# ASYMPTOTICS OF PARITY BIASES FOR PARTITIONS INTO DISTINCT PARTS VIA NAHM SUMS

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Abstract. For a random partition, one of the most basic questions is: what can one expect about the parts which arise? For example, what is the distribution of the parts of random partitions modulo $N$? Since most partitions contain a 1, and indeed many 1s arise as parts of a random partition, it is natural to expect a skew towards 1 $\pmod N$. This is indeed the case. For instance, Kim, Kim, and Lovejoy recently established "parity biases" showing how often one expects partitions to have more odd than even parts. Here, we generalize their work to give asymptotics for biases $\pmod N$ for partitions into distinct parts. The proofs rely on the Circle Method and give independently useful techniques for analyzing the asymptotics of Nahm-type $q$-hypergeometric series.

## 1. Introduction and statement of results

The study of the distribution of parts in integer partitions has a long history. For instance, many authors have studied counting functions for partitions with restrictions on the possible parts. Here, we are interested in the relative numbers of parts which are in different congruence classes. As most of the parts in most partitions tend to be small, for example, including many 1s and 2s, it is natural to expect that the parts are not equidistributed modulo $N \in \mathbb{N}_{\geq 2}$. For instance, Kim, Kim, and Lovejoy [12] showed the following parity bias for partitions modulo 2, where $p_o(n)$ (resp. $p_e(n)$) denotes the number of partitions of $n$ with a majority of odd parts (resp. even parts):

$$p_o(n) > p_e(n) \quad \text{for } n \neq 2, \qquad \lim_{n\to\infty} \frac{p_o(n)}{p_e(n)} = 1 + \sqrt{2}. \tag{1.1}$$

There are also biases for congruence classes modulo general $N$. A deep analysis of these was given in three related papers by Dartyge, Sarkozy, and Szalay [6, 7, 8], who proved lower bounds on biases for parts of partitions in congruence classes for positive proportions of partitions. This was generalized by Beckwith and Mertens [3], who turned these results into asymptotic formulas. Recently, the study of partition part biases for partitions into distinct parts was initiated by Kim, Kim, and Lovejoy [12]. They conjectured an analogue of the inequality in (1.1) which was recently proven in [2]. The goal of this paper is to refine these inequalities to precise, general, asymptotics.

For $N, \ell, b \in \mathbb{N}$, the generating function for the number $a_{N,\ell,b}(n)$ of partitions into $\ell$ distinct parts such that the size of each part is at least $b$ and is congruent to $b \pmod N$ is given by

$$\sum_{n\geq 0} a_{N,\ell,b}(n) q^n = \frac{q^{\frac{N\ell(\ell-1)}{2}+b\ell}}{(q^N;q^N)_\ell}.$$

Here $(a;q)_r := \prod_{j=0}^{r-1}(1-aq^j)$ with $r \in \mathbb{N}_0 \cup \{\infty\}$ is the usual *$q$-Pochhammer symbol.*

Let $N \in \mathbb{N}_{\geq 2}$, $K \in \mathbb{N}_0$, $1 \leq \alpha, \beta \leq N$, and $\alpha \neq \beta$. Denote by $d^{[K]}_{\alpha,\beta;N}(n)$ the number of partitions of $n$ into distinct parts such that there are more parts of size congruent to $\alpha \pmod N$ than parts of size congruent to $\beta \pmod N$ and such that the size of all parts are greater than $K$. The generating function of $d^{[K]}_{\alpha,\beta;N}(n)$ is given by (throughout we use bold letters for vectors)

$$\mathcal{D}^{[K]}_{\alpha,\beta;N}(q) := \sum_{n\geq 0} d^{[K]}_{\alpha,\beta;N}(n) q^n = \sum_{\boldsymbol{n}\in\mathcal{S}_{\alpha,\beta}} \frac{q^{N\cdot H(\boldsymbol{n})}}{\prod_{j=1}^{N} (q^N;q^N)_{n_j}}, \tag{1.2}$$

where

$$\mathcal{S}_{\alpha,\beta} := \left\{\boldsymbol{n} = (n_1,\dots,n_N)^T \in \mathbb{N}_0^N : n_\alpha > n_\beta\right\}.$$

Moreover $H\colon \mathbb{Z}^N \to \mathbb{Q}$ is given by

$$H(\boldsymbol{n}) := \tfrac{1}{2}\boldsymbol{n}^T\boldsymbol{n} + \boldsymbol{b}^T\boldsymbol{n}, \qquad \boldsymbol{b} := \left(\tfrac{1}{N}-\tfrac{1}{2},\quad \tfrac{2}{N}-\tfrac{1}{2},\quad \dots,\quad \tfrac{1}{2}\right)^T + \boldsymbol{e}$$

where

$$\boldsymbol{e} = \boldsymbol{e}^{[N,K]} = (e_1,\dots,e_N)^T := \left\lfloor \tfrac{K}{N}\right\rfloor \boldsymbol{1} + (1,\dots,1,0,\dots,0)^T$$

is such that $\sum_j e_j = K$ and $\boldsymbol{1} := (1,\dots,1)^T \in \mathbb{Z}^N$. Note that we have

$$\sum_{j=1}^{N} b_j = K + \tfrac{1}{2}. \tag{1.3}$$

In this paper we investigate a generalized parity bias modulo $N$ for partitions into distinct parts. Namely, given two congruence classes $\alpha, \beta \pmod N$, we consider the number of partitions of $n$ into distinct parts with more parts congruent to $\alpha \pmod N$ than to $\beta \pmod N$, and vice versa. We study the difference between these two counts, which has generating function

$$\sum_{n\geq 0}\left(d^{[K]}_{\alpha,\beta;N}(n) - d^{[K]}_{\beta,\alpha;N}(n)\right) q^n.$$

The case $N = 2$, $\alpha = 1$, $\beta = 2$, and $K = 0$ corresponds to the parity bias problem for partitions into distinct parts which Kim, Kim, and Lovejoy considered in [12] (the reader is also referred to [5, 11] for related works on partition parity biases modulo $N$). Our main result is as follows.

**Theorem 1.1.** *We have*

$$d^{[K]}_{\alpha,\beta;N}(n) = \frac{e^{\pi\sqrt{\frac{n}{3}}}}{2^{K+3}\cdot 3^{\frac{1}{4}} N^{N-1} n^{\frac{3}{4}}} \times \sum_{\substack{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^N \\ NH(\boldsymbol{\ell})\equiv n \pmod N}} \left(1 + \frac{N^2 - 2N[\ell_\alpha - \ell_\beta]_N + \beta - \alpha + N\left(e_\beta - e_\alpha\right)}{2\cdot 3^{\frac{1}{4}}\sqrt{N} n^{\frac{1}{4}}} + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

*as $n \to \infty$, where $[\ell]_N$ denotes the smallest positive integer congruent to $\ell \pmod N$.*

**Remark.** *Sums of the shape* (1.2)*, i.e., sums over (partial) lattices of $q$ raised to quadratic polynomials divided by products of Pochhammer symbols, have been the subject of many recent works. In particular, they are important in knot theory, algebraic $K$-theory, physics, and the intersections of these subjects with quantum modular forms (see e.g.* [4, 10, 13, 14]*, a more complete list of references and such connections is given in* [9]*). They also arise (thought not always named "Nahm sum") in $q$-series and combinatorics, for example, in* [1, 12]*. Recently, Garoufalidis and Zagier* [9] *investigated a general class of Nahm sums, but this does not allow us to study* (1.2) *particularly due to more general subset of a lattice which we consider. Our analytic methods extend their methods, and are analytically flexible. Thus, they may be more broadly useful in the study of such sums when they arise in applications to combinatorics or knot theory.*

If $N = 2$ or $N \geq 5$, then we can further simplify the asymptotic formula. In particular, in this case the asymptotic formula does not depend on the congruence class of $n \pmod N$.

**Theorem 1.2.** *Suppose that $N = 2$ or $N \geq 5$. Then we have, as $n \to \infty$,*

$$d^{[K]}_{\alpha,\beta;N}(n) = \frac{e^{\pi\sqrt{\frac{n}{3}}}}{2^{K+3} \cdot 3^{\frac{1}{4}} n^{\frac{3}{4}}} \left(1 + \frac{-N + \beta - \alpha + N\left(e_\beta - e_\alpha\right)}{2 \cdot 3^{\frac{1}{4}} \sqrt{N} n^{\frac{1}{4}}} + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

Kim, Kim, and Lovejoy [12, Section 6] conjectured that for $n \geq 20$ there are more partitions into distinct parts with more odd parts than even parts than vice versa. The conjecture was first proved [2] using combinatorial arguments, but an asymptotic formula for the parity bias had not been found. Theorem 1.2 gives a new asymptotic formula for the parity bias as a corollary.

**Corollary 1.3.** *We have, as $n \to \infty$,*

$$d^{[K]}_{1,2;2}(n) - d^{[K]}_{2,1;2}(n) = \frac{(-1)^K e^{\pi\sqrt{\frac{n}{3}}}}{2^{K+3}\sqrt{6}n} \left(1 + O\left(n^{-\frac{1}{4}}\right)\right).$$

*In particular, we have, as $n \to \infty$,*

$$\frac{d^{[K]}_{1,2;2}(n)}{d^{[K]}_{2,1;2}(n)} \to 1.$$

**Remark.** *Corollary 1.3 answers Problem 6.1 of* [2] *on explicit inequalities between these quantities (after a small correction). Problem 6.1 of* [2] *conjectured that for all $n \geq 14$:*

$$\begin{cases} d^{[1]}_{1,2;2}(n) > d^{[1]}_{2,1;2}(n) & \text{if } n \equiv 0 \pmod{2}, \\ d^{[1]}_{1,2;2}(n) < d^{[1]}_{2,1;2}(n) & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

*While this conjecture is true for many small values, it does not hold in general. Corollary 1.3 provides a modified version, showing that $d^{[1]}_{1,2;2}(n) < d^{[1]}_{2,1;2}(n)$ for $n$ sufficiently large.*

For $N \in \{3, 4\}$, the asymptotics of the parity bias can indeed depend on the residue classes $n \pmod{N}$. We use the following corollary to Theorem 1.1 to illustrate this phenomenon.

**Corollary 1.4.** *We have the asymptotic formulas, as $n \to \infty$,*

$$d^{[0]}_{1,2;3}(n) - d^{[0]}_{2,1;3}(n) = \begin{cases} \frac{e^{\pi\sqrt{\frac{n}{3}}}}{n}\left(\frac{1}{24} + O\left(n^{-\frac{1}{4}}\right)\right) & \text{if } n \equiv 0 \pmod{3}, \\ \frac{e^{\pi\sqrt{\frac{n}{3}}}}{n}\left(\frac{1}{6} + O\left(n^{-\frac{1}{4}}\right)\right) & \text{if } n \equiv 1 \pmod{3}, \\ \frac{e^{\pi\sqrt{\frac{n}{3}}}}{n}\left(-\frac{1}{12} + O\left(n^{-\frac{1}{4}}\right)\right) & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

The main tool for the proof of Theorem 1.1 is the Circle Method. We briefly outline our strategy. Firstly, we look at individual summands in (1.2), and derive two different asymptotic series for them as $q$ approaches a root of unity. The first such series is obtained by considering the summands as holomorphic functions in $z$ ($q = e^{-z}$), and deriving an asymptotic expansion as $z \to 0$ from the right half-plane. Unfortunately, the asymptotic expansion only holds for a narrow cone around the positive real line and is insufficient for the computation of the major arc contribution. Thus we derive a second asymptotic series by treating the real and imaginary parts of $z$ separately. This asymptotic series is no longer holomorphic in $z$, but holds in a sufficiently large region. To derive an asymptotic expansion for $\mathcal{D}^{[K]}_{\alpha,\beta;N}(q)$, we sum over the asymptotic series for the summands in (1.2), ignoring the summands with negligible contribution. An asymptotic expansion for $d^{[K]}_{\alpha,\beta;N}(n)$ is then obtained using the Circle Method. Theorem 1.1 is derived by explicit evaluation of the first two terms in the asymptotic expansion for $d^{[K]}_{\alpha,\beta;N}(n)$.

The paper is organized as follows. In Section 2, we give preliminaries on asymptotics, such as the asymptotic formula for $-\operatorname{Log}((q;q)_\infty)$ and the Euler–Maclaurin summation formula. In Section 3, we derive the two asymptotic expansions for summands appearing in (1.2). In Section 4, we prove bounds for the terms we ignore in the derivation of the asymptotic expansion of $\mathcal{D}^{[K]}_{\alpha,\beta;N}(q)$, as well as the minor arc contributions. In Section 5, we use the results obtained in previous sections to derive asymptotic expansions for $\mathcal{D}^{[K]}_{\alpha,\beta;N}(q)$ and $d^{[K]}_{\alpha,\beta;N}(n)$. In Section 6, we evaluate the asymptotic expansion of $d^{[K]}_{\alpha,\beta;N}(n)$ explicitly, and prove Theorems 1.1 and 1.2. In Section 7, we give numerical examples illustrating the biases we prove.


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## Notation

For the readers convenience we list the notation that is used in the paper. We always treat $N \in \mathbb{N}_{\geq 2}$, $K \in \mathbb{N}_0$ and $1 \leq \alpha, \beta \leq N$, $\alpha \neq \beta$ as fixed parameters.

- For $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$, we write $z = \varepsilon(1+iy)$ with $\varepsilon > 0$ and $y \in \mathbb{R}$, and $q = e^{-z} \in \mathbb{C}$.
- The constant $\lambda$ is fixed, real, and satisfies $-\frac{2}{3} < \lambda < -\frac{1}{2}$.
- For $\ell \in \mathbb{Z}$, we write $[\ell]_N$ to denote the smallest positive integer congruent to $\ell \pmod{N}$.
- We define the set $\mathcal{S}_{\alpha,\beta} := \{\boldsymbol{n} = (n_1, \ldots, n_N)^T \in \mathbb{N}_0^N : n_\alpha > n_\beta\}$.
- We write $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_N)$ for an element in $(\mathbb{Z}/N\mathbb{Z})^N$. We always assume $0 \leq \ell_j < N$.
- For $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_N) \in (\mathbb{Z}/N\mathbb{Z})^N$, we define $\mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}} := \{\boldsymbol{n} \in \mathcal{S}_{\alpha,\beta} : \boldsymbol{n} \equiv \boldsymbol{\ell} \pmod{N}\}$.
- We let $\mathcal{N}_{\varepsilon,\lambda} := \{\boldsymbol{n} \in \mathbb{N}_0^N : \boldsymbol{n} = \frac{\log(2)}{\varepsilon}\boldsymbol{1} + \boldsymbol{\mu}, |\boldsymbol{\mu}| \leq \varepsilon^\lambda\}$ and write $\boldsymbol{\mu} = \boldsymbol{n} - \frac{\log(2)}{\varepsilon}\boldsymbol{1}$.
- For $\boldsymbol{n} \in \mathbb{N}_0^N$, we define $\boldsymbol{u} \in \mathbb{C}^N$ entrywise by $n_j = \frac{\log(2)}{z} + \frac{u_j}{\sqrt{z}}$, giving a map $\boldsymbol{n} \mapsto \boldsymbol{u}$. The bijective image of $\mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}}$ (resp. $\mathcal{N}_{\varepsilon,\lambda}$) under the map $\boldsymbol{n} \mapsto \boldsymbol{u}$ is denoted $\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}$ (resp. $\mathcal{U}_{\varepsilon,\lambda}$). We also define $\boldsymbol{v} \in \mathbb{C}^N$ entrywise by $n_j = \frac{\log(2)}{\varepsilon} + \frac{v_j}{\sqrt{z}}$.
- The functions $C_r$, $\Lambda$, and $D_r$, are given as

$$\sum_{r\geq 0} C_r(\boldsymbol{u}) z^{\frac{r}{2}} := \exp(\phi(\boldsymbol{u}, z)), \qquad \text{with} \quad \phi(\boldsymbol{u}, z) := -\boldsymbol{b}^T\boldsymbol{u}\sqrt{z} - \frac{Nz}{24} + \sum_{j=1}^N \xi\left(\frac{u_j}{\sqrt{z}}, z\right),$$

$$\text{where} \qquad \xi(\nu, z) = -\sum_{r=2}^{R-1} \left(B_r(-\nu) - \delta_{r,2}\nu^2\right) \operatorname{Li}_{2-r}\left(\frac{1}{2}\right) \frac{z^{r-1}}{r!} + O\left(z^{R-1}\right) \quad (R \in \mathbb{N}),$$

$$\Lambda(y) := N\left(\frac{\pi^2}{6} - \frac{\log(2)^2(1+iy)^2}{2} - \operatorname{Li}_2\left(2^{-(1+iy)}\right)\right),$$

$$\sum_{r\geq 0} D_r(\boldsymbol{v}, y) z^{\frac{r}{2}} := \exp(\phi(\boldsymbol{v}, z)), \qquad \text{with} \quad \phi(\boldsymbol{v}, z) := -\boldsymbol{b}^T\boldsymbol{v}\sqrt{z} - \frac{Nz}{24} + \sum_{j=1}^N \xi_y\left(\frac{v_j}{\sqrt{z}}, z\right),$$

where $\xi_y(\nu, z) = -\sum_{r=2}^{R-1} \left(B_r(-\nu) - \delta_{r,2}\nu^2\right) \mathrm{Li}_{2-r}\left(2^{-(1+iy)}\right) \frac{z^{r-1}}{r!} + O\left(z^{R-1}\right) \quad (R \in \mathbb{N}).$

- The functions $\mathcal{D}^{[K]}_{\alpha,\beta;N}$ and $\mathcal{D}^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}$ are defined as

$$\mathcal{D}^{[K]}_{\alpha,\beta;N}(q) := \sum_{n\geq 0} d^{[K]}_{\alpha,\beta;N}(n)q^n = \sum_{\boldsymbol{n}\in\mathcal{S}_{\alpha,\beta}} \frac{q^{N\cdot H(\boldsymbol{n})}}{\prod_{j=1}^N (q^N;q^N)_{n_j}},$$

$$\mathcal{D}^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(q) := \sum_{\boldsymbol{n}\in\mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}}} \frac{q^{NH(\boldsymbol{n})}}{\prod_{j=1}^N (q^N;q^N)_{n_j}}.$$

- The functions $f^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}$, $g^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}$, $g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}$, and $g^{[K,2]}_{\alpha,\beta;N,\boldsymbol{\ell}}$ are given as

$$f^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) := \sum_{\boldsymbol{n}\in\mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}}} \frac{e^{-NH(\boldsymbol{n})z}}{\prod_{j=1}^N (e^{-Nz};e^{-Nz})_{n_j}}, \qquad g^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) := f^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}\left(\frac{z}{N}\right),$$

$$g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) := \sum_{\boldsymbol{n}\in\mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{N}_{\varepsilon,\lambda}} \frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^N (e^{-z};e^{-z})_{n_j}}, \quad g^{[K,2]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) := \sum_{\boldsymbol{n}\in\mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}}\setminus\mathcal{N}_{\varepsilon,\lambda}} \frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^N (e^{-z};e^{-z})_{n_j}}.$$

- For $\boldsymbol{u} \in \mathbb{C}^N$, we let $u_\alpha$ be the $\alpha$-th entry of $\boldsymbol{u}$, and $\boldsymbol{u}_{[\mathbf{1}]}$ be the remaining $N-1$ entries. For convenience, we write $\boldsymbol{u} = (\boldsymbol{u}_{[\mathbf{1}]}, u_\alpha)$. Analogously, we write $\boldsymbol{\mu}_{[\mathbf{1}]}$ for the corresponding $N-1$ entries of $\boldsymbol{\mu}$. For $1 \leq c \leq N$, $c \neq \alpha$, we let $u_c$ (resp. $u_\alpha$) be the $c$-th (resp. $\alpha$-th) entry of $\boldsymbol{u}$, and $\boldsymbol{u}_{[\mathbf{2}]}$ for the remaining $N-2$ entries. For convenience, we write $\boldsymbol{u} = (\boldsymbol{u}_{[\mathbf{1}]}, u_\alpha) = (\boldsymbol{u}_{[\mathbf{2}]}, u_c, u_\alpha)$.
- For fixed $\boldsymbol{\ell} \in (\mathbb{Z}/N\mathbb{Z})^N$, the sets $\mathfrak{u}_\alpha(\boldsymbol{u}_{[\mathbf{1}]})$, $\mathfrak{u}_c$, and $\mathcal{U}_{[1]}$ are defined as

$$\mathfrak{u}_\alpha\left(\boldsymbol{u}_{[\mathbf{1}]}\right) := \left\{u_\beta + t\sqrt{z} \ : \ t \in [\ell_\alpha - \ell_\beta]_N + N\mathbb{N}_0\right\}, \quad \mathfrak{u}_c := \left\{-\frac{\log(2)}{\sqrt{z}} + t\sqrt{z} \ : \ t \in \ell_c + N\mathbb{N}_0\right\},$$

$$\mathcal{U}_{[\mathbf{1}]} := \left\{\boldsymbol{u}_{[\mathbf{1}]} \in \left(-\frac{\log(2)}{\sqrt{z}} + \mathbb{R}\sqrt{z}\right)^{N-1} \ : \ \left|\boldsymbol{\mu}_{[\mathbf{1}]}\right| \leq \varepsilon^\lambda\right\}.$$

- The functions $G_j$ and $G_{j,\alpha}$ are given by

$$G_j(\boldsymbol{u}) := e^{\frac{\pi^2 N}{12z}} C_j(\boldsymbol{u}) e^{-\boldsymbol{u}^T\boldsymbol{u}}, \qquad G_{j,\alpha}\left(\boldsymbol{u}_{[\mathbf{1}]}\right) := \sum_{u_\alpha\in\mathfrak{u}_\alpha\left(\boldsymbol{u}_{[\mathbf{1}]}\right)} G_j\left(\boldsymbol{u}_{[\mathbf{1}]}, u_\alpha\right).$$

- The coefficients $E_{\boldsymbol{\ell},r}$, $V_{j,r}$ and $W_r$ are defined as, with $R \in \mathbb{N}$,

$$g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) =: \frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}} \sum_{r=0}^{R-1} E_{\boldsymbol{\ell},r} z^{\frac{r}{2}} + e^{\frac{\pi^2 N}{12\varepsilon}} O\left(\varepsilon^{N\left(\lambda+\frac{1}{2}\right)+3R\left(\lambda+\frac{2}{3}\right)}\right),$$

$$\sum_{\boldsymbol{u}\in\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{U}_{\varepsilon,\lambda}} G_j(\boldsymbol{u}) =: e^{\frac{\pi^2 N}{12z}} \sum_{r=-N}^{R-1} V_{j,r} z^{\frac{r}{2}} + O\left(\varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+2N+R-1)+\frac{N+R}{2}-1} e^{\frac{\pi^2 N}{12\varepsilon}}\right),$$

$$\sum_{u_\alpha\in\mathfrak{u}_\alpha\left(\boldsymbol{u}_{[\mathbf{1}]}\right)} G_j\left(\boldsymbol{u}_{[\mathbf{1}]}, u_\alpha\right) =: \frac{1}{N\sqrt{z}} \int_{u_\beta}^{u_\beta+\sqrt{z}\infty} G_j\left(\boldsymbol{u}_{[\mathbf{1}]}, u_\alpha\right) du_\alpha + \sum_{r=0}^{R-1} W_r G_j^{(r)}\left(\boldsymbol{u}_{[\mathbf{1}]}, u_\beta\right) z^{\frac{r}{2}}$$
$$+ O\left(\varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+R+1)+\frac{R-1}{2}} e^{\frac{\pi^2 N}{12\varepsilon}}\right).$$

## 2. Preliminaries

We first recall several special functions. For $s \in \mathbb{C}$, the *polylogarithm* is given by

$$\mathrm{Li}_s(w) := \sum_{n\geq 1} \frac{w^n}{n^s}, \quad |w| < 1.$$

For $w \in (0,1)$, define the *Rogers dilogarithm function* (shifted by a constant so that $L(1) = 0$)

$$L(w) := \mathrm{Li}_2(w) + \frac{1}{2}\log(w)\log(1-w) - \frac{\pi^2}{6}.$$

It is well-known (see [20, p.6]) that

$$L\left(\frac{1}{2}\right) = -\frac{\pi^2}{12}. \tag{2.1}$$

Let $B_r(x)$ denote the *$r$-th Bernoulli polynomial* defined via the generating function

$$\frac{te^{xt}}{e^t - 1} =: \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

The *$r$-th Bernoulli number* is given by $B_r := B_r(0)$.

We use $\ll, \gg$, and $\asymp$ to compare the size of quantities, without any assumption on their signs or arguments unless stated otherwise; e.g., for $y \in \mathbb{R}$, $y \ll 1$ means $-C \leq y \leq C$ for some $C > 0$.

We first give a basic estimate for $(q;q)_\infty^{-1}$; the proof of the following lemma is straightforward.

**Lemma 2.1.** *Let $\varepsilon > 0$ and $z := \varepsilon(1+iy)$. Suppose $y \ll \varepsilon^{-\frac{1}{2}+\delta}$ for some $\delta > 0$. Then we have*[1]

$$-\operatorname{Log}((q,q)_\infty) = \frac{\pi^2}{6z} + \frac{1}{2}\operatorname{Log}\left(\frac{z}{2\pi}\right) - \frac{z}{24} + \mathcal{E},$$

*as $z \to 0$, where the error term $\mathcal{E}$ satisfies $\mathcal{E} \ll z^L$ for all $L \in \mathbb{N}$.*

The following lemma is a special case of Lemma 2.1 of [9].

**Lemma 2.2.** *Let $z, w \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, $|w| < 1$, and $\nu \in \mathbb{C}$ such that $\nu z = o(1)$. Then*[2]

$$\operatorname{Log}\left(\left(we^{-\nu z}q;q\right)_\infty\right) = -\mathrm{Li}_2(w)\frac{1}{z} - \left(\nu + \frac{1}{2}\right)\operatorname{Log}(1-w) - \frac{\nu^2 z}{2}\frac{w}{1-w} + \psi_w(\nu,z).$$

*Here $\psi_w(\nu,z)$, for $R \in \mathbb{N}$, has an asymptotic expansion as $z \to 0$ with $\mathrm{Re}(z) > 0$*

$$\psi_w(\nu,z) = -\sum_{r=2}^{R-1}\left(B_r(-\nu) - \delta_{r,2}\nu^2\right)\mathrm{Li}_{2-r}(w)\frac{z^{r-1}}{r!} + O\left(z^{R-1}\right), \tag{2.2}$$

*with $\delta_{j,k}$ the Kronecker delta symbol. In particular, for every $n \in \mathbb{N}_0$ the coefficient of $\nu^n$ is $O(z^{\frac{2n}{3}})$.*

**Remark.** [9, Lemma 2.1] *is only stated for real $z$. However, a line-by-line check of the proof therein shows that the statement still holds for complex $z$.*

We also make use of the following version of the Euler–Maclaurin summation formula. The classical version of Euler–Maclaurin summation compares the sum $\sum_{n=0}^{m} F(n)$ to the integral $\int_0^m F(x)dx$; this version makes use of the same expressions but with a change of variables $F(x) = f(xz+a)$. A related "shifted" version of Euler–Maclaurin, can also be found in Proposition 3 of [19].

[1] Throughout, we take the principal branch for the logarithm.

[2] In [9, Lemma 2.1], there is a cyclic quantum dilogarithm term, which vanishes if $q$ approaches 1.

**Proposition 2.3.** *Let $a, z \in \mathbb{C}$. Let $f$ be a holomorphic function that is defined in $L(a,z) := \{tz + a : t \in \mathbb{R}_0^+\}$. Then we have, for all $m, R \in \mathbb{N}$,*

$$\sum_{n=0}^{m} f(nz+a) = \frac{1}{z}\int_a^{a+mz} f(x)dx + \frac{f(mz+a)+f(a)}{2}$$
$$+ \sum_{r=1}^{R} \frac{B_{2r}z^{2r-1}}{(2r)!}\left(f^{(2r-1)}(mz+a) - f^{(2r-1)}(a)\right) + O(1)z^{2R}\int_a^{a+mz}\left|f^{(2R+1)}(x)\right|dx.$$

*Furthermore, if $f$ has rapid decay on $L(a,z)$, then we have*

$$\sum_{n\geq 0} f(nz+a) = \frac{1}{z}\int_a^{a+z\infty} f(x)dx + \frac{f(a)}{2}$$
$$- \sum_{r=1}^{R} \frac{B_{2r}z^{2r-1}}{(2r)!} f^{(2r-1)}(a) + O(1)z^{2R}\int_a^{a+z\infty}\left|f^{(2R+1)}(x)\right|dx.$$

## 3. Asymptotic formulas for summands

3.1. **The setup.** We study the asymptotics as $z \to 0$, in the right half-plane, of the function

$$\frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^{N}(q;q)_{n_j}}. \tag{3.1}$$

For $z$ real, asymptotics for (3.1) were obtained in [9, 18]. To avoid confusion, we write $\varepsilon > 0$ for a real variable and denote by $z = \varepsilon(1+iy)$ a complex variable with $\mathrm{Re}(z) > 0$. For $-\frac{2}{3} < \lambda < -\frac{1}{2}$, we define, with $|\boldsymbol{\mu}|$ the (Euclidean) norm of the vector $\boldsymbol{\mu}$,

$$\mathcal{N}_{\varepsilon,\lambda} := \left\{\boldsymbol{n} \in \mathbb{N}_0^N : \boldsymbol{n} = \frac{\log(2)}{\varepsilon}\mathbf{1} + \boldsymbol{\mu},\ |\boldsymbol{\mu}| \leq \varepsilon^{\lambda}\right\}.$$

**Remark.** *There is a general philosophy behind considering this expansion. As in* [9] *(see page 3), and other references studying Nahm sums, natural vectors to "expand" around are solutions to the so-called Nahm equation (essentially, this corresponds to the saddle point). In our situation, $H(\boldsymbol{n})$ can be decomposed into a quadratic form piece and a linear piece, and the quadratic form piece is simply the form $\frac{1}{2}|\boldsymbol{n}|^2$. In this case, the Nahm equation (see (6) in* [9]*) becomes $1 - z_j = z_j$, ($1 \leq j \leq N$) which has unique solution in $(0,1)^N$ equal to $\frac{1}{2}\mathbf{1}$. As explained in Proposition 2.1 of* [9]*, one should center the expansion around $n_j \approx \frac{1}{\varepsilon}\log(\frac{1}{z_j})$, which in our case becomes $n_j \approx \frac{\log(2)}{\varepsilon}$ for all $j$. Thus, $\frac{\log(2)}{\varepsilon}\mathbf{1}$ is the saddle point we need to expand around, which, philosophically, explains the definition of $\mathcal{N}_{\varepsilon,\lambda}$. This also becomes apparent in the forthcoming proofs.*

In this section, we prove asymptotic formulas for (3.1), in the case $\boldsymbol{n} \in \mathcal{N}_{\varepsilon,\lambda}$.

3.2. **Narrow range estimates.** We prove asymptotic expansions for (3.1) if $\boldsymbol{n} \in \mathcal{N}_{\varepsilon,\lambda}$.

**Proposition 3.1.** *Let $\boldsymbol{n} \in \mathcal{N}_{\varepsilon,\lambda}$ and $\boldsymbol{u} \in \mathbb{C}^N$ be such that, for $1 \leq j \leq N$,*

$$n_j = \frac{\log(2)}{z} + \frac{u_j}{\sqrt{z}}. \tag{3.2}$$

*Suppose that $y \ll \varepsilon^{\delta}$ for some $\delta > 0$. Then we have*

$$\frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^{N}(q;q)_{n_j}} = \left(\frac{z}{\pi}\right)^{\frac{N}{2}} \frac{e^{\frac{\pi^2 N}{12z} - \frac{Nz}{24}}}{2^{K+\frac{1}{2}}} e^{-\boldsymbol{u}^T\boldsymbol{u} - \boldsymbol{b}^T\boldsymbol{u}\sqrt{z}} \prod_{j=1}^{N}\exp\left(\xi\left(\frac{u_j}{\sqrt{z}}, z\right)\right), \tag{3.3}$$

*where, for $R \in \mathbb{N}$, $\xi(\nu, z)$ has an asymptotic expansion as $z \to 0$*

$$\xi(\nu, z) = -\sum_{r=2}^{R-1} \left(B_r(-\nu) - \delta_{r,2}\nu^2\right) \mathrm{Li}_{2-r}\left(\frac{1}{2}\right) \frac{z^{r-1}}{r!} + O\left(z^{R-1}\right). \tag{3.4}$$

*Proof.* We write

$$\frac{1}{(q;q)_{n_j}} = \frac{\left(q^{n_j+1};q\right)_\infty}{(q;q)_\infty}. \tag{3.5}$$

By (3.2) we have $q^{n_j} = \frac{1}{2}e^{-u_j\sqrt{z}}$. Let $\nu_j := \frac{u_j}{\sqrt{z}}$. Since $\boldsymbol{n} \in \mathcal{N}_{\varepsilon,\lambda}$, we have, for some $|\mu_j| \leq \varepsilon^\lambda$,

$$\nu_j = n_j - \frac{\log(2)}{z} = \left(\frac{\log(2)}{\varepsilon} + \mu_j\right) - \frac{\log(2)}{\varepsilon(1+iy)} = \left(1 - \frac{1}{1+iy}\right)\frac{\log(2)}{\varepsilon} + \mu_j. \tag{3.6}$$

If $y \ll \varepsilon^\delta$, then the first term in (3.6) has size $\varepsilon^{\delta-1}$. Since $|\mu_j| \leq \varepsilon^\lambda$, we conclude that $\nu_j z = o(1)$. Thus we may apply Lemma 2.2 with $w = \frac{1}{2}$, and deduce that

$$\mathrm{Log}\left(\left(q^{n_j+1};q\right)_\infty\right) = -\frac{1}{z}\mathrm{Li}_2\left(\frac{1}{2}\right) + \log(2)\left(\nu_j + \frac{1}{2}\right) - \frac{\nu_j^2 z}{2} + \psi_{\frac{1}{2}}(\nu_j, z). \tag{3.7}$$

By (3.5), (3.7), and Lemma 2.1 we have, with $\mathcal{E}$ satisfying $\mathcal{E} \ll z^L$ for all $L \in \mathbb{N}$,

$$-\mathrm{Log}\left((q;q)_{n_j}\right) = -\mathrm{Li}_2\left(\frac{1}{2}\right)\frac{1}{z} + \log(2)\left(\nu_j + \frac{1}{2}\right) - \frac{\nu_j^2 z}{2} + \psi_{\frac{1}{2}}(\nu_j, z) + \frac{\pi^2}{6z} - \frac{1}{2}\mathrm{Log}\left(\frac{2\pi}{z}\right) - \frac{z}{24} + \mathcal{E}.$$

Now let $\xi(\nu_j, z) := \psi_{\frac{1}{2}}(\nu_j, z) + \mathcal{E}$. Since $\mathcal{E} \ll z^L$ for all $L \in \mathbb{N}$, it follows that $\xi(\nu_j, z)$ has the same asymptotic expansion as $\psi_{\frac{1}{2}}(\nu_j, z)$. So we may write

$$-\mathrm{Log}\left((q;q)_{n_j}\right) = -\frac{1}{2}\mathrm{Log}\left(\frac{\pi}{z}\right) - \frac{z}{24} + \left(\frac{\pi^2}{6} - \mathrm{Li}_2\left(\frac{1}{2}\right)\right)\frac{1}{z} + \log(2)\nu_j - \frac{\nu_j^2 z}{2} + \xi(\nu_j, z).$$

Summing over $j$ gives

$$-\sum_{j=1}^{N}\mathrm{Log}\left((q;q)_{n_j}\right) = -\frac{N}{2}\mathrm{Log}\left(\frac{\pi}{z}\right) - \frac{Nz}{24} + \left(\frac{\pi^2}{6} - \mathrm{Li}_2\left(\frac{1}{2}\right)\right)\frac{N}{z} + \sum_{j=1}^{N}\left(\log(2)\nu_j - \frac{\nu_j^2 z}{2} + \xi(\nu_j, z)\right). \tag{3.8}$$

Using (1.3), we note that

$$-H(\boldsymbol{n})z = -\frac{N\log(2)^2}{2z} - \frac{\log(2)}{\sqrt{z}}\sum_{j=1}^{N} u_j - \frac{1}{2}\boldsymbol{u}^T\boldsymbol{u} - \log(2)\left(K + \frac{1}{2}\right) - \boldsymbol{b}^T\boldsymbol{u}\sqrt{z}. \tag{3.9}$$

We combine (3.8) and (3.9) and get

$$\begin{aligned}\mathrm{Log}\left(\frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^{N}(q;q)_{n_j}}\right) = -\frac{N}{2}\mathrm{Log}\left(\frac{\pi}{z}\right) - \frac{Nz}{24} + \left(\frac{\pi^2}{6} - \mathrm{Li}_2\left(\frac{1}{2}\right)\right)\frac{N}{z} + \sum_{j=1}^{N}\left(\log(2)\nu_j - \frac{\nu_j^2 z}{2} + \xi(\nu_j, z)\right)\\ -\frac{N\log(2)^2}{2z} - \frac{\log(2)}{\sqrt{z}}\sum_{j=1}^{N} u_j - \frac{1}{2}\boldsymbol{u}^T\boldsymbol{u} - \log(2)\left(K + \frac{1}{2}\right) - \boldsymbol{b}^T\boldsymbol{u}\sqrt{z}.\end{aligned}$$

Using (2.1), we evaluate

$$-\left(\mathrm{Li}_2\left(\frac{1}{2}\right) + \frac{\log(2)^2}{2} - \frac{\pi^2}{6}\right)\frac{N}{z} = -L\left(\frac{1}{2}\right)\frac{N}{z} = \frac{\pi^2 N}{12z}.$$

Finally, using this and the fact that $\nu_j = \frac{u_j}{\sqrt{z}}$, we may rewrite

$$\operatorname{Log}\left(\frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^{N}(q;q)_{n_j}}\right)$$
$$= -\frac{N}{2}\operatorname{Log}\left(\frac{\pi}{z}\right) + \frac{\pi^2 N}{12z} - \frac{Nz}{24} - \boldsymbol{u}^T\boldsymbol{u} - \boldsymbol{b}^T\boldsymbol{u}\sqrt{z} - \log(2)\left(K+\frac{1}{2}\right) + \sum_{j=1}^{N}\xi\left(\frac{u_j}{\sqrt{z}}, z\right).$$

Finally, we obtain (3.3) by exponentiating. □

Now we use Proposition 3.1 to derive an asymptotic expansion for (3.1).

**Proposition 3.2.** *Assume the setup as in Proposition 3.1. If $y \ll \varepsilon^{\frac{1}{3}+\delta}$ for some $\delta > 0$, then we have, for $R \in \mathbb{N}$, uniformly in $\boldsymbol{u}$,*

$$\frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^{N}(q;q)_{n_j}} = \left(\frac{z}{\pi}\right)^{\frac{N}{2}} \frac{e^{\frac{\pi^2 N}{12z}} e^{-\boldsymbol{u}^T\boldsymbol{u}}}{2^{K+\frac{1}{2}}} \sum_{r=0}^{R-1} C_r(\boldsymbol{u}) z^{\frac{r}{2}} + O\left(\varepsilon^{3R\delta_1}\right) z^{\frac{N}{2}} e^{\frac{\pi^2 N}{12z}} e^{-\boldsymbol{u}^T\boldsymbol{u}},$$

*where $\delta_1 := \min\{\delta, \frac{2}{3}+\lambda\} > 0$. The $C_r(\boldsymbol{u})$ are defined as coefficients of the formal exponential*

$$\sum_{r\geq 0} C_r(\boldsymbol{u}) z^{\frac{r}{2}} := \exp(\phi(\boldsymbol{u},z)), \qquad \phi(\boldsymbol{u},z) := -\boldsymbol{b}^T\boldsymbol{u}\sqrt{z} - \frac{Nz}{24} + \sum_{j=1}^{N}\xi\left(\frac{u_j}{\sqrt{z}}, z\right), \tag{3.10}$$

*and $\xi(\nu, z)$ has the asymptotic expansion given in* (3.4).

*Proof.* Using (3.3) and (3.10), we have, as a formal power series,

$$\frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^{N}(q;q)_{n_j}} = \left(\frac{z}{\pi}\right)^{\frac{N}{2}} \frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}} e^{-\boldsymbol{u}^T\boldsymbol{u}} \sum_{r\geq 0} C_r(\boldsymbol{u}) z^{\frac{r}{2}}.$$

It remains to show that for $R \in \mathbb{N}$, uniformly in $\boldsymbol{u}$,

$$\left(\frac{z}{\pi}\right)^{\frac{N}{2}} \frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}} e^{-\boldsymbol{u}^T\boldsymbol{u}} \sum_{r\geq R} C_r(\boldsymbol{u}) z^{\frac{r}{2}} = O\left(\varepsilon^{3R\delta_1}\right) z^{\frac{N}{2}} e^{\frac{\pi^2 N}{12z}} e^{-\boldsymbol{u}^T\boldsymbol{u}}. \tag{3.11}$$

As $y \ll \varepsilon^{\frac{1}{3}+\delta}$, we have $z \asymp \varepsilon$. Since $\boldsymbol{n} \in \mathcal{N}_{\varepsilon,\lambda}$, we have $|\boldsymbol{\mu}| \leq \varepsilon^{\lambda}$, and it follows from (3.6) that $\nu_j \ll \varepsilon^{-\frac{2}{3}+\delta_1}$ and $|\boldsymbol{u}| \ll \varepsilon^{-\frac{1}{6}+\delta_1}$. Now we consider the asymptotic expansion of $\phi(\boldsymbol{u}, z)$. As $\psi_{\frac{1}{2}}(\nu_j, z)$ and $\xi(\nu_j, z)$ have the same asymptotic expansion (see the proof of Proposition 3.1), we use (2.2) and deduce for $S \geq 3$ that

$$\phi(\boldsymbol{u},z) = -\boldsymbol{b}^T\boldsymbol{u}\sqrt{z} - \frac{Nz}{24} - \sum_{j=1}^{N}\sum_{r=2}^{S-1}\left(B_r\left(-\frac{u_j}{\sqrt{z}}\right) - \delta_{r,2}\frac{u_j^2}{z}\right)\operatorname{Li}_{2-r}\left(\frac{1}{2}\right)\frac{z^{r-1}}{r!}$$
$$+ O\left(\varepsilon^{\frac{1}{3}(S-3)+S\delta_1}\right). \tag{3.12}$$

Uniformity in $\boldsymbol{u}$ follows as the implied constant in the error term is independent of $\boldsymbol{u}$.

Finally, we prove the proposition by showing that exponentiating (3.12) gives a well-defined asymptotic series. For this rewrite (3.12) as a formal power series in $z^{\frac{1}{2}}$

$$\phi(\boldsymbol{u},z) =: \sum_{m\geq 1} g_m(\boldsymbol{u}) z^{\frac{m}{2}}.$$

Then, since $\deg(B_m) = m$, it follows that $\deg(g_m) \leq m+2$. For $m \in \mathbb{N}$, we have

$$g_m(\boldsymbol{u}) z^{\frac{m}{2}} \ll \varepsilon^{(m+2)\left(-\frac{1}{6}+\delta_1\right)+\frac{m}{2}} \ll \varepsilon^{\frac{m-1}{3}+(m+2)\delta_1}.$$

It follows that for all $M \in \mathbb{N}$, we have $\sum_{m=1}^{M} g_m(\boldsymbol{u}) z^{\frac{m}{2}} = O(\varepsilon^{3\delta_1})$ uniformly in $\boldsymbol{u}$. We exponentiate (3.12) and obtain a formal expression

$$\exp(\phi(\boldsymbol{u}, z)) = \exp\left(\sum_{m\geq 1} g_m(\boldsymbol{u}) z^{\frac{m}{2}}\right) = \prod_{m\geq 1}\sum_{k\geq 0} \frac{g_m(\boldsymbol{u})^k z^{\frac{mk}{2}}}{k!} =: \sum_{r\geq 0} C_r(\boldsymbol{u}) z^{\frac{r}{2}},$$

with $C_0(\boldsymbol{u}) = 1$. We claim that $\deg(C_r) \leq 3r$ for all $r \in \mathbb{N}$. To see this, we observe that

$$\deg(C_r) \leq \max_{\sum_\ell m_\ell = r} \left\{ \sum_{(m_1, m_2, \dots)} \deg(g_{m_\ell}) \right\}.$$

Here the maximum is taken over sequences $(m_1, m_2, \dots)$ of non-negative integers satisfying $\sum_\ell m_\ell = r$. But since $\deg(g_m) \leq m + 2 \leq 3m$ for $m \in \mathbb{N}$, we deduce that $\deg(C_r) \leq 3r$ for all $r \in \mathbb{N}$ as claimed. From $|\boldsymbol{u}| \ll \varepsilon^{-\frac{1}{6}+\delta_1}$ it follows that for all $R \in \mathbb{N}$, uniformly in $\boldsymbol{u}$,

$$\exp(\phi(\boldsymbol{u}, z)) = \sum_{r=0}^{R-1} C_r(\boldsymbol{u}) z^{\frac{r}{2}} + O\left(\varepsilon^{3R\delta_1}\right).$$

This proves (3.11), and hence the proposition. ☐

3.3. **Wider range estimates.** To establish an alternative asymptotic expansion for (3.1), which is valid on the major arc, let $\mathcal{A} := (1 + \frac{2^{-(1+iy)}}{1-2^{-(1+iy)}}) I_N$, with $I_N$ the $N \times N$ identity matrix, and

$$\Lambda(y) := N\left(\frac{\pi^2}{6} - \frac{\log(2)^2(1+iy)^2}{2} - \mathrm{Li}_2\left(2^{-(1+iy)}\right)\right).$$

**Proposition 3.3.** *Let $\boldsymbol{n} \in \mathcal{N}_{\varepsilon,\lambda}$ and $\boldsymbol{v} \in \mathbb{C}^N$ be such that for $1 \leq j \leq N$,*

$$n_j = \frac{\log(2)}{\varepsilon} + \frac{v_j}{\sqrt{z}}. \tag{3.13}$$

*Suppose that $y \ll \varepsilon^{-1-\frac{3\lambda}{2}+\delta}$ for some $\delta > 0$. Then we have*

$$\frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^{N}(q;q)_{n_j}} = 2^{-\left(K+\frac{1}{2}\right)(1+iy)}\left(1 - 2^{-(1+iy)}\right)^{-\frac{N}{2} - \frac{1}{\sqrt{z}}\sum_{j=1}^{N} v_j}\left(\frac{z}{2\pi}\right)^{\frac{N}{2}}$$
$$\times \exp\left(\frac{\Lambda(y)}{z} - \frac{Nz}{24} - \frac{\log(2)(1+iy)}{\sqrt{z}}\sum_{j=1}^{N} v_j - \frac{1}{2}\boldsymbol{v}^T\mathcal{A}\boldsymbol{v} - \boldsymbol{b}^T\boldsymbol{v}\sqrt{z}\right)\prod_{j=1}^{N}\exp\left(\xi_y\left(\frac{v_j}{\sqrt{z}}, z\right)\right),$$

*where, for $R \in \mathbb{N}$, $\xi_y(\nu, z)$ has an asymptotic expansion, as $z \to 0$,*

$$\xi_y(\nu, z) = -\sum_{r=2}^{R-1}\left(B_r(-\nu) - \delta_{r,2}\nu^2\right)\mathrm{Li}_{2-r}\left(2^{-(1+iy)}\right)\frac{z^{r-1}}{r!} + O\left(z^{R-1}\right). \tag{3.14}$$

**Remark.** *Note the difference between $\boldsymbol{u}$ and $\boldsymbol{v}$ in (3.2) and (3.13). Writing $\boldsymbol{n}$ in terms of $\boldsymbol{v}$ (instead of $\boldsymbol{u}$) allows to derive an asymptotic expansion which holds for a wider range in $y$.*

*Proof of Proposition 3.3.* By (3.13), we have $q^{n_j} = 2^{-(1+iy)} e^{-v_j\sqrt{z}}$. Define $\nu_j := \frac{v_j}{\sqrt{z}}$. Then we have

$$\nu_j = n_j - \frac{\log(2)}{\varepsilon} = O\left(\varepsilon^{\lambda}\right).$$

If $y \ll \varepsilon^{-1-\frac{3\lambda}{2}+\delta}$, then $z \ll \varepsilon^{-\frac{3\lambda}{2}+\delta_2}$, where $\delta_2 := \min\{1 + \frac{3\lambda}{2}, \delta\} > 0$ and $\nu_j z = o(1)$.

Define $\omega := 2^{-(1+iy)}$. Observing that $|\omega| = |2^{-(1+iy)}| < 1$, we may apply Lemma 2.2 and deduce

$$\operatorname{Log}\left(\left(q^{n_j+1};q\right)_\infty\right) = -\frac{\operatorname{Li}_2(\omega)}{z} - \left(\nu_j + \frac{1}{2}\right)\operatorname{Log}(1-\omega) - \frac{\nu_j^2 z}{2}\frac{\omega}{1-\omega} + \psi_\omega(\nu_j, z).$$

Using Lemma 2.1 and (3.5), we sum over $j$ and get, with $\mathcal{E}$ satisfying $\mathcal{E} \ll z^L$ for all $L \in \mathbb{N}$,

$$-\sum_{j=1}^{N} \operatorname{Log}((q;q)_{n_j}) = -\frac{N\operatorname{Li}_2(\omega)}{z} + \sum_{j=1}^{N}\left(-\left(\nu_j + \frac{1}{2}\right)\operatorname{Log}(1-\omega) - \frac{\nu_j^2 z}{2}\frac{\omega}{1-\omega} + \psi_\omega(\nu_j, z)\right)$$
$$+ \frac{\pi^2 N}{6z} - \frac{N}{2}\operatorname{Log}\left(\frac{2\pi}{z}\right) - \frac{Nz}{24} + N\mathcal{E}.$$

Now let $\xi_y(\nu_j, z) := \psi_\omega(\nu_j, z) + \mathcal{E}$. As we have $\mathcal{E} \ll z^L$ for all $L \in \mathbb{N}$, it follows that $\xi_y(\nu_j, z)$ has the same asymptotic expansion as $\psi_\omega(\nu_j, z)$. Next, using (1.3), we have

$$-H(\boldsymbol{n})z = -\frac{N\log(2)^2(1+iy)}{2\varepsilon} - \frac{\log(2)(1+iy)}{\sqrt{z}}\sum_{j=1}^{N} v_j - \frac{1}{2}\boldsymbol{v}^T\boldsymbol{v} - \log(2)\left(K + \frac{1}{2}\right)(1+iy) - \boldsymbol{b}^T\boldsymbol{v}\sqrt{z}.$$

The rest of the computation is analogous to that of Proposition 3.1. □

Now we use Proposition 3.3 to derive another asymptotic expansion for (3.1); the proof of the proposition follows mutatis mutandis the proof of Proposition 3.2 and is omitted here.

**Proposition 3.4.** *Assume the setup in Proposition 3.3. If $y \ll \varepsilon^{-1-\frac{3\lambda}{2}+\delta}$ for some $\delta > 0$, then we have an asymptotic expansion for $R \in \mathbb{N}$, uniformly in $\boldsymbol{v}$,*

$$\frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^{N}(q;q)_{n_j}} = 2^{-\left(K+\frac{1}{2}\right)(1+iy)}\left(1 - 2^{-(1+iy)}\right)^{-\frac{N}{2}}\left(\frac{z}{2\pi}\right)^{\frac{N}{2}} e^{\frac{\Lambda(y)}{z}}$$
$$\times e^{-\frac{1}{2}\boldsymbol{v}^T\mathcal{A}\boldsymbol{v} + \frac{1}{\sqrt{z}}\left(-\log(2)(1+iy) - \operatorname{Log}\left(1-2^{-(1+iy)}\right)\right)\sum_{j=1}^{N} v_j} \sum_{r=0}^{R-1} D_r(\boldsymbol{v}, y) z^{\frac{r}{2}}$$
$$+ z^{\frac{N}{2}} e^{\frac{\Lambda(y)}{z}} e^{-\frac{1}{2}\boldsymbol{v}^T\mathcal{A}\boldsymbol{v} + \frac{1}{\sqrt{z}}\left(-\log(2)(1+iy) - \operatorname{Log}\left(1-2^{-(1+iy)}\right)\right)\sum_{j=1}^{N} v_j} O\left(\varepsilon^{2R\delta_2}\right),$$

*where $\delta_2 := \min\{1 + \frac{3\lambda}{2}, \delta\} > 0$, $D_r(\boldsymbol{v}, y)$ are defined as coefficients of the formal exponential*

$$\sum_{r\geq 0} D_r(\boldsymbol{v}, y) z^{\frac{r}{2}} := \exp(\phi(\boldsymbol{v}, z)), \qquad \phi(\boldsymbol{v}, z) := -\boldsymbol{b}^T\boldsymbol{v}\sqrt{z} - \frac{Nz}{24} + \sum_{j=1}^{N}\xi_y\left(\frac{v_j}{\sqrt{z}}, z\right),$$

*and $\xi_y(\frac{v_j}{\sqrt{z}}, z)$ has the asymptotic expansion given in* (3.14).

## 4. Error Estimates

In this section, we establish some error estimates which are used in Section 5.

**Proposition 4.1.** *For every $L \in \mathbb{N}$, as $\operatorname{Re}(z) \to 0$ in the right half-plane, we have*

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^N\setminus\mathcal{N}_{\varepsilon,\lambda}} \left|\frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^{N}(q;q)_{n_j}}\right| \ll \varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}.$$

*Proof.* First suppose that $z=\varepsilon$ is real, i.e., $y=0$. Fix $\boldsymbol{n}\in\mathbb{N}_0^N\setminus\mathcal{N}_{\varepsilon,\lambda}$, and let $\boldsymbol{\mu}\in\mathbb{R}^N$ be such that $n_j=\frac{\log(2)}{\varepsilon}+\mu_j$ for $1\le j\le N$. We use (3.5). Observe that $q^{n_j}=\frac{1}{2}e^{-\mu_j\varepsilon}$. Using the power series expansion $\log(1-x)=-\sum_{n\ge1}\frac{x^n}{n}$, we expand

$$\sum_{j=1}^N \log\left(\left(q^{n_j+1};q\right)_\infty\right)=-\sum_{j=1}^N\sum_{k\ge1}\frac{1}{k\cdot 2^k}\frac{e^{-k\mu_j\varepsilon}}{e^{k\varepsilon}-1}. \tag{4.1}$$

Since $e^x>1+x$ for all $x\neq0$ and $\frac{1}{e^x-1}>\frac{1}{x}-\frac{1}{2}$ for $x>0$, we have

$$\frac{e^{-k\mu_j\varepsilon}}{e^{k\varepsilon}-1}>(1-k\mu_j\varepsilon)\left(\frac{1}{k\varepsilon}-\frac{1}{2}\right)=\frac{1}{k\varepsilon}-\left(\mu_j+\frac{1}{2}\right)+\frac{k\mu_j\varepsilon}{2}.$$

Plugging this into (4.1) gives

$$\sum_{j=1}^N \log\left(\left(q^{n_j+1};q\right)_\infty\right)<-N\operatorname{Li}_2\left(\frac{1}{2}\right)\frac{1}{\varepsilon}+\sum_{j=1}^N\left(\log(2)\left(\mu_j+\frac{1}{2}\right)-\frac{\mu_j\varepsilon}{2}\right). \tag{4.2}$$

Combining (4.2) and Lemma 2.1 yields for $L\in\mathbb{N}$, as $\varepsilon\to0$,

$$\log\left(\frac{e^{-H(\boldsymbol{n})\varepsilon}}{\prod_{j=1}^N(q;q)_{n_j}}\right)<-H(\boldsymbol{n})\varepsilon+\frac{\pi^2N}{6\varepsilon}-\operatorname{Li}_2\left(\tfrac{1}{2}\right)\frac{N}{\varepsilon}-\frac{N}{2}\log\left(\frac{\pi}{\varepsilon}\right)-\frac{N\varepsilon}{24}+\left(\log(2)-\frac{\varepsilon}{2}\right)\sum_{j=1}^N\mu_j+N\mathcal{E}. \tag{4.3}$$

Next we compute

$$-H(\boldsymbol{n})\varepsilon=-\frac{\log(2)^2N}{2\varepsilon}-\log(2)\sum_{j=1}^N\mu_j-\boldsymbol{\mu}^T\boldsymbol{\mu}\frac{\varepsilon}{2}-\log(2)\sum_{j=1}^N b_j-\boldsymbol{b}^T\boldsymbol{\mu}\varepsilon. \tag{4.4}$$

Plugging (4.4) into (4.3) and using (2.1) gives

$$\log\left(\frac{e^{-H(\boldsymbol{n})\varepsilon}}{\prod_{j=1}^N(q;q)_{n_j}}\right)<\frac{\pi^2N}{12\varepsilon}+T, \tag{4.5}$$

where

$$T:=-\frac{N}{2}\log\left(\frac{\pi}{\varepsilon}\right)-\frac{N\varepsilon}{24}-\frac{\varepsilon}{2}\sum_{j=1}^N\mu_j-\boldsymbol{\mu}^T\boldsymbol{\mu}\frac{\varepsilon}{2}-\log(2)\sum_{j=1}^N b_j-\boldsymbol{b}^T\boldsymbol{\mu}\varepsilon+N\mathcal{E}.$$

If $\delta>0$, $l<-\frac{1}{2}-\delta$, and $|\boldsymbol{\mu}|>\varepsilon^l$, then $T$ is dominated by $-\boldsymbol{\mu}^T\boldsymbol{\mu}\frac{\varepsilon}{2}$ and there exists $\varepsilon_0>0$, independent of $l$, such that $T<-\frac{1}{4}\varepsilon^{2l+1}$ for $\varepsilon<\varepsilon_0$. Plugging this into (4.5) shows, for $\varepsilon<\varepsilon_0$, that

$$\frac{e^{-H(\boldsymbol{n})\varepsilon}}{\prod_{j=1}^N(q;q)_{n_j}}<e^{\frac{\pi^2N}{12\varepsilon}-\frac{\varepsilon^{2l+1}}{4}}. \tag{4.6}$$

Now consider the sum

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^N\setminus\mathcal{N}_{\varepsilon,\lambda}}\frac{e^{-H(\boldsymbol{n})\varepsilon}}{\prod_{j=1}^N(q;q)_{n_j}}=\sum_{r\ge1}\sum_{\substack{\boldsymbol{n}\in\mathbb{N}_0^N\setminus\mathcal{N}_{\varepsilon,\lambda}\\ \varepsilon^{-r+1}<|\boldsymbol{\mu}|\le\varepsilon^{-r}}}\frac{e^{-H(\boldsymbol{n})\varepsilon}}{\prod_{j=1}^N(q;q)_{n_j}}=:\sum_{r\ge1}\mathcal{R}(r). \tag{4.7}$$

For $\mathcal{R}(1)$, the sum is over $\boldsymbol{n}\in\mathbb{N}_0^N\setminus\mathcal{N}_{\varepsilon,\lambda}$ with $1<|\boldsymbol{\mu}|\le\varepsilon^{-1}$. Note that $\boldsymbol{n}\in\mathbb{N}_0^N\setminus\mathcal{N}_{\varepsilon,\lambda}$ implies $|\boldsymbol{\mu}|>\varepsilon^\lambda$, so we are really summing the terms with $\varepsilon^\lambda<|\boldsymbol{\mu}|\le\varepsilon^{-1}$. From (4.6), the summands in $\mathcal{R}(1)$ are bounded by $e^{\frac{\pi^2N}{12\varepsilon}-\frac{\varepsilon^{2\lambda+1}}{4}}$. Since $\mathcal{R}(1)$ contains $O(\varepsilon^{-N})$ terms, we conclude that for $\varepsilon<\varepsilon_0$

$$\mathcal{R}(1)\ll\frac{e^{\frac{\pi^2N}{12\varepsilon}-\frac{\varepsilon^{2\lambda+1}}{4}}}{\varepsilon^N}. \tag{4.8}$$

Now consider the summands in $\mathcal{R}(r)$ for $r \geq 2$. Since $|\boldsymbol{\mu}| > \varepsilon^{-r+1}$, it follows from (4.6) that the summand is bounded by $e^{\frac{\pi^2 N}{12\varepsilon} - \frac{\varepsilon^{3-2r}}{4}}$. As $\mathcal{R}(r)$ contains $O(\varepsilon^{-Nr})$ terms, we conclude that for $\varepsilon < \varepsilon_0$,

$$\mathcal{R}(r) \ll \frac{e^{\frac{\pi^2 N}{12\varepsilon} - \frac{\varepsilon^{3-2r}}{4}}}{\varepsilon^{Nr}}. \tag{4.9}$$

Plugging (4.8) and (4.9) into (4.7), we deduce that

$$\sum_{\boldsymbol{n} \in \mathbb{N}_0^N \setminus \mathcal{N}_{\varepsilon,\lambda}} \frac{e^{-H(\boldsymbol{n})\varepsilon}}{\prod_{j=1}^N (q;q)_{n_j}} = \sum_{r \geq 1} \mathcal{R}(r) \ll e^{\frac{\pi^2 N}{12\varepsilon}} \left( \frac{e^{-\varepsilon^{\frac{2\lambda+1}{4}}}}{\varepsilon^N} + \sum_{r \geq 2} \frac{e^{-\frac{\varepsilon^{3-2r}}{4}}}{\varepsilon^{Nr}} \right) \ll \varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}$$

for all $L \in \mathbb{N}$ as $\varepsilon \to 0$. This establishes the statement for $z$ real.

The general case follows from the trivial bound

$$\left| \frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^N (q;q)_{n_j}} \right| \leq \frac{e^{-H(\boldsymbol{n})\varepsilon}}{\prod_{j=1}^N (|q|\,;|q|)_{n_j}}. \qquad \square$$

The following proposition helps to establish the minor arc estimate for (1.2).

**Proposition 4.2.** *Let $\boldsymbol{n} \in \mathcal{N}_{\varepsilon,\lambda}$, and suppose that $\varepsilon^{-\delta} < |y| \leq \frac{\pi}{\varepsilon}$ for some $\delta > 0$ sufficiently small. Then we have, for all $L \in \mathbb{N}$,*

$$\frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^N (q;q)_{n_j}} \ll \varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}.$$

*Proof.* As $q = e^{-z} = e^{-\varepsilon(1+iy)}$, the assumption $\varepsilon^{-\delta} < |y| \leq \frac{\pi}{\varepsilon}$ implies that $\varepsilon^{1-\delta} < |\operatorname{Arg}(q)| \leq \pi$. For $1 \leq j \leq N$, define the sets

$$\mathcal{Q}_j := \left\{ q^k : 1 \leq k \leq n_j \right\}, \qquad \mathcal{Q}_j^- := \left\{ s \in \mathcal{Q}_j : \operatorname{Re}(s) \leq 0 \right\}, \qquad \mathcal{Q}_j^+ := \mathcal{Q}_j \setminus \mathcal{Q}_j^-.$$

If $\varepsilon^{1-\delta} < |\operatorname{Arg}(q)| \leq \pi$ and $\boldsymbol{n} \in \mathcal{N}_{\varepsilon,\lambda}$, then one can show that there exists $C > 0$, independent of $j$, such that $|\mathcal{Q}_j^-| \geq C n_j$ for $\varepsilon > 0$ sufficiently small.

Next we give an upper bound for $|(q;q)_{n_j}^{-1}|$. For this, we note that for $s \in \mathbb{C}$, we have $|1-s| > 1$ if $\operatorname{Re}(s) < 0$. It follows that

$$\left| \frac{1}{(q;q)_{n_j}} \right| \leq \prod_{\substack{1 \leq k \leq n_j \\ q^k \in \mathcal{Q}_j^+}} \frac{1}{|1-q^k|} \leq \prod_{\substack{1 \leq k \leq n_j \\ q^k \in \mathcal{Q}_j^+}} \frac{1}{1-|q|^k}.$$

As shown above for $\varepsilon > 0$ sufficiently small, we have $|\mathcal{Q}_j^-| \geq C n_j$. It follows that the number of terms in the product is bounded above by $|\mathcal{Q}_j^+| \leq (1-C) n_j$. As $(1-|q|^k)^{-1}$ decreases as $k$ increases, we obtain an upper bound

$$\left| \frac{1}{(q;q)_{n_j}} \right| \leq \prod_{\substack{1 \leq k \leq n_j \\ q^k \in \mathcal{Q}_j^+}} \frac{1}{1-|q|^k} \leq \prod_{k=1}^{\lfloor (1-C) n_j \rfloor} \frac{1}{1-|q|^k} = \frac{1}{(|q|;|q|)_{\lfloor (1-C) n_j \rfloor}}.$$

This yields

$$\left| \frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^N (q;q)_{n_j}} \right| \leq \frac{e^{-H(\boldsymbol{n})\varepsilon}}{\prod_{j=1}^N (|q|;|q|)_{\lfloor (1-C) n_j \rfloor}}.$$

Finally, if $\boldsymbol{n} \in \mathcal{N}_{\varepsilon,\lambda}$ and $\varepsilon > 0$ is sufficiently small, then we have

$$\frac{e^{-H(\boldsymbol{n})\varepsilon}}{\prod_{j=1}^N (|q|;|q|)_{\lfloor (1-C)n_j \rfloor}} \leq \frac{e^{-H(\lfloor (1-C)n_1\rfloor,\ldots,\lfloor (1-C)n_N\rfloor)\varepsilon}}{\prod_{j=1}^N (|q|;|q|)_{\lfloor (1-C)n_j \rfloor}} = o\left(\varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}\right)$$

for all $L \in \mathbb{N}$ by Proposition 4.1, since $(\lfloor (1-C)n_1\rfloor, \ldots, \lfloor (1-C)n_N\rfloor) \notin \mathcal{N}_{\varepsilon,\lambda}$. □

## 5. Asymptotics for $d^{[K]}_{\alpha,\beta;N}(n)$

We split the sum $\mathcal{D}^{[K]}_{\alpha,\beta;N}$ as

$$\mathcal{D}^{[K]}_{\alpha,\beta;N}(q) = \sum_{\boldsymbol{\ell} \in (\mathbb{Z}/N\mathbb{Z})^N} \mathcal{D}^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(q), \qquad \mathcal{D}^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(q) := \sum_{\boldsymbol{n} \in \mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}}} \frac{q^{NH(\boldsymbol{n})}}{\prod_{j=1}^N \left(q^N;q^N\right)_{n_j}}, \tag{5.1}$$

where $\mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}} := \{\boldsymbol{n} \in \mathcal{S}_{\alpha,\beta} : \boldsymbol{n} \equiv \boldsymbol{\ell} \pmod{N}\}$. We always pick the representative for $\boldsymbol{\ell} \in (\mathbb{Z}/N\mathbb{Z})^N$ with $0 \leq \ell_j < N$ $(1 \leq j \leq N)$. Let $\zeta_N := e^{\frac{2\pi i}{N}}$. Then we have

$$\mathcal{D}^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(\zeta_N e^{-z}) = \zeta_N^{NH(\boldsymbol{\ell})} f^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z),$$

where

$$f^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) := \sum_{\boldsymbol{n} \in \mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}}} \frac{e^{-NH(\boldsymbol{n})z}}{\prod_{j=1}^N \left(e^{-Nz};e^{-Nz}\right)_{n_j}}.$$

It is more convenient to investigate

$$g^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) := f^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}\left(\frac{z}{N}\right) = \sum_{\boldsymbol{n} \in \mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}}} \frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^N \left(e^{-z};e^{-z}\right)_{n_j}}.$$

We split

$$g^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) = g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) + g^{[K,2]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z), \tag{5.2}$$

where

$$g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) := \sum_{\boldsymbol{n} \in \mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{N}_{\varepsilon,\lambda}} \frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^N (e^{-z};e^{-z})_{n_j}}, \quad g^{[K,2]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) := \sum_{\boldsymbol{n} \in \mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}} \setminus \mathcal{N}_{\varepsilon,\lambda}} \frac{e^{-H(\boldsymbol{n})z}}{\prod_{j=1}^N (e^{-z};e^{-z})_{n_j}}.$$

**Proposition 5.1.** *If $y \ll \varepsilon^{1+\lambda+\delta}$ for some $\delta > 0$, then we have, for $R \in \mathbb{N}$ as $z \to 0$,*

$$g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) = \frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}} \sum_{r=0}^{R-1} E_{\boldsymbol{\ell},r} z^{\frac{r}{2}} + O\left(\varepsilon^{N\left(\lambda+\frac{1}{2}\right)+3R\left(\lambda+\frac{2}{3}\right)} e^{\frac{\pi^2 N}{12\varepsilon}}\right),$$

*where the $E_{\boldsymbol{\ell},r}$ are explicitly computable. In particular, for every $L \in \mathbb{N}$, we have as, $z \to 0$, with $R_0 = R_0(L) := \lceil \frac{L-N(\lambda+\frac{1}{2})}{3\lambda+2} \rceil$*

$$g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) = \frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}} \sum_{r=0}^{R_0-1} E_{\boldsymbol{\ell},r} z^{\frac{r}{2}} + O\left(\varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}\right).$$

We prove Proposition 5.1 in a series of lemmas. Let $\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}$ be the bijective image of $\mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}}$ under $\boldsymbol{n} \mapsto \boldsymbol{u}$ given in (3.2), and $\mathcal{U}_{\varepsilon,\lambda}$ the bijective image of $\mathcal{N}_{\varepsilon,\lambda}$ under the same map. Since $\boldsymbol{n} \mapsto \boldsymbol{u}$ takes $\mathbb{R}^N$ to $(-\frac{\log(2)}{\sqrt{z}} + \mathbb{R}\sqrt{z})^N$, we have $\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \subseteq (-\frac{\log(2)}{\sqrt{z}} + \mathbb{R}\sqrt{z})^N \subseteq \mathbb{C}^N$.

**Lemma 5.2.** *If $y \ll \varepsilon^{1+\lambda+\delta}$ for some $\delta > 0$ and $P(\boldsymbol{u})$ is a polynomial in $\boldsymbol{u}$, then, as $z \to 0$,*

$$\sum_{\boldsymbol{u}\in\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\setminus\mathcal{U}_{\varepsilon,\lambda}} \left|P(\boldsymbol{u})e^{-\boldsymbol{u}^T\boldsymbol{u}}\right| = O\left(\varepsilon^L\right), \quad \textit{and} \quad \int_{\left(-\frac{\log(2)}{\sqrt{z}}+\mathbb{R}\sqrt{z}\right)^N\setminus\mathcal{U}_{\varepsilon,\lambda}} \left|P(\boldsymbol{u})e^{-\boldsymbol{u}^T\boldsymbol{u}}\right| \boldsymbol{du} = O\left(\varepsilon^L\right).$$

*for all $L \in \mathbb{N}$.*

*Proof.* Let $\boldsymbol{\mu} \in \mathbb{R}^N$ be such that $n_j = \frac{\log(2)}{\varepsilon} + \mu_j$ for $1 \le j \le N$. We may rewrite

$$u_j = \frac{\log(2)\sqrt{z}}{\varepsilon}\left(1 - \frac{1}{1+iy}\right) + \mu_j\sqrt{z}. \tag{5.3}$$

For $y \ll \varepsilon^{1+\lambda+\delta}$, the first term in (5.3) has size $\ll \varepsilon^{\frac{1}{2}+\lambda+\delta}$. This implies that

$$\operatorname{Im}(u_j) \ll \varepsilon^{\frac{1}{2}+\lambda+\delta} + \varepsilon^{\frac{3}{2}+\lambda+\delta}\mu_j.$$

Meanwhile, if $\mu_j \gg \varepsilon^\lambda$, then we have $\operatorname{Re}(u_j) \gg \varepsilon^{\frac{1}{2}}\mu_j$. In particular, this implies that $\operatorname{Im}(u_j) \ll \varepsilon^\delta \operatorname{Re}(u_j)$. As $\boldsymbol{n} \notin \mathcal{N}_{\varepsilon,\lambda}$, we have $|\boldsymbol{\mu}| > \varepsilon^\lambda$, and in particular there is some $j$ such that $\mu_j \gg \varepsilon^\lambda$ for some $j$. Thus $-\boldsymbol{u}^T\boldsymbol{u}$ has negative real part of size $\gg \varepsilon^{1+2\lambda}$ as $\varepsilon \to 0$. As $1 + 2\lambda < 0$, this implies

$$\sum_{\boldsymbol{u}\in\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\setminus\mathcal{U}_{\varepsilon,\lambda}} \left|P(\boldsymbol{u})e^{-\boldsymbol{u}^T\boldsymbol{u}}\right| = O\left(\varepsilon^L\right), \quad \text{and} \quad \int_{\left(-\frac{\log(2)}{\sqrt{z}}+\mathbb{R}\sqrt{z}\right)^N\setminus\mathcal{U}_{\varepsilon,\lambda}} \left|P(\boldsymbol{u})e^{-\boldsymbol{u}^T\boldsymbol{u}}\right| \boldsymbol{du} = O\left(\varepsilon^L\right)$$

for all $L \in \mathbb{N}$, noting that $P(\boldsymbol{u})$ only has polynomial growth. ◻

Now we prove a bound on the size of the exponential factors appearing in Proposition 3.2.

**Lemma 5.3.** *If $y \ll \varepsilon^{1+\lambda+\delta}$ for some $\delta > 0$, and $\boldsymbol{u} \in \mathcal{U}_{\varepsilon,\lambda}$, then we have, as $z \to 0$,*

$$\left|e^{\frac{\pi^2 N}{12z}} e^{-\boldsymbol{u}^T\boldsymbol{u}}\right| \le e^{\frac{\pi^2 N}{12\varepsilon}}.$$

*Proof.* We first estimate the real and the imaginary parts of $\sqrt{z}$. Writing $\sqrt{z} := \varepsilon_0(1+iy_0)$, we have $\varepsilon = \varepsilon_0^2(1-y_0^2)$ and $y = \frac{2y_0}{1-y_0^2}$. Since $\varepsilon > 0$, we have $1 - y_0^2 > 0$, thus $|y_0| < 1$. So $0 < 1 - y_0^2 < 1$, and $y = \frac{2y_0}{1-y_0^2} \ge 2y_0$. Since $y \ll \varepsilon^{1+\lambda+\delta}$, we have $1-y_0^2 = 1+O(\varepsilon^{2+2\lambda+2\delta})$, and $\varepsilon_0 = \varepsilon^{\frac{1}{2}}(1+O(\varepsilon^{2+2\lambda+2\delta}))$.

We next bound $e^{\operatorname{Re}(-\boldsymbol{u}^T\boldsymbol{u})}$. For this, we compute

$$\operatorname{Re}\left(-\boldsymbol{u}^T\boldsymbol{u}\right) = \sum_{j=1}^N \operatorname{Re}\left(-u_j^2\right) = \sum_{j=1}^N \left(\operatorname{Im}(u_j)^2 - \operatorname{Re}(u_j)^2\right). \tag{5.4}$$

Since

$$u_j = -\frac{\log(2)}{\varepsilon_0}\left(1 - iy_0 - y_0^2 + O\left(y_0^3\right)\right) + n_j\varepsilon_0(1+iy_0),$$

it follows that

$$\operatorname{Im}(u_j) = \frac{\log(2)}{\varepsilon_0}\left(y_0 + O\left(y_0^3\right)\right) + n_j\varepsilon_0 y_0,$$

$$\operatorname{Im}(u_j)^2 = \frac{\log(2)^2}{\varepsilon_0^2}\left(y_0^2 + O\left(y_0^3\right)\right) + n_j\log(2)\left(y_0^2 + O\left(y_0^3\right)\right) + n_j^2\varepsilon_0^2 y_0^2. \tag{5.5}$$

Since $\boldsymbol{u} \in \mathcal{U}_{\varepsilon,\lambda}$, we deduce that $n_j = \frac{1}{\varepsilon}(\log(2) + O(\varepsilon^{1+\lambda}))$. Plugging this into (5.5), we get

$$\operatorname{Im}(u_j)^2 = \frac{\log(2)^2}{\varepsilon_0^2}\left(y_0^2 + O\left(y_0^3\right)\right) + \frac{1}{\varepsilon}\left(\log(2) + O\left(\varepsilon^{1+\lambda}\right)\right)\log(2)\left(y_0^2 + O\left(y_0^3\right)\right)$$
$$+ \frac{1}{\varepsilon^2}\left(\log(2) + O\left(\varepsilon^{1+\lambda}\right)\right)^2 \varepsilon_0^2 y_0^2.$$

Using $\varepsilon = \varepsilon_0^2(1-y_0^2)$, $(1-y_0^2)^{-1} = 1 + y_0^2 + O(y_0^4)$, and $y_0 \ll \varepsilon^{1+\lambda+\delta}$, we may expand

$$\begin{aligned}\operatorname{Im}(u_j)^2 &= \frac{\log(2)^2\left(1-y_0^2\right)}{\varepsilon}\left(y_0^2 + O\left(y_0^3\right)\right) + \frac{1}{\varepsilon}\left(\log(2) + O\left(\varepsilon^{1+\lambda}\right)\right)\log(2)\left(y_0^2 + O\left(y_0^3\right)\right) \\ &\qquad + \frac{1}{\varepsilon}\left(\log(2) + O\left(\varepsilon^{1+\lambda}\right)\right)^2 y_0^2\left(1 + y_0^2 + O\left(y_0^4\right)\right) \\ &= \frac{\log(2)^2 y_0^2}{\varepsilon}(1 + O(y_0)) + \frac{\log(2)^2 y_0^2}{\varepsilon}\left(1 + O\left(\varepsilon^{1+\lambda}\right)\right) + \frac{\log(2)^2 y_0^2}{\varepsilon}\left(1 + O\left(\varepsilon^{1+\lambda}\right)\right) \\ &= \frac{y_0^2}{\varepsilon}\left(3\log(2)^2 + O\left(\varepsilon^{1+\lambda}\right)\right) \leq \frac{y^2}{\varepsilon}\left(\frac{3}{4}\log(2)^2 + O\left(\varepsilon^{1+\lambda}\right)\right).\end{aligned}$$

It follows by (5.4) that

$$\operatorname{Re}\left(-\boldsymbol{u}^T\boldsymbol{u}\right) \leq \frac{Ny^2}{\varepsilon}\left(\frac{3}{4}\log(2)^2 + O\left(\varepsilon^{1+\lambda}\right)\right). \tag{5.6}$$

On the other hand, we have

$$\operatorname{Re}\left(\frac{\pi^2 N}{12z}\right) = \frac{\pi^2 N}{12\varepsilon}\operatorname{Re}\left(\frac{1}{1+iy}\right) = \frac{\pi^2 N}{12\varepsilon}\left(1 - y^2 + O\left(y^4\right)\right).$$

Combining this with (5.6) we obtain, for $\varepsilon > 0$ sufficiently small

$$\operatorname{Re}\left(\frac{\pi^2 N}{12z}\right) + \operatorname{Re}\left(-\boldsymbol{u}^T\boldsymbol{u}\right) = \frac{N}{\varepsilon}\left(\frac{\pi^2}{12} + y^2\left(\frac{3}{4}\log(2)^2 - \frac{\pi^2}{12} + O\left(\varepsilon^{1+\lambda}\right)\right) + O\left(y^4\right)\right) \leq \frac{\pi^2 N}{12\varepsilon}. \qquad \square$$

We next show the following lemma.

**Lemma 5.4.** *If $y \ll \varepsilon^{1+\lambda+\delta}$ for some $\delta > 0$, then for $R \in \mathbb{N}$ we have, as $z \to 0$,*

$$g_{\alpha,\beta;N,\boldsymbol{\ell}}^{[K,1]}(z) = \left(\frac{z}{\pi}\right)^{\frac{N}{2}} \frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}} \sum_{r=0}^{R-1} \sum_{\boldsymbol{u}\in\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{U}_{\varepsilon,\lambda}} e^{-\boldsymbol{u}^T\boldsymbol{u}} C_r(\boldsymbol{u}) z^{\frac{r}{2}} + O\left(\varepsilon^{N\left(\lambda+\frac{1}{2}\right)+3R\left(\lambda+\frac{2}{3}\right)} e^{\frac{\pi^2 N}{12\varepsilon}}\right).$$

*Proof.* Summing the asymptotics in Proposition 3.2, we obtain[3]

$$g_{\alpha,\beta;N,\boldsymbol{\ell}}^{[K,1]}(z) = \left(\frac{z}{\pi}\right)^{\frac{N}{2}} \frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}} \sum_{r=0}^{R-1} \sum_{\boldsymbol{u}\in\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{U}_{\varepsilon,\lambda}} e^{-\boldsymbol{u}^T\boldsymbol{u}} C_r(\boldsymbol{u}) z^{\frac{r}{2}} + z^{\frac{N}{2}} e^{\frac{\pi^2 N}{12z}} O\left(\varepsilon^{3R\left(\lambda+\frac{2}{3}\right)}\right) \sum_{\boldsymbol{u}\in\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{U}_{\varepsilon,\lambda}} e^{-\boldsymbol{u}^T\boldsymbol{u}}. \tag{5.7}$$

Since the summation over $\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}$ contains $\ll \varepsilon^{N\lambda}$ terms, it follows from Lemma 5.3 that the second term on the right-hand side of (5.7) is bounded as claimed. $\square$

For convenience, we set for $j \in \mathbb{N}_0$

$$G_j(\boldsymbol{u}) := e^{\frac{\pi^2 N}{12z}} C_j(\boldsymbol{u}) e^{-\boldsymbol{u}^T\boldsymbol{u}},$$

where the polynomial $C_j(\boldsymbol{u})$ is defined in (3.10); in particular, we have $\deg(C_j) \leq 3j$ (see the proof of Proposition 3.2). We next estimate $\sum_{\boldsymbol{u}\in\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{U}_{\varepsilon,\lambda}} G_j(\boldsymbol{u})$.

[3]Note that $\delta_1$ in Proposition 3.2 takes the value $\delta_1 = \lambda + \frac{2}{3}$. The error in (5.7) makes sense, because the error in Proposition 3.2 is uniform in $\boldsymbol{u}$.

**Lemma 5.5.** *If $y \ll \varepsilon^{1+\lambda+\delta}$ for some $\delta > 0$, then, for $R \in \mathbb{N}$, we have, as $z \to 0$,*

$$\sum_{\boldsymbol{u} \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}} G_j(\boldsymbol{u}) = e^{\frac{\pi^2 N}{12z}} \sum_{r=-N}^{R-1} V_{j,r} z^{\frac{r}{2}} + O\left(\varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+2N+R-1)+\frac{N+R}{2}-1} e^{\frac{\pi^2 N}{12\varepsilon}}\right).$$

*Proof.* For the first step, we consider the sum over $u_\alpha$, the $\alpha$-th component of $\boldsymbol{u}$. To emphasize this component, we write $\boldsymbol{u}_{[\boldsymbol{1}]}$ to denote the remaining $N-1$ components of $\boldsymbol{u}$, and $\boldsymbol{u} = (\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha)$. Note that this notation does not say that $u_\alpha$ is the $N$-th component of $\boldsymbol{u}$.

We decompose

$$\sum_{\boldsymbol{u} \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}} G_j(\boldsymbol{u}) = \sum_{\substack{\boldsymbol{u}_{[\boldsymbol{1}]} \in \mathbb{C}^{N-1} \\ \exists u_\alpha \in \mathbb{C}, \left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha\right) \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}}} \sum_{\substack{u_\alpha \in \mathfrak{u}_\alpha\left(\boldsymbol{u}_{[\boldsymbol{1}]}\right) \\ \left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha\right) \in \mathcal{U}_{\varepsilon,\lambda}}} G_j\left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha\right), \tag{5.8}$$

where

$$\mathfrak{u}_\alpha\left(\boldsymbol{u}_{[\boldsymbol{1}]}\right) := \left\{u_\beta + t\sqrt{z} \ : \ t \in [\ell_\alpha - \ell_\beta]_N + N\mathbb{N}_0\right\}.$$

If $\boldsymbol{u}_{[\boldsymbol{1}]}$ is such that there exists $u_\alpha \in \mathbb{C}$ with $(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha) \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}$, then we have

$$\mathfrak{u}_\alpha\left(\boldsymbol{u}_{[\boldsymbol{1}]}\right) = \left\{u_\alpha \in \mathbb{C} : \left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha\right) \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\right\},$$

so the decomposition (5.8) makes sense. We evaluate the sum variable by variable, starting with the innermost sum over $u_\alpha$. By Lemma 5.2, we may extend the inner sum in (5.8) by removing the condition $(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha) \in \mathcal{U}_{\varepsilon,\lambda}$. This introduces an error term of size $O(\varepsilon^L)e^{\frac{\pi^2 N}{12z}}$ for all $L \in \mathbb{N}$, which is negligible. By Proposition 2.3 with $a = u_\beta + [\ell_\alpha - \ell_\beta]_N\sqrt{z}$, $z \mapsto N\sqrt{z}$, we obtain

$$\begin{aligned}\sum_{u_\alpha \in \mathfrak{u}_\alpha\left(\boldsymbol{u}_{[\boldsymbol{1}]}\right)} G_j\left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha\right) = \frac{1}{N\sqrt{z}} \int_{u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}}^{u_\beta+\sqrt{z}\infty} G_j\left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha\right) du_\alpha + \frac{1}{2} G_j\left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\beta + [\ell_\alpha - \ell_\beta]_N\sqrt{z}\right) \\ - \sum_{r=1}^{R} \frac{B_{2r} N^{2r-1} z^{r-\frac{1}{2}}}{(2r)!} G_j^{(2r-1)}\left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\beta + [\ell_\alpha - \ell_\beta]_N\sqrt{z}\right) \\ + O(1) z^R \int_{u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}}^{u_\beta+\sqrt{z}\infty} \left|G_j^{(2R+1)}\left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha\right)\right| du_\alpha.\end{aligned} \tag{5.9}$$

First we estimate the second integral in (5.9). By a direct calculation, we see that, for $k \in \mathbb{N}$

$$G_j^{(k)}\left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha\right) = e^{\frac{\pi^2 N}{12z}} P_k(\boldsymbol{u}) e^{-\boldsymbol{u}^T \boldsymbol{u}},$$

where the derivative is taken with respect to $u_\alpha$ and where $P_k(\boldsymbol{u})$ is a polynomial of degree at most $3j + k$. By Lemma 5.2, we may restrict the second integral in (5.9) to $\mathcal{U}_{\varepsilon,\lambda}$, introducing an error of size $O(\varepsilon^L)e^{\frac{\pi^2 N}{12z}}$ for all $L \in \mathbb{N}$. Now we consider the part of the integral over $\mathcal{U}_{\varepsilon,\lambda}$, which has measure $\ll \varepsilon^{\lambda+\frac{1}{2}}$. For $\boldsymbol{u} \in \mathcal{U}_{\varepsilon,\lambda}$, we use $|\boldsymbol{u}| \ll \varepsilon^{\lambda+\frac{1}{2}}$ and Lemma 5.3 to conclude that

$$G_j^{(k)}\left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha\right) \ll \varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+k)} e^{\frac{\pi^2 N}{12\varepsilon}}. \tag{5.10}$$

Hence

$$\int_{u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}}^{u_\beta+\sqrt{z}\infty} \left|G_j^{(2R+1)}\left(\boldsymbol{u}_{[\boldsymbol{1}]}, u_\alpha\right)\right| du_\alpha \ll \varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+2R+2)} e^{\frac{\pi^2 N}{12\varepsilon}}. \tag{5.11}$$

Next we consider the first integral in (5.9). We split

$$\int_{u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}}^{u_\beta+\sqrt{z}\infty} G_j\left(\boldsymbol{u_{[1]}},u_\alpha\right)du_\alpha=\int_{u_\beta}^{u_\beta+\sqrt{z}\infty} G_j\left(\boldsymbol{u_{[1]}},u_\alpha\right)du_\alpha-\int_{u_\beta}^{u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}} G_j\left(\boldsymbol{u_{[1]}},u_\alpha\right)du_\alpha.$$

We keep the first integral on the right-hand side. For the second integral, we apply Proposition 2.3 (with $a=u_\beta$, $z\mapsto[\ell_\alpha-\ell_\beta]_N\sqrt{z})$ to obtain

$$\begin{aligned}\int_{u_\beta}^{u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}} G_j\left(\boldsymbol{u_{[1]}},u_\alpha\right)du_\alpha&=\frac{[\ell_\alpha-\ell_\beta]_N\sqrt{z}}{2}\left(G_j\left(\boldsymbol{u_{[1]}},u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}\right)+G_j\left(\boldsymbol{u_{[1]}},u_\beta\right)\right)\\&-\sum_{r=1}^{R}\frac{B_{2r}[\ell_\alpha-\ell_\beta]_N^{2r}z^r}{(2r)!}\left(G_j^{(2r-1)}\left(\boldsymbol{u_{[1]}},u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}\right)-G_j^{(2r-1)}\left(\boldsymbol{u_{[1]}},u_\beta\right)\right)\\&+O(1)z^{R+\frac{1}{2}}\int_{u_\beta}^{u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}}\left|G_j^{(2R+1)}\left(\boldsymbol{u_{[1]}},u_\alpha\right)\right|du_\alpha.\end{aligned}\tag{5.12}$$

Using (5.10), we conclude

$$\int_{u_\beta}^{u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}}\left|G_j^{(2R+1)}\left(\boldsymbol{u_{[1]}},u_\alpha\right)\right|\ll\varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+2R+1)+\frac{1}{2}}e^{\frac{\pi^2N}{12\varepsilon}}.\tag{5.13}$$

It remains to consider terms of the form $G_j^{(k)}(\boldsymbol{u_{[1]}},u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z})$ appearing in (5.9) and (5.12). We apply Taylor's Theorem and rewrite

$$\begin{aligned}G_j^{(k)}\left(\boldsymbol{u_{[1]}},u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}\right)&=\sum_{r=0}^{R-1}\frac{[\ell_\alpha-\ell_\beta]_N^r z^{\frac{r}{2}}}{r!}G_j^{(k+r)}\left(\boldsymbol{u_{[1]}},u_\beta\right)\\&+\frac{1}{(R-1)!}\int_{u_\beta}^{u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}}G_j^{(k+R)}\left(\boldsymbol{u_{[1]}},u_\alpha\right)\left(u_\alpha-u_\beta\right)^{R-1}du_\alpha.\end{aligned}\tag{5.14}$$

Using (5.10), we conclude that

$$\int_{u_\beta}^{u_\beta+[\ell_\alpha-\ell_\beta]_N\sqrt{z}}G_j^{(k+R)}\left(\boldsymbol{u_{[1]}},u_\alpha\right)\left(u_\alpha-u_\beta\right)^{R-1}du_\alpha\ll\varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+k+R)+\frac{R}{2}}e^{\frac{\pi^2N}{12\varepsilon}}.\tag{5.15}$$

Combining (5.9), (5.12), (5.14), and applying the error estimates (5.11), (5.13), and (5.15), we write

$$\begin{aligned}\sum_{u_\alpha\in\mathfrak{u}_\alpha\left(\boldsymbol{u_{[1]}}\right)}G_j\left(\boldsymbol{u_{[1]}},u_\alpha\right)&=\frac{1}{N\sqrt{z}}\int_{u_\beta}^{u_\beta+\sqrt{z}\infty}G_j\left(\boldsymbol{u_{[1]}},u_\alpha\right)du_\alpha+\sum_{r=0}^{R-1}W_rG_j^{(r)}\left(\boldsymbol{u_{[1]}},u_\beta\right)z^{\frac{r}{2}}\\&+O\left(\varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+R+1)+\frac{R-1}{2}}e^{\frac{\pi^2N}{12\varepsilon}}\right),\end{aligned}\tag{5.16}$$

where

$$W_0=\frac{1}{2}-\frac{[\ell_\alpha-\ell_\beta]_N}{N},\tag{5.17}$$

$$\begin{aligned}W_r=\left(\frac{1}{2}-\frac{[\ell_\alpha-\ell_\beta]_N}{2N}\right)\frac{[\ell_\alpha-\ell_\beta]_N^r}{r!}&+\sum_{t=1}^{\lceil\frac{r}{2}\rceil}\frac{B_{2t}\left([\ell_\alpha-\ell_\beta]_N^{2t}-N^{2t}\right)[\ell_\alpha-\ell_\beta]_N^{r-2t+1}}{(2t)!(r-2t+1)!N}\\&-\delta_{r\equiv1\pmod 2}\frac{B_{r+1}[\ell_\alpha-\ell_\beta]_N^{r+1}}{(r+1)!N},\qquad(r\geq1).\end{aligned}$$

Next we consider the summation over the other variables. We set

$$G_{j,\alpha}\left(\boldsymbol{u}_{[1]}\right) := \sum_{u_\alpha \in \mathfrak{u}_\alpha\left(\boldsymbol{u}_{[1]}\right)} G_j\left(\boldsymbol{u}_{[1]}, u_\alpha\right).$$

Let $1 \le c \le N$, $c \ne \alpha$, and consider the summation over $u_c$. To emphasize the $c$-th and the $\alpha$-th entries of $\boldsymbol{u}$, we write $\boldsymbol{u}_{[2]}$ to denote the remaining $N-2$ components of $\boldsymbol{u}$, and write $\boldsymbol{u} = (\boldsymbol{u}_{[2]}, u_c, u_\alpha)$. We are again abusing notation and this does not mean that $u_c, u_\alpha$ are the final components of $\boldsymbol{u}$. We write $\boldsymbol{u}_{[1]} = (\boldsymbol{u}_{[2]}, u_c)$. We consider

$$\sum_{\boldsymbol{u} \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}} G_j(\boldsymbol{u}) = \sum_{\substack{\boldsymbol{u}_{[2]} \in \mathbb{C}^{N-2} \\ \exists u_c, u_\alpha \in \mathbb{C}, \left(\boldsymbol{u}_{[2]}, u_c, u_\alpha\right) \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}}} \sum_{\substack{u_c \in \mathfrak{u}_c,\ u_\alpha \in \mathfrak{u}_\alpha\left(\boldsymbol{u}_{[2]}, u_c\right) \\ \left(\boldsymbol{u}_{[2]}, u_c, u_\alpha\right) \in \mathcal{U}_{\varepsilon,\lambda}}} G_j\left(\boldsymbol{u}_{[1]}, u_\alpha\right),$$

where

$$\mathfrak{u}_c := \left\{-\frac{\log(2)}{\sqrt{z}} + t\sqrt{z} \ :\ t \in \ell_c + N\mathbb{N}_0\right\}.$$

Analogous to the summation over $u_\alpha$, we may use Lemma 5.2 to extend the sum by removing the condition $(\boldsymbol{u}_{[2]}, u_c, u_\alpha) \in \mathcal{U}_{\varepsilon,\lambda}$, introducing an error of size $O(\varepsilon^L)e^{\frac{\pi^2 N}{12z}}$ for all $L \in \mathbb{N}$, which is negligible. So we may consider instead the sum

$$\sum_{\substack{u_c \in \mathfrak{u}_c \\ u_\alpha \in \mathfrak{u}_\alpha\left(\boldsymbol{u}_{[2]}, u_c\right)}} G_j\left(\boldsymbol{u}_{[2]}, u_c, u_\alpha\right) = \sum_{u_c \in \mathfrak{u}_c} G_{j,\alpha}\left(\boldsymbol{u}_{[2]}, u_c\right).$$

Again, we apply Proposition 2.3, and write

$$\begin{aligned}\sum_{u_c \in \mathfrak{u}_c} G_{j,\alpha}\left(\boldsymbol{u}_{[2]}, u_c\right) = \frac{1}{N\sqrt{z}} \int_{-\frac{\log(2)}{\sqrt{z}} + \ell_c\sqrt{z}}^{-\frac{\log(2)}{\sqrt{z}} + \sqrt{z}\infty} G_{j,\alpha}\left(\boldsymbol{u}_{[2]}, u_c\right) du_c + \frac{1}{2} G_{j,\alpha}\left(\boldsymbol{u}_{[2]}, -\frac{\log(2)}{\sqrt{z}} + \ell_c\sqrt{z}\right) \\ - \sum_{r=1}^{R} \frac{B_{2r} N^{2r-1} z^{r-\frac{1}{2}}}{(2r)!} G_{j,\alpha}^{(2r-1)}\left(\boldsymbol{u}_{[2]}, -\frac{\log(2)}{\sqrt{z}} + \ell_c\sqrt{z}\right) \\ + O(1) z^R \int_{-\frac{\log(2)}{\sqrt{z}} + \ell_c\sqrt{z}}^{-\frac{\log(2)}{\sqrt{z}} + \sqrt{z}\infty} \left|G_{j,\alpha}^{(2R+1)}\left(\boldsymbol{u}_{[2]}, u_c\right)\right| du_c. \quad (5.18)\end{aligned}$$

Thanks to the exponential decay of $e^{-\boldsymbol{u}^T \boldsymbol{u}}$, we have, as $z \to 0$ for all $r \in \mathbb{N}_0$ and $L \in \mathbb{N}$,

$$G_{j,\alpha}^{(r)}\left(\boldsymbol{u}_{[2]}, -\frac{\log(2)}{\sqrt{z}} + \ell_c\sqrt{z}\right) = O\left(\varepsilon^L\right) e^{\frac{\pi^2 N}{12z}}, \quad \int_{-\frac{\log(2)}{\sqrt{z}} - \sqrt{z}\infty}^{-\frac{\log(2)}{\sqrt{z}} + \ell_c\sqrt{z}} G_{j,\alpha}\left(\boldsymbol{u}_{[2]}, u_c\right) du_c = O\left(\varepsilon^L\right) e^{\frac{\pi^2 N}{12z}}. \quad (5.19)$$

Therefore, the lower boundary of the first integral in (5.18) can be extended to $-\frac{\log(2)}{\sqrt{z}} - \sqrt{z}\infty$, and all the terms except for the main term can be ignored introducing an error term of size $O(\varepsilon^L)e^{\frac{\pi^2 N}{12z}}$ for all $L \in \mathbb{N}$. Meanwhile, the error term in (5.18) has size

$$z^R \int_{-\frac{\log(2)}{\sqrt{z}} + \ell_c\sqrt{z}}^{-\frac{\log(2)}{\sqrt{z}} + \sqrt{z}\infty} \left|G_{j,\alpha}^{(2R+1)}\left(\boldsymbol{u}_{[2]}, u_c\right)\right| du_c = O\left(\varepsilon^{\left(\lambda + \frac{1}{2}\right)(3j + 2R + 2) + R + \lambda} e^{\frac{\pi^2 N}{12\varepsilon}}\right). \quad (5.20)$$

By taking $R$ sufficiently large, it follows from (5.18), (5.19), and (5.20) that

$$\sum_{u_c \in \mathfrak{u}_c} G_{j,\alpha}\left(\boldsymbol{u}_{[2]}, u_c\right) = \frac{1}{N\sqrt{z}} \int_{-\frac{\log(2)}{\sqrt{z}} + \mathbb{R}\sqrt{z}} G_{j,\alpha}\left(\boldsymbol{u}_{[2]}, u_c\right) du_c + O\left(\varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}\right)$$

for all $L \in \mathbb{N}$. Using the same argument, we sum over the other coordinates, and obtain that

$$\sum_{\boldsymbol{u} \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}} G_j(\boldsymbol{u}) = \frac{z^{\frac{1-N}{2}}}{N^{N-1}} \int_{\left(-\frac{\log(2)}{\sqrt{z}}+\mathbb{R}\sqrt{z}\right)^{N-1}} G_{j,\alpha}\left(\boldsymbol{u}_{[\mathbf{1}]}\right) \boldsymbol{du}_{[\mathbf{1}]} + O\left(\varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}\right)$$

for all $L \in \mathbb{N}$. Again, we may use Lemma 5.2 to restrict the integral to

$$\mathcal{U}_{[\mathbf{1}]} := \left\{ \boldsymbol{u}_{[\mathbf{1}]} \in \left(-\frac{\log(2)}{\sqrt{z}} + \mathbb{R}\sqrt{z}\right)^{N-1} : \left|\boldsymbol{\mu}_{[\mathbf{1}]}\right| \leq \varepsilon^{\lambda} \right\},$$

where $\boldsymbol{\mu}_{[\mathbf{1}]}$ is the $(N-1)$-tuple associated to $\boldsymbol{u}_{[\mathbf{1}]}$ via (5.3), with a negligible error. So we may write

$$\sum_{\boldsymbol{u} \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}} G_j(\boldsymbol{u}) = \frac{z^{\frac{1-N}{2}}}{N^{N-1}} \int_{\mathcal{U}_{[\mathbf{1}]}} G_{j,\alpha}\left(\boldsymbol{u}_{[\mathbf{1}]}\right) \boldsymbol{du}_{[\mathbf{1}]} + O\left(\varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}\right).$$

Applying the asymptotic expansion (5.16), we get

$$\begin{aligned}\sum_{\boldsymbol{u} \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}} G_j(\boldsymbol{u}) &= \frac{z^{-\frac{N}{2}}}{N^N} \int_{\mathcal{U}_{[\mathbf{1}]}} \int_{u_\beta}^{u_\beta+\sqrt{z}\infty} G_j\left(\boldsymbol{u}_{[\mathbf{1}]}, u_\alpha\right) du_\alpha \boldsymbol{du}_{[\mathbf{1}]} \\ &+ \sum_{r=0}^{R-1} \frac{z^{\frac{1-N+r}{2}}}{N^{N-1}} W_r \int_{\mathcal{U}_{[\mathbf{1}]}} G_j^{(r)}\left(\boldsymbol{u}_{[\mathbf{1}]}, u_\beta\right) \boldsymbol{du}_{[\mathbf{1}]} + O\left(\varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+R+1)+\frac{R-1}{2}} e^{\frac{\pi^2 N}{12\varepsilon}}\right) \int_{\mathcal{U}_{[\mathbf{1}]}} \boldsymbol{du}_{[\mathbf{1}]}. \quad (5.21)\end{aligned}$$

Since $\mathcal{U}_{[\mathbf{1}]}$ has measure $\ll \varepsilon^{(N-1)(\lambda+\frac{1}{2})}$, the error term in (5.21) has size

$$O\left(\varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+R+N)+\frac{R-1}{2}} e^{\frac{\pi^2 N}{12\varepsilon}}\right).$$

Now, we may use Lemma 5.2 again, to extend the integrals in (5.21) to $(-\frac{\log(2)}{\sqrt{z}} + \mathbb{R}\sqrt{z})^{N-1}$

$$\begin{aligned}\sum_{\boldsymbol{u} \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}} G_j(\boldsymbol{u}) &= \frac{z^{-\frac{N}{2}}}{N^N} \int_{\left(-\frac{\log(2)}{\sqrt{z}}+\mathbb{R}\sqrt{z}\right)^{N-1}} \int_{u_\beta}^{u_\beta+\sqrt{z}\infty} G_j\left(\boldsymbol{u}_{[\mathbf{1}]}, u_\alpha\right) du_\alpha \boldsymbol{du}_{[\mathbf{1}]} \\ &+ \sum_{r=0}^{R-1} \frac{z^{\frac{1-N+r}{2}}}{N^{N-1}} W_r \int_{\left(-\frac{\log(2)}{\sqrt{z}}+\mathbb{R}\sqrt{z}\right)^{N-1}} G_j^{(r)}\left(\boldsymbol{u}_{[\mathbf{1}]}, u_\beta\right) \boldsymbol{du}_{[\mathbf{1}]} + O\left(\varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+R+N)+\frac{R-1}{2}} e^{\frac{\pi^2 N}{12\varepsilon}}\right).\end{aligned}$$

Finally, since $G_j^{(r)}(\boldsymbol{u}_{[\mathbf{1}]}, u_\alpha)$ is holomorphic and has rapid decay, we may shift the path and write

$$\begin{aligned}\sum_{\boldsymbol{u} \in \mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}} \cap \mathcal{U}_{\varepsilon,\lambda}} G_j(\boldsymbol{u}) &= \frac{z^{-\frac{N}{2}}}{N^N} \int_{\mathcal{R}_{\alpha,\beta;N}} G_j\left(\boldsymbol{u}\right) \boldsymbol{du} \\ &+ \sum_{r=0}^{R-1} \frac{z^{\frac{1-N+r}{2}}}{N^{N-1}} W_r \int_{\mathbb{R}^{N-1}} G_j^{(r)}\left(\boldsymbol{u}_{[\mathbf{1}]}, u_\beta\right) \boldsymbol{du}_{[\mathbf{1}]} + O\left(\varepsilon^{\left(\lambda+\frac{1}{2}\right)(3j+R+N)+\frac{R-1}{2}} e^{\frac{\pi^2 N}{12\varepsilon}}\right),\end{aligned}$$

where $\mathcal{R}_{\alpha,\beta;N} := \{\boldsymbol{u} \in \mathbb{R}^N : u_\alpha \geq u_\beta\}$. So we may set

$$V_{j,-N} := \frac{e^{-\frac{\pi^2 N}{12z}}}{N^N} \int_{\mathcal{R}_{\alpha,\beta;N}} G_j(\boldsymbol{u}) \boldsymbol{du}, \quad (5.22)$$

$$V_{j,r} := \frac{e^{-\frac{\pi^2 N}{12z}}}{N^{N-1}} W_{r+N-1} \int_{\mathbb{R}^{N-1}} G_j^{(r+N-1)}\left(\boldsymbol{u}_{[\mathbf{1}]}, u_\beta\right) \boldsymbol{du}_{[\mathbf{1}]} \qquad (r \geq -N+1). \quad (5.23)$$

Note that $V_{j,r}$ does not depend on $z$. ☐

We are now ready to prove Proposition 5.1.

*Proof of Proposition 5.1.* We estimate the following expression, occuring in Lemma 5.4:

$$\left(\frac{z}{\pi}\right)^{\frac{N}{2}} \frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}} \sum_{r=0}^{R-1} \sum_{\boldsymbol{u}\in\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{U}_{\varepsilon,\lambda}} e^{-\boldsymbol{u}^T\boldsymbol{u}} C_r(\boldsymbol{u}) z^{\frac{r}{2}} = \frac{1}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}} \sum_{r=0}^{R-1} z^{\frac{N+r}{2}} \sum_{\boldsymbol{u}\in\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{U}_{\varepsilon,\lambda}} G_r(\boldsymbol{u}).$$

By Lemma 5.5, we have

$$z^{\frac{N+r}{2}} \sum_{\boldsymbol{u}\in\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{U}_{\varepsilon,\lambda}} G_r(\boldsymbol{u}) = e^{\frac{\pi^2 N}{12z}} \sum_{j=0}^{R-1-r} V_{r,j-N} z^{\frac{r+j}{2}} + O\left(\varepsilon^{\left(\lambda+\frac{1}{2}\right)(2r+N+R-1)+\frac{N+R}{2}-1} e^{\frac{\pi^2 N}{12\varepsilon}}\right).$$

Summing over $r$ gives

$$\sum_{r=0}^{R-1} z^{\frac{N+r}{2}} \sum_{\boldsymbol{u}\in\mathcal{T}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{U}_{\varepsilon,\lambda}} G_r(\boldsymbol{u}) = e^{\frac{\pi^2 N}{12z}} \sum_{r=0}^{R-1}\sum_{j=0}^{r} V_{j,r-j-N} z^{\frac{r}{2}} + O\left(\varepsilon^{3(R-1)\left(\lambda+\frac{2}{3}\right)+N(\lambda+1)-\frac{1}{2}} e^{\frac{\pi^2 N}{12\varepsilon}}\right).$$

Since $N \geq 2$ and $-\frac{2}{3} < \lambda < -\frac{1}{2}$, the error term $O(\varepsilon^{N(\lambda+\frac{1}{2})+3R(\lambda+\frac{2}{3})} e^{\frac{\pi^2 N}{12\varepsilon}})$ from Lemma 5.4 dominates. Setting

$$E_{\boldsymbol{\ell},r} := \sum_{j=0}^{r} V_{j,r-j-N} \tag{5.24}$$

gives the proposition. ☐

Now we show that the asymptotic expansion above can also be applied for larger values of $|y|$, with negligible error. We require the following technical lemma about $\Lambda(y)$.

**Lemma 5.6.** *Let $s(y) := \mathrm{Re}(\frac{\Lambda(y)}{1+iy} - \frac{\pi^2 N}{12})$. Then we have the following:*

(1) *We have $s(y) \leq 0$ for all $y \in \mathbb{R}$, and the equality holds if and only if $y = 0$.*
(2) *For any $y_0 > 0$, there exists $d > 0$ such that $s(y) < -d$ for all $|y| \geq y_0$.*
(3) *We have, as $y \to 0$,*

$$s(y) = N\left(\log(2)^2 - \frac{\pi^2}{12}\right) y^2 + O\left(y^4\right).$$

*Proof.* (1) and (2) are easily verified, and (3) is obtained by evaluating the Taylor series at $y = 0$. ☐

Next we show that the asymptotic expansion in Proposition 5.1 gives a good approximation to $g^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z)$ in the range $y \ll \varepsilon^{-1-\frac{3\lambda}{2}+\delta}$, which covers the major arc.

**Proposition 5.7.** *If $y \ll \varepsilon^{-1-\frac{3\lambda}{2}+\delta}$, then, for every $L \in \mathbb{N}$,*

$$g^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) = \frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}} \sum_{r=0}^{R_0-1} E_{\boldsymbol{\ell},r} z^{\frac{r}{2}} + O\left(\varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}\right)$$

*as $z \to 0$, where $E_{\boldsymbol{\ell},r}$ and $R_0 = R_0(L)$ are as in Proposition 5.1.*

*Proof.* We split as in (5.2). By Proposition 4.1, we have, for all $L \in \mathbb{N}$ as $z \to 0$,

$$g^{[K,2]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) \ll \varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}.$$

So it remains to estimate $g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}$. The proposition follows from the following claim:

$$e^{-\frac{\pi^2 N}{12\varepsilon}}\left(g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z)-\frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}}\sum_{r=0}^{R_0-1}E_{\boldsymbol{\ell},r}z^{\frac{r}{2}}\right)=O\left(\varepsilon^L\right).$$

From Proposition 3.4, if $y \ll \varepsilon^{-1-\frac{3\lambda}{2}+\delta}$, then we have an asymptotic expansion

$$\begin{aligned}e^{-\frac{\pi^2 N}{12\varepsilon}}g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z)&=2^{-\left(K+\frac{1}{2}\right)(1+iy)}\left(1-2^{-(1+iy)}\right)^{-\frac{N}{2}}\left(\frac{z}{2\pi}\right)^{\frac{N}{2}}e^{\frac{1}{\varepsilon}\left(\frac{\Lambda(y)}{1+iy}-\frac{\pi^2 N}{12}\right)} \\ &\times\sum_{\boldsymbol{n}\in\mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{N}_{\varepsilon,\lambda}}e^{-\frac{1}{2}\boldsymbol{v}^T\mathcal{A}\boldsymbol{v}+\frac{1}{\sqrt{z}}\left(-\log(2)(1+iy)-\mathrm{Log}\left(1-2^{-(1+iy)}\right)\right)\sum_{j=1}^N v_j}\sum_{r=0}^{R_0-1}D_r(\boldsymbol{v},y)z^{\frac{r}{2}} \\ &+z^{\frac{N}{2}}e^{\frac{1}{\varepsilon}\left(\frac{\Lambda(y)}{1+iy}-\frac{\pi^2 N}{12}\right)}O\left(\varepsilon^{2R_0\delta_2}\right)\sum_{\boldsymbol{n}\in\mathcal{S}_{\alpha,\beta;N,\boldsymbol{\ell}}\cap\mathcal{N}_{\varepsilon,\lambda}}e^{-\frac{1}{2}\boldsymbol{v}^T\mathcal{A}\boldsymbol{v}+\frac{1}{\sqrt{z}}\left(-\log(2)(1+iy)-\mathrm{Log}\left(1-2^{-(1+iy)}\right)\right)\sum_{j=1}^N v_j}.\end{aligned} \tag{5.25}$$

We claim that if $\varepsilon^{\frac{1}{2}-\delta} \ll y \ll \varepsilon^{-1-\frac{3\lambda}{2}+\delta}$ for some $\delta>0$, then we have, for all $L\in\mathbb{N}$,

$$e^{-\frac{\pi^2 N}{12\varepsilon}}g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z)=O\left(\varepsilon^L\right). \tag{5.26}$$

To prove (5.26), it suffices to show that the exponent

$$\frac{1}{\varepsilon}\left(\frac{\Lambda(y)}{1+iy}-\frac{\pi^2 N}{12}\right)-\frac{1}{2}\boldsymbol{v}^T\mathcal{A}\boldsymbol{v}+\frac{1}{\sqrt{z}}\left(-\log(2)(1+iy)-\mathrm{Log}\left(1-2^{-(1+iy)}\right)\right)\sum_{j=1}^N v_j$$

has negative real part of size $\gg \varepsilon^{-\delta_0}$ for some $\delta_0>0$; the other factors in (5.25) are bounded as $z\to 0$. By Lemma 5.6 (1) the real part of $\frac{1}{\varepsilon}\,\mathrm{Re}(\frac{\Lambda(y)}{1+iy}-\frac{\pi^2 N}{12})=\frac{s(y)}{\varepsilon}$ is negative. So it suffices to show that this exponent has sufficiently large size and dominates other exponents with positive real parts.

First consider the case $\varepsilon^{-\delta+\frac{1}{2}} \ll y \ll 1$ for some (sufficiently small) $\delta>0$. By Lemma 5.6 (3), we find that

$$\frac{s(y)}{\varepsilon} \gg \frac{y^2}{\varepsilon} \gg \varepsilon^{-2\delta}$$

as $\varepsilon\to 0$. Meanwhile, by computing the Taylor series expansion we have, as $y\to 0$,

$$\mathrm{Re}\left(-\log(2)(1+iy)-\mathrm{Log}\left(1-2^{-(1+iy)}\right)\right) \ll y^2.$$

As $|\boldsymbol{\nu}| \ll \varepsilon^{\lambda}$ (see the proof of Proposition 3.3), we conclude that

$$\mathrm{Re}\left(\frac{1}{\sqrt{z}}\left(-\log(2)(1+iy)-\mathrm{Log}\left(1-2^{-(1+iy)}\right)\right)\sum_{j=1}^N v_j\right) \ll \varepsilon^{\lambda}y^2.$$

Next we consider the exponent $-\frac{1}{2}\boldsymbol{v}^T\mathcal{A}\boldsymbol{v}$. For this we split into two subcases.

(1) Suppose that $\varepsilon^{-\delta+\frac{1}{2}} \ll y \ll \varepsilon^{\frac{1}{4}}$. Since $v_j$ is a real multiple of $\sqrt{z}$, the condition $y \ll \varepsilon^{\frac{1}{4}}$ implies that $-\frac{1}{2}\boldsymbol{v}^T\mathcal{A}\boldsymbol{v}$ has negative real part as $z\to 0$.
(2) Suppose $\varepsilon^{\frac{1}{4}} \ll y \ll 1$. In this case we have $|\boldsymbol{v}| \ll \varepsilon^{\frac{1}{2}+\lambda}$, and it follows that $-\frac{1}{2}\boldsymbol{v}^T\mathcal{A}\boldsymbol{v} \ll \varepsilon^{1+2\lambda}$.

In either case, the exponent $\frac{s(y)}{\varepsilon}$ has size $\gg \varepsilon^{-2\delta}$ and dominate other exponents appearing in (5.25) that have positive real parts. So we conclude that (5.26) holds if $\varepsilon^{-\delta+\frac{1}{2}} \ll y \ll 1$, by taking $\delta_0=2\delta$.

Next we consider the case $1 \ll y \ll \varepsilon^{-1-\frac{3\lambda}{2}+\delta}$ for some $\delta > 0$. We use the trivial bound

$$\mathrm{Re}\left(-\log(2)(1+iy) - \mathrm{Log}\left(1-2^{-(1+iy)}\right)\right) \ll 1.$$

It then follows from the bound $|\boldsymbol{\nu}| \ll \varepsilon^{\lambda}$ that

$$\mathrm{Re}\left(\frac{1}{\sqrt{z}}\left(-\log(2)(1+iy) - \mathrm{Log}\left(1-2^{-(1+iy)}\right)\right)\sum_{j=1}^{N} v_j\right) \ll \varepsilon^{\lambda}.$$

Meanwhile, by Lemma 5.6 (2), we have $\frac{s(y)}{\varepsilon} \gg \frac{1}{\varepsilon}$. Finally, as $|\boldsymbol{\nu}| \ll \varepsilon^{\lambda}$ and $z \ll \varepsilon^{-\frac{3\lambda}{2}+\delta_2}$, where $\delta_2 = \min\{1+\frac{3\lambda}{2}, \delta\} > 0$ (see the proof of Proposition 3.3), we deduce that $|\boldsymbol{v}| \ll \varepsilon^{\frac{\lambda}{4}+\frac{\delta_2}{2}}$, and hence

$$-\tfrac{1}{2}\boldsymbol{v}^T\mathcal{A}\boldsymbol{v} \ll \varepsilon^{\frac{\lambda}{2}+\delta_2}.$$

From the computation above, we have that $\frac{s(y)}{\varepsilon} \gg \frac{1}{\varepsilon}$, and this exponent dominates the other exponents. So (5.26) also holds.

For the expression $2^{-K-\frac{1}{2}}\pi^{-\frac{N}{2}}e^{\frac{\pi^2 N}{12z}-\frac{\pi^2 N}{12\varepsilon}}\sum_{r=0}^{R_0-1} E_{\boldsymbol{\ell},r} z^{\frac{r}{2}}$, we only have a single exponent, namely $\frac{\pi^2 N}{12z} - \frac{\pi^2 N}{12\varepsilon}$. This exponent has negative real part of size

$$\mathrm{Re}\left(\frac{\pi^2 N}{12z} - \frac{\pi^2 N}{12\varepsilon}\right) = \frac{\pi^2 N}{12\varepsilon}\left(\mathrm{Re}\left(\frac{1}{1+iy}\right) - 1\right) \gg \varepsilon^{\max\{1+2\lambda-2\delta,-1\}}$$

for $\varepsilon^{\frac{1}{2}-\delta} \ll y \ll \varepsilon^{-1-\frac{3\lambda}{2}+\delta}$. It follows that, for $\varepsilon^{\frac{1}{2}-\delta} \ll y \ll \varepsilon^{-1-\frac{3\lambda}{2}+\delta}$, we have, for all $L \in \mathbb{N}$,

$$\frac{e^{\frac{\pi^2 N}{12z}-\frac{\pi^2 N}{12\varepsilon}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}}\sum_{r=0}^{R_0-1} E_{\boldsymbol{\ell},r} z^{\frac{r}{2}} = O\left(\varepsilon^{L}\right).$$

On the other hand, if $y \ll \varepsilon^{1+\lambda+\delta}$, then, by Proposition 5.1, we have, for all $L \in \mathbb{N}$,

$$g^{[K,1]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) - \frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}}\sum_{r=0}^{R_0-1} E_{\boldsymbol{\ell},r} z^{\frac{r}{2}} \ll \varepsilon^{L} e^{\frac{\pi^2 N}{12\varepsilon}}.$$

Choosing $\delta > 0$ sufficiently small, the two cases cover the full range $y \ll \varepsilon^{-1-\frac{3\lambda}{2}+\delta}$, establishing the proposition. □

We are now ready to derive the asymptotic expansion of $d^{[K]}_{\alpha,\beta;N}(n)$.

**Theorem 5.8.** *We have for $R \in \mathbb{N}$, as $n \to \infty$,*

$$d^{[K]}_{\alpha,\beta;N}(n) = \frac{e^{\pi\sqrt{\frac{n}{3}}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}}\left(\sum_{r=0}^{R-1}\sum_{\substack{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^N\\ NH(\boldsymbol{\ell})\equiv n \pmod N}}\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor} c_{\frac{r}{2}-j,\frac{\pi}{2}\sqrt{\frac{N}{3}},j} E_{\boldsymbol{\ell},r-2j}\left(\frac{N}{n}\right)^{\frac{r+3}{4}} + O\left(n^{-\frac{R+3}{4}}\right)\right),$$

*where $E_{\boldsymbol{\ell},r}$ are as in Proposition 5.1 and*

$$c_{A,B,r} := \frac{\left(-\frac{1}{4B}\right)^r B^{A+\frac{1}{2}}\Gamma\left(A+r+\frac{3}{2}\right)}{2\sqrt{\pi}r!\Gamma\left(A-r+\frac{3}{2}\right)}.$$

*Proof.* By Cauchy's Theorem we have, for $n \in \mathbb{N}_0$,

$$d^{[K]}_{\alpha,\beta;N}(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\mathcal{D}^{[K]}_{\alpha,\beta;N}(q)}{q^{n+1}} dq,$$

where $\mathcal{C}$ is a circle centred at the origin inside the unit circle surrounding zero exactly once counter-clockwise. Using (5.1) and the change of variables $q = e^{-z}$, we write, for any $\varepsilon > 0$, with $\zeta_N := e^{\frac{2\pi i}{N}}$

$$d^{[K]}_{\alpha,\beta;N}(n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^N} \zeta_N^{(n-NH(\boldsymbol{\ell}))k}\frac{1}{2\pi i}\int_{\varepsilon-\pi i}^{\varepsilon+\pi i} g^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z)e^{\frac{nz}{N}}dz. \tag{5.27}$$

Let $\theta > 0$ be fixed. We split the integral into the *major arc* $\mathcal{C}_1(\varepsilon) := \{z = \varepsilon(1+iy) : |y| \le \theta\}$ and the *minor arc* $\mathcal{C}_2(\varepsilon) := \{z = \varepsilon(1+iy) : \theta < |y| \le \frac{\pi}{\varepsilon}\}$. Note that Proposition 5.7 applies for the whole major arc (because $\varepsilon^{-1-\frac{3\lambda}{2}+\delta} \gg 1$). So on the major arc, for every $L \in \mathbb{N}$, we have

$$g^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) - \frac{e^{\frac{\pi^2 N}{12z}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}}\sum_{r=0}^{R_0-1} E_{\boldsymbol{\ell},r} z^{\frac{r}{2}} \ll \varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}, \tag{5.28}$$

where $R_0 = R_0(L)$ is given as in Proposition 5.1. On the minor arc, we use Propositions 4.2 and 5.7 to obtain for all $L \in \mathbb{N}$,

$$g^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) \ll \varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}. \tag{5.29}$$

Let $A \ge 0$, $B > 0$. By [15, Lemma 3.7], we have, as $n \to \infty$,

$$\frac{1}{2\pi i}\int_{\mathcal{C}_1\left(\frac{B}{\sqrt{n}}\right)} z^A e^{\frac{B^2}{z}+nz}dz = n^{\frac{1}{4}(-2A-3)}e^{2B\sqrt{n}}\left(\sum_{r=0}^{R-1}\frac{c_{A,B,r}}{n^{\frac{r}{2}}} + O\left(n^{-\frac{R}{2}}\right)\right).$$

Hence we obtain the asymptotic expansion

$$\begin{aligned}\frac{2^{-K-\frac{1}{2}}\pi^{-\frac{N}{2}}}{2\pi i}&\int_{\mathcal{C}_1\left(\frac{\pi N}{2\sqrt{3n}}\right)} e^{\frac{\pi^2 N}{12z}}\sum_{r=0}^{R-1}E_{\boldsymbol{\ell},r}z^{\frac{r}{2}}e^{\frac{nz}{N}}dz\\ &= \frac{e^{\pi\sqrt{\frac{n}{3}}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}}\left(\sum_{r=0}^{R-1}\sum_{j=0}^{\left\lfloor\frac{r}{2}\right\rfloor} c_{\frac{r}{2}-j,\frac{\pi}{2}\sqrt{\frac{N}{3}},j}E_{\boldsymbol{\ell},r-2j}\left(\frac{N}{n}\right)^{\frac{r+3}{4}} + O\left(n^{-\frac{R+3}{4}}\right)\right).\end{aligned} \tag{5.30}$$

Meanwhile, if a function $h$ satisfies $h(z) \ll \varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}}$ as $\varepsilon \to 0$ for $L \ge 0$, then we have, as $n \to \infty$,

$$\frac{1}{2\pi i}\int_{\frac{\pi N}{2\sqrt{3n}}-\pi i}^{\frac{\pi N}{2\sqrt{3n}}+\pi i} h(z)e^{\frac{nz}{N}}dz = O\left(n^{-\frac{L}{2}}e^{\pi\sqrt{\frac{n}{3}}}\right), \tag{5.31}$$

and we use (5.31) to evaluate the minor arc integral and the error integral.

Let $R \in \mathbb{N}$. We set $L = \frac{R+3}{2}$, and we pick $R_0 = R_0(L)$ as in Proposition 5.1, and write

$$\begin{aligned}\frac{1}{2\pi i}&\int_{\mathcal{C}\left(\frac{\pi N}{2\sqrt{3n}}\right)} g^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z)e^{\frac{nz}{N}}dz\\ &= \frac{2^{-K-\frac{1}{2}}\pi^{-\frac{N}{2}}}{2\pi i}\int_{\mathcal{C}\left(\frac{\pi N}{2\sqrt{3n}}\right)} e^{\frac{\pi^2 N}{12z}}\sum_{r=0}^{R_0-1}E_{\boldsymbol{\ell},r}z^{\frac{r}{2}}e^{\frac{nz}{N}}dz + \frac{1}{2\pi i}\int_{\mathcal{C}\left(\frac{\pi N}{2\sqrt{3n}}\right)} h(z)e^{\frac{nz}{N}}dz,\end{aligned}$$

where we have $h(z) \ll \varepsilon^L e^{\frac{\pi^2 N}{12\varepsilon}} = \varepsilon^{\frac{R+3}{2}}e^{\frac{\pi^2 N}{12\varepsilon}}$ by (5.28) and (5.29). Using (5.30) and (5.31), the integrals above can be evaluated as

$$\frac{e^{\pi\sqrt{\frac{n}{3}}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}}\left(\sum_{r=0}^{R_0-1}\sum_{j=0}^{\left\lfloor\frac{r}{2}\right\rfloor} c_{\frac{r}{2}-j,\frac{\pi}{2}\sqrt{\frac{N}{3}},j}E_{\boldsymbol{\ell},r-2j}\left(\frac{N}{n}\right)^{\frac{r+3}{4}} + O\left(n^{-\frac{R_0+3}{4}}\right)\right) + O\left(n^{-\frac{R+3}{4}}e^{\pi\sqrt{\frac{n}{3}}}\right).$$

As the terms with $r \geq R$ also have size $O(n^{-\frac{R+3}{4}} e^{\pi\sqrt{\frac{n}{3}}})$, we may truncate the asymptotic expansion, and obtain

$$\frac{1}{2\pi i}\int_{\mathcal{C}\left(\frac{\pi N}{2\sqrt{3}n}\right)} g^{[K]}_{\alpha,\beta;N,\boldsymbol{\ell}}(z) e^{\frac{nz}{N}} dz = \frac{e^{\pi\sqrt{\frac{n}{3}}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}} \left( \sum_{r=0}^{R-1}\sum_{j=0}^{\left\lfloor \frac{r}{2}\right\rfloor} c_{\frac{r}{2}-j,\frac{\pi}{2}\sqrt{\frac{N}{3}},j} E_{\boldsymbol{\ell},r-2j}\left(\frac{N}{n}\right)^{\frac{r+3}{4}} + O\left(n^{-\frac{R+3}{4}}\right)\right).$$

Plugging this back into (5.27) gives

$$d^{[K]}_{\alpha,\beta;N}(n) = \frac{e^{\pi\sqrt{\frac{n}{3}}}}{2^{K+\frac{1}{2}}\pi^{\frac{N}{2}}N} \left( \sum_{r=0}^{R-1}\sum_{k=0}^{N-1}\sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^N} \zeta_N^{(n-NH(\boldsymbol{\ell}))k} \sum_{j=0}^{\left\lfloor \frac{r}{2}\right\rfloor} c_{\frac{r}{2}-j,\frac{\pi}{2}\sqrt{\frac{N}{3}},j} E_{\boldsymbol{\ell},r-2j}\left(\frac{N}{n}\right)^{\frac{r+3}{4}} + O\left(n^{-\frac{R+3}{4}}\right)\right).$$

The theorem follows, using orthogonality of roots of unity. □

## 6. Proof of Theorems 1.1 and 1.2

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* To prove Theorem 1.1, it suffices to determine the first two terms in Theorem 5.8. We compute

$$c_{0,\frac{\pi}{2}\sqrt{\frac{N}{3}},0} = \frac{N^{\frac{1}{4}}}{2\sqrt{2}\cdot 3^{\frac{1}{4}}}, \qquad c_{\frac{1}{2},\frac{\pi}{2}\sqrt{\frac{N}{3}},0} = \frac{1}{4}\sqrt{\frac{\pi N}{3}}.$$

We follow the proof of Proposition 5.1 to determine $E_{\boldsymbol{\ell},0}$ and $E_{\boldsymbol{\ell},1}$. To begin with, we compute the coefficients $C_0(\boldsymbol{u})$ and $C_1(\boldsymbol{u})$ in the formal power series expansion (3.10). We expand

$$\exp(\phi(\boldsymbol{u},z)) = \exp\left(-\boldsymbol{b}^T\boldsymbol{u}z^{\frac{1}{2}} - \frac{Nz}{24} - \sum_{j=1}^{N}\sum_{r\geq 2}\left(B_r\left(-\frac{u_j}{\sqrt{z}}\right) - \frac{\delta_{r,2}u_j^2}{z}\right)\mathrm{Li}_{2-r}\left(\frac{1}{2}\right)\frac{z^{r-1}}{r!}\right).$$

Noting that we have

$$\exp\left(-\sum_{j=1}^{N}\sum_{r\geq 2}\left(B_r\left(-\frac{u_j}{\sqrt{z}}\right) - \frac{\delta_{r,2}u_j^2}{z}\right)\mathrm{Li}_{2-r}\left(\frac{1}{2}\right)\frac{z^{r-1}}{r!}\right) = 1 + \sum_{j=1}^{N}\left(-\frac{u_j}{2} + \frac{u_j^3}{3}\right)\sqrt{z} + O(z),$$

$$\exp\left(-\boldsymbol{b}^T\boldsymbol{u}\sqrt{z}\right) = 1 - \sum_{j=1}^{N} b_j u_j\sqrt{z} + O(z) = 1 + \sum_{j=1}^{N}\left(-\frac{j}{N} + \frac{1}{2} - e_j\right)u_j\sqrt{z} + O(z),$$

we get

$$C_0(\boldsymbol{u}) = 1, \qquad C_1(\boldsymbol{u}) = \sum_{j=1}^{N}\left(-\left(\frac{j}{N} + e_j\right)u_j + \frac{u_j^3}{3}\right).$$

Next we compute the constants $V_{0,-N}$, $V_{0,1-N}$, and $V_{1,-N}$. For this we use the evaluations

$$e^{-\frac{\pi^2 N}{12z}}\int_{\mathcal{R}_{\alpha,\beta;N}} G_0(\boldsymbol{u})d\boldsymbol{u} = \int_{\mathcal{R}_{\alpha,\beta;N}} e^{-\boldsymbol{u}^T\boldsymbol{u}}d\boldsymbol{u} = \frac{\pi^{\frac{N}{2}}}{2},$$

$$e^{-\frac{\pi^2 N}{12z}}\int_{\mathbb{R}^{N-1}} G_0\left(\boldsymbol{u}_{[\mathbf{1}]}, u_\beta\right)d\boldsymbol{u}_{[\mathbf{1}]} = \int_{\mathbb{R}^{N-1}} e^{-u_\beta^2 - \boldsymbol{u}_{[\mathbf{1}]}^T\boldsymbol{u}_{[\mathbf{1}]}}d\boldsymbol{u}_{[\mathbf{1}]} = \frac{\pi^{\frac{N-1}{2}}}{\sqrt{2}},$$

where $\mathcal{R}_{\alpha,\beta;N} := \{\boldsymbol{u}\in\mathbb{R}^N : u_\alpha \geq u_\beta\}$. Meanwhile, we compute

$$e^{-\frac{\pi^2 N}{12z}}\int_{\mathcal{R}_{\alpha,\beta;N}} G_1(\boldsymbol{u})d\boldsymbol{u} = \int_{\mathcal{R}_{\alpha,\beta;N}} C_1(\boldsymbol{u})e^{-\boldsymbol{u}^T\boldsymbol{u}}d\boldsymbol{u}$$

$$= -\sum_{j=1}^{N}\left(\frac{j}{N}+e_j\right)\int_{\mathcal{R}_{\alpha,\beta;N}} u_j e^{-\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{du} + \frac{1}{3}\sum_{j=1}^{N}\int_{\mathcal{R}_{\alpha,\beta;N}} u_j^3 e^{-\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{du}. \quad (6.1)$$

We consider the first integral on the right-hand side. If $j \notin \{\alpha,\beta\}$, then we have

$$\int_{\mathcal{R}_{\alpha,\beta;N}} u_j e^{-\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{du} = \int_{u_\alpha>u_\beta} e^{-u_\alpha^2-u_\beta^2}du_\alpha du_\beta \int_{\mathbb{R}^{N-2}} u_j e^{-\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{du} = 0$$

because the rightmost integral is anti-symmetric with respect to $u_j$. On the other hand, we have

$$\int_{\mathcal{R}_{\alpha,\beta;N}} u_\alpha e^{-\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{du} = \int_{u_\alpha>u_\beta} u_\alpha e^{-u_\alpha^2-u_\beta^2}du_\alpha du_\beta \int_{\mathbb{R}^{N-2}} e^{-\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{du} = \frac{\pi^{\frac{N-1}{2}}}{2\sqrt{2}},$$
$$\int_{\mathcal{R}_{\alpha,\beta;N}} u_\beta e^{-\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{du} = \int_{u_\alpha>u_\beta} u_\beta e^{-u_\alpha^2-u_\beta^2}du_\alpha du_\beta \int_{\mathbb{R}^{N-2}} e^{-\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{du} = -\frac{\pi^{\frac{N-1}{2}}}{2\sqrt{2}}.$$

The second integral on the right of (6.1) is invariant under interchanging $u_\alpha$ and $u_\beta$. Hence

$$\sum_{j=1}^{N}\int_{\mathcal{R}_{\alpha,\beta;N}} u_j^3 e^{-\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{du} = \frac{1}{2}\sum_{j=1}^{N}\int_{\mathbb{R}^N} u_j^3 e^{-\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{du} = 0,$$

since the integral is anti-symmetric. Plugging into (6.1), it follows that

$$e^{-\frac{\pi^2 N}{12z}}\int_{\mathcal{R}_{\alpha,\beta;N}} G_1(\boldsymbol{u})\boldsymbol{du} = \left(\frac{\beta-\alpha}{N}+(e_\beta-e_\alpha)\right)\frac{\pi^{\frac{N-1}{2}}}{2\sqrt{2}}.$$

It follows from (5.22) and (5.23) (and (5.17) for $W_0$) that

$$V_{0,-N} = \frac{\pi^{\frac{N}{2}}}{2N^N}, \quad V_{0,1-N} = \frac{\pi^{\frac{N-1}{2}}}{\sqrt{2}N^{N-1}}\left(\frac{1}{2}-\frac{[\ell_\alpha-\ell_\beta]_N}{N}\right), \quad V_{1,-N} = \frac{\pi^{\frac{N-1}{2}}(\beta-\alpha+N(e_\beta-e_\alpha))}{2\sqrt{2}N^{N+1}}.$$

Finally, using (5.24), we compute

$$E_{\boldsymbol{\ell},0} = V_{0,-N} = \frac{\pi^{\frac{N}{2}}}{2N^N},$$
$$E_{\boldsymbol{\ell},1} = V_{0,1-N}+V_{1,-N} = \frac{\pi^{\frac{N-1}{2}}}{2\sqrt{2}N^{N-1}}\left(1-\frac{2[\ell_\alpha-\ell_\beta]_N}{N}+\frac{\beta-\alpha+N(e_\beta-e_\alpha)}{N^2}\right).$$

Theorem 1.1 then follows from Theorem 5.8. □

To prove Theorem 1.2, we need the following lemma, which follows by a direct calculation.

**Lemma 6.1.** *Let $N \geq 5$, $1 \leq \alpha,\beta \leq N$, and $r,\ell_\alpha,\ell_\beta \in \mathbb{Z}/N\mathbb{Z}$. Then*

$$\#\left\{\boldsymbol{\ell}_{[\boldsymbol{2}]} \in (\mathbb{Z}/N\mathbb{Z})^{N-2} : NH\left(\boldsymbol{\ell}_{[\boldsymbol{2}]},\ell_\alpha,\ell_\beta\right) \equiv r \pmod N\right\} = N^{N-3},$$

*where $\boldsymbol{\ell}_{[\boldsymbol{2}]} \in (\mathbb{Z}/N\mathbb{Z})^{N-2}$ runs through the indices $j \notin \{\alpha,\beta\}$.*

Now we are ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* The case $N = 2$ can be verified directly from Theorem 1.1. Now suppose that $N \geq 5$. By Lemma 6.1, for every $r,\ell_\alpha,\ell_\beta \in \mathbb{Z}/N\mathbb{Z}$, there exist $N^{N-3}$ tuples $\boldsymbol{\ell} \in (\mathbb{Z}/N\mathbb{Z})^N$ such that $NH(\boldsymbol{\ell}) \equiv r \pmod N$. So we may evaluate the inner sum in Theorem 1.1 as follows:

$$\sum_{\substack{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^N \\ NH(\boldsymbol{\ell})\equiv r \pmod N}} \left(1+\frac{N^2-2N[\ell_\alpha-\ell_\beta]_N+\beta-\alpha+N\left(e_\beta-e_\alpha\right)}{2\cdot 3^{\frac{1}{4}}\sqrt{N}}n^{-\frac{1}{4}}\right)$$

$$= N^{N-1}\left(1+\frac{-N+\beta-\alpha+N\left(e_\beta-e_\alpha\right)}{2\cdot 3^{\frac{1}{4}}\sqrt{N}}n^{-\frac{1}{4}}\right). \qquad \square$$

## 7. Numerical examples

We provide numerical data supporting our statements. All computations were done in PARI/GP and the plots were created in Sage [16, 17].

7.1. **Numerical data for $N = 2$.** We provide some numerical data for the parity bias problem.

**Example.** Let $N = 2$, $\{\alpha, \beta\} = \{1, 2\}$, and $K \in \{0, 1\}$, corresponding to the claims in [2, 12]. The numbers $d^{[K]}_{1,2;2}(n)$, $d^{[K]}_{2,1;2}(n)$ and their difference for $0 \le n \le 50$ are given in Table 1 for $K = 0$ and in Table 2 for $K = 1$. The differences $d^{[K]}_{1,2;2}(n) - d^{[K]}_{2,1;2}(n)$ are plotted for $0 \le n \le 100$ in Figure 1. For $K = 0$ we observe, in accordance with [2, 12], that $d^{[0]}_{1,2;2}(n) > d^{[0]}_{2,1;2}(n)$ for $n \ge 20$. For $K = 1$, we note that the numbers for $13 \le n \le 29$ agree with the claim in [2], Problem 6.1, i.e.,

$$d^{[1]}_{1,2;2}(n) - d^{[1]}_{2,1;2}(n)\begin{cases} > 0 & \text{if } n \text{ is even,}\\ < 0 & \text{if } n \text{ is odd,}\end{cases}$$

whereas numerics suggests that for all $n \ge 29$ we have $d^{[1]}_{1,2;2}(n) - d^{[1]}_{2,1;2}(n) < 0$. This was checked numerically up to $n \le 10000$ which took 5 minutes using PARI/GP on an Apple M1 Pro chip.

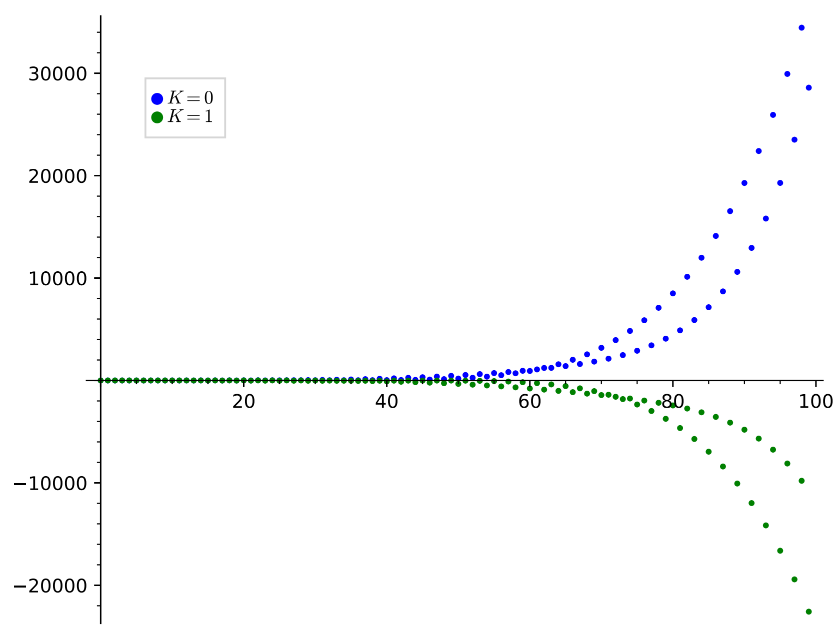


Figure 1. $d^{[K]}_{1,2;2}(n) - d^{[K]}_{2,1;2}(n)$ for $0 \le n \le 100$, $K \in \{0, 1\}$

Furthermore, the numbers $(d^{[K]}_{1,2;2}(n) - d^{[K]}_{2,1;2}(n))ne^{-\pi\sqrt{\frac{n}{3}}}$ are plotted in Figure 2 for $10 \le n \le 5000$. Supporting Corollary 1.3, the figure suggests that they converge to $(-1)^K 2^{\frac{7}{2}-K}3^{-\frac{1}{2}}$, independently of $n \pmod 2$.

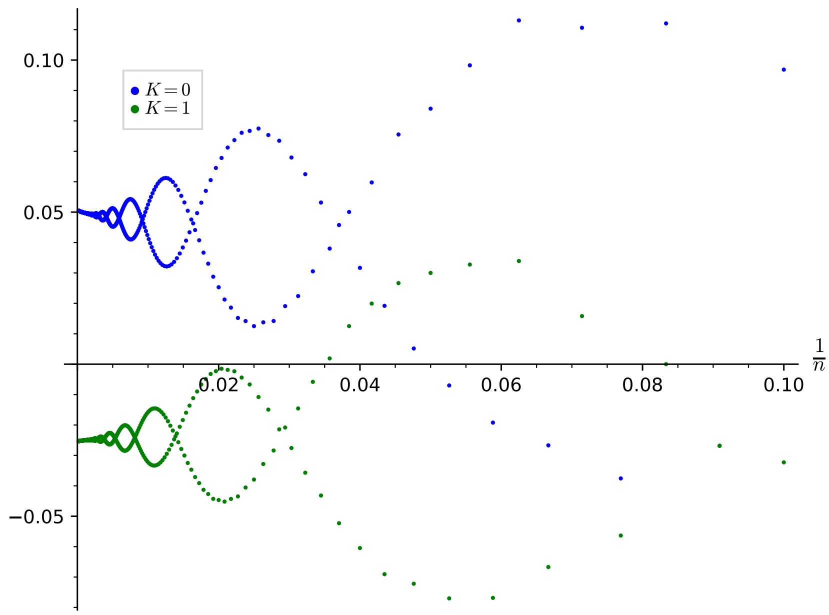


FIGURE 2. $(\frac{1}{n},\ (d^{[K]}_{1,2;2}(n) - d^{[K]}_{2,1;2}(n))ne^{-\pi\sqrt{\frac{n}{3}}})$ for $10 \le n \le 5000$, $K \in \{0,1\}$

| $n$ | $d^{[0]}_{1,2;2}(n)$ | $d^{[0]}_{2,1;2}(n)$ | $d^{[0]}_{1,2;2}(n) - d^{[0]}_{2,1;2}(n)$ |
|---|---|---|---|
| 1 | 1 | 0 | 1 |
| 2 | 0 | 1 | −1 |
| 3 | 1 | 0 | 1 |
| 4 | 1 | 1 | 0 |
| 5 | 1 | 0 | 1 |
| 6 | 2 | 2 | 0 |
| 7 | 1 | 1 | 0 |
| 8 | 4 | 2 | 2 |
| 9 | 2 | 2 | 0 |
| 10 | 6 | 3 | 3 |
| 11 | 3 | 4 | −1 |
| 12 | 9 | 4 | 5 |
| 13 | 5 | 7 | −2 |
| 14 | 12 | 5 | 7 |
| 15 | 9 | 11 | −2 |
| 16 | 17 | 7 | 10 |
| 17 | 14 | 16 | −2 |
| 18 | 22 | 10 | 12 |
| 19 | 22 | 23 | −1 |
| 20 | 29 | 15 | 14 |
| 21 | 33 | 32 | 1 |
| 22 | 38 | 21 | 17 |
| 23 | 48 | 43 | 5 |
| 24 | 50 | 32 | 18 |
| 25 | 68 | 57 | 11 |
| 26 | 65 | 45 | 20 |
| 27 | 95 | 74 | 21 |
| 28 | 86 | 66 | 20 |
| 29 | 128 | 96 | 32 |
| 30 | 113 | 92 | 21 |
| 31 | 172 | 123 | 49 |
| 32 | 149 | 129 | 20 |
| 33 | 226 | 157 | 69 |
| 34 | 197 | 175 | 22 |
| 35 | 295 | 199 | 96 |
| 36 | 260 | 239 | 21 |
| 37 | 379 | 253 | 126 |
| 38 | 342 | 316 | 26 |
| 39 | 485 | 320 | 165 |
| 40 | 449 | 419 | 30 |
| 41 | 613 | 406 | 207 |
| 42 | 587 | 544 | 43 |
| 43 | 773 | 514 | 259 |
| 44 | 762 | 704 | 58 |
| 45 | 967 | 652 | 315 |
| 46 | 987 | 898 | 89 |
| 47 | 1206 | 825 | 381 |
| 48 | 1269 | 1142 | 127 |
| 49 | 1497 | 1045 | 452 |
| 50 | 1623 | 1435 | 188 |

TABLE 1. Numerics for $K = 0$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d^{[1]}_{1,2;2}(n)$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 2 | 1 | 4 | 1 | 6 | 2 | 9 | 3 |
| $d^{[1]}_{2,1;2}(n)$ | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 2 | 1 | 3 | 2 | 4 | 4 | 5 | 7 | 6 | 11 |
| $d^{[1]}_{1,2;2}(n) - d^{[1]}_{2,1;2}(n)$ | 0 | −1 | 1 | −1 | 1 | −2 | 1 | −1 | 0 | −1 | −1 | 0 | −3 | 1 | −5 | 3 | −8 |

| $n$ | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d^{[1]}_{1,2;2}(n)$ | 12 | 5 | 16 | 9 | 20 | 14 | 26 | 22 | 32 | 33 | 40 | 48 | 50 | 67 | 63 | 93 | 79 | 125 |
| $d^{[1]}_{2,1;2}(n)$ | 8 | 16 | 11 | 23 | 14 | 32 | 20 | 43 | 27 | 57 | 39 | 74 | 54 | 95 | 76 | 121 | 103 | 153 |
| $d^{[1]}_{1,2;2}(n) - d^{[1]}_{2,1;2}(n)$ | 4 | −11 | 5 | −14 | 6 | −18 | 6 | −21 | 5 | −24 | 1 | −26 | −4 | −28 | −13 | −28 | −24 | −28 |

| $n$ | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d^{[1]}_{1,2;2}(n)$ | 101 | 166 | 129 | 216 | 166 | 279 | 215 | 354 | 278 | 448 | 360 | 559 | 467 | 695 | 603 |
| $d^{[1]}_{2,1;2}(n)$ | 143 | 191 | 191 | 239 | 257 | 297 | 338 | 369 | 444 | 458 | 572 | 569 | 737 | 705 | 935 |
| $d^{[1]}_{1,2;2}(n) - d^{[1]}_{2,1;2}(n)$ | −42 | −25 | −62 | −23 | −91 | −18 | −123 | −15 | −166 | −10 | −212 | −10 | −270 | −10 | −332 |

Table 2. Numerics for $K = 1$.

7.2. **Numerical data for $N = 3$.** We give numerical data to illustrate Corollary 1.4.

**Example.** We consider $N = 3$ and $K = 0$. The first 17 values for $d^{[0]}_{1,2;3}(n)$ and $d^{[0]}_{1,2;3}(n)$, and their difference are listed in Table 3. Figure 3 depicts the difference $d^{[0]}_{1,2;3}(n) - d^{[0]}_{2,1;3}(n)$ for $0 \leq n \leq 100$. Moreover, the numbers $(d^{[0]}_{1,2;3}(n) - d^{[0]}_{2,1;3}(n))ne^{-\pi\sqrt{\frac{n}{3}}}$ are plotted for $10 \leq n \leq 1000$ in Figure 4. As pointed out above, we observe that the asymptotics of the difference indeed depend on $n \pmod 3$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d^{[0]}_{1,2;3}(n)$ | 1 | 0 | 0 | 2 | 1 | 0 | 4 | 2 | 0 | 8 | 4 | 1 | 14 | 8 | 2 | 24 | 14 |
| $d^{[0]}_{2,1;3}(n)$ | 0 | 1 | 0 | 0 | 2 | 0 | 1 | 4 | 0 | 2 | 8 | 0 | 4 | 14 | 1 | 8 | 24 |
| $d^{[0]}_{1,2;3}(n) - d^{[0]}_{2,1;3}(n)$ | 1 | −1 | 0 | 2 | −1 | 0 | 3 | −2 | 0 | 6 | −4 | 1 | 10 | −6 | 1 | 16 | −10 |

Table 3. Numerics for $N = 3$, $K = 0$.

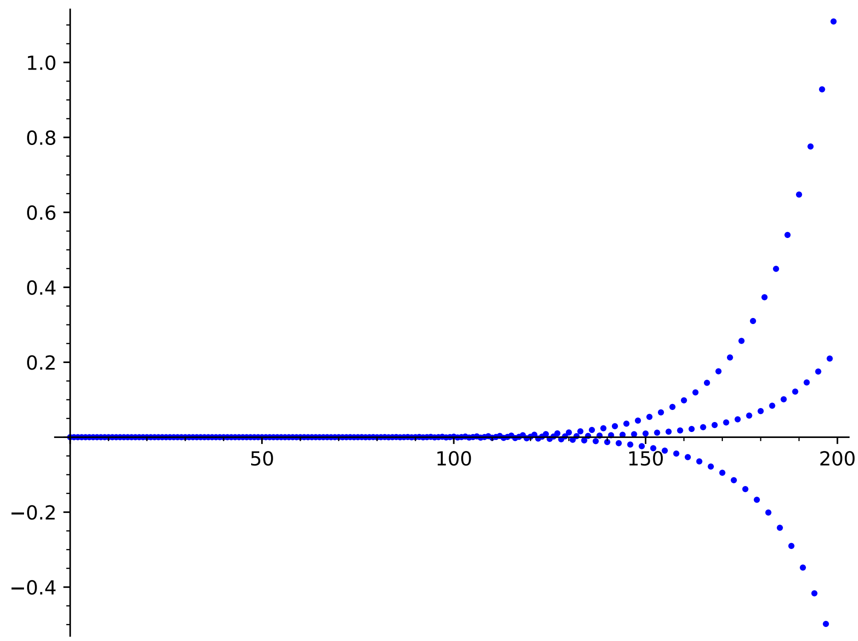


FIGURE 3. $d^{[0]}_{1,2;3}(n) - d^{[0]}_{2,1;3}(n)$ for $0 \le n \le 100$

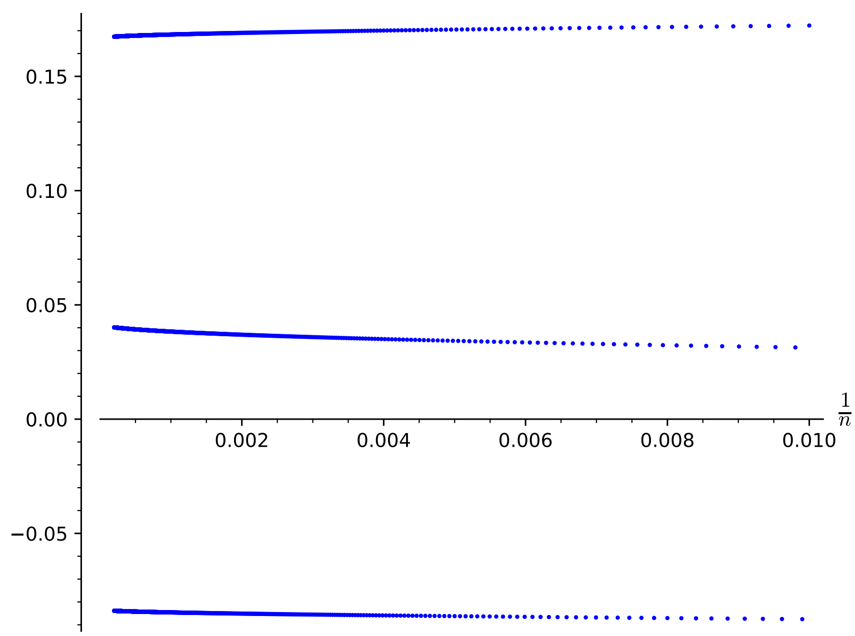


FIGURE 4. $(\frac{1}{n},\ (d^{[0]}_{1,2;3}(n) - d^{[0]}_{2,1;3}(n))ne^{-\pi\sqrt{\frac{n}{3}}})$ for $10 \le n \le 1000$

## REFERENCES

[1] G. Andrews, *Questions and conjectures in partition theory*, Am. Math. Monthly **93** (1986), 708–711.

[2] K. Banerjee, S. Bhattacharjee, M. Dastidar, P. Mahanta, and M. Saikia, *Parity biases in partitions and restricted partitions*, European J. Combin. **103** (2022).

[3] O. Beckwith and M. Mertens, *The number of parts in certain residue classes of integer partitions*, Research in Number Theory **1:11** (2015).
[4] F. Calegari, S. Garoufalidis, and D. Zagier, *Bloch groups, algebraic K-theory, units and Nahm's Conjecture*, to appear in Annales Scientifiques de l'École Normale Supérieure.
[5] S. Chern, *Further results on biases in integer partitions*, Bull. Korean Math. Soc. **59:1** (2022), 111–117.
[6] N. Dartyge and A. Sarkozy, *Arithmetic Properties of summands of partitions II*, Ramanujan J. **10** (2005), 383–394.
[7] N. Dartyge, A. Sarkozy, and M. Szalay, *On the distribution of the summands of partitions in residue classes*, Acta Math. Hungar. **109** (2005), 215–237.
[8] N. Dartyge, A. Sarkozy, and M. Szalay, *On the distribution of the summands of unequal partitions in residue classes*, Acta Math. Hungar. **110** (2006), 323–335.
[9] S. Garoufalidis and D. Zagier, *Asymptotics of Nahm sums at roots of unity*, Ramanujan J. **55** (2021), 219–238.
[10] S. Garoufalidis and D. Zagier, *Knots and their related q-series*, arXiv:2304.09377, 2023.
[11] B. Kim and E. Kim, *Biases in integer partitions*. Bull. Aust. Math. Soc. **104:2** (2021), 177–186.
[12] B. Kim, E. Kim, and J. Lovejoy, *Parity bias in partitions*. European J. Combin. **89** (2020), 103159.
[13] M. Kontsevich and Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, Commun. Number Theory Phys. **5** no. 2 (2011), 231–352.
[14] W. Nahm, *Conformal field theory and torsion elements of the Bloch group* In: Frontiers in number theory, physics, and geometry. II, pages 67–132. Springer, Berlin, 2007.
[15] H. Ngo and R. Rhoades, *Integer partitions, probabilities and quantum modular forms*. Res. Math. Sci. **4:17** (2017).
[16] The PARI Group, *PARI/GP version 2.15.0*, Univ. Bordeaux, 2022, `http://pari.math.u-bordeaux.fr`.
[17] The Sage developers *SageMath, the Sage mathematics software system (Version 9.6)*, 2022, `https://www.sagemath.org`.
[18] M. Vlasenko and S. Zwegers, *Nahm's conjecture: asymptotic computations and counterexamples*, Commun. Number Theory Phys. **5** (2021), 617–642.
[19] D. Zagier, *The Mellin transform and related analytic techniques*, Appendix to E. Zeidler, Quantum Field Theory I: Basics in mathematics and physics. A bridge between mathematicians and physicists, 305–323. Springer, Berlin (2006).
[20] D. Zagier, *The dilogarithm function*, In: P. Cartier, P. Moussa, B. Julia, P. Vanhove (eds), Frontiers in number theory, physics, and geometry II. Springer, Berlin, Heidelberg (2007).

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