renormalisation* of f of *type* n.

We state our results here for C^r maps, where $r \geq 2$. If f is only $C^{1+\epsilon}$ the same results hold only the conjugacies are $C^{1+\epsilon}$.

Theorem 3.2.1 *Let f be a C^r map for some $r \geq 2$ with positive topological entropy and a finite critical set. Exactly one of the following statements holds:*

- *$HD(\mu) < 1$ for all measures of maximal entropy μ.*
- *There exists a restrictive interval J of period k such that $f^k_{|J}$ has finite post-critical set PC, f^k is C^r conjugate to a continuous piecewise linear map of constant slope on $J \setminus PC$ and $\mu(J) > 0$ for some measure of maximal entropy μ.*

Remark: If one weakens the condition that the critical set should be finite to that f has a finite partition into intervals of monotonicity, then the post-critical set need no longer be finite. The conjugacy will be C^r on a dense collection of open subintervals of J.

The condition that f has positive topological entropy is necessary. Indeed a map which is the identity on $[-1, 0]$ and a Feigenbaum map on $[0, 1]$ has zero topological entropy and an absolutely continuous invariant measure.

For non-renormalisable maps a more dynamical version is possible.

Corollary 3.2.2 *Let f be a C^r map, for some $r \geq 2$, without non-trivial restrictive intervals. Exactly one of the following statements holds:*

- *$HD(\mu) < 1$ for measure of maximal entropy μ.*
- *f has finite post-critical set PC and f is C^r conjugate to a continuous piecewise linear map of constant slope on $I \setminus PC$.*

The (symmetric) *tent map* $T_s : [0, 1] \to [0, 1]$, for $0 < s \leq 2$, is defined by

$$\begin{aligned} T_s(x) &= sx, & 0 \leq x \leq \frac{1}{2}, \\ T_s(x) &= s(1-x), & \frac{1}{2} < x \leq 1. \end{aligned}$$

If $1 < s \leq \sqrt{2}$, the tent map T_s provides a simple example of a map which is renormalisable of type 2 (see figure 3.1).

In the unimodal case we achieve a complete dynamical classification of maps where the Hausdorff dimension of the measure of maximal entropy is one.

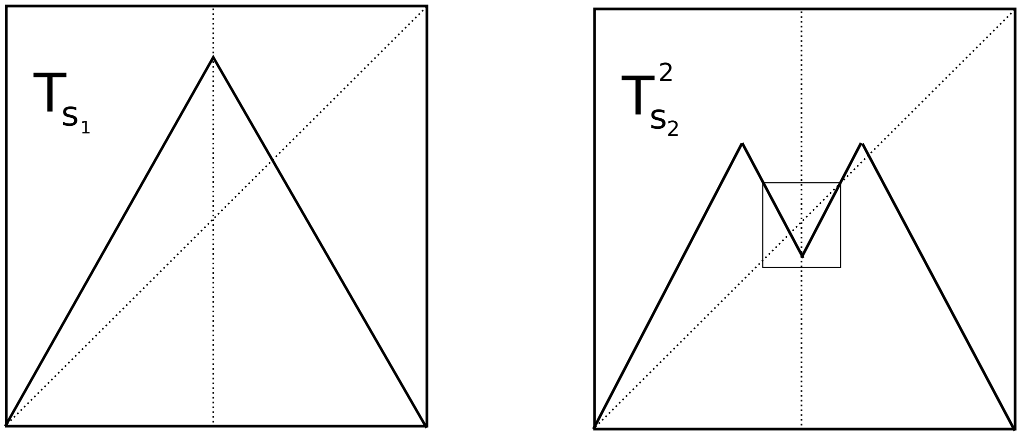

Figure 3.1: Tent maps. T_{s_2} is renormalisable of type 2.

Definition 3.2.3 *A $C^{1+\epsilon}$ map f is* unimodal *if it has exactly one turning point, no other critical points in the interior of I, exactly two fixed points and $f(\partial I) \subset \partial I$. A unimodal C-map f is called C^r* pre-Chebyshev *if*

- *f is exactly k times renormalisable, for some $0 \leq k < \infty$, and each renormalisation is of type 2 (i.e. of Feigenbaum type);*
- *Let $f^{2^k}_{|J} : J \to J$ be the resultant non-renormalisable map. Then there is a C^r conjugacy h on the interior of J from $f^{2^k}_{|J}$ to the full tent map T_2;*

If f is pre-Chebyshev then the C^r conjugacy h extends to a topological conjugacy on I from f to a symmetric tent map T_s of slope $\pm s$, where the entropy of f is $\log s$. Then if f is k times renormalisable, $s = 2^{(2^{-k})}$. Also remark that the full tent map T_2 is analytically conjugate to the Chebyshev quadratic polynomial $4x(1-x)$.

Theorem 3.2.4 *Suppose f is a unimodal C^r map, for some $r \geq 2$, and that f has postive topological entropy and measure of maximal entropy μ. Exactly one of the following statements holds:*

- *$HD(\mu) < 1$.*
- *f is C^r pre-Chebyshev.*

3.2.2 Proofs

We shall make use of the following result of Milnor and Thurston (1977), also Parry (1966), see [6] II.8.

Fact 3.2.5 *Assume that $f : I \to I$ is a continuous, piecewise (strictly) monotone map with positive topological entropy $h_t(f)$ and let $s = \exp(h_t(f))$. Then there exists a continuous, piecewise linear map $T : [0,1] \to [0,1]$ with slope $\pm s$, and a continuous, monotone increasing map $h : I \to [0,1]$ which is a semi-conjugacy between f and T, i.e.*

$$h \circ f = T \circ h.$$

Proposition III.4.3 of [6] adds the following properties for C^1 multimodal maps without wandering intervals (which will follow from transitivity):

Fact 3.2.6 *Let X be the set of points which are eventually mapped into a restrictive interval or into the basin of a periodic attractor.*

1. *The semi-conjugacy h can collapse an interval only if it is contained in X. Basins of periodic attractors are certainly collapsed.*
2. *The union of X and the set of points which are mapped eventually into an interval $V \neq I$ with $f(V) \subset V$ is dense in I. Moreover f is conjugate to T on the complement of X.*
3. *If T has a periodic turning point c then $h^{-1}(c)$ is a restrictive interval.*

In particular, if f is transitive then the semi-conjugacy is a conjugacy.
Proof of Theorem 3.2.1: According to [13], [14], measures of maximal entropy form a convex set with a finite number of (ergodic) extremal points. A measure of maximal entropy of Hausdorff dimension one therefore has an ergodic component of dimension one.

We deal with the easy direction first. Suppose that there is a C^r conjugacy from a restrictive interval to a piecewise linear map of constant slope as per the statement of the theorem. Given the restrictive interval J of period k and μ such that $\mu(J) > 0$ from the statement, we can assume without loss of generality that μ is ergodic. Then the support of μ is $\bigcup_{i=0}^{k-1} f^k(J)$. The measure $\mu_{|J}$ when normalised is the measure of maximal entropy for $f^k_{|J} : J \to J$ and coincides with the C^r pullback of the absolutely continuous measure of maximal entropy for the piecewise linear map. Thus μ is absolutely continuous and $\mathrm{HD}(\mu) = 1$.

Now consider a measure of maximal entropy of Hausdorff dimension one. Let μ be an ergodic component of dimension one. Since the topological entropy of f is positive, the Lyapunov exponent of μ is positive. We can regard $f : I \to I$ as the restriction of a continuous cusp map defined on a larger interval which has a finite partition into intervals of monotonicity (since f has). By Corollary 1.9.3, μ is absolutely continuous and the support of μ is a finite union of intervals on which μ is equivalent to Lebesgue and on which f is transitive. Then Theorem 1.9.10 implies that there is a C^r induced Markov map ϕ for f (Definition 1.9.9), with range an interval V say contained in a connected component J of the support of μ.

There is an iterate of f, call it $g = f^k$, which maps J to J, and μ, when normalised, is the measure of maximal entropy for g. By transitivity, g is at most finitely renormalisable, so we can assume $g : J \to J$ is non-renormalisable. Now g is a continuous transitive map of J, so the semi-conjugacy ρ from g to a piecewise linear map T given by Fact 3.2.5 is actually a conjugacy. Thus the push-forward of μ, $\rho_*\mu$ is (when renormalised) a measure of maximal entropy for T. But T is a transitive piecewise linear map, so it has a unique measure of maximal entropy which is equivalent to Lebesgue measure. By uniqueness, $\rho_*\mu$ is absolutely continuous and indeed equivalent to Lebesgue measure λ.

It follows that ρ and ρ^{-1} are absolutely continuous. Indeed, $\lambda(A) = 0 \iff \mu(A) = 0 \iff \rho_*\mu(\rho(A)) = 0 \iff \lambda(\rho(A)) = 0$.

Pushing forward ϕ by ρ we get a C^r induced Markov map ψ for T, so $\rho \circ \phi = \psi \circ \rho$. By Corollary 3.1.18, ρ is C^r on W. Let x be in W, $n > 0$ and $t := |DT|$. Then

$$D(\rho \circ g^n)(x) = D\rho(g^n(x)) \cdot Dg^n(x) = D(T^n \circ \rho)(x) = t^n \cdot D\rho(x),$$

so if $Dg^n(x) \neq 0$ then

$$D\rho(g^n(x)) = \frac{t^n}{Dg^n(x)} D\rho(x) \tag{3.2}$$

exists and is non-zero. A similar calculation gives existence and continuity of $D^i\rho(g^n(x))$ where $Dg^n(x) \neq 0$ for all finite $i \leq r$.

Applying Lemma 1.5.2, there exists a k such that $\bigcup_{j=0}^k g^j(W) = J$. Thus ρ is C^r on $J \setminus \bigcup_{j=1}^k g^j(Crit)$ where $Crit$ denotes the critical set of g. On the other hand, by (3.2), $D\rho$ is singular on the post-critical set $PC = \bigcup_{j\geq 1} g^j(Crit)$. Thus the post-critical set for g is finite, completing the proof of Theorem 3.2.1. □

We now show that if f (and thus g) has no inflection points, then $PC = \partial J$. This is equivalent to showing that $g(Crit \cup \partial X) \subset \partial X$. Recall $g : J \to J$ is non-renormalisable.

Lemma 3.2.7 *For each non-renormalisable transitive C^1 map without inflection points $f' : I \to I$, for each point y in the interior of I, there exists an infinite sequence $y = y_0, y_1, \ldots$ satisfying $f'(y_{i+1}) = y_i$, $Df'(y_i) \neq 0$ and $y_i \neq y_j$ for all $i \neq j$.*

Proof: Firstly note that there exists x in the interior of I with $f'(x) = y$ and $Df'(x) \neq 0$, for otherwise there is a contradiction with transitivity and non-renormalisability (all critical points are turning points). We can in this way construct a sequence which satisfies the conditions of the lemma unless y is part of a periodic orbit without non-critical (or boundary) preimages (other than itself). The partition induced by this periodic orbit defines restrictive intervals, contradicting non-renormalisability. □

Lemma 3.2.8 *Suppose g has no inflection points, then $PC = \partial J$,*

Proof: By transitivity, $g(Crit) \cup g^2(Crit) \supset \partial X$. It remains to show $PC \subset \partial X$.

Suppose there exists $y \in PC \setminus \partial X$. Then $Dh(y) = Dh(y_i) = \infty$ for all i, where y_i is given by Lemma 3.2.7. But from above, the conjugacy is C^r on all of X except the finite post-critical set. The contradiction proves the lemma. □

Remark: In this case all branches map *onto* the domain and a simple calculation gives that if the slope is $\pm t$, then T has $t - 1$ turning points.

Proof of Corollary 3.2.2: It follows immediately from Theorem 3.2.1. □

Fact 3.2.9 *([17]) For the symmetric tent map T_s with slope $\pm s$ and turning point c:*

1. *If $\sqrt{2} < s \leq 2$ then for every open interval J there exists an n such that $T_s^n(J) \supset [T_s^2(c), T_s(c)]$. In particular T_s has no restrictive intervals.*

2. *If integer k is such that $\sqrt{2} < s^{2^k} \leq 2$ then T_s has exactly k restrictive intervals $I_k \subset I_{k-1} \subset \cdots \subset I_0 = I$ containing the turning point. I_i has period 2^i for $i = 0, 1, \ldots, k$. For each $x \in I$ there exists an integer n such that either $T_s^n(x) \in I_k$ or such that $T_s^n(x)$ is one of the k periodic orbits that remain outside of the forward orbit of I_k.*

Proof of Theorem 3.2.4: A unimodal map has no inflection points. Suppose $\mathrm{HD}(\mu) = 1$. Then μ is absolutely continuous so the semi-conjugacy h to the symmetric tent map T given by Fact 3.2.5 collapses no restrictive intervals and h is a conjugacy on I. By unicity of measures of maximal entropy for unimodal maps, μ is the pullback of the absolutely continuous invariant measure for T. Arguing as above there is a restrictive interval containing the critical point c on which h is a C^r conjugacy and on which the renormalisation of T is the full tent map T_2. Applying Fact 3.2.9 gives the remainder of the theorem. □

3.3 Positive Lyapunov exponent

The induced expansion due to positive Lyapunov exponent permits us to show some conjugacy results.

The following result was proven by Bruin and Luzzatto [4] for C^3 interval maps with finitely many non-flat critical points. In our setting, C^3 cusp maps with negative Schwarzian derivative may not have poles, but may have infinitely many critical points and (critical) discontinuities and the critical points are not necessarily non-flat.

Theorem 3.3.1 *Let f and g be C^3 cusp maps with negative Schwarzian derivative and suppose f is conjugate to g by a homeomorphism h. Let μ be an ergodic invariant probability measure for f and $h_*\mu$ the push-forward of μ.*

Then the Lyapunov exponent of μ with respect to f is positive if and only if the Lyapunov exponent of $h_*\mu$ is positive with respect to g.

Proof: If μ is atomic and has positive Lyapunov exponent then it is repelling on both sides of any atom. This topological property is preserved by h, so image atoms are topologically repelling. The negative Schwarzian derivative prevents the existence of topologically repelling (on both sides) parabolic periodic points, so the atoms for g are hyperbolic repelling, and thus have positive Lyapunov exponent. Henceforth we assume that μ is non-atomic.

Let $(Y, F, \overline{\mu})$ be the natural extension of (f, μ) and let k_0, A, m_A be as per Section 1.6 where $0 < \eta < \chi$. By Lemma 1.6.6 there exists a regularly returning open interval W such that $m_A(W) > 0$ and $|W| < k_0^{-1}$. Set $W_A := A \cap \Pi^{-1}W$. Let

$$W' := B(\Pi y', k_0)$$

for some $y' \in W_A$. If $F^n y \in W_A$ then there exists an interval V_n containing Πy such that f^n maps V_n diffeomorphically onto W', and W' compactly contains W so $h(W')$ compactly contains $h(W)$. Since g has negative Schwarzian derivative, there exists a constant C such that, for any n, if g^n maps an interval J' onto $h(W')$ diffeomorphically and g^n maps $J \subset J'$ onto $h(W)$ then the distortion of g^n on J is bounded by C. Thus there exists a $\delta > 0$ and $\lambda > 1$ such that if $|J| < \delta$ then $|Dg^n_{|J}| > \lambda$. Since h is continuous, there exists a $\delta' > 0$ such that for any subinterval V of I with $|V| < \delta'$, $|h(V)| < \delta$.

By Theorem 1.4.1, there exists $N > 0$ such that for all $n \geq N$ and all $y \in Y$, if $F^n(y) \in W_A$ then there exists $V \ni \Pi y$ with $|V| < \delta'$ such that f^n maps V diffeomorphically onto W'. But $|h(V)| < \delta$ and g^n maps $h(V)$ diffeomorphically onto $h(W')$, so $|Dg^n_{|J}| > \lambda$ where $J \subset h(V)$ is mapped diffeomorphically onto $h(W)$.

For $y \in Y$ denote by $n_k = n_k(y)$ the k^{th} entry time of y into W_A. On some set of full measure X with respect to $\overline{\mu}$,

$$\lim_{k\to\infty} \frac{k}{n_k} = m_A(W) > 0.$$

Let $y \in X$ and consider $|Dg^{n_{kN}}(h(\Pi y))|$. From the above, this is greater than λ^k since $n_{N(k+1)} - n_{kN} \geq N$. Thus

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log |Dg^n(h(\Pi y))| &\geq \lim_{k\to\infty} \frac{1}{n_{kN}} \log \lambda^k \\
&\geq \lim_{k\to\infty} \frac{kN}{N n_{kN}} \log \lambda = \frac{m_A(W)}{N} \log \lambda > 0,
\end{aligned}$$

and we conclude that the Lyapunov exponent is greater than zero almost everywhere as required. □

We have not seen the following result before. Nevertheless, it is fairly straightforward to prove.

Theorem 3.3.2 *Let f and g be cusp maps and suppose f is conjugate to g by a homeomorphism h. Let μ be an ergodic invariant probability measure for f, $h_*\mu$ the push-forward of μ and suppose both measures have positive finite Lyapunov exponent.*

If h preserves multipliers then h preserves the Lyapunov exponent.

Proof: Let χ_f and χ_g be the Lyapunov exponents of f and g respectively. Suppose $\chi_f \neq \chi_g$ and let positive η be smalller than $\frac{1}{2}|\chi_f - \chi_g|$.

Let $(Y_f, F, \overline{\mu})$ be the natural extension for (f, μ) and $(Y_g, G, h_*\overline{\mu})$ the natural extension for $(g, h_*\mu)$. There is a set of full measure $X_1 \subset Y_f$ such that for almost every $y \in X_1$, for any neighbourhood V of Πy, there is an infinite increasing sequence n_k such that $\Pi F^{-n_k} y \in V$, by ergodicity. Theorem 1.4.1 gives another subset $X \subset X_1$ of full measure such that, for all $y \in X$, there is a sequence V_n such that f^n maps $V_n \ni \Pi F^{-n} y$ diffeomorphically onto V with distortion bounded by $\log 2$, and such that

$$\rho_f(y)^{-1} e^{n(\chi_f - \eta)} < |Df^n(\Pi F^{-n} y)| < \rho_f(y) e^{n(\chi_f + \eta)},$$

where ρ_f is a measurable function on X with $0 < \rho_f(y) < \infty$. Then V_{n_k} contains a periodic point p_k such that

$$2^{-1} \rho_f(y)^{-1} e^{n_k(\chi_f - \eta)} < |Df^{n_k}(p_k)| < 2\rho_f(y) e^{n_k(\chi_f + \eta)}.$$

Continuing, $h(X)$ is a subset of Y_g of full measure. Applying Theorem 1.4.1 again, there is a set $X_g \subset h(X)$ of full measure with the same estimates but with ρ_g in place of ρ_f and $h(p_k)$ in place of p_k. Since h preserves multipliers,

$$|Df^{n_k}(p_k)| = |Dg^{n_k}(h(p_k))|,$$

so

$$2^{-1} \rho_f(y)^{-1} e^{n_k(\chi_f - \eta)} < 2\rho_g(y) e^{n_k(\chi_g + \eta)}$$

and

$$2^{-1} \rho_g(y)^{-1} e^{n_k(\chi_g - \eta)} < 2\rho_f(y) e^{n_k(\chi_f + \eta)}.$$

These hold for k (and thus n_k) arbitrarily large, and $\eta < \frac{1}{2}|\chi_f - \chi_g|$, so we conclude that $\chi_f = \chi_g$. □

Evidently the entropy is always preserved by conjugacies. Applying the Dynamical Volume Lemma will give the following:

Theorem 3.3.3 *Let f and g be cusp maps with finite partitions into intervals of monotonicity. Suppose f is conjugate to g by a homeomorphism h. Let μ be an ergodic invariant probability measure for f, $h_*\mu$ the push-forward of μ and suppose both measures have positive finite Lyapunov exponent.*

If h preserves multipliers then h preserves the Hausdorff dimension of the measure.

Proof: By Theorem 3.3.2, f and g have the same Lyapunov exponent, and entropy is a topological invariant so f and g have the same entropy. By Theorem 1.7.2,

$$\mathrm{HD}(\mu) = \frac{h_\mu}{\chi_\mu} = \frac{h_{h_*\mu}}{\chi_{h_*\mu}} = \mathrm{HD}(h_*\mu).$$

□

Theorem 3.3.4 *Let f and g be cusp maps with finite partitions into intervals of monotonicity. Suppose f is conjugate to g by a homeomorphism h. Let μ be an ergodic invariant probability measure for f, $h_*\mu$ the push-forward of μ and suppose both measures have positive finite Lyapunov exponent.*

If h preserves multipliers and μ is absolutely continuous then $h_*\mu$ is absolutely continuous.

Proof: Since μ is absolutely continuous, $\mathrm{HD}(\mu) = 1$. By Theorem 3.3.3, $\mathrm{HD}(h_*\mu) = 1$. We apply Corollary 1.9.3 to conclude that $h_*\mu$ is absolutely continuous. □

We can actually do better than this and show that h itself is absolutely continuous. Later we shall improve on this once more by showing that h is $C^{1+\epsilon}$ on a large domain.

Theorem 3.3.5 *Let f and g be cusp maps with finite partitions into intervals of monotonicity and only a finite number of discontinuities. Suppose f is conjugate to g by a homeomorphism h. Let μ be an ergodic invariant probability measure for f, $h_*\mu$ the push-forward of μ and suppose both measures have positive finite Lyapunov exponent.*

If μ and $h_*\mu$ are both absolutely continuous then h is absolutely continuous on the support of μ. The support of μ is a finite union of intervals.

Proof: By Theorem 1.9.6, the support of μ is a finite union of intervals on which μ is equivalent to Lebesgue. The same holds for $h_*\mu$. The result follows immediately. □

3.4 Conjugacies with the Chebyshev polynomial $4x(1-x)$

We first noticed the Chebyshev polynomial $4x(1-x)$ as a special case when trying to prove our results in [8]. See also [26]. More recently, while working with cusp maps the question of invariant measures with infinite Lyapunov exponent, or such that the logarithm of the derivative is non-integrable, arose naturally. Topological Chebyshev maps are a natural source of examples as (topologically) the dynamics is well understood, being the same as that of the full tent map T_2. It turns out that smooth (except at the boundary) conjugacies with $4x(1-x)$ provide an interesting set of examples.

In Theorem 1.9.7 and Corollary 1.9.8, we showed that for $C^{1+\epsilon}$ maps there are no absolutely continuous invariant measures with positive Lyapunov exponent if the critical points are too flat. Here we find unimodal maps with absolutely continuous invariant measures with positive entropy and *very* flat critical points. This is strange, but a contradiction is avoided because these maps have singularities at the boundary and the logarithm of the derivative is non-integrable.

Topological Chebyshev maps will provide examples of maps whose measures of maximal entropy have infinite Lyapunov exponent.

Chebyshev cusp maps

In Chapter 1 we developed the theory of cusp maps with invariant measures with positive finite Lyapunov exponent. One could ask do such measures exist in a non-trivial way. They do. We also noted that if the Lyapunov exponent did not exist, or if it were infinite, then the proofs would not work. There are measures, and even ergodic absolutely continuous invariant probability measures with smooth densities, for which the Lyapunov exponent does not exist. Our goal here is to present a few examples of such maps and measures, formulated in the following proposition.

Proposition 3.4.1 *Let TC denote the class of cusp maps or restrictions of cusp maps which are topologically conjugate by a conjugacy h to the full tent map T_2. Let TC^∞ denote the subclass of* *such maps such that h is a C^∞ diffeomorphism on the interior of the interval. For $f \in TC$ let μ_f denote the pullback by h of the ergodic absolutely continuous invariant probability measure for T_2. For $f \in TC$, μ_f is the measure of maximal entropy. If $f \in TC^\infty$ then μ_f is also absolutely continuous.*

Then there exist

1. *$f \in TC^\infty$ with poles at the boundary such that the Lyapunov exponent of μ_f is 2;*
2. *$f \in TC^\infty$ such that the Lyapunov exponent of μ_f does not exist;*
3. *$f \in TC^\infty$ such that f has a smooth parabolic fixed point at the boundary and the Lyapunov exponent of μ_f is 2;*
4. *$f \in TC$ such that μ_f has positive infinite Lyapunov exponent.*

For each $\alpha > 0$ let h_α be a C^∞ homeomorphism of $[0,1]$ with the following properties:

- $|Dh_\alpha| > 0$ except at 0 and at 1;
- on a neighbourhood of 0 one has $h_\alpha(x) = e^{-x^{-\alpha}}$;
- the graph of h has a point of central symmetry at $(\frac{1}{2}, \frac{1}{2})$, in other words $h(1-x) = 1-h(x)$.

Recall the definition of the full tent map $T := T_2 : [0,1] \to [0,1]$:

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}; \\ 2-2x & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

For each $\alpha > 0$ define $g_\alpha : [0,1] \to [0,1]$ by

$$g_\alpha(x) := h_\alpha \circ T \circ h_\alpha^{-1}.$$

Then g_α is conjugate to the tent map, symmetric and has non-zero, non-infinite derivative everywhere except for at $0, \frac{1}{2}$, and 1 (figure 3.2). Denote $\frac{1}{2}$ by c. Of course, $g(0) = 0, g(c) = 1, g(1) = 0$. Let us calculate the derivative near 0 which simplifies beautifully!

Firstly note that, near zero, $h^{-1}(x) = (-\log x)^{-\frac{1}{\alpha}}$, so

$$Dh^{-1}(x) = \frac{1}{\alpha}(-\log x)^{\frac{-1-\alpha}{\alpha}} \frac{1}{x}.$$

Clearly $|DT| = 2$. Next, $Dh(x) = \alpha x^{-1-\alpha} e^{-x^{-\alpha}}$. Putting these together using the chain rule,

$$\begin{aligned} Dg_\alpha(x) &= Dh(2h^{-1}(x))2Dh^{-1}(x) \\ &= \frac{\alpha}{\alpha} 2^{-1-\alpha}(-\log x)^{\frac{1+\alpha}{\alpha}} e^{-2^{-\alpha}(-\log x)^1} 2(\log x)^{\frac{-1-\alpha}{\alpha}} \frac{1}{x} \\ &= 2^{-\alpha} x^{2^{-\alpha}} \frac{1}{x} \\ &= 2^{-\alpha} x^{2^{-\alpha}-1}. \end{aligned}$$

One can double-check using the conjugacy that near zero, $g_\alpha(x) = x^{2^{-\alpha}}$. The derivative near 1 is the same but with negative sign. We shall see soon that there is a transition in behaviour at $\alpha = 1$ which corresponds to a cusp at 0 of root type $= \frac{1}{2}$.

Now let us look at the derivative near the critical point. Here $Dh_\alpha \neq 0$ and we can assume it is almost constant $\approx Dh_\alpha(c) =: l^{-1}$, say. If x is close to c then $h_\alpha^{-1}(c)$ is close to $\frac{1}{2}$ too, so it gets mapped by T to a point close to 1. We get

$$\begin{aligned}
|Dg_\alpha(x+c)| &= \left|Dh_\alpha(Th_\alpha^{-1}(x+c))2Dh_\alpha^{-1}(x+c)\right| \\
&\approx Dh_\alpha(2l|x|)2l \\
&= \alpha(2l|x|)^{-1-\alpha}e^{-(2l|x|)^{-\alpha}}.
\end{aligned}$$

We have seen in Corollary 1.9.8, that at $\alpha = 1$ there is some sort of a transition in other maps with this critical behaviour (polynomial times $e^{-|y|^{-\alpha}}$).

The tent map an absolutely continuous invariant measure of maximal entropy which is Lebesgue measure. We can pull this back by h_α to get the measure of maximal entropy for g_α. This measure μ is absolutely continuous since h_α is smooth and is given by $d\mu(x) = Dh_\alpha^{-1}(x)dx$. Let us try integrating $\log|Dg_\alpha|$ near 0:

$$\begin{aligned}
\int \log|Dg_\alpha(x)|Dh^{-1}(x)\,dx &\approx \int (\log 2^{-\alpha} + (2^{-\alpha}-1)\log x)(\frac{1}{\alpha}(-\log x)^{\frac{-1-\alpha}{\alpha}}\frac{1}{x})\,dx \\
&= \int \log 2^{-\alpha} Dh^{-1}(x) - (2^{-\alpha}-1)\frac{1}{\alpha}(-\log x)^{\frac{-1}{\alpha}}\frac{1}{x}\,dx \\
&= C_0 + \int C_1 \frac{1}{x(-\log x)^{\frac{1}{\alpha}}}\,dx,
\end{aligned}$$

where C_0 and C_1 are positive constants. This last integral, well known in France as an *intégrale de Bertrand*, is finite if and only if $\alpha < 1$.

Near c, since Dh is approximately a positive constant, $\log|Dg_\alpha|$ will be integrable with respect to μ if and only if it integrable with respect to Lebesgue. It is easy to see that it is integrable with respect to Lebesgue if and only if $\alpha < 1$.

Thus g_α has an absolutely continuous invariant probability measure with **finite** Lyapunov exponent if $\alpha < 1$. In this case, since h_α is smooth everywhere bar the boundary, every periodic point other than 0 has multiplier 2^k where k is the period. In fact the Lyapunov exponent is equal to 2. This follows from the existence of the Lyapunov exponent at almost every point — using the conjugacy one can show that $\frac{1}{n}\log|Dg_\alpha^n(x)|$ approaches 2 infinitely often almost everywhere.

If $\alpha \geq 1$ then g_α has an **absolutely continuous invariant probability measure** whose **Lyapunov exponent does not exist**. At the critical point $\int \log|Dg_\alpha|d\mu$ is $-\infty$; at 0 it is $+\infty$. If we write $\underline{\chi}(x)$ and $\overline{\chi}(x)$ for the lower and upper Lyapunov exponents of g_α at x, then, almost everywhere,

$$\underline{\chi}(x) \leq 2 \leq \overline{\chi}(x).$$

Now we conjugate in the other direction. For each $\alpha > 0$ define $f_\alpha : [0,1] \to [0,1]$ by

$$f_\alpha(x) := h_\alpha^{-1} \circ T \circ h_\alpha.$$

Again, f_α is symmetric, conjugate to the tent map and the derivative is non-zero and finite except possibly at $0, c, 1$. In fact the norm of the derivative is positive and finite everywhere except for at c (figure 3.2). Let us calculate Df_α near 0.

$$Df_\alpha(x) = Dh_\alpha^{-1}(2e^{-x^{-\alpha}})2\alpha x^{-\alpha-1}e^{-x^{-\alpha}}$$

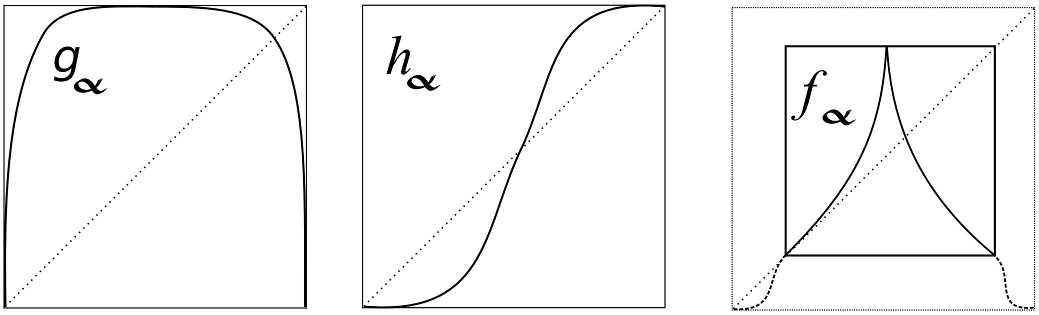

Figure 3.2: Graphs of g_α, h_α and f_α.

$$\begin{aligned} &= \alpha^{-1}(-\log 2 + x^{-\alpha})^{\frac{-1-\alpha}{\alpha}} \frac{1}{2} e^{x^{-\alpha}} 2\alpha x^{-\alpha-1} e^{-x^{-\alpha}} \\ &= (-\log 2 + x^{-\alpha})^{\frac{-1-\alpha}{\alpha}} x^{-\alpha-1} \\ &= (1 - x^{\alpha}\log 2)^{\frac{-1-\alpha}{\alpha}}. \end{aligned}$$

Thus $Df_\alpha(0) = 1$ and there is a parabolic fixed point at zero. It is straightforward to check that, for all i such that $2 \le i < \alpha + 1$, $D^i f_\alpha(0)$ exists and $D^i f_\alpha(0) = 0$. Also, f_α is C^∞ near zero if and only if $\alpha > 0$ is a natural number, in which case $D^{\alpha+1} f_\alpha(0) > 0$.

Near the critical point we have

$$\begin{aligned} |Df_\alpha(c+x)| &= \left|Dh_\alpha^{-1}(Th_\alpha(x+c))2Dh_\alpha(x+c)\right| \\ &\approx Dh_\alpha^{-1}(2l^{-1}|x|)2l^{-1} \\ &= 2l^{-1}\alpha^{-1}\left(-\log(l^{-1}2|x|)\right)^{\frac{-1-\alpha}{\alpha}} 2^{-1}l|x|^{-1} \\ &= \alpha^{-1}|x|^{-1}\left(-\log(l^{-1}2|x|)\right)^{\frac{-1-\alpha}{\alpha}}. \end{aligned}$$

Hence the derivative does indeed tend to infinity at c. Moreover, $\frac{1}{Df_\alpha}$ is C^ϵ near c, so an extension of f_α is an cusp map.

Pulling back Lebesgue measure, we get an absolutely continuous invariant probability measure μ for f_α with C^∞ density $Dh(x)$. The Lyapunov exponent here is integrable and equal to 2, **despite the existence of a smooth parabolic point in the support of** μ.

Now we show that there is a cusp map f in TC whose measure of maximal entropy has infinite Lyapunov exponent. Let $f : [0,1] \to [0,1]$ be a map in TC with poles at the boundary and at the preimage c of the turning point $\frac{1}{2}$, such that on U a neighbourhood of zero, $f(x) = x^\alpha$ for some α with $0 < \alpha < 1$. See figure 3.3.

Let $f_1 := f_{|[0,c]}$ and let p, N be such that $p = f_1^{-N}(c) \in U$ and set $p_n := f_1^{-n}(p)$. Then

$$p_n = p^{\frac{1}{\alpha^n}}.$$

On $[p_{n+1}, p_n]$ one has $\alpha p_{n+1}^{\alpha-1} \le Df \le \alpha p_n^{\alpha-1}$, and $\mu_f([p_{n+1}, p_n] = 2^{-(n+N+1)}$ since f is conjugate to the full tent map T_2. So, on $[p_{n+1}, p_n]$,

$$\log\alpha + (\alpha-1)\frac{1}{\alpha^{n+1}}\log p \le \log Df \le \log\alpha + (\alpha-1)\frac{1}{\alpha^n}\log p.$$

Subtracting $\log\alpha$ and integrating,

$$(\alpha-1)\log p\sum_{i=0}^{\infty}\frac{1}{\alpha^{i+1}}2^{-(i+N+1)} \le -\mu_f([0,p])\log\alpha + \int_0^p \log Df\, d\mu_f \le (\alpha-1)\log p\sum_{i=0}^{\infty}\frac{1}{\alpha^i}2^{-(i+N+1)}$$

so, near zero, $\int \log Df \, d\mu_f$ is finite if $\alpha > \frac{1}{2}$ and positive infinite if $\alpha \leq \frac{1}{2}$. By definition, the derivative is bounded away from zero, so, if $\alpha \leq \frac{1}{2}$, then

$$\int_{[0,1]} \log |Df| \, d\mu = +\infty.$$

□

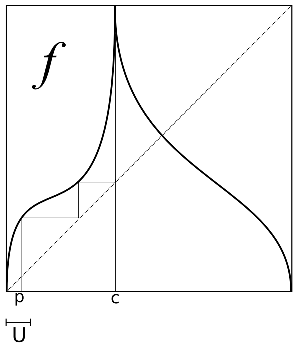

Figure 3.3: Graph of f.

4

Dimension theory

Contents

4.1 Fractal dimensions

The complicated nature of the long-term behaviour of dynamical systems leads one to consider many objects which are not smooth. Often to these objects one can assign numbers in an intelligent manner which characterise some aspect of the system. Notions of dimension are closely related to these numbers and appear regularly throughout this work. We provide necessary definitions of useful concepts in this section.

Everything in this section is well-known and only a small portion of the general dimension theory of dynamical systsems. Good references are [30], [42] and [36], for example.

Lyapunov exponent

Let $f : K \to K$ be a C^1 map of a forward invariant bounded set K. The *Lyapunov exponent* $\chi(x)$ of a point x in K is defined by

$$\chi(x) := \lim_{n\to\infty} -\frac{1}{n} \log |Df^n(x)|$$

where this limit exists. If the limit does not exist one can define lower and upper Lyapunov exponents:

$$\underline{\chi}(x) = \liminf_{n\to\infty} -\frac{1}{n}\log|Df^n(x)|;$$

$$\overline{\chi}(x) = \limsup_{n\to\infty} -\frac{1}{n}\log|Df^n(x)|.$$

Away from the critical orbits the Lyapunov exponent of a point is invariant: $\chi(x) = \chi(f(x))$. If μ is an invariant probability measure one defines the *(characteristic) Lyapunov exponent of* μ, χ_μ, by

$$\chi_\mu = \int_K \log|Df(x)|d\mu$$

when the integral is defined. If K is compact then Df is bounded so $\chi_m u \in [-\infty, +\infty)$. If the integral is undefined (which can happen for cusp maps) then we say *the Lyapunov exponent of* μ *does not exist.*

If μ is ergodic and the Lyapunov exponent of μ exists then $\chi_\mu = \chi(x)$ almost everywhere.

Hausdorff dimension

Let $U \subset \mathbb{R}^N$. Recall the *diameter* of U is written as $|U|$ and means the supremum of the distance between pairs of points of U.

The *Hausdorff dimension* of a subset A of $\mathbb{R}^N$, written $\mathrm{HD}(A)$, can be defined as follows:

$$\mathrm{HD}(A) := \inf\{\delta : \inf_{\mathcal{U}} \sum_{U\in\mathcal{U}} |U|^\delta = 0\}$$

where the inner infemum is taken over all countable covers $\mathcal{U}$ of A. The Hausdorff dimension of a subset of $\mathbb{R}^N$ always exists.

Any reader unfamiliar with the Hausdorff dimension should take the time to verify the following straightforward properties:

- If $A \subset B$ then $\mathrm{HD}(A) \leq \mathrm{HD}(B)$;
- if A is countable then $\mathrm{HD}(A) = 0$;
- if A is an open subset of a k-dimensional smooth submanifold, then $\mathrm{HD}(A) = k$;
- $\mathrm{HD}(\bigcup_{i=1}^\infty A_i) = \sup_i \mathrm{HD}(A_i)$;
- if $f : A \to \mathbb{R}^N$ is Lipschitz then $\mathrm{HD}(f(A)) \leq \mathrm{HD}(A)$.

In particular, the Hausdorff dimension of the set of rational numbers is zero since they are countable, but that of its closure is one.

Box dimensions

Given a bounded set A and a number $\delta > 0$, let N_δ be the minimal number of sets (or equivalently *boxes*) of diameter equal to δ needed to cover A. Define the *upper Box dimension* by

$$\overline{\mathrm{BD}}(A) = \limsup_{\delta\to 0} \frac{\log N_\delta}{-\log\delta}$$

and the lower Box dimension $\underline{\mathrm{BD}}(A)$ analogously. If the two quantities coincide then the common quantity is called the *Box dimension*, also known as the Minkowski dimension or capacity.

It follows from the definitions that $\mathrm{HD}(A) \leq \underline{\mathrm{BD}}(A) \leq \overline{\mathrm{BD}}(A)$. In general these quantities do not coincide, but they do for some classes of sets defined by dynamical systems. It is interesting to find examples where Hausdorff and Box dimensions do not coincide.

Pointwise dimension

The prior examples of dimension were purely geometrical in that they depended only on the location of points in space. Often invariant measures can be used to describe dynamics on a set and conformal measures can give a better picture of the geometry of a set at small scales.

Given a Borel probability measure μ supported on a subset A of $\mathbb{R}^N$, the *lower and upper pointwise dimensions* at x in A are given by

$$\underline{d}_\mu(x) = \liminf_{\epsilon \searrow 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon} \text{ and } \overline{d}_\mu(x) = \limsup_{\epsilon \searrow 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon}.$$

If the upper and lower pointwise dimensions coincide at x then the common value is called the *pointwise dimension* at x and denote it $d_\mu(x)$.

Dimension of a measure

Given a given a Borel probability measure μ supported on a subset A of $\mathbb{R}^N$, the *Hausdorff dimension of* μ is defined as

$$\mathrm{HD}(\mu) = \inf\{\mathrm{HD}(X) : \mu(A \setminus X) = 0\}.$$

The principle use of the notions of pointwise dimension is to estimate the dimension of measures:

Fact 4.1.1 *Let μ be a Borelian probability measure supported on a subset of $\mathbb{R}^N$. If almost everywhere with respect to μ one has $\underline{d}_\mu \geq d$ for some d then*

$$HD(\mu) \geq d.$$

If almost everywhere with respect to μ one has $\overline{d}_\mu \leq d$ for some d then

$$HD(\mu) \leq d.$$

Hence, if almost everywhere one has

$$\lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = d$$

(so the limit exists almost everywhere) then $HD(\mu) = d$.

Dynamical Volume Lemmas

In dimension one, real or complex, a very elegant relation between entropy, Lyapunov exponent and Hausdorff dimension of measures (with positive entropy h_μ) often exists and takes the form

$$\mathrm{HD}(\mu) = \frac{h_\mu}{\chi_\mu}.$$

For diffeomorphisms in higher dimensions the work of Ledrappier and Young [19], [20] was seminal. We show this relation for a class of cusp maps in Theorem 1.7.2.

Hyperbolic dimension and Dynamical dimension

Take a compact set J and a C^1 (usually at least $C^{1+\epsilon}$) map $f : J \to J$. The *Hyperbolic dimension* of J, denoted HypD(J), is defined as the supremum of the Hausdorff dimensions of all hyperbolic subsets of J. Recall a subset K is hyperbolic if it is forward invariant, compact and if there is an iterate f^n of f such that $|Df^n_{|K}| > 1$.

The *Dynamical dimension* of J, denoted DD(J), is the supremum of HD(μ) as μ runs over all ergodic invariant Borel probability measures supported on J with positive finite Lyapunov exponent.

We include the word finite in order to deal with cusp maps. For usually considered smooth interval maps and rational complex maps, the Lyapunov exponent is automatically finite. Strangely, the more expanding case of infinite Lyapunov exponent is harder to deal with than the finite case.

Absolutely continuous invariant measures are appreciated by physicists as they are considered to adequately describe the real-world behaviour of a system, or at least provide clues. If J is an interval and f has an absolutely continuous invariant measure with positive Lyapunov exponent, then this measure obviously has Hausdorff dimension 1, so in this case $\mathrm{DD}(J) = 1$. If f does not have an absolutely continuous invariant measure, the next best case is that $\mathrm{DD}(J) = 1 = \mathrm{HypD}(J)$ (often Dynamical and Hyperbolic dimensions coincide).

4.2 Hyperbolic dimension

Recall that the Hyperbolic dimension of a set is the supremum of Hausdorff dimensions of hyperbolic subsets, and so is bounded from above by the Hausdorff dimension of the set. In order to get a lower bound and (when possible) show coincidence of dimensions, which is the main objective here, one technique is to find a simpler expanding induced map and study it instead.

Given a C^2 interval map $f : I \to I$ and its completely invariant Julia set $\mathcal{J}$, if one can find a full expanding induced Markov map then it has eternally bounded distortion and one can apply the results of Subsubsection 4.3.3 to show that $\mathrm{HypD}(\mathcal{J}) = 1$. The idea is to construct a conformal measure for the induced map and use the nice scaling properties of conformal measures to get a lower bound for the Hausdorff dimension of the points which stay in the domain of the Markov map forever. By the construction of the conformal measure and a simple application of the Mañé Hyperbolicity Theorem, the lower bound will also be a lower bound for the Hyperbolic dimension. This technique also works for fractal sets.

We shall continue with the presentation of a preprint submitted for publication. The results in the preprint overcome the only difficulty in treating finitely renormalisable quadratic maps and lay the foundations for dealing with multimodal maps with complicated critical behaviour. The last section contains a discussion of the situation for maps from the quadratic family $f_a(x) = ax(1-x)$.

4.3 Hyperbolic Dimension for Interval Maps

What follows in this section is a preprint submitted for publication. Sections have been renamed subsections and the references are common with those of the thesis. The original can be found on the author's website.

4.3.1 Introduction

The coincidence of different notions of fractal dimensions has been a topic of interest for some time. The hyperbolic dimension is defined as the supremum of Hausdorff dimensions of hyperbolic subsets. In general the dynamics of hyperbolic sets is well understood. If Hausdorff and hyperbolic dimensions coincide then one may say that the hyperbolic subsets approximate, in some sense, the original set. We study this problem in the cadre of C^2 maps of an interval I.

Definition 4.3.1 *An interval map $f : I \to I$ is* of class D *(we also say f is a* D-map*) if*

- *f is of class C^2;*
- *the critical set $Crit = \{c \in I : Df(c) = 0\}$ is of finite cardinality;*
- *there are at most a finite number of parabolic periodic points of any one period;*
- *all critical points are C^2* non-flat*, i.e. for each $c \in Crit$, there is a C^2 diffeomorphism ψ and an $l > 1$ such that locally $f(x) = \pm|\psi(x-c)|^l + f(c)$;*
- *$f : \partial I \to \partial I$.*

One can define the Fatou set of an interval map as the set of points for which there is a neighbourhood on which iterates of f form an equicontinuous family. For C^2 maps this coincides with the interior of the basins of attraction of periodic attractors. The *Julia set* $\mathcal{J}$ is the complement of the Fatou set. See [6] for details.

Given a continuous interval map $f : I \to I$ and subset $K \subset I$, let $\mathcal{W}$ be the collection of connected components of $I \setminus K$. Following [7] we define the collection of sets $\tau(K)$ as follows:

$$\tau(K) = \{W \in \mathcal{W} : f(W) \cap K \neq \emptyset\}.$$

Definition 4.3.2 *Given a C^1 interval map $f : I \to I$ we say K is a τ-set for f if*

- *K is a forward-invariant (i.e. $f(K) \subset K$) compact subset of the Julia set of f;*
- *the cardinality of $\tau(K)$ is finite;*
- *for all $W \in \tau(K)$, ∂W is pre-periodic.*

The *omega-limit set* $\omega(x)$ of a point x is defined by

$$\omega(x) = \bigcap_{N>0} \overline{\{f^n(x) : n \geq N\}}$$

and that of a set S by

$$\omega(S) = \bigcup_{x \in S} \omega(x).$$

A point x for D-map f is *non-recurrent* if $x \notin \omega(x)$.

Theorem 4.3.3 *Let K be a τ-set for a D-map f such that each critical point of f contained in K is non-recurrent. Then the Hausdorff and hyperbolic dimensions of K coincide.*

Corollary 4.3.4 *Suppose $f : I \to I$ is a D-map such that each critical point is non-recurrent. Then the Hausdorff and hyperbolic dimensions of the Julia set $\mathcal{J}$ of f coincide, i.e.*

$$HD(\mathcal{J}) = HypD(\mathcal{J}).$$

Proof: The Julia set is completely invariant so $\tau(\mathcal{J}) = \emptyset$ and $\mathcal{J}$ is a τ-set. Apply the theorem to conclude. □

Definition 4.3.5 *A parabolic periodic point p of period q for interval map f shall be called* flat *if one can write*

$$f^{2q}(x) = x + \psi(x),$$

where $\lim_{x \to p} \psi(x)(x-p)^{-n} = 0$ *for all* $n \geq 0$.

For C^∞ maps this condition on ψ is equivalent to $D^n\psi(p) = 0$ for all $n \geq 0$.

Theorem 4.3.6 *Suppose for a D-map f there exists a closed interval U with the following properties:*

- *there exists a flat parabolic point of period q such that $p \in \partial U$;*
- *U contains no critical points of f^{2q};*
- $\frac{f^{2q}(x)-p}{x-p} > 1$ *for all $x \in U$;*
- *there exists $V = [y, f^{2q}(y)]$ contained in U and an integer n such that $f^n(V) \supset U$.*

Then the hyperbolic dimension of the Julia set of f is equal to 1.

Corollary 4.3.7 *There exists D-maps such that $HypD(\mathcal{J}) = 1$ but $\lambda(\mathcal{J}) = 0$, where λ denotes Lebesgue measure.*

Of course, the dynamics away from a forward-invariant compact set does not affect the dynamics on the set itself. This motivates the following definition.

Definition 4.3.8 *Let K be a compact set of an interval I. Two maps f and g are K-*equivalent *if there is a neighbourhood V of K such that $f_{|V} = g_{|V}$.*

Lemma 4.3.9 *Let f and g be K-equivalent maps of class C^1. Then the hyperbolic dimension of K with respect to f and the hyperbolic dimension of K with respect to g coincide.*

Proof: A set $K' \subset K$ is hyperbolic for f if and only if K' is also hyperbolic for g. □

One can thus look for equivalent maps which are the best-behaved away from the set of interest K.

Definition 4.3.10 *Let K be a compact subset contained in the interior of an interval I. For integers $n \geq 0$ we say f is* of class $M_n(K)$ *if*

- *f is a D-map of I;*
- *K is a τ-set for f;*
- *there are exactly n critical points in K;*
- *each critical point in K is non-recurrent;*
- *all critical $c \in (I \setminus K)$ satisfy $f(c) \in \partial I$;*
- *both $x \in \partial I$ are hyperbolic, attracting, fixed points.*

The following result is slightly more general than Theorem 4.3.3 which one recovers by applying Lemma 4.3.12.

Theorem 4.3.11 *Given a compact subset K of an interval, let f be K-equivalent to a map of class $M_n(K)$ for some $n \geq 0$. Then the Hausdorff and hyperbolic dimensions of K coincide.*

We denote by $\mathrm{HypD}(K)$ the hyperbolic dimension of K. The Hausdorff dimension [30] of a set K, denoted here as $\mathrm{HD}(K)$, is defined as follows:

$$\mathrm{HD}(K) = \inf\{t \geq 0 : \lim_{\varepsilon \searrow 0} (\inf_{\mathcal{U}} \sum_{i=1}^{\infty} (\mathrm{diam} U_i)^t) < \infty\},$$

where $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ is a countable cover of K by sets U_i of diameter less than ε.

We say a set K is *hyperbolic* if it is forward-invariant and compact and there exists an iterate of f such that $|Df^n_{|K}| > 1$. A point p of period q is *parabolic* if and only if $|Df^q(p)| = 1$.

In [24] the *Mañé Hyperbolicity Theorem* is proven: for D-maps, if all periodic points of a forward-invariant compact K are hyperbolic repelling, with K not containing critical points, then K is hyperbolic.

In the real setting, Urbańksi [41] and Hofbauer [12] have studied weakly expanding systems (systems f, K such that $|Df|_{|K} \geq 1$ with equality only at a finite number of fixed points) in some detail. We allow the presence of critical points. Even without critical points in K, Theorem 4.3.3 treats a wider class of dynamical systems: for example, if set K contains a parabolic fixed point p for f and some point $x \in K$ satisfies $f(x) = p$ and $|Df(x)| < 1$ then no iterate of $f_{|K}$ is weakly expanding since $|Df^n(x)| = |Df(x)| < 1$. Both [41] and [12] also have extra conditions on behaviour near parabolic points.

For a survey of results about dimensions of Julia sets of rational maps of the Riemann sphere see [42]. Theorem 4.5 of [42] states that for rational maps with parabolic points and without recurrent critical points the Hausdorff and hyperbolic dimensions of the Julia set coincide. The bulk of the work was carried out in [40]. For the same class of rational maps, it is proven in [33] that either the Julia set is the whole Riemann sphere or its upper Box dimension is < 2. We show that the analagous result is not true for real maps: there exist Julia sets of Lebesgue measure zero but of hyperbolic dimension equal to 1, see Corollary 4.3.7.

If f is a D-map with two fixed points p and q without a critical point inbetween, then all points in $[p, q] \cap \mathcal{J}$ are fixed and repelling on at least one side. By definition of D-maps, the number of parabolic fixed points is at most countable. The number of repelling fixed points is at most countable since f is C^2. Thus $\mathrm{HD}([p, q] \cap \mathcal{J}) = 0$.

On the other hand, allowing an uncountable number of parabolic points can lead to strange Julia sets. Given any Cantor set in an interval, one can define a monotone increasing C^2 function g such that $\frac{1}{2} < Dg < 2$, g leaves the Cantor set fixed and each point of the Cantor set is parabolic with respect to g and attracting on one side and repelling on the other. If there are no other fixed points in the interior of the interval then the Cantor set coincides with the Julia set of g, yet there are no hyperbolic subsets.

The condition $f : \partial I \to \partial I$ is standard and ensures that branches of our induced maps are full. Merely taking ∂I pre-periodic is sufficient: one can extend f to a larger interval and add an element to $\tau(K)$.

The Julia set $\mathcal{J}$ of a function is completely invariant and so $\tau(\mathcal{J})$ is empty. If the Julia set contains recurrent critical points, upon removing all inverse images of suitable small neighbourhoods of the recurrent critical points, a τ-set disjoint from the recurrent critical points remains. Similarly, if one subsequently removes all inverse images of an appropriate neighbourhood of parabolic points and any remaining critical points, then a hyperbolic τ-set remains. Any hyperbolic subset of $\mathcal{J}$ is contained in such a set.

The next subsection contains a miscellany of results and definitions used in the rest of the paper. Section 4.3.3 uses conformal measures to give a dimension estimate for expanding induced maps. Then come subsections giving dimension estimates for parabolic and critical dynamics. Section 4.3.6 ties these results together to give a short proof of Theorem 4.3.11 which combined with Lemma 4.3.12 proves Theorem 4.3.3. Finally in Section 4.3.7 we prove Theorem 4.3.6 and show how to construct a map which satisfies the claim of Corollary 4.3.7.

4.3.2 Preliminaries

Lemma 4.3.12 *Suppose K is a τ-set for D-map f, and that all critical points in K are non-recurrent. Let $n \geq 0$ be the number of critical points in K. Then there exists a map g of class $M_n(K)$ such that g is K-equivalent to f.*

Proof: Let V be finite open cover of K such that $\overline{V} \cap (\mathrm{Crit}(f) \setminus K) = \emptyset$. Working on a larger interval if necessary, by modifying f outside of V one can obtain a D-map g which is K-equivalent to f and which sends all critical points not in K to the hyperbolically attracting boundary.

It remains to show that K is a τ-set for g, which reduces to showing that for all $W \in \tau_g(K)$, ∂W is preperiodic (here $\tau_g(K)$ is $\tau(K)$ for the map g). Suppose there exists a positive n such that $f^n(W) \cap K \neq \emptyset$. Then $f^{n-1}(W) \in \tau_f(K)$. Also $f^{n-1} : \partial W \to \partial f^{n-1} W$ by forward invariance of K, so ∂W is preperiodic. Otherwise, for all n, $f^n(W) \cap K = \emptyset$. The absence of wandering intervals and forward invariance of K then imply that the size of iterates $|f^n(W)|$ is bounded away from 0 so ∂W must be preperiodic. □

The following is proved in [9]:

Fact 4.3.13 *Köbe principle: Let I be a compact interval and $f : I \to I$ be a C^2 map with all critical points C^2 non-flat. Then there exists a continuous increasing function σ, $\sigma(0) = 0$, with the following property. If $J \subset T$ are open intervals and $n \in N$ is such that f^n is a diffeomorphism on T then, for every $x, y \in J$, we have*

$$\frac{Df^n(x)}{Df^n(y)} \geq \frac{e^{-\sigma(\max_{i=0}^{n-1} |f^i(T)|) \cdot \sum_{i=0}^{n-1} |f^i(J)|}}{(1 + \nu(f^n(J), f^n(T)))^2}, \tag{4.1}$$

where for open intervals $\overline{A} \subset B$, $\nu(A, B) = \frac{|A|}{dist(A, \partial B)}$.

The bounding quantity (or its inverse) in inequality (4.1) is called a distortion bound. Note that if the $f^i(J)$ are disjoint then the sum of the lengths of these intervals is bounded by $|I|$. They will be disjoint for first entry and first return maps:

Definition 4.3.14 *An interval U is* regularly returning *if and only if $f^n(\partial U) \cap U = \emptyset$ for every $n > 0$.*

Its nice property is if A, B are intervals, $f^n(A) = f^m(B) = U, n < m$ and $A \cap B \neq \emptyset$, then $B \subset A$. The *first entry time* to U, $e(x)$, for a point x is defined as

$$e(x) = \inf\{k \geq 0 : f^k(x) \in U\}.$$

The *first return time* to U, $r(x)$, for a point $x \in U$ is defined as

$$r(x) = \inf\{k \geq 1 : f^k(x) \in U\}.$$

The first entry map ψ_U for U is defined by $\psi_U(x) = f^{e(x)}(x)$ where $e(x)$ is defined. Similarly the first return map ϕ_U for U is defined as follows: $\phi_U(x) = f^{r(x)}(x)$. This is only defined where $r(x)$ is defined.

If $J \ni x$ is a connected component of the domain of ϕ_U for regularly returning interval U, then the intervals $J, f(J), \dots, f^{r(x)-1}(J)$ are pairwise disjoint, and $r(x)$ is constant on J. This disjointedness property makes first return (and similarly first entry) maps easy to work with.

When applying the Köbe principle, to bound $\max |f^i(T)|$ one makes use of the Contraction Principle [43].

Fact 4.3.15 *Let f be a C^2 map with a finite number of C^2 non-flat, non-periodic critical points. Then there exists a continuous increasing function γ, $\gamma(t) \to 0$ as $t \to 0$, such that if f^n maps open interval A diffeomorphically onto B, B disjoint from immediate basins of periodic attractors, then $|A| < \gamma(|B|)$.*

The remaining $\nu(f^n(J), f^n(T))$ tends to 0 as $|f^n(J)|$ does if $f^n(J)$ stays away from $\partial f^n(T)$. In this paper $f^n(J)$ shall be a regularly returning interval compactly contained in some interval disjoint from the forward orbit of the critical set.

A differentiable map ϕ is *expanding* if $|D\phi| \geq \lambda > 1$, for some constant λ, everywhere on the domain of definition of ϕ.

Lemma 4.3.16 *Let I be a compact interval and $f : I \to I$ be a C^2 map with all critical points C^2 non-flat. Suppose $U \subset I$ is a regularly returning interval such that the first return map ϕ to U is expanding, and that there exists an open interval $V \supset \overline{U}$ such that every branch of ϕ extends (as an iterate of f) to a diffeomorphism onto V.*

Then there exists a constant $C > 1$ such that

$$\frac{D\phi^n(x)}{D\phi^n(y)} \leq C$$

for all x, y in the same connected component of the domain of ϕ^n.

Proof: Let J be a connected component of the domain of ϕ and n_J the number satisfying $f^{n_J}_{|J} = \phi_{|J}$. Since U is regularly returning, $\{f^i(J)\}_{i=0}^{n_J-1}$ is a pairwise disjoint collection of intervals. Hence

$$\sum_{i=k}^{n_J-1} |f^i(J)| \leq |I|$$

for all $k = 0, 1, \dots, n_J - 1$. Applying the Köbe principle, there exists a constant $C_0 > 1$ independent of the connected component J such that

$$\frac{Df^k(x)}{Df^k(y)} \leq C_0$$

for all x, y in $f^{n_J-k}(J)$, for all $k = 1, 2, \dots, n_J$, for all connected components J of the domain of ϕ.

Let $g : W \to g(W)$ be a diffeomorphism of an interval W with distortion bounded by C_0, i.e. such that $\frac{Dg(x)}{Dg(y)} \leq C_0$ for all x, y in W. Let L be a subinterval of W. Writing $M = \sup_W |Dg(x)|$, it follows that $|g(L)| \geq \frac{M}{C_0}|L|$ and $|g(W)| \leq M|W|$, so

$$\frac{|g(L)|}{|g(W)|} \geq \frac{|L|}{C_0|W|}.$$

Hence, if $C_0|g(L)| \leq \alpha|g(W)|$ then $|L| \leq \alpha|W|$, for any $\alpha > 0$. Thus, if J is a connected component of the domain of ϕ and L is a subinterval of J such that $C_0|\phi(L)| \leq \alpha|U|$ then

$$|f^i(L)| \leq \alpha|f^i(J)|$$

for all $i = 0, 1, \ldots, n_J$, and so

$$\sum_{i=0}^{n_J-1} |f^i(L)| \leq \alpha \sum_{i=0}^{n_J-1} |f^i(J)|. \tag{4.2}$$

Let $\lambda > 1$ satisfy $|D\phi| \geq \lambda$ everwhere on its domain. There is an $N > 0$ such that $\lambda^N > C_0$. Let A be a connected component of the domain of ϕ^{N+n} for some $n > 0$. Since $|D\phi| \geq \lambda$, we have

$$|\phi^k(A)| \leq \lambda^{-N}\lambda^{k-n}|U|$$

for all $k \leq N + n$. Let $J(k)$ denote the connected component of the domain of ϕ containing $\phi^{k-1}(A)$. One has $C_0|\phi(\phi^{k-1}(A))| \leq C_0\lambda^{-N}\lambda^{k-n}|U| \leq \lambda^{k-n}|U|$, so by (4.2), we deduce, for $k = 1, \ldots, N + n$, that

$$\sum_{i=0}^{n_{J(k)}-1} |f^i(\phi^{k-1}(A))| \leq \lambda^{k-n} \sum_{i=0}^{n_{J(k)}-1} |f^i(J(k))| \leq \lambda^{k-n}|I|.$$

Let m be such that $f^m_{|A} = \phi^{N+n}_{|A}$. It follows from the previous line that

$$\begin{aligned} \sum_{i=0}^{m-1} |f^i(A)| &\leq \sum_{k=1}^{N+n} \lambda^{k-n}|I| \\ &= \lambda^{-n}\frac{\lambda^{N+n+1} - \lambda}{\lambda - 1}|I| \\ &\leq \frac{\lambda^{N+1}}{\lambda - 1}|I| =: C_1 \end{aligned}$$

say. Apply the Köbe principle one last time to get

$$\frac{Df^m(x)}{Df^m(y)} \geq \frac{e^{\sigma(|U|)C_1}}{(1 + \nu(U, V))^2} =: C^{-1}$$

say, for all x, y in A. The constant C not depending on A or on n, we conclude that for all $n > 0$,

$$\frac{D\phi^n(x)}{D\phi^n(y)} \leq C$$

for all x, y in the same connected component of the domain of ϕ^n. $\square$

Lemma 4.3.17 *Let f be a C^1 map of the interval I with a finite number of critical points. For any set $J \subset I$ one has $HD(f(J)) = HD(J)$.*

Proof: Let B_δ be the set of all points of I less than some $\delta > 0$ away from Crit. On B_δ the derivative of f is bounded away from 0. Any cover of $J \setminus B_\delta$ disjoint from B_δ is mapped by f to a cover of $f(J \setminus B_\delta)$ with bounded distortion. It follows that $\mathrm{HD}(J \setminus B_\delta) = \mathrm{HD}(f(J \setminus B_\delta))$ for all $\delta > 0$. Write $J = \bigcup_{\delta>0}(J \cap B_\delta) \cup \mathrm{Crit}$ and the result follows. $\square$

Lemma 4.3.18 *Let f be a C^1 map of the interval I with a finite number of critical points. For any subset $J \subset I$ one has $HD(\bigcup_{n\geq 0} f^{-n}(J)) = HD(J)$.*

Proof: It suffices to show that $\mathrm{HD}(f^{-n}(J)) \leq \mathrm{HD}(J)$ for each n. But f^n is C^1 and has a finite number of critical points and sends $f^{-n}(J)$ into J. Apply Lemma 4.3.17 to conclude. $\square$

Definition 4.3.19 *The* forward orbit *$O^+(x)$ of a point x is defined by*

$$O^+(x) = \{f^n(x) : n \geq 0\}$$

and that of a set S by

$$O^+(S) = \bigcup_{x\in S} O^+(x).$$

Definition 4.3.20 *Call a critical point c* primary *if $c \notin \overline{O^+(f(Crit))}$, and* primary with respect to a set *K if $c \notin \overline{O^+(f(Crit \cap K))}$.*

Lemma 4.3.21 *Given a continuous map f, if $y \in \omega(x)$ then $\omega(y) \subset \omega(x)$.*

Proof: Follows from the definition and continuity. $\square$

Corollary 4.3.22 *For a continuous map f, if $y \in \omega(x)$ and $z \in \omega(y)$ then $z \in \omega(x)$.*

Corollary 4.3.23 *For a continuous map f, if $y \in \overline{O^+(x)}$ and $z \in \overline{O^+(y)}$ then $z \in \overline{O^+(x)}$.*

Proof: Note that $\omega(x) = \bigcap_{n>0} \overline{O^+(f^n(x))}$. $\square$

Lemma 4.3.24 *Suppose there is a finite, strictly positive number of critical points in a set K, and that all these critical points are non-recurrent. Then there exists a critical point which is primary with respect to K.*

Proof: Consider all sequences $(c_1c_2\ldots)$ such that $c_i \in \mathrm{Crit}\cap K$ and $c_{i+1} \in \overline{O^+(f(c_i))}$. We claim that there is a sequence of maximal length (possibly equal to 1). For otherwise there would be a sequence of length greater than #Crit, and two occurrences of the same critical c in the same sequence, together with Corollary 4.3.23, imply c is not non-recurrent, contradiction.

Now consider a maximal such sequence $(c_1 \ldots c_r)$. We claim c_1 is primary with respect to K. For if $c_1 \in \overline{O^+(f(c_0))}$ say for some $c_0 \in K\cap$ Crit, then $(c_0c_1 \ldots c_r)$ is a longer such sequence, contradicting maximality. $\square$

4.3.3 Dimension Estimate for Induced Maps

Let $A \subset B$ be measurable subsets of an interval endowed with the usual subspace topology. Let t be a real number and $g : A \to B$ a C^1 function.

Definition 4.3.25 *A measure m on B is called* t-conformal *if*

$$m(g(C)) = \int_C |Dg(x)|^t dm, \tag{4.3}$$

whenever C is measurable, $g(C)$ is measurable and $g_{|C}$ is injective. It is also a probability measure *if $m(B) = 1$.*

Let $\{J_i\}_{i=1}^{\infty}$ be a countable collection of pairwise-disjoint, open subintervals of an open interval U satisfying $\overline{J_i} \cap \partial U = \emptyset$ for all i.

Let $\phi : \bigcup_{i \geq 1} J_i \to U$ be a C^1 (or C^2) function such that for each i, the restriction $\phi_{|J_i}$ is a diffeomorphism between J_i and U. Suppose also that ϕ is expanding, i.e. there exists λ such that $|D\phi| \geq \lambda > 1$, and that on each branch $\sup_{x \in J_i} |D\phi| < M_i$ for some constant $M_i < \infty$.

Let $Z_n = \bigcup_{i=1}^{n} J_i$, $Z = \bigcup_{n=1}^{\infty} Z_n$, and set

$$\begin{aligned} L_n &= \{x \in Z_n : \phi^j(x) \in Z_n \ \forall j \geq 0\} \\ &= \bigcap_{j=0}^{\infty} \phi^{-j}(Z_n) \\ &= \bigcap_{j=0}^{\infty} \phi^{-j}(Z_n \cap \phi^{-1}(\overline{Z_n})) \\ \text{and } L &= \{x \in Z : \phi^j(x) \in Z \ \forall j \geq 0\}. \end{aligned}$$

Let ϕ_n be the restriction of ϕ to Z_n. Then L_n is completely invariant with respect to ϕ_n and is compact, as a countable intersection of closed sets, thus it is hyperbolic. It is long-known [15] that there exists an atom-free t_n-conformal probability measure m_n for $\phi_n : Z_n \to U$ with support on L_n, where t_n is the Hausdorff dimension of L_n.

Lemma 4.3.26 *There exists a t-conformal probability measure m for $\phi : Z \to U$ where*

$$t = \lim_{n \to \infty} HD(L_n).$$

Proof: Let m be a weak limit point of the t_n-conformal probability measures m_n for $\phi_n : Z_n \to U$. We first show t-conformality of m for $\phi : J_i \to I$ for each i. This is a standard convergence argument.

Suppose $A \subset J_i$ satisfies $m(\partial A) = m(\partial \phi(A)) = 0$. Then, using conformality of m_n for $\phi : J_i \to U$ when $n > i$,

$$m(\phi(A)) = \lim_{n \to \infty} m_n(\phi(A)) = \lim_{n \to \infty} \int_A |D\phi|^{t_n} dm_n = \int_A |D\phi|^t dm.$$

Now we deal with subsets A compactly contained in J_i. For $\varepsilon > 0$ consider an open cover of A by a countable family of sets $\{B_j\}$ satisfying $m(\partial B_j) = m(\partial \phi(B_j)) = 0$ and $B_j \subset J_i$ and

$$m(\bigcup_j B_j \setminus A) < \frac{\varepsilon}{M_i}.$$

We can now define inductively a partition $\{C_j\}$ of $\bigcup_j B_j$ as follows:

$$C_1 = B_1, C_{j+1} = B_{j+1} \setminus \bigcup_{k=1}^{j} B_j.$$

We have shown m is t-conformal for each C_j. Thus

$$\begin{aligned} m(\phi(A)) &\leq m(\bigcup_j \phi(C_j)) = \sum_j m(\phi(C_j)) \\ &= \sum_j \int_{C_j} |D\phi|^t dm = \int_{\bigcup_j C_j} |D\phi|^t dm \\ &= \int_A |D\phi|^t dm + \int_{\bigcup_j C_j} |D\phi|^t dm \leq \int_A |D\phi|^t dm + \varepsilon. \end{aligned}$$

This is true for all $\varepsilon > 0$ so $m(\phi(A)) \leq \int_A |D\phi|^t dm$. In the other direction, using the same partition,

$$\begin{aligned} m(\phi(A)) &= \sum_j (m(\phi(C_j)) - m(\phi(C_j \setminus A))) \\ &\geq \sum_j \int_{C_j} |D\phi|^t dm - \sum_j \int_{C_j \setminus A} |D\phi|^t dm \\ &\geq \int_A |D\phi|^t dm - \varepsilon. \end{aligned}$$

If A is open and $\partial A \cap \partial J_i \neq \emptyset$, let $\overline{C_j} \subset A$ be increasing subsets of A such that $\bigcup_{j\geq 1} C_j = A$.

$$\lim_{j\to\infty} m(\phi(A \setminus C_j)) = \lim_{j\to\infty} m(A \setminus C_j) = 0$$

implies m is t-conformal for A. We conclude m is t-conformal for $\phi : J_i \to U$ for all i.

Each open set containing points of L contains an interval which is sent by some iterate of ϕ onto U, and thus is of positive measure, if U is. We show that for any J_i compactly contained in U, $m(\overline{J_i}) \geq \frac{1}{M_i}$:

$$\begin{aligned} m_n(\phi(J_i)) &= m_n(U) = 1 \\ &= \int_{J_i} |D\phi|^{t_n} dm_n \\ &\leq M_i^{t_n} m_n(J_i) \\ &\leq M_i m_n(J_i) \\ m_n(J_i) &\geq \frac{1}{M_i}. \end{aligned}$$

Letting $n \to \infty$, $m(\overline{J_i}) \geq \frac{1}{M_i}$, as required, so m is not trivial (zero) on U.

Defining $m' = \frac{1}{m(U)} m$ we obtain a t-conformal probability measure as required. □

Proposition 4.3.27 *Suppose that there is a constant C such that for all n and all x, y belonging to a connected component of the domain of ϕ^n,*

$$\frac{D\phi^n(x)}{D\phi^n(y)} \leq C.$$

Then the Hausdorff dimension of L is equal to $t = \lim_{n\to\infty} HD(L_n)$.

Proof: Of course $L \supset L_n$ so $\mathrm{HD}(L) \geq t$. Let m be the measure from the preceding lemma. Consider a pairwise-disjoint, countable cover of L by $\{C_j\}_{j\geq 1}$ where for each j there exists $n_j > 0$ such that $\phi^{n_j} : C_j \to U$ is a diffeomorphism. For every $\varepsilon > 0$ there exists such a cover with $\mathrm{diam}(C_j) < \varepsilon$ for all j since $|D\phi^{-1}| \leq \lambda^{-1} < 1$. From above, the distortion of ϕ^{n_j} on each C_j is bounded by some constant C independent of j and ε.

We find a finite bound for $\sum_{j\geq 1} \mathrm{diam}(C_j)^t$ independently of ε which shows $\mathrm{HD}(L) \leq t$:

$$\begin{aligned} m(U) &= \int_{C_j} |D\phi^{n_j}(x)|^t dm \\ &\leq C^t |D\phi^{n_j}(x_0)|^t m(C_j). \\ \mathrm{diam}(U)^t &= \left(\int_{C_j} |D\phi^{n_j}(x)| d\lambda \right)^t \\ &\geq C^{-t} |D\phi^{n_j}(x_0)|^t \mathrm{diam}(C_j)^t \end{aligned}$$

for some fixed $x_0 \in C_j$. Dividing, we find

$$\frac{\mathrm{diam}(C_j)^t}{m(C_j)} \leq \frac{C^{2t}\mathrm{diam}(U)^t}{m(U)}$$

which gives

$$\mathrm{diam}(C_j)^t \leq \frac{C^{2t}\mathrm{diam}(U)^t}{m(U)} \cdot m(C_j).$$

Since m is a probability measure and the C_j are pairwise disjoint the sum of $m(C_j) \leq 1$ and

$$\sum_{j\geq 1} \mathrm{diam}(C_j)^t \leq \frac{C^{2t}\mathrm{diam}(U)^t}{m(U)} < \infty,$$

as required. □

We shall now apply these results to induced maps. Let K be a forward invariant compact set for a D-map f.

Definition 4.3.28 *We say (ϕ, Z, K) is an* induced Markov system *if there is an open interval U such that for all $W \in \tau(K)$, one has $U \cap O^+(\partial W) = \emptyset$ and there is a collection of intervals $\{J_i\}_{i=1}^{\infty}$ with $Z = \bigcup_i J_i$ and a map $\phi : Z \to U$ such that the following hold:*

- *$K \cap J_i \neq \emptyset$ for all i;*
- *on each J_i there is an $n_i > 0$ such that $\phi_{|J_i} = f^{n_i}_{|J_i}$;*
- *$\overline{J_i} \cap \partial U = \emptyset$ for all i;*

- *$\phi : J_i \to U$ is a diffeomorphism onto U;*
- *there is a constant C such that for all $n > 0$ and all x, y belonging to a same connected component of the domain of ϕ^n,*
$$\frac{D\phi^n(x)}{D\phi^n(y)} \leq C.$$

As before, let $Z_n = \bigcup_{i=1}^{n} J_i$, $Z = \bigcup_{n=1}^{\infty} Z_n$, and set

$$\begin{aligned} L_n &= \{x \in Z_n : \phi^j(x) \in Z_n \ \forall j \geq 0\} \\ \text{and } L &= \{x \in Z : \phi^j(x) \in Z \ \forall j \geq 0\}. \end{aligned}$$

We can then apply Proposition 4.3.27 to get

$$\mathrm{HD}(L) = \lim_{n\to\infty} \mathrm{HD}(L_n). \tag{4.4}$$

This will be interesting if we show that $L \subset K$ and that L_n is contained in a hyperbolic set.

Lemma 4.3.29 *Let (ϕ, Z, K) be an induced Markov system for a D-map f and let $L = \{x \in Z : \phi^j(x) \in Z \ \forall j \geq 0\}$. Then $L \subset K$.*

Proof: By definition, $K \cap Z$ is non-empty. Suppose $x \in Z$ satisfies $\phi(x) \in K$. Let i be such that $x \in J_i$. Let n_i be such that $\phi_{|J_i} = f^{n_i}_{|J_i}$. By definition, for all $W \in \tau(K)$ the forward orbit of ∂W under f is disjoint from U. Thus for all $0 \leq n < n_i$, we have $f^n(J_i) \cap \partial W = \emptyset$. Again by definition, $J_i \cap K \neq \emptyset$, so for all $0 \leq n < n_i$, we have $f^n(J_i) \cap W = \emptyset$ and thus $x \in K$. Therefore $K \cap Z$ is backward-invariant with respect to ϕ. The set of all inverse images of x by ϕ is contained in K and is dense in L. By compactness of K, $L \subset K$. □

Lemma 4.3.30 *For all $n > 0$, the set L_n considered above is contained in a hyperbolic set for f.*

Proof: The set
$$\bigcup_{i=1}^{n} \bigcup_{j=0}^{n_i} f^j(J_i \cap L_n)$$
is a forward invariant compact and is at a positive distance from all critical and parabolic points. By the Mañé Hyperbolicty Theorem [24] it is a hyperbolic subset of K containing L_n. □

Proposition 4.3.31 *Let (ϕ, Z, K) be an induced Markov system for a D-map f and let $L = \{x \in Z : \phi^j(x) \in Z \ \forall j \geq 0\}$. Then HD($L$) $\leq$ HypD(K).*

Proof: Each L_n is contained in a hyperbolic set so $\mathrm{HD}(L_n) \leq \mathrm{HypD}(K)$. Apply equation (4.4) to conclude. □

4.3.4 Parabolic Estimates

Let K be a compact subset of the interval and let f be K-equivalent to a function of class $M_0(K)$. Our goal in this subsection is to show that the set of points in K whose forward orbits accumulate on the parabolic periodic points has Hausdorff dimension less than the hyperbolic dimension of K.

Definition 4.3.32 *For a D-map f and forward-invariant set K, denote by $Rec(x,K)$ the set of points $y \in K$ such that $x \in \omega(y)$. If S is a set then*

$$Rec(S,K) = \bigcup_{x \in S} Rec(x,K).$$

Proposition 4.3.33 *Let K be a compact set and f a map of class $M_0(K)$ with a parabolic periodic point p such that $f(p) = p$ and $Df(p) = 1$. Then*

$$HD(Rec(p,K)) \leq HypD(K).$$

Proof: We shall construct an expanding induced map and apply the estimates of the Section 4.3.3.

Recall that $\tau(K)$ is finite and for all $W \in \tau(K), \partial W$ is preperiodic. Thus the set of points

$$S = \bigcup_{n \geq 0} \bigcup_{W \in \tau(K)} f^n(\partial W)$$

is finite. Since f is of class $M_0(K)$, all critical points are mapped directly to the attracting boundary ∂I.

Let U be a small, regularly returning, open interval whose closure is disjoint from $S \cup \partial I \cup \mathrm{Crit}$ and such that $p \in \partial U$. If there is a fixed point q in U then $\mathrm{HD}((p,q)) = 0$. Henceforth, suppose there is no fixed point in U and that $f(U) \setminus U \neq \emptyset$, so p is topologically repelling in U.

Now U is compactly contained I and all branches of the first return map to U, or indeed any subinterval of I, extend to diffeomorphisms onto the interior of I. By Fact 4.3.13, there exists a constant $C > 1$ such that the first return map to any regularly returning interval contained in U has distortion bounded by C. Indeed, one can take

$$C^{-1} = \frac{e^{-\sigma(|I|)|I|}}{(1 + \nu(U,I))^2}.$$

Denote by g the branch of f^{-1} which fixes p. Since p is a parabolic point,

$$\lim_{n \to \infty} \frac{|g^n(U) \setminus g^{n+1}(U)|}{|g^{n+1}(U)|} = 0.$$

Thus there exists $N > 0$ such that if $n \geq N$ then

$$\frac{|g^n(U) \setminus g^{n+1}(U)|}{|g^{n+1}(U)|} \leq \frac{1}{2C^2}.$$

Let V be the largest of the intervals $g^n(U) \setminus g^{n+1}(U)$ where $n \geq N$, realised for $n = n_0$, and let T be the interval $g^{n_0+1}(U)$. Then $\frac{|V|}{|T|} < \frac{1}{2C^2}$. Denote by ψ_V the first entry map to V and by ψ_T the first entry map to T. Both of these maps have distortion bounded by C.

Consider the map $\phi = \psi_T \circ \psi_{V|T}$, where $\psi_{V|T}$ is the restriction of ψ_V to T. The domain of ϕ is contained in T, and ϕ has distortion bounded by C^2. Let J be a connected component of the domain of ϕ. Then $J \subset g^n(U) \setminus g^{n+1}(U)$ for some $n > n_0$, so

$$|J| \leq |V| \leq \frac{1}{2C^2}|T|.$$

J is mapped by ϕ onto T, so there exists $x \in J$ such that $|D\phi(x)| \geq 2C^2$. For all $y \in J$ one deduces

$$|D\phi(y)| \geq \frac{1}{C^2} 2C^2 = 2,$$

so $|D\phi| \geq 2$ everywhere on its domain of definition. By Lemma 4.3.16, there is a bound C' on the distortion of branches of ϕ^n independent of the $n \geq 1$.

Denote by Z the union of the connected components of the domain of ϕ which have non-empty intersection with K. Then (ϕ, Z, K) is an induced Markov system. Let

$$L = \{x \in K \cap T : \phi^n(x) \in Z \text{ for all } n \geq 0\}.$$

By Proposition 4.3.31, $\mathrm{HD}(L) \leq \mathrm{HypD}(K)$. Note that L contains all points in $\mathrm{Rec}(p, K) \cap T$ whose forward orbits accumulate on p in T.

The same argument for the other side of the parabolic point p combined with Lemma 4.3.18 allows us to conclude that $\mathrm{HD}(\mathrm{Rec}(p, K)) < \mathrm{HypD}(K)$. □

Theorem 4.3.34 *Let K be a compact subset of the interval and let f be K-equivalent to a function of class $M_0(K)$. Then*

$$HD(K) = HypD(K).$$

Proof: By Lemma 4.3.9 we can assume f is of class $M_0(K)$. Iterates f^n of f are then of class $M_0(K)$. Hyperbolic dimensions with respect to f and with respect to iterates f^n coincide. Let p be a parabolic point of period n say. Then $p, f(p), \ldots, f^{n-1}$ are all fixed points of the map f^{2n} with $Df^{2n}(f^i(p)) = 1$. Denote by $\pi(K)$ the parabolic points contained in a set K. We apply Proposition 4.3.33 for the map f^{2n} for each $n > 0$ to get that

$$\mathrm{HD}\left(\bigcup_{p \in \pi(K)} \mathrm{Rec}(p, K)\right) \leq \mathrm{HypD}(K).$$

Consider $S = K \setminus (\bigcup_{p \in \pi(K)} \mathrm{Rec}(p, K))$. It remains to show that $\mathrm{HD}(S) \leq \mathrm{HypD}(K)$. For $\delta > 0$ let B_δ be the δ-neighbourhood of the parabolic set $\pi(K)$, i.e. the set of points at a distance $< \delta$ from $\pi(K)$. Put

$$K_\delta = K \setminus \bigcup_{i \geq 0} f^{-i}(B_\delta).$$

One can then write

$$S = \bigcup_{\delta > 0} K_\delta,$$

so

$$\mathrm{HD}(S) = \sup_{\delta > 0} \mathrm{HD}(K_\delta).$$

Each K_δ is hyperbolic by the Mañé Hyperbolicity Theorem and every hyperbolic set is contained in a K_δ for some $\delta > 0$. □

4.3.5 Critical Estimates

Let K be a compact subset of the interval and let f be K-equivalent to a function of class $M_n(K)$. Our goal in this subsection is to show that the set of points whose forward orbits accumulate on the critical set has Hausdorff dimension less than the hyperbolic dimension of K. Recall that for a point x, $\mathrm{Rec}(x, K)$ was defined in Definition 4.3.32.

Proposition 4.3.35 *Let K be a τ-set for a D-map f with a primary critical point c. Then*

$$HD(Rec(c, K)) \leq HypD(K).$$

Proof: As per the previous subsection, we shall construct an expanding induced map and apply the results of Section 4.3.3.

Since c is primary, there exists an open, regularly returning neighbourhood V' of c such that every branch of the first return map to V' is a diffeomorphism onto V'. Similarly, for all regularly returning intervals $U \subset V'$, every branch of the first return map to U is a diffeomorphism onto U which extends to a diffeomorphism onto V'. Of course, the domain of the first return map to U does not contain c.

If $\mathrm{HD}(\mathrm{Rec}(c, K)) \neq 0$ then branches of the first return map to V' accumulate on c. By Fact 4.3.13, for all $C_0 > 1$ there exists $\delta > 0$ such that if V is a regularly returning, open neighbourhood of c and $|V| < \delta$ then the first return map has distortion bounded by C_0. Fix C_0 with $1 < C_0 < \frac{4}{3}$ and a corresponding δ.

An elementary argument gives that C^2 non-flatness of the critical points implies that f is almost symmetric near each critical point. That is, if f_-^{-1} is the branch of f^{-1} sending $f(c)$ to c such that $f_-^{-1}(x) \leq c$ for all x in its domain, and $\varepsilon > 0$, then

$$\lim_{\varepsilon \to 0} \left| \frac{c - f_-^{-1}(f(c + \varepsilon))}{\varepsilon} \right| = 1.$$

Then there exists a $\delta' > 0$ such that if V is a neighbourhood of c, $f(\partial V)$ is a point (such a V is sometimes called symmetric) and $|V| < \delta'$, then for all intervals $W \subset V$ which do not contain c, $\frac{|W|}{|V|} < \frac{2}{3}$.

Consider such a V which is regularly returning and also satisfies $|V| < \delta$. Let W be a connected component of the domain of the first return map ϕ_V to V. Then $\frac{|W|}{|V|} < \frac{2}{3}$ and ϕ_V sends W onto V so there is a point $x \in W$ such that $|D\phi_V(x)| > \frac{3}{2}$. The distortion bound then implies that $|D\phi_V| > \frac{9}{8}$ on W and thus everywhere on its domain.

Since ϕ_V satisfies $|D\phi_V| \geq \frac{9}{8} > 1$ we can apply Lemma 4.3.16 so there exists a bound C on the distortion of branches of ϕ_V^n independent of $n \geq 1$.

Recall that $\tau(K)$ is finite and for all $W \in \tau(K), \partial W$ is preperiodic. Thus the set of points

$$S = \bigcup_{n \geq 0} \bigcup_{W \in \tau(K)} f^n(\partial W)$$

is finite.

Let $U \subset V$ be a small, regularly returning, open interval disjoint from S and such that $c \in \partial U$. Also assume that ∂U is not in the domain of ϕ_V, so that branches of the first return map ϕ_U to U accumulate on ∂U.

Denote by Z the union of the connected components of the domain of ϕ_U which have non-empty intersection with K. Then (ϕ_U, Z, K) is an induced Markov system. Let

$$L = \{x \in K \cap T : \phi_U^n(x) \in Z \text{ for all } n \geq 0\}.$$

By Proposition 4.3.31, $\mathrm{HD}(L) \leq \mathrm{HypD}(K)$. Note that L contains all points in $\mathrm{Rec}(c, K) \cap U$ whose forward orbits accumulate on p in U.

The same argument for the other side of the critical point c combined with Lemma 4.3.18 allows us to conclude that $\mathrm{HD}(\mathrm{Rec}(c, K)) < \mathrm{HypD}(K)$. □

Theorem 4.3.36 *For all integers $n \geq 0$ the following statement holds. Let K be a compact subset of the interval and let f be K-equivalent to a function of class $M_n(K)$, then*

$$HD(Rec(Crit, K)) \leq HypD(K).$$

Proof: We shall provide a proof by induction. Note that the statement of the theorem is trivially true if $n = 0$. Suppose that the statement holds for all $n \leq N - 1$. We must show that it holds for $n = N$. Consider a compact set K and map f such that f is K-equivalent to a function of class $M_N(K)$. By Lemma 4.3.9 we can assume f is of class $M_N(K)$. By Lemma 4.3.24 f has a primary critical point c. Now apply Proposition 4.3.35 to get

$$\mathrm{HD}(\mathrm{Rec}(c, K)) \leq \mathrm{HypD}(K).$$

Let $S = \mathrm{Rec}(\mathrm{Crit}, K) \setminus \mathrm{Rec}(c, K)$. It thus remains to show that $\mathrm{HD}(S) \leq \mathrm{HypD}(K)$. For $\delta > 0$ let B_δ be the ball of radius δ around the primary critical point c and let

$$K_\delta = K \setminus \bigcup_{i \geq 0} f^{-i}(B_\delta).$$

One can then write

$$S = \bigcup_{\delta > 0} \mathrm{Rec}(\mathrm{Crit}, K_\delta) \cup \bigcup_{i \geq 0} f^{-i}(c),$$

so

$$\mathrm{HD}(S) = \sup_{\delta > 0} \mathrm{HD}(\mathrm{Rec}(\mathrm{Crit}, K_\delta)).$$

Since c is primary, for sufficiently small δ the ball B_δ is disjoint from $O^+(f(\mathrm{Crit}))$. Thus f is K_δ-equivalent to a map of class $M_{N-1}(K_\delta)$. Applying the inductive hypothesis gives

$$\mathrm{HD}(\mathrm{Rec}(\mathrm{Crit}, K_\delta)) \leq \mathrm{HypD}(K_\delta) \leq \mathrm{HypD}(K).$$

Therefore $\mathrm{HD}(S) \leq \mathrm{HypD}(K)$ as required. □

4.3.6 Proof of Theorem 4.3.11

Given a compact subset K of an interval, let f be K-equivalent to a map of class $M_n(K)$. We wish to show that the Hausdorff and hyperbolic dimensions of K coincide. By Lemma 4.3.9 we can assume f is of class $M_n(K)$. We need the following lemmas:

Lemma 4.3.37 *Let U be a connected component of $I \setminus K$ and let $y \in \partial U \cap K$. Then y is preperiodic.*

Proof: If $f^i(U) \in \tau(K)$ for some $i \geq 0$ then $f^i(y)$ is preperiodic, so y is and we are done. Henceforth we assume $f^i(U) \cap K = \emptyset$ for all $i \geq 0$, so $f^i(U) \notin \tau(K)$ for any i.

Any critical points in $I \setminus K$ are mapped directly into the attracting boundary since f is of class $M_n(K)$. By the absence of wandering intervals ([6], theorem A, p 267), either some iterate of U contains a critical point of f (in $I \setminus K$) or has non-empty intersection with an immediate basin of attraction of a periodic attractor ([6], lemma II.3.1). In either case, we get that there exists an $m > 0$ and a $p > 0$ such that $f^m(U)$ has non-empty intersection with the immediate basin of a periodic attractor A of period p say. Now $f^m(y) \in K$, so it is not contained in any basin of attraction. The boundary ∂A is periodic of period p or $2p$. If $f^m(y) \in \partial A$ we are done; otherwise $f^m(U)$ contains a point x of ∂A. This point acts as an anchor: $f^{m+2kp}(U) \ni x$ for all $k \geq 0$. If k is not preperiodic then $f^{m+p}(U) \cup f^{m+2p}(U) \ni f^m(k)$, contradicting $f^i(U) \cap K = \emptyset$ for all $i \geq 0$. □

Lemma 4.3.38 *Preperiodic points are dense in the Julia set of f.*

Proof: This actually hold for all maps of class D. Lebesgue almost every point either is contained in the basin of attraction of a periodic point or accumulates on a critical point. The boundary of the basins of attraction consists of preperiodic points (periodic points at the boundary of the immediate basins). Thus preperiodic points are dense anywhere the Julia set is not connected.

Now let U be an interval contained in the Julia set. It suffices to show that $\overline{U}$ contains a preperiodic point. By Fact 4.3.15, the size of $f^i(U)$ is larger than some constant ϵ for all $i \geq 0$. Thus $X := \overline{\bigcup_{i\geq 0} f^i(U)}$ is a finite union of intervals which get mapped into each other by f, so at least one of them contains a periodic point. We claim that there exists an $m > 0$ such that $\bigcup_{i=0}^{m} f^i(\overline{U}) = X$.

Indeed, suppose otherwise. Since f is continuous and has only a finite number of critical points, there is an $n > 0$ such that $X \setminus \bigcup_{i=0}^{n} f^i(\overline{U})$ is a finite collection of intervals mapped monotonically into each other by f. But then they cannot be contained in the Julia set, which is a contradiction since the Julia set is completely invariant set which thus contains all iterates of U. □

Lemma 4.3.39 *For each $\delta > 0$ there exists a neighbourhood V_δ of Crit$\cap K$, which satisfies the following conditions:*

- *each connected component of V_δ contains at least one critical point;*
- *every point $x \in V_\delta \cap K$ satisfies dist$(x, \mathit{Crit} \cap K) \leq \delta$;*
- *the boundary of V_δ is made up entirely of preperiodic points contained in $K \cap \partial I$.*

Proof: Let c be in Crit$\cap K$. Let J be a connected component of $I \setminus \{c\}$. If $K \cap J \cap B(c, \delta) = \emptyset$, let p be the point in $J \cap (K \cup \partial I)$ which is closest to c, so $(p, c) \cap K = \emptyset$. By Lemma 4.3.37 p is preperiodic. If $K \cap J \cap B(c, \delta) \neq \emptyset$ then $J \cap B(c, \delta)$ either contains connected components of $I \setminus K$ or (non-exclusively) contains an interval contained in K. Either way, there is a preperiodic point p contained in $K \cap J \cap B(c, \delta)$. One finds in the same way a point p' for the other side of c.

Define V_δ as the union of intervals (p, p') constructed in this manner around each critical point in Crit$\cap K$. It satisfies the claims of the lemma. □

We now conclude the proof of Theorem 4.3.11. Let $S = K \setminus \mathrm{Rec}(\mathrm{Crit}, K)$. By Theorem 4.3.36 it only remains to show that $\mathrm{HD}(S) \leq \mathrm{HypD}(K)$.

$$K_\delta = K \setminus \bigcup_{i\geq 0} f^{-i}(V_\delta),$$

where V_δ is given by the preceding lemma for δ small. Due to the properties of V_δ, K_δ is a τ-set: we have only added at most a finite number of sets to $\tau(K)$, each being a connected component of V_δ with preperiodic boundary. One can then write

$$S = \bigcup_{\delta > 0} K_\delta \cup \bigcup_{i \geq 0} f^{-i}(\mathrm{Crit} \cap K),$$

so

$$\mathrm{HD}(S) = \sup_{\delta > 0} \mathrm{HD}(K_\delta).$$

It now suffices to show that $\mathrm{HD}(K_\delta) = \mathrm{HypD}(K_\delta)$, since $K_\delta \subset K$. But f is K_δ-equivalent to a map of class $M_0(K_\delta)$. Applying Theorem 4.3.34 gives the required result.

4.3.7 Lebesgue 0, Hausdorff 1

Recall the cadre of Theorem 4.3.6:

We suppose for D-map f that there exists a closed interval U with the following properties:

- there exists a flat parabolic point of period q such that $p \in \partial U$;
- U contains no critical points of f^{2q};
- $\frac{f^{2q}(x)-p}{x-p} > 1$ for all $x \in U$;
- there exists $V = [y, f^{2q}(y)]$ contained in U and an integer n such that $f^n(V) \supset U$.

Consider the set $W = U \cup f(V) \cup \cdots \cup f^{n-1}(V)$. The collection of points K that stay in W forever form a non-empty, forward-invariant, compact set. We can then study hyperbolic subsets of W. The Denker-Urbanski construction (similarly to Section 4.3.3) on increasing hyperbolic subsets of K gives in the limit a t-conformal probability measure m on K with $t = \mathrm{HypD}(K) \leq 1$. We shall show $t = 1$.

For simplicity, let $F = f^{2q}$, let g be the branch of F^{-1} which fixes p and suppose, without loss of generality, that $p \leq x$ for all $x \in U$.

Since m is conformal and $f^{2n}(V)$ covers K it follows that $m(V) > 0$. One has $\sum_{n\geq 0} |g^n(V)| \leq |U|$ so by Fact 4.3.13 there is a constant C such that

$$\frac{DF^n(x)}{DF^n(y)} < C$$

for all $x, y \in g^n(V)$ for all $n > 0$. Thus by t-conformality (as per Section 4.3.3)

$$|g^n(V)|^t \leq \frac{C|V|^t}{m(V)} m(g^n(v)).$$

Summing both sides

$$\sum_{n\geq 0} |g^n(V)|^t \leq \frac{C|V|^t}{m(V)} < \infty. \tag{4.5}$$

Up as far as here is well-known (see [41]). Let

$$b(x) = \sup\{g^n(y) : g^n(y) < x\}.$$

Then $b(x)$ is constant on the interior of each interval $g^n(V)$ and

$$|g^n(V)|^t = |F(g^n(y)) - g^n(y)|^t = \int_{g^n(V)} |F(b(x)) - b(x)|^{t-1}.$$

Summing one gets

$$\sum_{n\geq 0} |g^n(V)|^t = \int_p^{F(y)} |F(b(x)) - b(x)|^{t-1} < \infty,$$

where the finiteness of the sum and integral comes from inequality (4.5).

Now for each positive number r, by flatness of the parabolic point, there exists a $\delta > 0$ such that for all $x \in (p, p+\delta]$ one has $x < F(x) < x + (x-p)^r$. Also $b(x) < x$, thus

$$|F(b(x)) - b(x)| < (x-p)^r.$$

Since $t-1 \leq 0$ we have

$$\int_p^{F(y)} |F(b(x)) - b(x)|^{t-1} \geq \int_p^{p+\delta} (x-p)^{r(t-1)}$$

which is integrable only if $r(t-1) > -1$. But this means $t > \frac{r-1}{r}$ and this holds for all positive numbers r so $t \geq 1$. Thus $t = 1$ and Theorem 4.3.6 is proven. □

To finish, we show that such maps can exist. Let $a > 0$ and on the interval $[-a, 1+\frac{a}{3}]$ let the function f satisfy:

- $f(-a) = -a$;
- $f(x) < x$ for all x such that $-a < x < 0$;
- if $x \in [0, \frac{1}{3}]$ then $f(x) = x + e^{3-\frac{1}{x}+\log\frac{2}{3}}$;
- if $x \in [\frac{2}{3}, 1+\frac{a}{3}]$ then $f(x) = 3 - 3x$.
- on $[-a, 1+\frac{a}{3}]$ the function f is C^2 and has only one critical point, necessarily contained in $[\frac{1}{3}, \frac{2}{3}]$, and this critical point is non-flat.

Then f is a D-map with a flat parabolic point at 0 which satisfies Theorem 4.3.6 - one can take $U = [0, \frac{1}{3}]$. Then $f(U) = f([\frac{2}{3}, 1]) = [0, 1]$. The critical point is in the basin of attraction of the fixed point at a, and the Julia set for f is the complement of this basin of attraction. It is well-known and easy to show that the Julia set is of Lebesgue measure 0, for example by showing that density points cannot exist.

We have constructed a D-map satisfying Corollary 4.3.7, that is, a map whose Julia set has hyperbolic (and therefore Hausdorff) dimension equal to 1 but is of Lebesgue measure 0.

4.4 The quadratic family

We consider the family of maps $\mathcal{Q} = \{f_a(x) = ax(1-x)\}$ where $a \in [2, 4]$. These maps have been very widely studied. Many properties of this family of maps are shared by arbitrary full families of unimodal maps with quadratic critic points and negative Schwarzian derivative. All quadratic maps have negative Schwarzian derivative so one can apply the Koebe Principle to get good distortion bounds for iterates. That the critical point is quadratic is essential to get the decay of geometry necessary to prove the following result.

Fact 4.4.1 *Let $f \in \mathcal{Q}$ be a finitely renormalisable map such that all periodic points are hyperbolic repelling.*

There exists an interval U, pairwise disjoint subintervals U_i, and a full expanding C^2 Markov map $\phi : \bigcup_i U_i \to U$ with eternally bounded distortion such that, for each i, $\phi_{|U_i}$ is an iterate of f.

Proof: This follows immediately from [10], Theorem 2. □

We refer to Sections 3.1 and Section 4.3 for definitions of concepts used here.

Theorem 4.4.2 *Let $f \in \mathcal{Q}$ be finitely renormalisable with Julia set $\mathcal{J}$. Then*

$$HypD(\mathcal{J}) = HD(J).$$

Proof: Quadratic maps have negative Schwarzian derivative so if f has a parabolic periodic point or a hyperbolic attracting periodic point then the critical point is in the basin of attraction of the periodic point by theorem II.6.1 of [6], due to Singer. In this case one can apply Corollary 4.3.4.

Otherwise all periodic points are hyperbolic repelling. In the terminology of Section 4.3, $\tau(\mathcal{J}) = \emptyset$, so the map ϕ given by Fact 4.4.1 forms an induced Markov system $(\phi, \bigcup_i U_i, \mathcal{J})$ for f. Now ϕ is full, so the set of points which stay (under iteration by ϕ) in its domain forever has Hausdorff dimension one. Thus, by Proposition 4.3.31 the Hyperbolic dimension of $\mathcal{J}$ is greater than or equal to 1, and thus equals 1, the Hausdorff dimension of $\mathcal{J}$. □

For infinitely renormalisable quadratic maps, the situation is less clear. The Hausdorff dimension of the Julia set in this case is always 1. For the Feigenbaum map, any hyperbolic subset consists purely of periodic and preperiodic points and so the Hyperbolic dimension is 0. For other maps "of bounded type", the Hyperbolic dimension should be positive but strictly less than 1. For maps "of unbounded type", the Hyperbolic dimension can equal 1, but whether it does or not may depend on whether the renormalisations have "unbounded geometry". This is a project for the future.

Since the techniques developed allow for a quick and easy proof, we include the following:

Theorem 4.4.3 *Let f be a an cusp map with an ergodic invariant probability measure μ with positive finite Lyapunov exponent.*

The Hyperbolic dimension of the Julia set $\mathcal{J}$ of f is greater than or equal to $HD(\mu)$.

Proof: Let $(Y, F, \overline{\mu})$ be the natural extension of (f, μ) and let k_0, A, m_A be as per Section 1.6 where $0 < \eta < \chi$. By Lemma 1.6.6 there exists a regularly returning open interval W with non-periodic boundary such that $m_A(W) > 0$ and $|W| < k_0^{-1}$. Set $W_A := A \cap \Pi^{-1}W$.

Almost every point x is the projection of a point y in Y which enters W_A infinitely often. If $F^n y \in W_A$ then there exists an interval V_n containing Πy such that f^n maps V_n diffeomorphically onto W with distortion bounded by $\log 4$ and $|Df^n| > k_0^{-1}e^{n(\chi-\delta)}$ everywhere on V_n. Thus, given any $\lambda > 1$, taking n large, almost every point x is contained in an interval V_x which is mapped by an iterate f^{n_x} of f with derivative greater than λ such that

$$\left|\log|Df^{n_x}(w)| - \log|Df^{n_x}(w')|\right| < \log 4 \left(\frac{|f^{n_x}(w) - f^{n_x}(w')|}{|W|}\right)^{\epsilon}$$

for all $w, w' \in V_x$. Since W is regularly returning, all such pairs of intervals are nested or disjoint, so there is a countable disjoint collection $\mathcal{V} = \{V_i\}_{i=1}^{\infty}$ of positive measure such that no $V \in \mathcal{V}$ satisfies $\partial V \cap \partial W \neq \emptyset$ and $\mu(W \setminus \bigcup_i V_i) = 0$. Denote by n_V the unique number such that $f^{n_V} : V \to W$ is a diffeomorphism (so $n_V = n_x$ for some $x \in V$) for each $V \in \mathcal{V}$. Define $\phi : \bigcup_{i>0} V_i \to W$ as the induced map such that $\phi_{|V} = f^{n_V}$.

Now ϕ is an expanding $C^{1+\epsilon}$ Markov map and so has eternally bounded distortion. By ergodicity, almost every point in W stays in the domain of ϕ forever under iteration by ϕ; denote the set of all such points by L. Then $\mathrm{HD}(\mu_{|W} \leq \mathrm{HD}(L)$. By Lemma 1.5.2, it follows that $\mathrm{HD}(\mu) = \mathrm{HD}(\mu_{|W}) \leq HD(L)$. Again we have $(\phi, \bigcup_i V_i, \mathcal{J})$ is an induced Markov system, so Proposition 4.3.31 gives

$$\mathrm{HD}(L) \leq \mathrm{HypD}(\mathcal{J}).$$

Thus $\mathrm{HD}(\mu) \leq \mathrm{HypD}(\mathcal{J})$ as required. □

5

Expanding cocycles for interval maps

Contents

5.1 Summary

In this self-contained chapter we give a cocycle expansivity result for C^2 multimodal interval maps with non-flat critical points obtaining new information on orbits passing through critical neighbourhoods.

Résumé

Cocycles expansives pour des transformations de l'intervalle. On obtient des informations nouvelles pour des orbites qui passent par des voisinages critiques pour des applications multimodales de l'intervalle, en démontrant que, pour des cocycles bien adaptées, ces applications sont expansives.

5.2 Version française abrégée

Pour les systèmes dynamiques réels en dimension un, le théorème d'hyperbolicité de Mañe [24] est d'une grande importance. Il dit que, loin des points critiques, la dynamique est hyperbolique, et donc se comporte bien.

Notre résultat décrit à la fois la dynamique hyperbolique et la dynamique non-hyperbolique en obtenant des informations nouvelles sur les orbites qui passent arbitrairement proche des points critiques. On démontre que les applications sont expansives pour des cocycles que l'on définit dans cet article.

Soit f une transformation de l'intervalle I de classe C^2 et sans point périodique. Notons Df la dérivée de f. On dit que f a un point critique en c, ou $c \in \mathrm{Crit}(f)$, si $Df(c) = 0$. On suppose que la cardinalité de $\mathrm{Crit}(f)$ est finie, et que, pour chaque $c \in \mathrm{Crit}(f)$, il existe un difféomorphisme ψ de classe C^2, un $\beta > 1$ et un voisinage de c sur lequel (5.1) est vrai.

On définit $D_{\delta,W}f$ par (5.2-5.3) et $D_{\delta,W}f^n(x)$, par récurrence, par $D_{\delta,W}f^n(x) = D_{\delta,W}f^{n-1}(f(x)) \cdot D_{\delta,W}f(x)$. Soit $\mathcal{A}$ la réunion des bassins d'attraction immédiats des attracteurs périodiques et soit $\mathcal{P}$ un voisinage des points périodiques paraboliques.

Théorème 5.2.1 *Pour chaque $\delta > 0$, il existe des voisinages arbitrairement petits W de Crit(f) et des nombres $N_0 \geq 1, \lambda > 1$ tels que pour $n \geq N_0$*

$$|D_{\delta,W}f^n(x)| \geq \lambda > 1$$

pour tout x tel que, pour tout i avec $0 \leq i \leq n$, $f^i(x) \notin \mathcal{A} \cup \mathcal{P}$.

Corollaire 5.2.2 *Le théorème reste vrai si on prend $W = \{x \in I : |Df(x)| < \delta\}$.*

Proof: D'après le théorème, il existe $N_0 \geq 1$, $\lambda > 1$ et un petit voisinage $W' \subset W$ tels que, pour $n \geq N_0$, $|D_{\delta,W'}f^n(x)| \geq \lambda > 1$. Si $W \supset W'$ alors $|D_{\delta,W}f(x)| \geq |D_{\delta,W'}f(x)|$, donc $|D_{\delta,W}f^n(x)| \geq \lambda > 1$, ce qu'il fallait démontrer. □

Soient $U \subset I$ et $\epsilon(x) = \inf\{n \geq 0 : f^n(x) \in U\}$. Alors l'application de première entrée, ψ_U, pour un ensemble U, est définie, à l'endroit où $\epsilon(f(x))$ l'est, par $\psi_U(x) = f^{\epsilon(x)}(x)$. On définit l'application de premier retour ϕ_U par $\phi_U = \psi_U \circ f_{|U}$.

On va esquisser la démonstation du théorème, laissant les démonstrations rigoureuses pour la partie en anglais qui suit.

On fixe un voisinage de Crit(f) tel que (5.4) est vraie et tel que l'application de première entrée pour ce voisinage a la distortion bornée par $4e$. Puis on démontre la proposition centrale.

Proposition 5.2.3 *Pour chaque $\lambda > 1$, pour chaque point critique $c \in$Crit(f) $\setminus \mathcal{A}$ il existe un voisinage V_* de c tel que l'application de premier retour ϕ_{V_*} satisfait $|D_{\delta,W}\phi_{V_*}| \geq \lambda$.*

Ceci est assez délicat. Utilisant un résultat de [43], on peut démontrer que, pour chacun de ces points critiques, il existe $\alpha > 0$ et des voisinages V_c de c, arbitrairement petits, tels que la dérivée $|D\psi_{V_c}(x)| > \alpha$ pour un certain x dans le domaine central V_* de ψ_{V_c}. Puis il faut passer à l'application de premier retour pour V_*. En écrivant $\phi^n_{V_c}(x) = \phi_{V_*}(x)$, l'inégalité (5.4) donne (5.5). Ces propriétés, combinées avec la borne sur la distortion, impliquent (5.6), et, si l'on prend V_c assez petit, on a la proposition.

Ensuite, on fait un peu d'analyse combinatoire. On sait (de [24]) que les segments d'orbites assez grands qui ne passent pas par des voisinages périodiques sont expansifs. On a démontré que les applications de premier retour pour des voisinages critiques sont expansives pour $D_{\delta,W}$.

Si on prend une très longue orbite, soit, à un moment, elle passe beaucoup de temps en dehors des voisinages critiques, soit elle passe beaucoup de fois par des voisinages critiques. Dans le deuxième cas, par le principe des tiroirs, l'orbite passe beaucoup de fois par le voisinage d'un certain point critique. On peut alors découper l'orbite en une partie qui correspond à beaucoup d'itérations de l'application de premier retour pour ce voisinage (donc très expansive) et en deux autres parties. Dans le premier cas, on découpe aussi l'orbite, en une partie où elle passe beaucoup de temps en dehors des voisinages critiques (avec beaucoup d'expansion) et en deux autres parties.

Il suffit alors de démontrer qu'il y a une borne inférieure positive pour $D_{\delta,W}f^n$ sur chaque orbite. Pour ce faire, on découpe l'orbite en un nombre bornée de parties, avec une borne qui ne dépend que de la cardinalité de Crit(f). Certaines parties sont expansives, soit grâce aux applications de premier retour, soit grâce au temps longs passé en dehors des voisinages critiques. Les autres parties sont de longueur bornée uniformément.

Donc, pour les orbites assez longues (pour n grand) l'expansion l'emporte.

5.3 Introduction

The Mañe Hyperbolicity Theorem [24] is central to smooth one-dimensional discrete dynamical systems, implying that dynamics disjoint from critical neighbourhoods is hyperbolic (and thus straightforward).

Our result covers both the hyperbolic and the more interesting non-hyperbolic dynamics, obtaining new information for orbits which pass arbitrarily close to the critical set. We define appropriate derivative cocycles and show that these are expanding.

Let f be a C^2 map of the interval I to itself with a finite number of critical points denoted $\mathrm{Crit}(f)$ and no parabolic periodic points. We denote the derivative of f by Df and say a point is *critical* at c if $Df(c) = 0$. We suppose the critical points are non-flat, that is for x near $c \in \mathrm{Crit}(f)$, we can write

$$f(x) = \pm|\psi(x)|^{\beta} + f(c), \tag{5.1}$$

where ψ is C^2, $\psi(c) = 0$ and $\beta > 1$.

We define a δ-W derivative cocycle $D_{\delta,W}$ for f as follows:

$$\begin{aligned} D_{\delta,W} f(x) &= \delta, \ x \in W, & (5.2)\\ D_{\delta,W} f(x) &= Df(x), \ x \notin W. & (5.3) \end{aligned}$$

$D_{\delta,W} f^n(x)$ is defined inductively as $D_{\delta,W} f^{n-1}(f(x)) \cdot D_{\delta,W} f(x)$. Denote by $\mathcal{A}$ the union of immediate basins of attraction of periodic attractors and by $\mathcal{P}$ some small neighbourhood of the parabolic periodic points (q-periodic points p such that $|Df^q(p)| = 1$). The following is the main result of the chapter.

Theorem 5.3.1 *Given any $\delta > 0$ there exist arbitrarily small neighbourhoods W of Crit(f) and $N_0 \geq 1, \lambda > 1$ such that for each $n \geq N_0$*

$$|D_{\delta,W} f^n(x)| \geq \lambda > 1$$

for every x satisfying $f^i(x) \notin \mathcal{A} \cup \mathcal{P}$ for all $0 \leq i \leq n$.

Corollary 5.3.2 *The theorem holds for $W = \{x \in I : |Df(x)| < \delta\}$.*

Proof: According to the theorem there are $N_0 \geq 1$, $\lambda > 1$ and small $W' \subset W$ such that, for $n \geq N_0$, $|D_{\delta,W'} f^n(x)| \geq \lambda > 1$. If $W \supset W'$ then $|D_{\delta,W} f(x)| \geq |D_{\delta,W'} f(x)|$, thus $|D_{\delta,W} f^n(x)| \geq \lambda > 1$, as required. □

Before continuing to the proof, we define first entry and first return maps. An open interval U is called *regularly returning* for f if $f^n(\partial U) \cap U = \emptyset$ for all $n \geq 0$. Define the integer-valued entry time $\epsilon(x)$ for x to U as $\epsilon(x) = \inf\{n \geq 0 : f^n(x) \in U\}$ where the infimum exists.

Then the first entry map for U, ψ_U, is defined (where $\epsilon(x)$ is) by $\psi_U(x) = f^{\epsilon(x)}(x)$. The first return map to U, ϕ_U, is defined on U (where $\epsilon(f(x))$ is) by $\phi_U(x) = \psi_U(f(x))$.

5.4 Proofs

Let W' be a neighbourhood of $\mathrm{Crit}(f)$ such that $|Df_{|W'}(x)| < \frac{\delta}{4e}$ (where e is the base of the natural logarithm) for all $x \in W'$, let W be an open neighbourhood of $\mathrm{Crit}(f)$ compactly contained in W' and let $0 < \gamma = \mathrm{dist}(W, \partial W')$. Then

$$\left|\frac{D_{\delta,W'} f(x)}{Df(x)}\right| > \frac{\delta}{\frac{\delta}{4e}} > 4e \text{ for all } x \in W'. \tag{5.4}$$

We shall use the following Koebe Principle, proved in [9], in what follows:

Fact 5.4.1 *Koebe principle: Let I be a compact interval and $f : I \to I$ be a C^2 map with all critical points C^2 non-flat. Then there exists a continuous increasing function σ, $\sigma(0) = 0$, with the following property. If $J \subset T$ are open intervals and $n \in N$ is such that f^n is a diffeomorphism on T then, for every $x, y \in J$, we have*

$$\frac{Df^n(x)}{Df^n(y)} \geq \frac{e^{-\sigma(\max_{i=0}^{n-1}|f^i(T)|)\cdot\sum_{i=0}^{n-1}|f^i(J)|}}{(1+\nu(f^n(J), f^n(T)))^2},$$

where for open intervals $\overline{A} \subset B$, $\nu(A, B) = \frac{|A|}{dist(A,\partial B)}$.

Fix positive $\eta < \mathrm{dist}(\partial W, \mathrm{Crit}(f))$ small enough such that $\sigma(\eta) < 1$. By the Contraction Principle [43], [6], given η there exists $\varepsilon > 0$ such that every open interval mapped by f^n, for some $n \geq 0$, to an interval (not in the immediate basin of a periodic attractor) of size less than or equal to ε is of size less than or equal to η.

Similarly there exists ε_0 such that every interval mapped by f^n, some $n \geq 0$, to an interval $\leq \varepsilon_0$ is of size $\leq \frac{1}{3}\varepsilon$ and smaller than γ.

Let V_c be a regularly returning neighbourhood of critical point c of size $\leq \varepsilon_0$. Let U be a connected component of the domain of the first entry map ψ_{V_c} to V_c, with $\psi_{V_c|U} = f^n$. Then $U, f(U), \ldots, f^n(U)$ are all of size $\leq \frac{1}{3}\varepsilon$ (since $|f^{n-i}(f^i(U))| \leq \varepsilon_0$).

Let k be such that $U, f(U), \ldots, f^{k-1}(U)$ are all disjoint from W but $f^k(U) \cap W \neq \emptyset$. Since $|f^k(U)| \leq \frac{1}{3}\varepsilon$, we can choose an interval T of size $\leq \varepsilon$ containing $f^k(U)$ such that $\nu(f^k(U), T) \leq 1$. Then T is also sufficiently small that there is an interval $T_0 \supset U$ with $f^k : T_0 \to T$ a diffeomorphism. This follows since $\max_{0 \leq i \leq k} |f^i(T_0)| \leq \eta$ and $f^i(T_0) \supset f^i(U)$ and $f^i(U)$ is disjoint from W, so $f^i(T_0)$ does not contain any critical point.

We can then apply the Koebe Principle which gives distortion bound

$$\frac{Df^k(x)}{Df^k(y)} \geq \frac{e^{-\sigma(\eta)\cdot 1}}{(1+\nu(f^k(U), T))^2} \geq \frac{\frac{1}{e}}{(1+1)^2} = \frac{1}{4e}$$

for all $x, y \in U$.

For intervals $A \subset B$ one says A is *α-well inside* B if $\mathrm{dist}(A, \partial B) \geq \alpha|A|$.

Proposition 5.4.2 *Given $\lambda > 1$, for each critical point c not in the basin of attraction of a periodic attractor, there exists a neighbourhood V_* of c such that the first return map ϕ_{V_*} is λ-expanding with respect to the δ-W-derivative.*

Proof: By Theorem A' in [43] there exist $\alpha > 0$ depending only on f and arbitrarily small V_c as above which are regularly returning and satisfy one of the following for the first return map ϕ_{V_c}:

1. V_c is a restrictive interval (the central domain is V_c) and $|\phi_{V_c}(V_c)| \geq \alpha|V_c|$; we set $V_* = V_c$.

2. The central domain $V_* \ni c$ is α-well inside V_c.

3. The critical point is non-recurrent so either there is a neighbourhood of c so the first return map is not defined or there are arbitrarily small, in particular α-well inside, regularly returning intervals V_* containing c defined by boundary points of connected components of the domain of the first return map to V_c.

In case (1) it follows immediately (since $\int_{V_*} |D\phi_{V_*}(x)|' dx \geq \alpha |V_*|$) that $\max_{x \in V_*} |D\phi_{V_*}(x)| > \alpha$.

Now we treat cases (2), (3). The first return map to V_*, ϕ_{V_*}, can be written at each x in its domain as a composition of $n = n(x)$ iterates of the first return map to V_c,

$$\phi_{V_*}(x) = \phi^n_{V_c}(x).$$

Let $U \subset V_*$ be the connected component of the domain of $\phi^n_{V_c}$ (note not of ϕ_{V_*}) containing x. Then $\phi^n_{V_c}(\partial U) \subset \partial V_c$ and $\phi^n_{V_c}(x) \in V_*$ so $\phi^n_{V_c}(U)$ contains a connected component of $V_c \setminus V_*$ and as such is of size $\geq \alpha |V_*|$, i.e.

$$|\phi^n_{V_c}(U)| \geq \alpha |V_*| \geq \alpha |U|$$

$$\Rightarrow \max_{x \in U} |D\phi^n_{V_c}(x)| > \alpha.$$

Now $\phi^n_{V_c}$ restricted to U ($U = V_* = V_c$ in the renormalizable case (1) above) can be written as f^m for some m. Points in the orbit of U where $f^k(U) \cap W \neq \emptyset$ split up the orbit of U defined by f^m into N_W (say) sections outside of W. Each section outside of W contributes distortion of at most $4e$ from above. When $f^k(U) \cap W \neq \emptyset$ (and here we use inequality (5.4) and $|f^k(U)| < \gamma$ which implies $f^k(U) \subset W'$),

$$\frac{|D_{\delta, W'} f(f^k(x))|}{|Df(f^k(x)|} \geq 4e,$$

thus, setting $\delta_{V_c} = \max_{x \in V_c} |Df(x)|$, we have

$$|D_{\delta, W'} \phi^n_{V_c}(x)| \geq (4e)^{N_W - 1} \frac{\delta}{\delta_{V_c}} |D\phi^n_{V_c}(x)|. \tag{5.5}$$

This, together with the distortion bound and $\max_{x \in U} |D\phi^n_{V_c}(x)| > \alpha$, implies

$$\min_{x \in U} |D_{\delta, W'} \phi^n_{V_c}(x)| \geq \alpha \cdot \frac{1}{(4e)^{N_W}} \cdot (4e)^{N_W - 1} \frac{\delta}{\delta_{V_c}} = \frac{\alpha \delta}{4e} \frac{1}{\delta_{V_c}}, \tag{5.6}$$

This holds for all connected components of the domain of ϕ_{V_*} and so for all of the domain of ϕ_{V_*}. Constants α and δ are fixed, and choosing V_c sufficiently small $\frac{1}{\delta_{V_c}}$ is big, $\frac{\alpha\delta}{4e} \frac{1}{\delta_{V_c}} \geq \lambda$, and the proposition is proven. $\square$

Recall that $\mathcal{A}$ is the union of immediate basins of periodic attractors, and that $\mathcal{P}$ is some small neighbourhood of the parabolic periodic points of f. The following Fact is theorem III.5.1 in book [6] (see also [24]).

Fact 5.4.3 *Given $\lambda > 1$ and an open neighbourhood V of Crit(f), there exists N_V such that if $x, f(x), \dots, f^n(x) \in I \setminus (V \cup \mathcal{A} \cup \mathcal{P})$ and $n \geq N_V$ then*

$$|Df^n(x)| \geq \lambda.$$

Proof: [Proof of Theorem 5.3.1:] Given $\lambda > 1$ we fix regularly returning neighbourhoods $V_1, \dots, V_p$ of the p (say) critical points such that the first return map to each of them is λ-expanding (by Proposition 5.4.2).

Let $V = \bigcup_{i=1}^p V_p$. Then by Fact 5.4.3 there exists N_V such that if $n \geq N_V$ then $|Df^n(x)| > \lambda$ for all x satisfying $x, f(x), \dots, f^n(x) \in I \setminus (V \cup \mathcal{A} \cup \mathcal{P})$. Let $\nu = \inf\{|Df(x)| : x \in I \setminus (V \cup \mathcal{A}) > 0\}$.

Suppose we have an orbit $x, \ldots, f^n(x)$ in $I \setminus (\mathcal{A} \cup \mathcal{P})$. We set a coding a_j for this orbit where $a_j = 0$ if $f^j(x) \notin V$ and $a_j = i$ if $f^j(x) \in V_i$. There are $p+1$ characters $0, 1, \ldots, p$. In general for most j, a_j will equal 0.

Case 1: There are more than $C \cdot p + 1$ non-zero values a_j for some C. (We fix C at the end of this Case). This implies, according to the Pigeon-Hole Principle, the existence of an $i \in \{1, \ldots, p\}$ such that i appears at least $C+1$ times, which corresponds to at least C first returns to V_i. In this part of the orbit there is therefore λ^C expansion.

Let us look to the left of the first occurrence of i in the coding. The non-zero terms are from a set of $p-1$ characters $\{1, \ldots, p\} \setminus \{i\}$. Let i_1 be the rightmost such term. Then the sequence from the leftmost occurrence of i_1 to the rightmost corresponds to a composition of ($k \geq 0$) first returns to U_{i_1} and has thus "expansion" $\geq \lambda^k \geq 1$.

Now to the left of the leftmost i_1, the non-zero terms come from a set $\{1, \ldots, p\} \setminus \{i, i_1\}$ of $p-2$ characters. Continuing inductively, one can split up the sequence left of the first (leftmost) i into at most $p-1$ sections corresponding to compositions of first return maps to critical neighbourhoods and at most p sections of the form $a00\ldots0$ (where $a \in \{1, \ldots, p\} \setminus \{i\}$ for all apart from possibly the leftmost section where $a = 0$ if $x \notin V$) separating the "first return" sections. Each section $a00\ldots0$ can be at most ν^{N_V} contracting, since if there are more than $N_V - 1$ zeros we can apply Proposition 5.4.2 and get expansion of λ.

Arguing similarly to the right of the rightmost occurrence of i, we deduce for the whole sequence that

$$|D_{\delta,W} f^n(x)| \geq (\nu^{N_V})^p \cdot \lambda^C \cdot (\nu^{N_V})^p = \nu^{2N_V p} \cdot \lambda^C.$$

Since ν, N_V, λ and p are fixed constants, we can fix C sufficiently large that the expression on the right is greater than $\lambda > 1$, so $|D_{\delta,W} f^n(x)| \geq \lambda$.

Case 2: There are less than $C \cdot p + 1$ non-zero terms in the coding sequence. Suppose $n > N_V K(C \cdot p + 1)$ for some K, then there is a (maximal) block of at least $N_V K$ consecutive zeros in the sequence. Dividing the block into K parts and applying Fact 5.4.3, we get expansion of λ^K.

As before, to the left of this block one can bound contraction by $(\nu^{N_V})^p$ and likewise to the right. Thus for a sequence (orbit) of this type, we have

$$|D_{\delta,W} f^n(x)| > (\nu^{N_V})^p \cdot \lambda^K \cdot (\nu^{N_V})^p = \nu^{2N_V p} \cdot \lambda^K.$$

There exists K_0 such that, replacing K by K_0, the above expression is greater than λ, so $|D_{\delta,W} f^n(x)| \geq \lambda$.

Now let $N_0 = N_V K_0(C \cdot p + 1)$. If $n \geq N_0$ either Case 1 or Case 2 holds which concludes the proof of Theorem 5.3.1. □

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Résumé

Ce travail de thèse se situe dans le domaine de la dynamique unidimensionnelle. Dans le premier chapitre on définit une classe d'applications de l'intervalle, lisses par morceaux. Ces applications peuvent admettre des singularités où la norme de la dérivée tend vers l'infini. Pour des mesures avec exposant de Lyapunov positif, on construit une variété instable dans l'extension naturelle et on démontre un lemme de volume. Une telle mesure qui est absolument continue par rapport à la mesure de Lebesgue est engendrée par une application de Markov induite.

Dans le deuxième chapitre on étudie des applications holomorphes rationnelles. Le résultat principal est qu'une mesure avec exposant de Lyapunov positif est engendrée par une application de Markov induite.

Dans le troisième chapitre, on classifie les applications lisses de l'intervalle ayant une mesure d'entropie maximale qui est absolument continue par rapport à la mesure de Lebesgue. Ceci est l'analogue d'un résultat de Zdunik dans le cadre des applications rationnelles. D'autres résultats sur les conjugaisons sont aussi démontrés.

Dans le quatrième chapitre, on démontre que, pour des applications lisses de l'intervalle dont les points critiques sont non récurrents, la dimension hyperbolique et la dimension de Hausdorff des ensembles de Julia coïncident.

Dans le dernier chapitre on définit un cocycle pour des applications lisses de l'intervalle et on démontre que ce cocycle croît exponentiellement. Le cocycle coïncide avec la dérivée en dehors d'un voisinage de l'ensemble critique ; à l'intérieur de ce voisinage c'est une constante positive.

Mots-clés: ergodique, Lyapunov, hyperbolique, dimension, singularité, entropie, induite.

Abstract

This doctoral thesis is in the domain of one-dimensional dynamics. In the first chapter we study piecewise smooth interval maps which may have singularities where the derivative tends to infinity. For measures with positive Lyapunov exponent we construct the unstable manifold in the natural extension and prove a dynamical volume lemma. If such a measure is absolutely continuous with respect to Lebesgue measure then it is generated by an induced Markov map.

In the second chapter we study rational maps of the Riemann sphere. The main result is that a measure with positive Lyapunov exponent which is absolutely continuous with respect to a conformal measure is generated by an induced Markov map.

In the third chapter a classification is obtained of smooth interval maps whose measure of maximal entropy is absolutley continuous with respect to Lebesgue measure. This is the analogue of a result of Zdunik for rational maps. Other conjugacy results are also shown.

In the fourth chapter we show that for smooth interval maps with non-recurrent critical points the hyperbolic and Hausdorff dimension of the Julia sets coincide.

In the final chapter we define a cocyle for interval maps and show that with respect to this cocycle the dynamics is expanding.

Keywords: ergodic, Lyapunov, hyperbolic, dimension, singularity, cusp, entropy, induced.