Ricci Flow Singularity Formation

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Notation and Conventions

- For quantities A and B, the notation $A \lesssim B$ means there exists a constant $C > 0$ such that $A \leq CB$. $A \lesssim_{a,b,c} B$ is used to indicate that the constant C depends on a, b, c. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. $A \sim_{a,b,c} B$ is defined analogously.
- We write "$A \gg 1$ (depending on a, b, c)" to mean "A is sufficiently large depending on a, b, c," or, more precisely, "$A > A_*$ for some constant $A_*(a, b, c) > 1$."
  An analogous definition applies to "$0 < \varepsilon \ll 1$ (depending on a, b, c)."

- All manifolds M are smooth (C^∞) and boundaryless $\partial M = \emptyset$. A **closed manifold** is a compact (and boundaryless) manifold.
- For $\Omega, \Omega' \subseteq M$, we write $\Omega \Subset \Omega'$ to mean the closure $\overline{\Omega}$ is a compact subset of Ω'.
- For $f : M \to \mathbb{R}$ and $\Gamma \in \mathbb{R}$, $\{f < \Gamma\}$ denotes the set $\{x \in M : f(x) < \Gamma\}$.

- For a manifold M, $T^k_l(M)$ denotes the bundle of $\binom{k}{l}$-tensors on M and $\mathcal{T}^k_l(M)$ denotes the set of $\binom{k}{l}$-tensor fields. We also denote, for example, $T^k(M) = T^k_0(M)$ for simplicity. For concreteness, $\mathcal{T}_1(M)$ is the set of vector fields on M and $\mathcal{T}^1(M)$ is the set of covector fields or 1-forms on M.
  $S^2(M) \subseteq T^2(M)$ denotes the subbundle of symmetric $\binom{2}{0}$-tensors on M and $\mathcal{S}^2(M)$ the set of symmetric $\binom{2}{0}$-tensor fields.
- Let (M, g) be a Riemannian manifold and S, T be tensors. The notation $S \# T$ denotes some linear combination of tensors formed by permuting the indices of $S \otimes T$ and taking traces thereof (without respect to any metric). The notation $S * T$ denotes some linear combination of tensors formed by taking traces (with respect to g), raising/lowering indices (with respect to g), and permuting indices of $S \otimes T$.

Chapter 1

Introduction

Geometric flows refer to processes where a geometric object changes in time according to its geometry. The Ricci flow in particular has had many applications in geometry and topology, most notably with Perelman's solution of the Poincaré conjecture. However, geometric flows generally form singularities in finite time, and applications require a precise understanding of these singularities. Indeed, the solution of the Poincaré conjecture required a precise understanding of 3D Ricci flow singularities.

In this mini-course, we will examine singularities of Ricci flow. The broad goal will be to show that asymptotically conical gradient shrinking Ricci solitons appear as singularities for general Ricci flows, and to develop fundamental techniques in the analysis of nonlinear parabolic PDEs.

1.1 Definition and Fundamental Properties

Definition 1.1 (Ricci Flow)**.** Let M be a manifold. A smooth collection of Riemannian metrics $(g(t))_{t\in(0,T)}$ on M evolves by **Ricci flow** if

$$\frac{\partial g}{\partial t} = -2Rc \qquad \forall t \in (0,T) \tag{RF}$$

where $Rc = Rc_{g(t)}$ is the Ricci curvature tensor of $g(t)$.

Remark 1.2. Let (M^n, g) be a Riemannian manifold and let $(x^1, \dots, x^n) : U \subseteq M \to \mathbb{R}^n$ be local coordinates. Let

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) : U \to \mathbb{R}$$

denote the components of the metric in these coordinates.

If the coordinates are **harmonic**

$$\text{i.e.} \qquad \Delta_g x^i \equiv 0 \qquad (\forall i),$$

then the components Rc_{ij} of the Ricci tensor $Rc = Rc_{ij}\, dx^i \otimes dx^j$ satisfy

$$Rc_{ij} = -\frac{1}{2}\Delta_g(g_{ij}) + Q_{ij}(g, \partial g) \qquad (\forall i, j)$$

where $Q_{ij}(g, \partial g)$ is some expression involving at most 1st-order derivatives of the functions $(g_{ab})_{1\le a,b\le n}$.

In particular, the Ricci flow equation (RF) becomes

$$\partial_t(g_{ij}) = \Delta_g(g_{ij}) - 2Q_{ij}(g, \partial g).$$

In this way, the Ricci flow can informally be regarded as a nonlinear heat equation for the Riemannian metric g.

Theorem 1.3 (Short-Time Existence and Uniqueness)**.** *Let (M^n, g_0) be a closed Riemannian manifold. Then there exists $T > 0$ and a smooth Ricci flow $(g(t))_{t\in[0,T)}$ such that $g(0) = g_0$.*

Moreover, if $(g(t))_{t\in[0,T)}$ and $(\tilde{g}(t))_{t\in[0,\tilde{T})}$ are smooth Ricci flows on M with the same initial data $g(0) = \tilde{g}(0) = g_0$, then $g(t) = \tilde{g}(t)$ for all $t \in [0, \min(T, \tilde{T}))$.

Remark 1.4. Let (M^n, g_0) be a closed Riemannian manifold. By the short-time existence and uniqueness of Ricci flow (Theorem 1.3), we can define the **(maximal) Ricci flow starting from** g_0 to be the unique smooth Ricci flow $(g(t))_{t\in[0,T)}$ with initial data $g(0) = g_0$ such that any other smooth Ricci flow with the same initial data is a restriction of $g(t)$ to a possibly smaller time interval.

Remark 1.5. Unlike the heat equation, the Ricci flow is a so-called *degenerate* parabolic equation (see Subsection 2.1.1 below). The existence and uniqueness of Ricci flows therefore requires a great deal of care to prove.

Remark 1.6. There is a broad class of research projects that aim to generalize Theorem 1.3 by weakening the assumptions to allow for:

1. non-compact spaces M,
2. non-smooth spaces M,
3. non-smooth initial metrics g_0, or
4. "weak" Ricci flows.

For example, here's a curious open problem:

Question 1.7. *For a general smooth, complete, non-compact, Riemannian 3-manifold (M^3, g_0) with*

$$Rc_{g_0} \geq 0 \qquad \text{on } M,$$

does there exists a unique smooth complete Ricci flow starting from g_0?

There is an affirmative answer to this question if (M, g_0) is additionally assumed to have bounded curvature $\sup_M |Rm| < \infty$ and we restrict the class of Ricci flows $g(t)$ to those with bounded curvature for each time t [Shi89].

As with any PDE, it is important to understand its symmetries or invariances.

Proposition 1.8 (Symmetries of Ricci Flow)**.** *If $(M^n, g(t))$ is a Ricci flow, then so is:*

1. *(Time Shift Invariance) $g(\cdot + t_0)$ for any fixed $t_0 \in \mathbb{R}$,*
2. *(Diffeomorphism Invariance) $\phi^* g(t)$ for any fixed diffeomorphism $\phi : M \to M$, and*
3. *(Parabolic Scaling Invariance) $\lambda^2 g\left(\frac{t}{\lambda^2}\right)$ for any fixed $\lambda \neq 0$.*

Additionally, if $(M, g(t)), (\tilde{M}, \tilde{g}(t))$ are Ricci flows, then the product metric $g(t) \oplus \tilde{g}(t)$ is a Ricci flow on $M \times \tilde{M}$.

Example 1.9. If (M, g_0) is an Einstein manifold, (i.e. $Rc_{g_0} = \lambda g_0$ for some constant $\lambda \in \mathbb{R}$), then

$$g(t) = (1 - 2\lambda t)g_0$$

is a Ricci flow with $g(0) = g_0$.

Notice that, when $\lambda > 0$ (e.g. spheres with the round metric), the metric shrinks. In particular, the Ricci flow starting from an Einstein metric with $\lambda > 0$ does not exist for all positive times $t > 0$. On the other hand, when $\lambda < 0$ (e.g. hyperbolic space), the metric expands.

1.1.1 Ricci Flow Singularities

The shrinking sphere example shows that, while Ricci flows exist for short-time, Ricci flows may become singular in finite time. Let's make our notion of finite-time singularity precise.

Definition 1.10. A Ricci flow $(g(t))_{t\in(0,T)}$ on a closed manifold M is said to develop a **finite-time singularity at** $T < \infty$ if, for every $\varepsilon > 0$, $g(t)$ does not extend to a smooth Ricci flow defined for $t \in (0, T + \varepsilon)$.

We have the following alternative characterization of singularities.

Theorem 1.11 ([Ham95, Theorem 8.1])**.** *If a Ricci flow $(g(t))_{t\in(0,T)}$ on a closed manifold M develops a finite-time singularity at $T < \infty$, then*

$$\lim_{t\nearrow T} \sup_M |Rm_{g(t)}|_{g(t)} = \infty.$$

In particular, Theorem 1.11 rules out the possibility that a Ricci flow has ∇Rm blowing up while Rm stays bounded.

1.2 Ricci Solitons

Definition 1.12. A **Ricci soliton** (M^n, g, X, λ) is a Riemannian manifold (M^n, g), a vector field X on M, and a constant $\lambda \in \mathbb{R}$ such that

$$Rc + \frac{1}{2}\mathcal{L}_X g = \lambda g. \tag{1.1}$$

A Ricci soliton is said to be:

1. **shrinking** if $\lambda > 0$,
2. **steady** if $\lambda = 0$, and
3. **expanding** if $\lambda < 0$.

A **gradient Ricci soliton** is a Ricci soliton (M^n, g, X, λ) such that $X = \nabla f$ for some smooth function $f : M \to \mathbb{R}$. In this case, f is called a **soliton potential function** and equation (1.1) becomes

$$Rc + \nabla^2 f = \lambda g. \tag{1.2}$$

Example 1.13. If (M^n, g) is an Einstein metric $Rc = \lambda g$, then $(M^n, g, 0, \lambda)$ is a gradient Ricci soliton.

From a flow perspective, Ricci solitons correspond to Ricci flow solutions that evolve self-similarly by scaling and diffeomorphisms.

Proposition 1.14. *If (M, g, X, λ) is a complete[1] Ricci soliton and the vector field X is complete, then there exists a Ricci flow $(M, g(t))_{t\in I}$ such that $g(0) = g$ and*

$$g(t) = (1 - 2\lambda t)\phi_t^* g \qquad (\forall t \in I) \tag{1.3}$$

where $\phi : M \times I \to M$, $\phi(x, t) = \phi_t(x)$ is the smooth family of diffeomorphisms satisfying

$$\begin{cases} \partial_t \phi_t = \frac{1}{1-2\lambda t} X \circ \phi_t & \text{on } M \times I, \\ \phi_0 = Id_M & \text{on } M, \end{cases} \tag{1.4}$$

and $I = \{t \in \mathbb{R} \colon 1 - 2\lambda t > 0\}$.

Conversely, if $(M, g(t))_{t\in I}$ is a Ricci flow defined on some interval $I \ni 0$ and there exist a constant $\lambda \in \mathbb{R}$ and a smooth family of diffeomorphisms $\phi : M \times I \to M$ such that

$$g(t) = (1 - 2\lambda t)\phi_t^* g(0) \qquad (\forall t \in I),$$

then $(M, g(0), X, \lambda)$ is Ricci soliton with vector field X given by $X = \partial_t \phi_t|_{t=0}$.

Ricci solitons are also important because they model finite-time singularities of general Ricci flows.

Theorem 1.15 (Informal Statement, [Bam21])**.** *If a Ricci flow $(g(t))_{t\in[0,T)}$ develops a finite-time singularity at $T < \infty$, then, for any sequence $\lambda_i \searrow 0$, the sequence of Ricci flows given by*

$$g_i(\tau) := \frac{1}{\lambda_i} g(T + \lambda_i \tau) \qquad (\tau < 0)$$

subsequentially converges (in some weak sense) to a gradient shrinking Ricci soliton (which is possibly non-smooth).

Example 1.16. The following are examples of solitons:

1. Euclidean space $(\mathbb{R}^n, g_{Euc}, X = 0, \lambda = 0)$ is a steady soliton since it's Ricci-flat Einstein.
   - This construction generalizes to any Ricci-flat manifold.
2. Euclidean space $(\mathbb{R}^n, g_{Euc})$ can also be regarded as a gradient shrinking/expanding Ricci soliton with potential function $f(x) = \pm\frac{1}{4}\operatorname{dist}^2_{Euc}(x, \mathbf{0})$ and $\lambda = \pm\frac{1}{2}$.
   - This construction generalizes to Ricci-flat cones $(\mathcal{C}, g_{\mathcal{C}})$.
3. Cylinders $(\mathbb{S}^k \times \mathbb{R}^{n-k}, g_{\mathbb{S}^k} \oplus g_{Euc})$ are gradient shrinking solitons with potential function $f(\sigma, z) = \frac{1}{4}\operatorname{dist}^2_{Euc}(z, \mathbf{0})$ and $\lambda = \frac{1}{2}$.
   - This construction generalizes to the case where the sphere factor $(S^k, g_{\mathbb{S}^k})$ is replaced by an Einstein manifold with positive scalar curvature.
4. (FIK Soliton) There is a gradient shrinking Kähler-Ricci soliton on the total space M^4 of the tautological line bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$ over $\mathbb{CP}^n$ [FIK03]. It's asymptotic to a cone at infinity.
5. If $\hat{g}$ is a Riemannian metric on S^3 such that the cone metric

   $$g_{\mathcal{C}} = dr^2 + r^2\hat{g} \qquad \text{on } \mathcal{C}(S^3) = \mathbb{R}_{>0} \times S^3$$

   has non-negative scalar curvature, then there exists a gradient expanding Ricci soliton $(\mathbb{R}^4, g, f, -\frac{1}{2})$ with g asymptotic to $g_{\mathcal{C}}$ at infinity [BC25].

Remark 1.17. There is a general research program that aims to construct examples of Ricci solitons.

[1] A Ricci soliton is said to be **complete** if the underlying Riemannian manifold (M, g) is complete.

1.2.1 Properties of Gradient Solitons

Proposition 1.18 (Soliton Identities)**.** *Let (M^n, g, f, λ) be a connected[2] gradient Ricci soliton $Rc + \nabla^2 f = \lambda g$. Then there exists a constant $C \in \mathbb{R}$ such that, throughout M,*

$$R + \Delta f = \lambda n \tag{1.5}$$
$$\Delta f - |\nabla f|^2 + 2\lambda f = C \tag{1.6}$$
$$\text{and } R + |\nabla f|^2 = 2\lambda f + \lambda n - C. \tag{1.7}$$

Proof. Taking trace of $Rc + \nabla^2 f = \lambda g$ gives the first equation (1.5).

To prove the second equation (1.6), first take the divergence of $Rc + \nabla^2 f = \lambda g$ and use the contracted Bianchi identity and (1.5) to deduce

$$0 = \operatorname{div} Rc + \Delta\nabla f = \frac{1}{2}\nabla R + \Delta\nabla f = -\frac{1}{2}\nabla\Delta f + \Delta\nabla f.$$

We also compute that

$$\begin{aligned}
\Delta\nabla f &= \nabla_a\nabla^a\nabla_j f \\
&= \nabla_a\nabla_j\nabla^a f \\
&= \nabla_j\nabla_a\nabla^a f + R_{aj}{}^a{}_b\nabla^b f \\
&= \nabla_j\Delta f + R_{jb}\nabla^b f \\
&= \nabla_j\Delta f - \nabla_j\nabla_b f\nabla^b f + \lambda g_{jb}\nabla^b f \\
&= \nabla_j\Delta f - \frac{1}{2}\nabla_j|\nabla f|^2 + \lambda\nabla_j f.
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &= -\frac{1}{2}\nabla\Delta f + \Delta\nabla f \\
&= -\frac{1}{2}\nabla\Delta f + \nabla\Delta f - \frac{1}{2}\nabla|\nabla f|^2 + \lambda\nabla f \\
&= \frac{1}{2}\nabla\left(\Delta f - |\nabla f|^2 + 2\lambda f\right).
\end{aligned}$$

Equation (1.6) then follows from the fact M is connected.

Finally, equation (1.7) follows from equations (1.5)–(1.6). □

Remark 1.19. When $\lambda \neq 0$, one can translate the potential function f to $f + C'$ so that (1.7) becomes $R + |\nabla f|^2 = 2\lambda f$.

Henceforth, we will focus on the following class of gradient shrinking Ricci solitons.

Definition 1.20. We'll say $(M^n, \overline{g}, f)$ is a **shrinker** if $(M^n, \overline{g}, f)$ is a complete, connected, non-flat, gradient, shrinking Ricci soliton normalized such that

$$Rc + \nabla^2 f = \frac{1}{2}g \quad \text{and} \quad R + |\nabla f|^2 = f \qquad \text{on } M. \tag{1.8}$$

[2]A soliton is **connected** if the underlying manifold M is connected.

For shrinkers, equations (1.5) and (1.6) become

$$R + \Delta f = \frac{n}{2} \quad \text{and} \tag{1.9}$$

$$\Delta f - |\nabla f|^2 + f = \frac{n}{2}. \tag{1.10}$$

Proposition 1.21 (Properties of Shrinkers)**.** *Let $(M^n, \overline{g}, f)$ be a shrinker. Then it satisfies the following properties:*

1. *([Che09, Corollary 2.5]) Throughout M, the scalar curvature $R > 0$ and $f = R + |\nabla f|^2 > 0$ is positive.*
2. *([CZ10, Theorem 1.1]) If $p_0 \in M$, then there are constants $c_\pm \in \mathbb{R}$ depending on the shrinker $(M^n, \overline{g}, f)$ and p_0 such that*

$$\frac{1}{4}\left(\mathrm{dist}_{\overline{g}}(p_0, x) - c_-\right)^2 \leq f(x) \leq \frac{1}{4}\left(\mathrm{dist}_{\overline{g}}(p_0, x) + c_+\right)^2 \qquad \forall x \in M. \tag{1.11}$$

Exercise 1.22. *Let $(M^n, \overline{g}, f)$ be a shrinker. Use Proposition 1.21 to show that, for any $\Gamma \in \mathbb{R}$, the sub-level set $\{x \in M \colon f(x) \leq \Gamma\}$ is compact.*

Exercise 1.23. *Let $(M^n, \overline{g}, f)$ be a shrinker and let $(\phi_t : M \to M)_{t<1}$ be the family of diffeomorphisms given by (1.4), that is,*

$$\partial_t \phi_t = \frac{1}{1-t}\overline{\nabla} f \circ \phi_t \quad (\forall t < 1), \qquad \phi_0 = Id.$$

Prove that

$$0 \leq (1-t)\partial_t(\phi_t^* f) \leq \phi_t^* f \qquad \text{on } M \times (-\infty, 1)$$

and

$$\frac{1}{1-t} f(x) \leq f(\phi_t(x)) \leq f(x) \qquad \forall (x,t) \in M \times (-\infty, 0].$$

1.3 Asymptotically Conical Manifolds

The collection of shrinkers generally will be too broad for our analysis, so we will restrict to a certain subcollection. The first natural choice would be to consider compact (and boundaryless) shrinkers, but it is unclear there exist such spaces.

Question 1.24. *Does there exist a compact (and boundaryless) gradient shrinking Ricci soliton that is not Einstein and not Kähler?*

Instead, we will consider the subcollection of shrinkers that are non-compact and look like cones near infinity.

Definition 1.25. Let (Σ, g_Σ) be a Riemannian manifold. The **cone** over (Σ, g_Σ) is the Riemannian manifold $(\mathcal{C} = \mathcal{C}(\Sigma), g_{\mathcal{C}})$ given by

$$\mathcal{C} = (0, \infty) \times \Sigma, \qquad g_{\mathcal{C}} = dr^2 + r^2 g_\Sigma.$$

(Σ, g_Σ) is called the **link** of the cone $\mathcal{C}$.

Definition 1.26. Given (Σ, g_Σ), a Riemannian manifold (M, g) is said to be **asymptotic to the cone** $(\mathcal{C} = \mathcal{C}(\Sigma), g_{\mathcal{C}})$ over (Σ, g_Σ) if there exists a compact subset $K \subseteq M$ and a diffeomorphism

$$\psi : (1, \infty) \times \Sigma \to M \setminus K$$

such that

$$\lim_{r\to\infty} \sup_{\sigma\in\Sigma} r^l |\nabla^l_{g_{\mathcal{C}}}(\psi^* g - g_{\mathcal{C}})|_{g_{\mathcal{C}}} = 0 \qquad \text{for all } l \in \mathbb{N}.$$

We'll say a Riemannian manifold (M, g) is **asymptotically conical (AC)** if there exists some closed (but not necessarily connected) Riemannian manifold (Σ, g_Σ) such that (M, g) is asymptotic to the cone over (Σ, g_Σ).

In what follows, we shall primarily work with asymptotically conical shrinkers $(M^n, \overline{g}, f)$. The combination of the asymptotically conical and shrinker structure gives these spaces some nice properties worth highlighting.

Proposition 1.27 (Quadratic Curvature Decay)**.** *If $(M^n, \overline{g}, f)$ is an AC shrinker, then there exists $C > 0$ such that*

$$|\overline{Rm}|_{\overline{g}} \leq \frac{C}{f} \qquad \text{on } M.$$

Proof. Fix $p_0 \in M$ and let $r = \mathrm{dist}_{\overline{g}}(\cdot, p_0)$ denote the distance function to p_0. Since $(M^n, \overline{g}, f)$ is asymptotically conical, its curvature decays like $\frac{1}{r^2}$ near infinity. Combined with the fact that $f > 0$ grows like r^2 near infinity (Proposition 1.21), it follows that

$$|\overline{Rm}|_{\overline{g}} \lesssim \frac{1}{1 + r^2} \lesssim \frac{1}{f}.$$

□

Remark 1.28. Let $(M, \overline{g}, f)$ be an AC shrinker, $p_0 \in M$, and $r = \mathrm{dist}_{\overline{g}}(\cdot, p_0)$ denote the distance function to p_0. Proposition 1.21 states $f \approx \frac{1}{4} r^2$ near infinity. The asymptotically conical structure of $(M, \overline{g})$ can be used to show that $\mathrm{Vol}(B(p_0, r)) \sim r^n$ near infinity. These properties allow one to explicitly estimate integrals involving f.

Kotschwar-Wang showed the following rigidity statement which says that AC shrinkers are essentially determined by their asymptotic cone.

Theorem 1.29 ([KW15])**.** *Let $(M_1, \overline{g}_1, f_1)$ and $(M_2, \overline{g}_2, f_2)$ be two AC shrinkers. If they are asymptotic to the same cone, then they are isometric outside a compact set and their universal covers are isometric.*

Corollary 1.30. *No AC shrinker is asymptotic to a Ricci-flat cone.*

Proof. Suppose an AC shrinker were asymptotic to a Ricci-flat cone. [KW15] proved a stronger version of the rigidity statement Theorem 1.29 that applies to incomplete gradient shrinking Ricci solitons. Since the Ricci-flat cone is a (incomplete) gradient shrinking Ricci soliton, this rigidity result then implies the AC shrinker is Ricci-flat outside a compact set. That, however, contradicts the fact that shrinkers have positive scalar curvature (Proposition 1.21). □

There is also a convenient characterization of which shrinkers are asymptotically conical due to Munteanu-Wang.

Theorem 1.31 ([MW17])**.** *If $(M, \overline{g}, f)$ is a non-compact shrinker such that $\overline{Rc}$ limits to 0 at infinity, then $(M, \overline{g})$ is asymptotically conical.*

Chapter 2

Linear Analysis

The remainder of the mini-course will be dedicated to understanding Ricci flows of metrics that are *close to* an asymptotically conical shrinker metric. We will ultimately prove a simplified version of the main result from [Sto26], which says that, given any AC shrinker, there exists a Ricci flow on a closed manifold that forms a finite-time singularity modeled on the given AC shrinker.

2.1 Linearized Ricci Flow

2.1.1 The Linearization of the Ricci Tensor

To understand the Ricci flows close to a shrinker, we require the linearization of the Ricci flow equation (RF) at a shrinker and thus the linearization of the Ricci tensor.

Unfortunately, the function $g \mapsto Rc_g$ is not a nice a operator because it is not a so-called elliptic operator. A quick way to see this is through the diffeomorphism invariance of the Ricci tensor. Consider a family of diffeomorphisms ϕ_s with $\phi_0 = \mathrm{Id}$ and $\partial_s|_{s=0}\phi_s = X \in \mathcal{T}_1(M)$. Then

$$Rc_{\phi_s^* g} = \phi_s^* Rc_g$$

and taking $\partial_s|_{s=0}$ of both sides gives

$$dRc_g[\mathcal{L}_X g] = \mathcal{L}_X Rc_g.$$

If $g \mapsto Rc_g$ were elliptic, then $dRc_g[\mathcal{L}_X g]$ would be 3rd-order in the vector field X. However, $\mathcal{L}_X Rc_g$ is only 1st-order in X. Thus, $g \mapsto Rc_g$ is not elliptic, and the Ricci flow isn't a strictly parabolic equation.

Because Rc isn't an elliptic operator, it is often more convenient to perform what's called *DeTurck's trick* and consider a closely related operator.

Definition 2.1. Let $\overline{g}, g$ be Riemannian metrics on a manifold M. The **DeTurck vector field** $B_{\overline{g}}(g) \in \mathcal{T}_1(M)$ is the vector field on M defined, in local coordinates, by

$$B_{\overline{g}}(g)^k = g^{ij}\left(\Gamma(g)_{ij}^k - \Gamma(\overline{g})_{ij}^k\right)$$

where $\Gamma(g)_{ij}^k, \Gamma(\overline{g})_{ij}^k$ denote the Christoffel symbols of the metrics $g, \overline{g}$ respectively.

Exercise 2.2. *Recall that Christoffel symbols are not the components of a tensor field on M. Nonetheless, show that the local coordinate expression for $B_{\overline{g}}(g)$ does define a well-defined vector field on M.*

Lemma 2.3 ([Shi89, Lemma 2.1])**.** *Fix a Riemannian manifold $(M^n, \overline{g})$. If g is a Riemannian metric on M and $h = g - \overline{g}$, then*

$$\begin{aligned}-2Rc_g + \mathcal{L}_{B_{\overline{g}}(g)}g = &-2\overline{Rc}_{ij} + g^{ab}\overline{\nabla}_a\overline{\nabla}_b h_{ij} + 2\overline{R}_i{}^k{}_j{}^l h_{kl} - \overline{Rc}_j^p h_{ip} - \overline{Rc}_i^p h_{jp}\\ &+ \overline{R}_{ja}{}^p{}_b\left(\overline{g}_{ip}\tilde{h}^{ab} + \hat{h}^{ab}h_{ip}\right) + \overline{R}_{ia}{}^p{}_b\left(\overline{g}_{jp}\tilde{h}^{ab} + \hat{h}^{ab}h_{jp}\right)\\ &+ g^{ab}g^{pq}\left(\overline{\nabla}h\#\overline{\nabla}h\right)_{abpqij}\end{aligned}$$

where the tensor fields $\hat{h}, \tilde{h}$ are defined by

$$g^{ab} = \overline{g}^{ab} - \hat{h}^{ab} \quad \text{and} \quad \hat{h}^{ab} = \overline{g}^{ak}\overline{g}^{bl}h_{kl} + \tilde{h}^{ab},$$

the indices on the curvature terms are raised/lowered with respect to $\overline{g}$, and $\overline{\nabla}h\#\overline{\nabla}h$ here denotes some linear combination of tensors of the form $\overline{\nabla}h \otimes \overline{\nabla}h$ with its entries permuted.

Remark 2.4. Explicitly, $\overline{\nabla}h\#\overline{\nabla}h$ in this case is given by

$$\begin{aligned}2(\overline{\nabla}h\#\overline{\nabla}h)_{abpqij} = (\overline{\nabla}_i h_{pa})(\overline{\nabla}_j h_{qb}) + 2(\overline{\nabla}_a h_{jp})(\overline{\nabla}_q h_{ib}) - 2(\overline{\nabla}_a h_{jp})(\overline{\nabla}_b h_{iq})\\ - 2(\overline{\nabla}_j h_{pa})(\overline{\nabla}_b h_{iq}) - 2(\overline{\nabla}_i h_{pa})(\overline{\nabla}_b h_{jq}).\end{aligned} \tag{2.1}$$

Remark 2.5. If $|h|_{\overline{g}} \ll 1$ is small, then

$$|\hat{h}|_{\overline{g}} \sim |h|_{\overline{g}} \qquad \text{and} \qquad |\tilde{h}|_{\overline{g}} \sim |h|^2_{\overline{g}}.$$

In particular, if $|h|_{\overline{g}} \ll 1$, then

$$-2Rc_g + \mathcal{L}_{B_{\overline{g}}(g)} = -2\overline{Rc} + \overline{\Delta}h + \overline{Rm} * h + \mathcal{Q}(h)$$

where

$$|\mathcal{Q}(h)|_{\overline{g}} \lesssim \left(|h|_{\overline{g}} + |\overline{\nabla}h|_{\overline{g}} + |\overline{\nabla}^2 h|_{\overline{g}}\right)^2.$$

2.1.2 Linearized Ricci Flow Near a Shrinker

Let $(M^n, \overline{g}, f)$ be a shrinker. The Ricci flow starting from $\overline{g}$ is given by

$$(1-t)\phi_t^*\overline{g}$$

for the smooth family of diffeomorphisms $\phi_t : M \to M$ satisfying $\partial_t\phi_t = \frac{1}{1-t}\overline{\nabla}f \circ \phi_t$ (Theorem 1.14). We wish to consider nearby Ricci flows. To that end, it will be convenient to transform the Ricci flow into an equation for which $\overline{g}$ is a stationary solution and which has a well-behaved linearization at $\overline{g}$.

Proposition 2.6 (Informal Statement, see [Sto26, Appendix B] for details)**.** *Let $(M^n, \overline{g}, f)$ be an AC shrinker and $(1-t)\phi_t^*\overline{g}$ its corresponding Ricci flow.*

If $G(t)$ is a Ricci flow on M that is suitably close to $(1-t)\phi_t^*\overline{g}$, then there exists a family of diffeomorphisms $\psi_t : M \to M$ such that

$$g(t) := \frac{1}{1-t}(\phi_t^{-1})^*\psi_t^* G(t)$$

satisfies the ***rescaled and reparametrized Ricci flow (rrRF)***

$$(1-t)\partial_t g = -2Rc_g + \mathcal{L}_{B_{\overline{g}}(g)}g + \mathcal{L}_{\overline{\nabla}f}g + g. \tag{rrRF}$$

Conversely, if $g(t)$ satisfies (rrRF) *and remains suitably close to $\overline{g}$, then there exists a family of diffeomorphisms $\psi_t : M \to M$ such that $G(t) = (1-t)(\psi_t^{-1})^*\phi_t^* g(t)$ solves Ricci flow* (RF).

In light of Proposition 2.6, it will suffice for our purposes to consider (rrRF). By defining a rescaled time variable

$$\tau = -\ln(1-t), \qquad d\tau = \frac{1}{1-t}dt,$$

the rescaled and reparametrized Ricci flow (rrRF) is equivalent to

$$\partial_\tau g = -2Rc_g + \mathcal{L}_{B_{\overline{g}}(g)}g + \mathcal{L}_{\overline{\nabla} f}g + g. \tag{rrRF}$$

Note that the shrinker $g(\tau) \equiv \overline{g}$ defines a stationary solution of (rrRF).

Unlike Ricci flow (RF), the rescaled and reparametrized Ricci flow (rrRF) has the advantage of being a parabolic equation.

Theorem 2.7. *Let $(M^n, \overline{g}, f)$ be a shrinker.*

If $g(\tau) = \overline{g} + h(\tau)$ is a smooth family of Riemmanian metrics such that

$$\partial_\tau g = -2Rc_g + \mathcal{L}_{B_{\overline{g}}(g)}g + \mathcal{L}_{\overline{\nabla} f}g + g, \tag{rrRF}$$

then

$$\partial_\tau h = \overline{\Delta} h - \overline{\nabla}_{\overline{\nabla} f} h + 2\overline{Rm}_i{}^a{}_j{}^b h_{ab} + \mathcal{E}(h) \tag{2.2}$$

where the indices on $\overline{Rm}$ are raised/lowered with respect to $\overline{g}$,

$$\mathcal{E}(h) = -\hat{h}^{ab}\overline{\nabla}_a\overline{\nabla}_b h + \overline{Rm} * \tilde{h} + \overline{Rm} * \hat{h} * h + (\overline{g}+h)^{-1} * (\overline{g}+h)^{-1} * \overline{\nabla} h * \overline{\nabla} h, \tag{2.3}$$

the tensor fields $\hat{h}, \tilde{h}$ are defined by

$$g^{ab} = \overline{g}^{ab} - \hat{h}^{ab} \quad \text{and} \quad \hat{h}^{ab} = \overline{g}^{ak}\overline{g}^{bl}h_{kl} + \tilde{h}^{ab}, \tag{2.4}$$

and the notation $S * T$ denotes some linear combination of tensors formed from taking traces (with respect to $\overline{g}$), raising/lowering indices (with respect to $\overline{g}$), and permuting indices of $S \otimes T$.

Proof. The proof follows from a direct application of Lemma 2.3, the shrinker equations (1.8), and Exercise 2.8 below. □

Exercise 2.8. *Let (M, g) be a Riemannian manifold, $h \in \mathcal{S}^2(M)$ a symmetric 2-tensor field on M, and $X \in \mathcal{T}_1(M)$ a vector field on M. Prove that*

$$\mathcal{L}_X h = \nabla_X h + h(\nabla_\bullet X, \bullet) + h(\bullet, \nabla_\bullet X).$$

In particular, show that if $(M, \overline{g}, f)$ is a shrinker and $h \in \mathcal{S}^2(M)$, then

$$\mathcal{L}_{\overline{\nabla} f} h_{ij} = \overline{\nabla}_{\overline{\nabla} f} h_{ij} + h_{ij} - h_{ik}\overline{Rc}_j^k - h_{jk}\overline{Rc}_i^k.$$

2.2 Spectral Analysis

In this section, we examine the linear part of (2.2) more closely.

Definition 2.9. Let $(M^n, \overline{g}, f)$ be a shrinker.

The **drift Laplacian** $\overline{\Delta}_f$ is the operator

$$\overline{\Delta}_f := \overline{\Delta} - \overline{\nabla}_{\overline{\nabla} f}$$

where $\overline{\Delta}$ is the metric Laplacian $\overline{\Delta} = \overline{\mathrm{div}}(\overline{\nabla}\bullet) = \overline{g}^{ab}\overline{\nabla}_a\overline{\nabla}_b\bullet$.

The **weighted Lichnerowicz Laplacian** $\overline{\Delta}_f + 2\overline{Rm}$ is the operator on symmetric 2-tensor fields $h \in \mathcal{S}^2(M)$ given by

$$\overline{\Delta}_f h + 2\overline{Rm}[h] := \overline{\Delta} h_{ij} - \overline{\nabla}_{\overline{\nabla} f} h_{ij} + 2\overline{Rm}_i{}^a{}_j{}^b h_{ab}. \tag{2.5}$$

in local coordinates.

The divergence theorem shows the drift Laplacian $\overline{\Delta}_f$ is symmetric with respect to the weighted volume measure $e^{-f}dV$ of the shrinker.

Lemma 2.10. *Let $(M^n, \overline{g}, f)$ be a shrinker with volume measure dV.*

If $h, k \in \mathcal{S}^2(M)$ are smooth and compactly supported, then

$$\int_M \langle \overline{\Delta}_f h, k\rangle_{\overline{g}} e^{-f} dV = -\int_M \langle \overline{\nabla} h, \overline{\nabla} k\rangle_{\overline{g}} e^{-f} dV. \tag{2.6}$$

Using Lemma 2.10 with $h = k$, it follows that the weighted Lichnerowicz Laplacian arises as the first variation of the energy functional

$$E[h] := \frac{1}{2}\left(-\int_M |\overline{\nabla} h|^2 e^{-f} dV + \int_M \langle 2\overline{Rm}[h], h\rangle e^{-f} dV\right).$$

The energy E consists of a Dirichlet energy term $\int |\overline{\nabla} h|^2 e^{-f} dV$ and zeroth order term $\int \langle 2Rm[h], h\rangle e^{-f} dV$. For an AC shrinker, the weighted volume form near infinity informally looks like

$$e^{-f} dV \approx e^{-\frac{1}{4}r^2} r^{n-1} dA$$

where $r = \mathrm{dist}_{\overline{g}}(p_0, \cdot)$ is a radial distance function (see Proposition 1.21). In particular, AC shrinkers have finite weighted volume $\int_M e^{-f} dV < \infty$.

Therefore, the weighted Lichnerowicz Laplacian $\overline{\Delta}_f + 2Rm$ should in some sense behave like a linear elliptic operator $\Delta + c$ on a bounded domain $\Omega \subseteq \mathbb{R}^n$. This intuition indeed holds true for the spectrum of $\overline{\Delta}_f + 2Rm$.

Theorem 2.11 (Spectral Basis, [Sto26, Theorem 2.18])**.** *Let $(M^n, \overline{g}, f)$ be an AC shrinker and let $L^2_f(M)$ denote the Hilbert space of sections $h : M \to S^2M$ with $\int_M |h|^2_{\overline{g}} e^{-f} dV < \infty$.*

There exists an L^2_f-orthonormal basis $\{h_j\}_{j=1}^\infty$ of $L^2_f(M)$ such that, for every $j \in \mathbb{N}$, h_j is a smooth eigenfunction of $\overline{\Delta}_f + 2\overline{Rm}$ with eigenvalue $\lambda_j \in \mathbb{R}$,

$$\overline{\Delta}_f h_j + 2Rm[h_j] = \lambda_j h_j.$$

Moreover, any eigenvalue occurs with finite multiplicity and

$$\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j \geq \cdots \to -\infty.$$

Note that some eigenvalues λ_j may be positive or 0. However, there are only finitely many j such that $\lambda_j \geq 0$. We can even describe some eigenmodes explicitly.

Proposition 2.12. *Let $(M^n, \overline{g}, f)$ be a shrinker. Then*

$$\overline{\Delta}_f \overline{Rc} + 2\overline{Rm}[\overline{Rc}] = \overline{Rc} \qquad \text{and}$$
$$\overline{\Delta}_f \overline{\nabla}^2 f + 2\overline{Rm}[\overline{\nabla}^2 f] = 0.$$

Remark 2.13. Let h_j denote eigenfunctions with eigenvalues λ_j as in Theorem 2.11. If the rescaled and reparametrized Ricci flow of $g(\tau) = \overline{g} + h(\tau)$ had no nonlinear terms, that is,

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \cancel{\mathcal{E}(h)},$$

then

$$h(\tau) = \sum_j c_j e^{\lambda_j \tau} h_j$$

would be a solution defined for all $\tau \in \mathbb{R}$. In particular, $h(\tau) = e^{\lambda_j \tau} h_j$ with $\lambda_j < 0$ would give a solution such that $g(\tau) \to \overline{g}$ as $\tau \to \infty$.

The goal of the mini-course will be to show that one perturb this approximate solution to an actual solution of (rrRF) (and thus the Ricci flow) with similar dynamics. To do so requires a collection of PDE bounds that are the subject of subsequent chapters.

Chapter 3

Local Estimates

In this chapter, we develop estimates for rescaled and reparametrized Ricci flows that are close to an asymptotically conical shrinker. The main result of this chapter occurs in Section 3.4. It provides C^2 estimates for the solution on a large pre-compact set assuming coarser global closeness to the shrinker.

Throughout, we assume the following conditions:

Assumption 3.1.
- $(M^n, \overline{g}, f)$ is an AC shrinker,
- $\{h_j\}_{j\in\mathbb{N}}$ is an L^2_f-orthonormal basis of eigenmodes with eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ as in Theorem 2.11,
- $\lambda_* < 0$ is a constant such that $\lambda_* \notin \{\lambda_j\}_{j=1}^{\infty}$, and
- $K \in \mathbb{N}$ is the unique index such that $\lambda_K > \lambda_* > \lambda_{K+1}$.

Additionally, all constants will implicitly depend on $M^n, \overline{g}, f$ and λ_*.

Let us establish some notation used throughout the remainder of the mini-course.

Definition 3.2. Let $(M^n, \overline{g}, f)$ be a shrinker.

For a smooth tensor field T defined on an open subset $\Omega \subseteq M$, we define C^k-norms

$$\|T\|_{C^k(\Omega)} := \sum_{j=0}^{k} \sup_{\Omega} |\overline{\nabla}^j T|, \tag{3.1}$$

weighted L^2-norms

$$\|T\|_{L^2_f(\Omega)} := \left(\int_\Omega |T|^2 e^{-f} dV\right)^{1/2} \tag{3.2}$$

and weighted Sobolev norms

$$\|T\|_{H^k_f(\Omega)} := \left(\sum_{j=0}^{k} \int_\Omega |\overline{\nabla}^j T|^2 e^{-f} dV\right)^{1/2}. \tag{3.3}$$

For an interval $I \subseteq \mathbb{R}$ and a smooth tensor field T defined on $\Omega \times I \subseteq M \times \mathbb{R}$, define $L^2H^k_f$-norms

$$\|T\|_{L^2H^k_f(\Omega\times I)} := \left(\int_I \|T(\tau)\|^2_{H^k_f(\Omega)} d\tau\right)^{1/2} = \left(\sum_{j=0}^{k} \int_{\Omega\times I} |\overline{\nabla}^j T|^2 e^{-f} dV d\tau\right)^{1/2}, \tag{3.4}$$

and $L^\infty H^k_f$-norms

$$\|T\|_{L^\infty H^k_f(\Omega\times I)} := \operatorname*{ess\,sup}_{\tau\in I} \|T(\tau)\|_{H^k_f(\Omega)}. \tag{3.5}$$

Remark 3.3. The above "norms" are always defined for smooth tensor fields, but they could be $+\infty$. So, strictly speaking, they are *not* norms. They will, however, define norms if one restricts to compactly supported tensor fields for example.

One could consider associated Banach spaces formed from completions with respect to these norms, but we will typically not require such spaces.

3.1 Evolution Equations for Derivatives

Lemma 3.4. *Assume Assumption 3.1, $\Omega \Subset M$ is open, $\tau_1 < \tau_2$, and h is a smooth solution of*

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E} \qquad \text{on } \Omega \times (\tau_1, \tau_2).$$

Then, for any $m \in \mathbb{N}_{\geq 0}$, the derivatives $\overline{\nabla}^m h$ satisfy evolution equations of the form

$$\partial_\tau \overline{\nabla}^m h = \overline{\Delta}_f \overline{\nabla}^m h + \sum_{l=0}^{m+1} B_l^m * \overline{\nabla}^l h - \overline{\operatorname{div}}_f \left(\hat{h} * \overline{\nabla}^{m+1} h\right) \qquad \text{on } \Omega \times (\tau_1, \tau_2)$$

where the tensor fields B_l^m satisfy the following estimate: if $0 < \varepsilon \ll 1$ is sufficiently small (depending on m), then

$$\|h(\tau)\|_{C^{m+1}(\Omega)} < \varepsilon \implies \sum_{l=0}^{m+1} \|B_l^m\|_{C^0(\Omega)} \leq C(m, \Omega).$$

Proof. For the proof, we will omit overlines to simplify the notation. It will be understood that all derivatives and curvatures are taken with respect to $\overline{g}$.

For the $m = 0$ case, first rewrite the 2nd-order term in $\mathcal{E}$ as

$$\begin{aligned} -\hat{h}^{ab}\nabla_a\nabla_b h_{ij} =& -\nabla_a\left(\hat{h}^{ab}\nabla_b h_{ij}\right) + \nabla_a\hat{h}^{ab}\nabla_b h_{ij} + \hat{h}^{ab}\nabla_b h_{ij}\nabla_a f - \hat{h}^{ab}\nabla_b h_{ij}\nabla_a f \\ =& -\overline{\operatorname{div}}_f(\hat{h} * \nabla h) + \nabla\hat{h} * \nabla h - \hat{h} * \nabla f * \nabla h. \end{aligned}$$

The evolution equation for h can then be written as

$$\partial_\tau h = \Delta_f h + B_1^0 * \nabla h + B_0^0 * h - \operatorname{div}_f(\hat{h} * \nabla h).$$

where

$$\begin{aligned} B_1^0 &= \nabla h + \hat{h} * \nabla h + \hat{h} * \hat{h} * \nabla h + \nabla\hat{h} + \hat{h} * \nabla f, \\ B_0^0 &= Rm + Rm * \hat{h} + Rm * T, \text{ and} \\ T^{akbl} &= g^{ak}(g+h)^{bl} - g^{ak}g^{bl}. \end{aligned}$$

Taking ∇^m of this evolution equation and commuting derivatives then gives

$$\begin{aligned} \partial_\tau \nabla^m h =& \nabla^m \Delta_f h - \nabla^m \operatorname{div}_f(\hat{h} * \nabla h) + \nabla^m(B_1^0 * \nabla h) + \nabla^m(B_0^0 * h) \\ =& \Delta_f \nabla^m h + \sum_{l=0}^{m} \nabla^l Rm * \nabla^{m-l} h \\ &+ \sum_{l=1}^{m} \nabla f * \nabla^{l-1} Rm * \nabla^{m-l} h + \sum_{l=1}^{m} \nabla^{l+1} f * \nabla^{m-l+1} h \\ &+ \sum_{l=0}^{m} \nabla^l B_1^0 * \nabla^{m-l+1} h + \sum_{l=0}^{m} \nabla^l B_0^0 * \nabla^{m-l} h \\ &- \operatorname{div}_f\left(\sum_{l=0}^{m} \nabla^l \hat{h} * \nabla^{m-l+1} h\right) + \sum_{l=0}^{m-1} \nabla^l Rm * \nabla^{m-1-l}(\hat{h} * \nabla h) \end{aligned}$$

where we also used the soliton equation $\nabla^2 f = -Rc + \frac{1}{2}g$ to simplify the commutator of ∇^m and div_f. This evolution equation can be written as

$$\partial_\tau \nabla^m h = \Delta_f \nabla^m h - \mathrm{div}_f\left(\hat{h} * \nabla^{m+1} h\right) + \sum_{l=0}^{m+1} B_l^m * \nabla^l h$$

where the tensors B_l^m are continuous functions of $h, \nabla h, \ldots, \nabla^{m+1} h$, and $x \in M$. Since $\Omega \Subset M$, it follows that

$$\|h\|_{C^{m+1}(\Omega)} < \varepsilon \ll 1 \implies \sum_{l=0}^{m+1} \|B_l^m\|_{C^0(\Omega)} \leq C(m, \Omega).$$

□

3.2 Local Interior Integral Estimates

Lemma 3.5. *Assume Assumption 3.1. Let $m \in \mathbb{N}_{\geq 0}, \Omega \Subset \Omega' \Subset M$ be open, $\tau_0 \leq \tau_1 < \tau_2$, and h be a smooth solution of*

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E} \qquad \text{on } \Omega' \times [\tau_0, \tau_2].$$

1. *If $\tau_0 < \tau_1$, then there exists $\varepsilon > 0$ (depending on m) and $C > 0$ (depending on $m, \Omega, \Omega', \tau_1 - \tau_0$) such that*

$$\sup_{\tau \in [\tau_0, \tau_2]} \|h(\tau)\|_{C^{m+1}(\Omega')} < \varepsilon$$
$$\implies \sup_{\tau \in [\tau_1, \tau_2]} \|\overline{\nabla}^m h(\tau)\|_{L^2_f(\Omega)} + \|h\|_{L^2 H_f^{m+1}(\Omega \times [\tau_1, \tau_2])} < C\|h\|_{L^2 H_f^m(\Omega' \times [\tau_0, \tau_2])}. \quad (3.6)$$

2. *If $\tau_0 = \tau_1$, then there exists $\varepsilon > 0$ (depending on m) and $C > 0$ (depending on m, Ω, Ω') such that*

$$\sup_{\tau \in [\tau_0, \tau_2]} \|h(\tau)\|_{C^{m+1}(\Omega')} < \varepsilon$$
$$\implies \sup_{\tau \in [\tau_0, \tau_2]} \|\overline{\nabla}^m h(\tau)\|_{L^2_f(\Omega)} + \|h\|_{L^2 H_f^{m+1}(\Omega \times [\tau_0, \tau_2])}$$
$$< C\left(\|h\|_{L^2 H_f^m(\Omega' \times [\tau_0, \tau_2])} + \|h(\tau_0)\|_{H_f^m(\Omega')}\right). \quad (3.7)$$

Proof. For the proof, we will omit overlines to simplify the notation. It will be understood that all derivatives and curvatures are taken with respect to $\overline{g}$.

By Lemma 3.4,

$$\partial_\tau \nabla^m h = \Delta_f \nabla^m h + \sum_{l=0}^{m+1} B_l^m * \nabla^l h - \mathrm{div}_f(\hat{h} * \nabla^{m+1} h).$$

Consider first the case where $\tau_1 > \tau_0$. There exists a smooth bump function $\eta : M \times \mathbb{R} \to [0, 1]$ such that

$$\eta \equiv 1 \text{ on } \Omega \times [\tau_1, \tau_2], \qquad \mathrm{supp}(\eta) \subseteq \Omega' \times (\tau_0, +\infty),$$

and the derivatives $\nabla\eta, \partial_\tau \eta$ are bounded by constants depending on $m, \Omega, \Omega', \tau_1 - \tau_0$. We compute and estimate the time derivative of

$$\int \eta^2 |\nabla^m h|^2 e^{-f} = \int_M \eta^2 |\nabla^m h|^2 e^{-f} dV_g.$$

In what follows,

- $C > 0$ is a constant that depends on m (and the soliton $M^n, \overline{g}, f$),
- $C' > 0$ is a constant that depends on $m, \Omega, \Omega', \tau_1 - \tau_0$ (and the soliton), and
- $C'_\delta > 0$ is a constant that depends on $\delta, m, \Omega, \Omega', \tau_1 - \tau_0$ (and the soliton).

These constants may change from line to line.

$$
\begin{aligned}
&\frac{d}{d\tau}\frac{1}{2}\int \eta^2 |\nabla^m h|^2 e^{-f} \\
=\ &\int \eta^2 \langle \nabla^m h, \partial_\tau \nabla^m h\rangle e^{-f} + \int \eta \partial_\tau \eta |\nabla^m h|^2 e^{-f} \\
&\text{(incorporate the evolution equation for } \nabla^m h) \\
=\ &\int \eta^2 \langle \nabla^m h, \Delta_f h\rangle e^{-f} + \sum_{l=0}^{m+1} \int \eta^2 \langle \nabla^m h, B_l^m * \nabla^l h\rangle e^{-f} \\
&- \int \eta^2 \langle \nabla^m h, \operatorname{div}_f(\hat{h} * \nabla^{m+1} h)\rangle e^{-f} + \int \eta \partial_\tau \eta |\nabla^m h|^2 e^{-f} \\
&\text{(apply the divergence theorem)} \\
=\ &-\int \eta^2 |\nabla^{m+1} h|^2 e^{-f} - \int 2\eta \nabla\eta * \nabla^m h * \nabla^{m+1} h\, e^{-f} + \sum_{l=0}^{m+1} \int \eta^2 \langle \nabla^m h, B_l^m * \nabla^l h\rangle e^{-f} \\
&+ \int \eta^2 * \nabla^{m+1} h * \hat{h} * \nabla^{m+1} h e^{-f} + \int \eta \nabla\eta * \nabla^m h * \hat{h} * \nabla^{m+1} h e^{-f} + \int \eta \partial_\tau \eta |\nabla^m h|^2 e^{-f} \\
\le\ &-\int \eta^2 |\nabla^{m+1} h|^2 e^{-f} + C\int \eta |\nabla\eta| |\nabla^m h| |\nabla^{m+1} h| e^{-f} + C\sum_{l=0}^{m+1} \int \eta^2 |\nabla^m h| |B_l^m| |\nabla^l h| e^{-f} \\
&+ C\int \eta^2 |\hat{h}| |\nabla^{m+1} h|^2 e^{-f} + C\int \eta |\nabla\eta| |\nabla^m h| |\hat{h}| |\nabla^{m+1} h| e^{-f} + \int \eta\eta_\tau |\nabla^m h|^2 e^{-f} \\
&\text{(apply Young's inequality)} \\
\le\ &\left(-1 + C\delta + C\|\hat{h}\|_{C^0(\Omega')}\right) \int \eta^2 |\nabla^{m+1} h|^2 e^{-f} \\
&+ C'_\delta \left(1 + \sum_{l=0}^{m+1} \|B_l^m\|^2_{C^0(\Omega')} + \|\hat{h}\|^2_{C^0(\Omega')}\right) \int \left(|\nabla\eta|^2 + \eta^2 + \eta\eta_\tau\right) \sum_{l=0}^{m} |\nabla^l h|^2 e^{-f} \\
&\text{(use properties of } \eta) \\
\le\ &\left(-1 + C\delta + C\|\hat{h}\|_{C^0(\Omega')}\right) \int \eta^2 |\nabla^{m+1} h|^2 e^{-f} + C'_\delta \left(1 + \sum_{l=0}^{m+1} \|B_l^m\|^2_{C^0(\Omega')} + \|\hat{h}\|^2_{C^0(\Omega')}\right) \|h\|^2_{H_f^m(\Omega')}.
\end{aligned}
$$

The parameter $\delta = \delta(m) > 0$ can be taken small enough so that $C\delta < \frac{1}{4}$. Additionally, if $0 < \varepsilon \ll 1$ is sufficiently small (depending on m and the soliton), then

$$
\|h\|_{C^{m+1}(\Omega')} < \varepsilon \qquad \Longrightarrow \qquad C\|\hat{h}\|_{C^0(\Omega')} < \frac{1}{4} \quad \text{and} \quad \sum_{l=0}^{m+1} \|B_l^m\|^2_{C^0(\Omega')} < C'
$$

by Lemma 3.4. Under these assumptions, it now follows that

$$
\frac{1}{2}\frac{d}{d\tau}\int \eta^2 |\nabla^m h|^2 e^{-f} + \frac{1}{2}\int \eta^2 |\nabla^{m+1} h|^2 e^{-f} \le C' \|h\|^2_{H_f^m(\Omega')}. \tag{3.8}
$$

Integrating (3.8) in time from τ_0 to $\tau \in [\tau_1, \tau_2]$ and using properties of η gives

$$
\sup_{\tau\in[\tau_1,\tau_2]} \frac{1}{2}\int_\Omega |\nabla^m h(\tau)|^2 e^{-f} \le C' \|h\|^2_{L^2 H_f^m(\Omega'\times[\tau_0,\tau_2])}. \tag{3.9}
$$

Integrating (3.8) in time from τ_0 to τ_2 and using properties of η gives

$$\frac{1}{2}\int_{\tau_1}^{\tau_2}\int_{\Omega}|\nabla^{m+1}h|^2e^{-f}dVd\tau \leq C'|h\|^2_{L^2H^m_f(\Omega'\times[\tau_0,\tau_2])}. \tag{3.10}$$

The statement of the lemma now follows from summing (3.9)–(3.10) and adding the quantity $\frac{1}{2}\|h\|^2_{L^2H^m_f(\Omega\times[\tau_1,\tau_2])}$ to both sides.

This completes the proof in the case where $\tau_1 > \tau_0$. The case where $\tau_1 = \tau_0$ is left as an exercise to the reader.

Exercise 3.6. *Prove Lemma 3.5 in the case where $\tau_0 = \tau_1$.*
(Hint: Mimic the proof for $\tau_0 < \tau_1$, but with a different bump function η.)

□

By iterating Lemma 3.5 over a suitable sequence of domains $\Omega = \Omega_0 \Subset \Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega_m = \Omega' \Subset M$ and a similar sequence of time intervals, we can replace the $L^2H^m_f$-norms on the right-hand sides of (3.6), (3.7) with $L^2L^2_f$-norms. The proof is left as an exercise to the reader.

Corollary 3.7 ($L^\infty H^m_f(Small) + L^2H^{m+1}_f(Small) \lesssim L^2L^2_f(Big) + L^2H^m_f(Initial)$)**.** *Assume Assumption 3.1. Let $m \in \mathbb{N}_{\geq 0}, \Omega \Subset \Omega' \Subset M$ be open, $\tau_0 \leq \tau_1 < \tau_2$, and h be a smooth solution of*

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E} \qquad \text{on } \Omega' \times [\tau_0, \tau_2].$$

1. *If $\tau_0 < \tau_1$, then there exists $\varepsilon > 0$ (depending on m) and $C > 0$ (depending on $m, \Omega, \Omega', \tau_1 - \tau_0$) such that*

$$\sup_{\tau\in[\tau_0,\tau_2]}\|h(\tau)\|_{C^{m+1}(\Omega')} < \varepsilon$$
$$\implies \|h\|_{L^\infty H^m_f(\Omega\times[\tau_1,\tau_2])} + \|h\|_{L^2H^{m+1}_f(\Omega\times[\tau_1,\tau_2])} < C\|h\|_{L^2L^2_f(\Omega'\times[\tau_0,\tau_2])}. \tag{3.11}$$

2. *If $\tau_0 = \tau_1$, then there exists $\varepsilon > 0$ (depending on m) and $C > 0$ (depending on m, Ω, Ω') such that*

$$\sup_{\tau\in[\tau_0,\tau_2]}\|h(\tau)\|_{C^{m+1}(\Omega')} < \varepsilon$$
$$\implies \|h\|_{L^\infty H^m_f(\Omega\times[\tau_0,\tau_2])} + \|h\|_{L^2H^{m+1}_f(\Omega\times[\tau_0,\tau_2])}$$
$$< C\left(\|h\|_{L^2L^2_f(\Omega'\times[\tau_0,\tau_2])} + \|h(\tau_0)\|_{H^m_f(\Omega')}\right). \tag{3.12}$$

Exercise 3.8. *Prove Corollary 3.7.*

3.3 Local Interior C^m Estimates

There also holds an estimate analogous to Corollary 3.7, but using C^m-norms instead of weighted Sobolev norms. Its proof is based on Schauder-type estimates, so we omit the proof and give only the statement.

Lemma 3.9 ($C^m(Small) \lesssim C^0(Big) + C^{m+1}(Initial)$)**.** *Assume Assumption 3.1. Let $m \in \mathbb{N}_{\geq 0}, \Omega \Subset \Omega' \Subset M$ be open, $\tau_0 \leq \tau_1 < \tau_2$, and h be a smooth solution of*

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E} \qquad \text{on } \Omega' \times [\tau_0, \tau_2].$$

1. *If $\tau_0 < \tau_1$, then there exists $\varepsilon > 0$ (depending on m) and $C > 0$ (depending on $m, \Omega, \Omega', \tau_1 - \tau_0$) such that*

$$\|h\|_{C^0(\Omega' \times [\tau_0, \tau_2])} < \varepsilon \quad \Longrightarrow \quad \sup_{\tau \in [\tau_1, \tau_2]} \|h\|_{C^m(\Omega)} < C \|h\|_{C^0(\Omega' \times [\tau_0, \tau_2])}. \tag{3.13}$$

2. *If $\tau_0 = \tau_1$, then there exists $\varepsilon > 0$ (depending on m) and $C > 0$ (depending on m, Ω, Ω') such that*

$$\begin{aligned} &\|h\|_{C^0(\Omega' \times [\tau_0, \tau_2])} + \|h(\tau_0)\|_{C^{m+1}(\Omega')} < \varepsilon \\ &\qquad \Longrightarrow \sup_{\tau \in [\tau_0, \tau_2]} \|h\|_{C^m(\Omega)} < C \left(\|h\|_{C^0(\Omega' \times [\tau_0, \tau_2])} + \|h(\tau_0)\|_{C^{m+1}(\Omega')} \right). \end{aligned} \tag{3.14}$$

3.4 Local C^2 Estimate

We now proceed to state and prove the main estimate of this chapter. It yields C^2 estimates on a large pre-compact set assuming weaker bounds on the solution and its initial data. The idea of the proof is to combine the integral estimates with the Sobolev inequality to achieve C^2 estimates.

Theorem 3.10. *Assume Assumption 3.1, $\tau_0 < \tau_1$, and h is a smooth solution of*

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E} \qquad \text{on } M \times [\tau_0, \tau_1]. \tag{3.15}$$

If

$$\|h(\tau)\|_{C^0(M)} \leq \varepsilon \qquad \forall \tau \in [\tau_0, \tau_1] \quad (\textit{Coarse } C^0 \textit{ Estimate}) \tag{3.16}$$

$$\|h(\tau)\|_{L^2_f(M)} \leq \mu e^{\lambda_* \tau} \qquad \forall \tau \in [\tau_0, \tau_1] \quad (\textit{Fine } L^2_f \textit{ Estimate}) \tag{3.17}$$

$$\|h(\tau_0)\|_{C^{4n+2}(\{f < 3\Gamma\})} \leq \mu e^{\lambda_* \tau_0} \qquad (\textit{Fine Local Initial Data Estimate}) \tag{3.18}$$

then

$$\|h(\tau)\|_{C^2(\{f < \Gamma\})} \leq C_\Gamma \mu e^{\lambda_* \tau} \qquad \forall \tau \in [\tau_0, \tau_1] \qquad (\textit{Fine Local } C^2 \textit{ Estimate}) \tag{3.19}$$

for some constant C_Γ (depending only on Γ, the soliton $(M^n, \overline{g}, f)$, and λ_*), so long as $\Gamma \gg 1$ is sufficiently large (depending only on the soliton $(M^n, \overline{g}, f)$ and λ_*), $0 < \varepsilon \ll 1$ is sufficiently small (depending only on Γ, the soliton, and λ_*), and $\tau_0 \gg 1$ is sufficiently large (depending only on Γ, μ, the soliton, and λ_*)

Proof. For the proof, we will omit overlines to simplify the notation. It will be understood that all derivatives and curvatures are taken with respect to $\overline{g}$.

For $i \in \mathbb{N}$, define the open subsets

$$\Omega_i := \{f < i \cdot \Gamma\} \subseteq M.$$

If $\Gamma \gg 1$ is sufficiently large (depending only on the soliton $M^n, \overline{g}, f$), then $\Omega_i \Subset M$ for all $i \in \mathbb{N}$ and

$$\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset \cdots \Subset \Omega_i \Subset \ldots.$$

Let $\tau \in [\tau_0, \tau_1]$. There are two cases to consider based on whether or not $\tau \geq \tau_0 + 1$.

Case 1: (Long-Time) $\tau \geq \tau_0 + 1$

The Local Interior C^m Estimate (Lemma 3.9) and the assumed Coarse C^0 Estimate (3.16) imply

$$\sup_{[\tau-\frac{1}{2},\tau]} \|h\|_{C^{4n+1}(\Omega_2)} \lesssim_\Gamma \|h\|_{C^0(\Omega_3\times[\tau-1,\tau])} \leq \varepsilon \tag{3.20}$$

so long as $0 < \varepsilon \ll 1$ is sufficiently small. In particular, estimate (3.20) implies the hypotheses of the Local Interior Integral Estimate (3.11) hold when $0 < \varepsilon \ll 1$ is sufficiently small depending on Γ. Therefore,

$$\begin{aligned}
\|h(\tau)\|_{C^2(\Omega_1)} &\lesssim_\Gamma \|h(\tau)\|_{H^{4n}_f(\Omega_1)} && \text{(Sobolev Inequality)}\\
&\lesssim_\Gamma \|h\|_{L^2L^2_f(\Omega_2\times[\tau-\frac{1}{2},\tau])} && \text{(Corollary 3.7, (3.11))}\\
&\leq \|h\|_{L^\infty L^2_f(\Omega_2\times[\tau-\frac{1}{2},\tau])}\\
&\lesssim \mu e^{\lambda_*\tau} && \text{(by the Fine } L^2_f \text{ Estimates (3.17)).}
\end{aligned}$$

Case 2: (Short-Time) $\tau_0 \leq \tau \leq \tau_0 + 1$

Lemma 3.9, the assumed coarse C^0 estimate (3.16), and the assumed initial data estimate (3.18) imply

$$\sup_{[\tau_0,\tau]} \|h\|_{C^{4n+1}(\Omega_2)} \lesssim_\Gamma \|h\|_{C^0(\Omega_3\times[\tau_0,\tau])} + \|h(\tau_0)\|_{C^{4n+2}(\Omega_3)} \leq \varepsilon + \mu e^{\lambda_*\tau_0} \tag{3.21}$$

so long as $0 < \varepsilon \ll 1$ is sufficiently small. In particular, estimate (3.21) implies the hypotheses of the local interior integral estimate (3.12) hold when $0 < \varepsilon \ll 1$ is sufficiently small depending on Γ, and $\tau_0 \gg 1$ is sufficiently large depending on Γ and μ (and the soliton and $\lambda_* < 0$). Therefore,

$$\begin{aligned}
\|h(\tau)\|_{C^2(\Omega_1)} &\lesssim_\Gamma \|h(\tau)\|_{H^{4n}_f(\Omega_1)} && \text{(Sobolev Inequality)}\\
&\lesssim_\Gamma \|h\|_{L^2L^2_f(\Omega_2\times[\tau_0,\tau])} + \|h(\tau_0)\|_{H^{4n}_f(\Omega_2)} && \text{(Corollary 3.7, (3.12))}\\
&\lesssim_\Gamma \|h\|_{L^\infty L^2_f(\Omega_2\times[\tau_0,\tau])} + \|h(\tau_0)\|_{C^{4n+2}(\Omega_3)}\\
&\lesssim \mu e^{\lambda_*\tau}
\end{aligned}$$

where the last line follows from the assumed fine L^2_f estimate (3.17), the assumed initial data estimate (3.18), and the fact that $\tau_0 \leq \tau \leq \tau_0 + 1$ in this case.

Combining the results of the two cases proves the theorem. □

Chapter 4

Estimates Near Infinity

In this chapter, we continue to assume Assumption 3.1. The previous chapter obtained C^2 bounds on a large pre-compact set. In this chapter, we seek to extend those estimates to global estimates.

The methods for this chapter, however, are completely different from the previous chapter. To obtain estimates near infinity, we shall construct supersolutions that, by the maximum principle, control the C^0 norm of the solution. A rescaling argument then bootstraps that C^0 estimate to a C^2 estimate.

4.1 Supersolutions

Lemma 4.1. *Assume Assumption 3.1. Let $\Omega \subseteq M$ be open, $\tau_1 < \tau_2$, and h be a smooth solution of*

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E}(h) \qquad \text{on } \Omega \times [\tau_1, \tau_2]. \tag{4.1}$$

Then there exist $C, \varepsilon > 0$ (both depending only on the soliton $(M^n, \overline{g}, f)$) such that

$$\begin{aligned} &|h|_{\overline{g}} \leq \varepsilon \quad \text{on } \Omega \times [\tau_1, \tau_2] \\ &\qquad \implies \partial_\tau |h|^2_{\overline{g}} \leq (1 + C|h|_{\overline{g}})\overline{\Delta}|h|^2_{\overline{g}} - \overline{\nabla}_{\overline{\nabla} f}|h|^2_{\overline{g}} + C|\overline{Rm}|_{\overline{g}}|h|^2_{\overline{g}} \quad \text{on } \Omega \times [\tau_1, \tau_2]. \end{aligned} \tag{4.2}$$

Proof. Throughout the proof, $C > 0$ will be a positive constant depending only on the soliton $(M^n, \overline{g}, f)$ and which may change from line to line. Similarly, $0 < \varepsilon \ll 1$ will be a small positive constant depending only on the soliton $(M^n, \overline{g}, f)$ and which may change from line to line.

First, if $|h|_{\overline{g}} \leq \varepsilon$, then the metrics $g = \overline{g} + h$ and $\overline{g}$ are comparable, namely,

$$\frac{1}{2}\overline{g} \leq g \leq 2\overline{g}.$$

It also follows from the definition of $\tilde{h}, \hat{h}$ (2.4) that

$$|\hat{h}|_{\overline{g}} \leq C|h|_{\overline{g}} \quad \text{and} \quad |\tilde{h}|^2_{\overline{g}} \leq C|h|^2_{\overline{g}} \qquad \text{when } |h|_{\overline{g}} \leq \varepsilon.$$

Using the definitions of the error term $\mathcal{E}(h)$ (2.3) and $\hat{h}$ (2.4), it will be convenient to

rewrite

$$\begin{aligned}
&\overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E}(h)\\
&= \overline{g}^{ab}\overline{\nabla}_a\overline{\nabla}_b h - \overline{\nabla}_{\overline{\nabla} f} h + \overline{Rm} * h\\
&\quad - \hat{h}^{ab}\overline{\nabla}_a\overline{\nabla}_b h + \overline{Rm} * \tilde{h} + \overline{Rm} * \hat{h} * h + (\overline{g}+h)^{-1} * (\overline{g}+h)^{-1} * \overline{\nabla} h * \overline{\nabla} h \quad \text{(by (2.3))}\\
&= (\overline{g}+h)^{ab}\overline{\nabla}_a\overline{\nabla}_b h - \overline{\nabla}_{\overline{\nabla} f} h + \overline{Rm} * h\\
&\quad + \overline{Rm} * \tilde{h} + \overline{Rm} * \hat{h} * h + (\overline{g}+h)^{-1} * (\overline{g}+h)^{-1} * \overline{\nabla} h * \overline{\nabla} h \qquad \text{(by (2.4))}.
\end{aligned}$$

These estimates therefore yield that, on $\Omega \times [\tau_1, \tau_2]$,

$$\begin{aligned}
&\frac{1}{2}\partial_\tau |h|^2_{\overline{g}}\\
&= \langle h, \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E}(h)\rangle_{\overline{g}}\\
&= \langle h, (\overline{g}+h)^{ab}\overline{\nabla}_a\overline{\nabla}_b h\rangle_{\overline{g}} - \langle h, \overline{\nabla}_{\overline{\nabla} f} h\rangle_{\overline{g}} + \overline{Rm} * h * h\\
&\quad + \overline{Rm} * \tilde{h} * h + \overline{Rm} * \hat{h} * h * h + (\overline{g}+h)^{-1} * (\overline{g}+h)^{-1} * \overline{\nabla} h * \overline{\nabla} h * h\\
&= \frac{1}{2}(\overline{g}+h)^{ab}\overline{\nabla}_a\overline{\nabla}_b |h|^2_{\overline{g}} - (\overline{g}+h)^{ab}\langle \overline{\nabla}_a h, \overline{\nabla}_b h\rangle_{\overline{g}} - \frac{1}{2}\overline{\nabla}_{\overline{\nabla} f}|h|^2_{\overline{g}} + \overline{Rm} * h * h\\
&\quad + \overline{Rm} * \tilde{h} * h + \overline{Rm} * \hat{h} * h * h + (\overline{g}+h)^{-1} * (\overline{g}+h)^{-1} * \overline{\nabla} h * \overline{\nabla} h * h\\
&\le \frac{1}{2}(\overline{g}^{ab} - \hat{h}^{ab})\overline{\nabla}_a\overline{\nabla}_b |h|^2_{\overline{g}} - (\overline{g}+h)^{ab}\langle \overline{\nabla}_a h, \overline{\nabla}_b h\rangle_{\overline{g}} - \frac{1}{2}\overline{\nabla}_{\overline{\nabla} f}|h|^2_{\overline{g}} + C|\overline{Rm}|_{\overline{g}}|h|^2_{\overline{g}} \quad \text{(by (2.4))}\\
&\quad + C|\overline{Rm}|_{\overline{g}}|\tilde{h}|_{\overline{g}}|h|_{\overline{g}} + C|\overline{Rm}|_{\overline{g}}|\hat{h}|_{\overline{g}}|h|^2_{\overline{g}} + C|(\overline{g}+h)^{-1}|^2_{\overline{g}}|\overline{\nabla} h|^2_{\overline{g}}|h|_{\overline{g}}\\
&\le \frac{1}{2}(1 + C|h|_{\overline{g}})\overline{g}^{ab}\overline{\nabla}_a\overline{\nabla}_b |h|^2_{\overline{g}} - \frac{1}{2}|\overline{\nabla} h|^2_{\overline{g}} - \frac{1}{2}\overline{\nabla}_{\overline{\nabla} f}|h|^2_{\overline{g}} + C|\overline{Rm}|_{\overline{g}}|h|^2_{\overline{g}}\\
&\quad + C|\overline{Rm}|_{\overline{g}}|h|^3_{\overline{g}} + C|\overline{\nabla} h|^2_{\overline{g}}|h|_{\overline{g}}\\
&\le \frac{1}{2}(1 + C|h|_{\overline{g}})\overline{\Delta}|h|^2_{\overline{g}} + \left(-\frac{1}{2} + C|h|_{\overline{g}}\right)|\overline{\nabla} h|^2_{\overline{g}} - \frac{1}{2}\overline{\nabla}_{\overline{\nabla} f}|h|^2_{\overline{g}} + C|\overline{Rm}|_{\overline{g}}|h|^2_{\overline{g}}
\end{aligned}$$

where the last two inequalities use the facts that $|h|_{\overline{g}} \le \varepsilon, |\hat{h}|_{\overline{g}} \lesssim |h|_{\overline{g}}, |\tilde{h}|_{\overline{g}} \lesssim |h|^2_{\overline{g}}$, and $\overline{g} + h \sim g$. For $0 < \varepsilon \ll 1$ sufficiently small (depending only on the soliton $(M^n, \overline{g}, f)$),

$$|h|_{\overline{g}} \le \varepsilon \quad \Longrightarrow \quad \left(-\frac{1}{2} + C|h|_{\overline{g}}\right)|\overline{\nabla} h|^2_{\overline{g}} \le \left(-\frac{1}{2} + C\varepsilon\right)|\overline{\nabla} h|^2_{\overline{g}} \le 0,$$

and so we deduce in this case that

$$\partial_\tau |h|^2_{\overline{g}} \le (1 + C|h|)\overline{\Delta}|h|^2_{\overline{g}} - \overline{\nabla}_{\overline{\nabla} f}|h|^2_{\overline{g}} + C|\overline{Rm}||h|^2 \qquad \text{on } \Omega \times [\tau_1, \tau_2].$$

□

On a shrinker $(M^n, \overline{g}, f)$, the potential function f behaves like $\text{dist}^2_{\overline{g}}$ (Proposition 1.21) and satisfies the PDE $\overline{\Delta}_f f = -f + \frac{n}{2}$ (1.10). These properties make f a convenient tool to construct supersolutions to (4.2). To construct supersolutions, the following lemma will be convenient.

Lemma 4.2. *Let $(M^n, \overline{g}, f)$ be a shrinker and let $\kappa \in \mathbb{R}$. Then*

$$\begin{aligned}
\overline{\Delta}(f^\kappa) &= \frac{n}{2}\kappa f^{\kappa-1} - \overline{R}\kappa f^{\kappa-1} + \kappa(\kappa-1)f^{\kappa-1}\frac{|\overline{\nabla} f|^2_{\overline{g}}}{f} \text{ and}\\
\overline{\Delta}_f(f^\kappa) &= \frac{n}{2}\kappa f^{\kappa-1} - \kappa f^\kappa + \kappa(\kappa-1)f^{\kappa-1}\frac{|\overline{\nabla} f|^2_{\overline{g}}}{f}.
\end{aligned}$$

Proof. The proof follows from a straightforward computation using the identities (1.8)–(1.10). □

Lemma 4.3 (Suspersolution 1)**.** *Assume Assumption 3.1 and let $C_0 > 0$. For $0 < a < 1$, define $u : M \to \mathbb{R}$ by*

$$u \doteqdot a - \frac{a}{\sqrt{f}}.$$

Then u satisfies

$$\partial_\tau u \geq \left(1 + C_0\sqrt{u}\right)\overline{\Delta}u - \overline{\nabla}_{\overline{\nabla} f} u + C_0|\overline{Rm}|_{\overline{g}} u \qquad \text{on } \{f > \Gamma\} \times \mathbb{R} \text{ and} \tag{4.3}$$

$$\frac{a}{2} < u < a \qquad \text{on } \{f > \Gamma\} \times \mathbb{R} \tag{4.4}$$

if $\Gamma \gg 1$ is sufficiently large depending only on C_0 and the soliton $(M^n, \overline{g}, f)$.

Remark 4.4. How might one guess that these are supersolutions?

Let's ignore nonlinear terms and simply consider

$$\overline{\Delta}u - \overline{\nabla}_{\overline{\nabla} f} u + C_0|\overline{Rm}|_{\overline{g}} u.$$

Plugging in $u = 1$ gives

$$\overline{\Delta}u - \overline{\nabla}_{\overline{\nabla} f} u + C_0|\overline{Rm}|_{\overline{g}} u = C_0|\overline{Rm}|_{\overline{g}} \lesssim \frac{1}{f}$$

where the estimate follows from the quadratic curvature decay of an AC shrinker.

For an AC shrinker, the soliton equations (1.8) and positive scalar curvature (Proposition 1.21) show

$$0 < \frac{|\overline{\nabla} f|^2_{\overline{g}}}{f} = \frac{f - \overline{R}}{f} < 1.$$

Thus, Lemma 4.2 gives

$$\left|\overline{\Delta} f^{-1/2}\right| \lesssim f^{-3/2},$$

$$-\overline{\nabla}_{\overline{\nabla} f} f^{-1/2} = +\frac{1}{2} f^{-3/2} |\overline{\nabla} f|^2_{\overline{g}} = \frac{1}{2} f^{-3/2}\left(f - \overline{R}\right) = \frac{1}{2} f^{-1/2} + O(f^{-5/2}), \text{ and}$$

$$\left|C_0|\overline{Rm}|_{\overline{g}} f^{-1/2}\right| \lesssim f^{-3/2}.$$

By taking a suitable linear combination of 1 and $f^{-1/2}$, the $-\overline{\nabla}_{\overline{\nabla} f} f^{-1/2} \approx \frac{1}{2} f^{-1/2}$ terms dominate near infinity and yield the correct sign. The remaining assumptions $\Gamma \gg 1$ and $0 < a < 1$ are added simply to ensure these $f^{-1/2}$ terms indeed dominate.

Proof. We omit overlines in this proof to simplify the notation.

A straightforward but tedious computation using Lemma 4.2 shows

$$\begin{aligned}
&\partial_\tau u - (1 + C_0\sqrt{u})\Delta u + \nabla_{\nabla f} u - C_0|Rm|u \\
&= f^{-\frac{1}{2}} \cdot \frac{a}{2} + f^{-1} a\left[-C_0|Rm|f\right] + f^{-\frac{3}{2}} a\left[-\frac{n}{4} + \frac{3}{4}\frac{|\nabla f|^2}{f} + C_0|Rm|f\right] \\
&\quad + f^{-\frac{3}{2}} a C_0\sqrt{u}\left[-\frac{n}{4} + \frac{R}{2} + \frac{3}{4}\frac{|\nabla f|^2}{f}\right] \\
&\geq \frac{a}{2} f^{-\frac{1}{2}} - f^{-1} a C_0\left(\sup_M |Rm|f\right) + f^{-3/2} a\left[-\frac{n}{4}\right] + f^{-3/2} a C_0\sqrt{u}\left[-\frac{n}{4}\right]
\end{aligned}$$

where the last inequality uses the facts that $R, f > 0$ on M (Proposition 1.21) and the shrinker equations (1.8) to deduce

$$0 \leq \frac{|\nabla f|^2}{f} = \frac{f - R}{f} < 1.$$

By the definition of u,

$$u < a < 1 \qquad \text{on } M$$

and

$$u > a - \frac{a}{\sqrt{\Gamma}} > \frac{a}{2} > 0 \qquad \text{on } \{(x,\tau)\colon f(x) > \Gamma\}$$

so long as $\Gamma \gg 1$. Thus, on the region where $f > \Gamma$,

$$\begin{aligned}
&\partial_\tau u - (1 + C_0\sqrt{u})\Delta u + \nabla_{\nabla f} u - C_0|Rm|u \\
&\geq \frac{a}{2} f^{-\frac{1}{2}} - f^{-1} a C_0 \left(\sup_M |Rm| f\right) - f^{-3/2} a \frac{n}{4} - f^{-3/2} a C_0 \sqrt{a} \frac{n}{4} \\
&\geq \frac{a}{2} f^{-\frac{1}{2}} - f^{-\frac{1}{2}} \frac{a C_0 (\sup_M |Rm| f)}{\sqrt{\Gamma}} - f^{-\frac{1}{2}} \frac{a \frac{n}{4}}{\Gamma} - f^{-\frac{1}{2}} \frac{a C_0 \sqrt{a} \frac{n}{4}}{\Gamma} \\
&\geq a f^{-\frac{1}{2}} \left[\frac{1}{2} - \frac{C_0 (\sup_M |Rm| f)}{\sqrt{\gamma}} - \frac{n}{4\Gamma} - \frac{n C_0}{4\Gamma}\right]
\end{aligned}$$

This last quantity is positive when $\Gamma \gg 1$ is sufficiently large depending only on C_0, n, and $\sup_M |Rm| f < \infty$. □

4.2 Higher Order Interior Estimates At Infinity

Lemma 4.5. *Assume Assumption 3.1. Let $\tau_0 < \tau_1$ and assume h is smooth solution of*

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E}(h) \qquad \text{on } M \times [\tau_0, \tau_1].$$

If

$$\begin{aligned}
&\|h(\tau)\|_{C^0(\{f>\Gamma\})} \leq \varepsilon \quad \forall \tau \in [\tau_0, \tau_1] && (\textit{Coarse } C^0 \textit{ Estimate Near } \infty) \\
&\textit{and } \|h(\tau_0)\|_{C^3(\{f>\Gamma\})} \leq \varepsilon && (\textit{Coarse Initial } C^3 \textit{ Estimate Near } \infty),
\end{aligned}$$

then

$$\|h(\tau)\|_{C^2(\{f>4\Gamma\})} \leq C\varepsilon \qquad \forall \tau \in [\tau_0, \tau_1]$$

for some $C = C(M^n, \overline{g}, f) > 0$ so long as $0 < \varepsilon \ll 1$ is sufficiently small and $\Gamma \gg 1$ is sufficiently large (both depending only on the soliton $(M^n, \overline{g}, f)$).

Ideally, one could prove this lemma simply by appealing to interior regularity estimates $\|u\|_{C^2(small)} \lesssim \|u\|_{C^0(big)}$ for solutions of linear parabolic equations $\partial_\tau u = a^{ij}(x,t)\partial_i\partial_j u + b^i(x,t)\partial_i u + c(x,t)u$. There are however two problems with this approach:

1. The $\mathcal{E}(h)$ term makes the evolution equation for h **nonlinear**.
2. The constant implicit in "$\lesssim$" generally depends on the C^0-norms of the coefficients a^{ij}, b^i, c. However, the evolution equation for h has a 1st-order term $-\overline{\nabla}_{\overline{\nabla} f} h = -\overline{\nabla}^i f \overline{\nabla}_i h$, and $|\overline{\nabla} f|$ is **not bounded** on an AC shrinker M. Indeed, the soliton equation (1.8) implies $|\overline{\nabla} f|^2 = f - R \sim \text{dist}^2$ near infinity.

The first issue can be remedied by a more robust interior regularity estimate that handles nonlinearities like $\mathcal{E}(h)$ when h is small (see [Sto26, Appendix C]). The second issue is handled by a *rescaling argument*, which is a useful technique in the analysis of PDEs. In order to gain experience with rescaling arguments, we shall give a proof of the above lemma under the simplifying assumption that $\mathcal{E}(h) \equiv 0$.

Proof (when $\mathcal{E}(h) \equiv 0$). Assume additionally that $\mathcal{E}(h) \equiv 0$ so that h satisfies the linear PDE

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] \qquad \text{on } M \times [\tau_0, \tau_1].$$

Fix $\tau_* \in (\tau_0, \tau_1]$. Consider the family $\mathcal{H}(s)$ of symmetric 2-tensor fields on M given by

$$\mathcal{H}(s) := (1-s)\phi_s^*[h(\tau_* - \ln(1-s))], \qquad (1 - e^{\tau_* - \tau_0} \le s \le 1 - e^{\tau_* - \tau_1})$$

where $\phi_s : M \to M$ is the family of diffeomorphisms for the soliton as in (1.4), that is,

$$\partial_s \phi_s = \frac{1}{1-s}\overline{\nabla} f \circ \phi_s, \qquad \phi_0 = \mathrm{Id}_M.$$

Then $\mathcal{H}$ satisfies

$$\begin{aligned}
&\partial_s \mathcal{H} \\
&= -\phi_s^* h + (1-s)\phi_s^* \mathcal{L}_{\partial_s \phi_s \circ \phi_s^{-1}} h + (1-s)\phi_s^* \partial_\tau h \cdot \left(\frac{1}{1-s}\right) \\
&= -\phi_s^* h + \phi_s^* \mathcal{L}_{\overline{\nabla} f} h + \phi_s^* \partial_\tau h \\
&= \phi_s^* \left[\overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{L}_{\overline{\nabla} f} h - h\right] \\
&= \phi_s^* \left[\overline{\Delta} h - \cancel{\overline{\nabla}_{\overline{\nabla} f} h} + 2\overline{Rm}[h] + \cancel{\overline{\nabla}_{\overline{\nabla} f} h} + \cancel{K} + \overline{Rc} * h - \cancel{K}\right] \qquad \text{(by Exercise 2.8)}
\end{aligned}$$

where the expression in square brackets is evaluated at time $\tau_* - \ln(1-s)$. Closer inspection of the $2\overline{Rm}[h] + \overline{Rc} * h$ term above shows that it is of the form

$$2\overline{Rm}[h] + \overline{Rc} * h = \overline{g}^{-1} \# \overline{R} \# h$$

where $\overline{R} = R_{\overline{g}} \in \mathcal{T}^3_1(M)$ here denotes the Riemann curvature as a $\binom{3}{1}$-tensor field.

It now follows from using scaling and pullback properties that

$$\begin{aligned}
\partial_s \mathcal{H} &= \phi_s^* \left[\overline{\Delta} h + \overline{g}^{-1} \# \overline{R} \# h\right] \\
&= (1-s)\Delta_{(1-s)\phi_s^* \overline{g}} \phi_s^* h + (1-s)((1-s)\phi_s^* \overline{g})^{-1} \# R_{(1-s)\phi_s^* \overline{g}} \# \phi_s^* h \\
&= \Delta_{\hat{g}} \mathcal{H} + \hat{g}^{-1} \# \hat{R} \# \mathcal{H}
\end{aligned}$$

where $\hat{g}(s) := (1-s)\phi_s^* \overline{g}$.

Using the fact that f grows quadratically at infinity (Proposition 1.21) and the curvature decays quadratically, one can show the following:

Exercise 4.6. *If $\Gamma \gg 1$ is sufficiently large depending only on the soliton $(M^n, \overline{g}, f)$, then, for any $p \in M$ such that $f(p) > \Gamma$,*

$$B_{\overline{g}}(p, 1) \subseteq \left\{\frac{1}{2} f(p) < f < 2f(p)\right\}$$

and $B_{\overline{g}}(p, 1) \subseteq M$ contains no points conjugate to p.

Now, fix $p_* \in M$ such that $f(p_*) > 4\Gamma$ and denote $s_0 := \max\{-1, 1 - e^{\tau_* - \tau_0}\}$. We consider

$$\partial_s \mathcal{H} = \Delta_{\hat{g}} \mathcal{H} + \hat{g}^{-1} \# \hat{R} \# \mathcal{H} \qquad \text{on } B_{\overline{g}}(p_*, 1) \times [s_0, 0]. \tag{4.5}$$

By Exercise 4.6, $B_{\overline{g}}(p_*, 1)$ has no points conjugate to p_*, and so the exponential map $\overline{\exp}_{p_*}$ is a local diffeomorphism onto $B_{\overline{g}}(p_*, 1)$. After pulling back by this exponential map, the PDE (4.5) for $\mathcal{H}$ becomes a linear parabolic PDE system on the Euclidean domain $B_{Euc}(\mathbf{0}, 1) \times [s_0, 0]$. Moreover, the coefficients of this PDE can be bounded by constants that depend on $(M^n, \overline{g}, f)$ but *are independent of* p_*, τ_*. Therefore, interior Schauder estimates for linear parabolic PDEs imply

$$\|\mathcal{H}\|_{C^{2,\alpha}(B_{\overline{g}}(p_*,1/2)\times[-1/4,0])} \leq C \|\mathcal{H}\|_{C^0(B_{\overline{g}}(p_*,1)\times[-1,0])} \qquad (\text{when } s_0 = -1)$$

and

$$\|\mathcal{H}\|_{C^{2,\alpha}(B_{\overline{g}}(p_*,1/2)\times[s_0,0])} \leq C \left(\|\mathcal{H}\|_{C^0(B_{\overline{g}}(p_*,1)\times[s_0,0])} + \|\mathcal{H}(s_0)\|_{C^{2,\alpha}(B_{\overline{g}}(p_*,1))} \right) \qquad (\text{when } -1 < s_0 < 0)$$

where in both cases the constant $C = C(M^n, \overline{g}, f)$ is *independent of* p_*, τ_*. In the case where $s_0 = -1$, using this estimate and undoing the pullback and rescaling gives the following estimate for h

$$\begin{aligned}
&|\overline{\nabla} h|_{\overline{g}}(p_*, \tau_*) + |\overline{\nabla}^2 h|_{\overline{g}}(p_*, \tau_*) \\
&= |\hat{\nabla} \mathcal{H}|_{\hat{g}}(p_*, 0) + |\hat{\nabla}^2 \mathcal{H}|_{\hat{g}}(p_*, 0) \\
&\leq C \sup_{B_{\overline{g}}(p_*,1)\times[-1,0]} |\mathcal{H}|_{\hat{g}} \\
&\leq C \sup_{\{\frac{1}{2} f(p_*) < f < 2 f(p_*)\} \times [-1,0]} |\mathcal{H}|_{\hat{g}} && \text{(Exercise 4.6)} \\
&= C \sup_{\substack{s \in [-1,0], \\ x \in \phi_s(\{\frac{1}{2} f(p_*) < f < 2 f(p_*)\})}} |h|_{\overline{g}}(x, \tau_* - \ln(1-s)) \\
&\leq C \sup_{\{\frac{1}{4} f(p_*) < f < 4 f(p_*)\} \times [\tau_0, \tau_*]} |h|_{\overline{g}}(x, \tau) && \text{(Exercise 1.23)} \\
&\leq C\varepsilon.
\end{aligned}$$

A similar argument also applies in the case that $-1 < s_0 < 0$ to show that

$$|\overline{\nabla} h|_{\overline{g}}(p_*, \tau_*) + |\overline{\nabla} h|_{\overline{g}}(p_*, \tau_*) \leq 2C\varepsilon.$$

Since $(p_*, \tau_*) \in \{f > 4\Gamma\} \times (\tau_0, \tau_1]$ was arbitrary and the constant C above is independent of (p_*, τ_*), we have therefore obtained that

$$\|h\|_{C^2(\{f > 4\Gamma\})} \leq 2C\varepsilon \qquad \forall \tau \in [\tau_0, \tau_1].$$

□

4.3 An Additional Supersolution and Interior Estimate

Lemma 4.7 (Supersolution 2)**.** *Assume Assumption 3.1 and let $C_0, \gamma > 0$. For $A, \kappa > 0$, define $u : M \to \mathbb{R}$ by*

$$u \doteqdot e^{-\kappa\tau} \left(A f^\kappa - A f^{\kappa - \frac{1}{2}} \right).$$

Then u satisfies

$$\partial_\tau u \geq \left(1 + C_0\sqrt{u}\right)\overline{\Delta}u - \overline{\nabla}_{\overline{\nabla} f}u + C_0|\overline{Rm}|_{\overline{g}}u \quad \text{on } \{(x,\tau) \in M \times [\tau_0,\infty)\colon \Gamma < f(x) < \gamma e^\tau\} \tag{4.6}$$

if $\Gamma \gg 1$ is sufficiently large (depending only on C_0, κ, and the soliton $(M^n, \overline{g}, f)$) and $A\gamma^\kappa < 1$.

Lemma 4.8 (Interior Estimate 2)**.** *Assume Assumption 3.1. Let $A, \kappa > 0$, let $\tau_0 < \tau_1$, and assume h is smooth solution of*

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E}(h) \qquad \text{on } M \times [\tau_0, \tau_1].$$

If

$$|h|_{\overline{g}} \leq Ae^{-\kappa\tau} f^\kappa \qquad \text{on } \{(x,\tau)\colon \Gamma < f(x) < \gamma e^\tau,\ \tau_0 \leq \tau \leq \tau_1\}$$
$$\text{and } \sum_{k=0}^{3} |\overline{\nabla}^k h|_{\overline{g}} \leq Ae^{-\kappa\tau_0} f^\kappa \qquad \text{on } \{(x,\tau_0)\colon \Gamma < f(x) < \gamma e^{\tau_0}\},$$

then

$$|\overline{\nabla} h|_{\overline{g}} + |\overline{\nabla}^2 h|_{\overline{g}} \leq CAe^{-\kappa\tau} f^\kappa \qquad \text{on } \{(x,\tau)\colon 4\Gamma < f(x) < \frac{1}{4}\gamma e^\tau,\ \tau_0 \leq \tau \leq \tau_1\}$$

for some $C = C(M^n, \overline{g}, f, \kappa) > 0$ so long as $\Gamma \gg 1$ is sufficiently large and $A\gamma^\kappa \ll 1$ is sufficiently small (both depending only on the soliton $(M^n, \overline{g}, f)$).

4.4 Global C^2 Estimate

We now combine the supersolutions and interior estimates from this chapter to improve the main estimate from the previous chapter (Theorem 3.10) to a C^2 estimate on *all of* M.

Theorem 4.9. *Assume Assumption 3.1, $\tau_0 < \tau_1$, and h is a smooth solution of*

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E} \qquad \text{on } M \times [\tau_0, \tau_1]. \tag{4.7}$$

If

$$\|h(\tau)\|_{C^0(M)} \leq \varepsilon \qquad \forall \tau \in [\tau_0, \tau_1] \qquad (\textit{Coarse } C^0 \textit{ Estimate}) \tag{4.8}$$
$$\|h(\tau)\|_{L^2_f(M)} \leq \mu e^{\lambda_* \tau} \qquad \forall \tau \in [\tau_0, \tau_1] \qquad (\textit{Fine } L^2_f \textit{ Estimate}), \tag{4.9}$$

and the initial data satisfies the strong estimate

$$\sum_{k=0}^{4n+2} |\overline{\nabla}^k h(\tau_0)|_{\overline{g}} \leq \mu_0 \min\left\{1, e^{\lambda_* \tau_0} f^{|\lambda_*|}\right\} \qquad \text{on } M, \tag{4.10}$$

then, h satisfies the global C^2 estimate

$$|h|_{\overline{g}} + |\overline{\nabla} h|_{\overline{g}} + |\overline{\nabla}^2 h|_{\overline{g}} \leq \min\left\{\frac{\varepsilon}{2}, e^{\lambda_* \tau} f^{|\lambda_*|}\right\} \qquad \text{on } M \times [\tau_0, \tau_1] \tag{4.11}$$

so long as $0 < \varepsilon \ll 1$ is sufficiently small (depending only on the soliton $(M^n, \overline{g}, f)$ and λ_*), $0 < \mu \ll 1$ is sufficiently small (depending only on the soliton and λ_*), $0 < \mu_0 \ll 1$ is sufficiently small (depending only on the soliton, λ_*, and ε), and $\tau_0 \gg 1$ is sufficiently large (depending only on the soliton, $\lambda_*, \varepsilon, \mu, \mu_0$).

Remark 4.10. The following proof will make use of the maximum principle for parabolic equations. Since the underlying manifold M is non-compact, it requires some care to justify the use of the maximum principle. We will ignore those details for the proof and simply invoke the maximum principle.

Proof. Throughout the proof, $C = C(M^n, \overline{g}, f, \lambda_*) > 0$ denotes a positive constant depending only on the soliton $(M^n, \overline{g}, f)$ and λ_*, and which may change from line to line.

Fix $\Gamma = \Gamma(M^n, \overline{g}, f, \lambda_*) \gg 1$ to be a constant large enough so that the results of the last two chapters apply with this choice of Γ. Let $0 < \delta = \delta(M^n, \overline{g}, f, \lambda_*, \varepsilon) \leq \varepsilon \ll 1$ be a small constant to be determined.

The assumption on the initial data implies

$$\|h(\tau_0)\|_{C^{4n+2}(\{f<3\Gamma\})} \leq \mu_0 e^{\lambda_* \tau_0} (3\Gamma)^{|\lambda_*|}.$$

Theorem 3.10 therefore implies

$$\|h(\tau)\|_{C^2(\{f<\Gamma\})} \leq C_\Gamma \max\{\mu, \mu_0 (3\Gamma)^{|\lambda_*|}\} e^{\lambda_* \tau} \leq \delta \qquad \forall \tau \in [\tau_0, \tau_1] \tag{4.12}$$

if $0 < \varepsilon \ll 1$ is sufficiently small (depending on Γ) and $\tau_0 \gg 1$ is sufficiently large (depending on $\Gamma, \mu, \mu_0, \delta$).

Lemma 4.1 implies

$$\partial_\tau |h|^2_{\overline{g}} \leq (1 + C_0 |h|_{\overline{g}}) \overline{\Delta} |h|^2_{\overline{g}} - \overline{\nabla}_{\overline{\nabla} f} |h|^2_{\overline{g}} + C_0 |\overline{Rm}|_{\overline{g}} |h|^2_{\overline{g}} \qquad \text{on } M \times [\tau_0, \tau_1]$$

for some $C_0 = C_0(M^n, \overline{g}, f) > 0$ when $0 < \varepsilon \ll 1$. By Lemma 4.3, the function $u = 2\delta^2 - \frac{2\delta^2}{\sqrt{f}}$ is a supersolution to this equation on

$$\{f > \Gamma\} \times [\tau_0, \tau_1]$$

so long as $\Gamma \gg 1$. Additionally, (4.12) implies that, along the lateral boundary $\{f = \Gamma\} \times [\tau_0, \tau_1]$,

$$|h|^2_{\overline{g}} \leq \delta^2 \leq u.$$

The assumption on the initial data implies that, on $\{f > \Gamma\} \times \{\tau_0\}$,

$$|h|^2_{\overline{g}} \leq \mu_0^2 \leq \delta^2 \leq u$$

if $0 < \mu_0 \leq \delta$. Therefore, the maximum principle implies that

$$|h|_{\overline{g}} \leq \sqrt{u} \leq \sqrt{2}\delta \qquad \text{on } \{f > \Gamma\} \times [\tau_0, \tau_1].$$

Combining this estimate with (4.12) gives the global C^0 bound

$$|h|_{\overline{g}} \leq \sqrt{2}\delta \qquad \text{on } M \times [\tau_0, \tau_1]. \tag{4.13}$$

The assumption on the initial data also implies

$$\|h(\tau_0)\|_{C^3(\{f > \frac{1}{4}\Gamma\})} \leq \mu_0 \leq \delta$$

if $0 < \mu_0 \leq \delta$. The interior estimate at infinity (Lemma 4.5) now applies to show

$$\|h(\tau)\|_{C^2(\{f>\Gamma\})} \leq C\delta \qquad \forall \tau \in [\tau_0, \tau_1]$$

when $0 < \delta \leq \varepsilon \ll 1$ and $\Gamma \gg 1$. Combining with (4.12) yields

$$\sum_{k=0}^{2} |\overline{\nabla}^k h|_{\overline{g}} \leq C\delta \leq \frac{\varepsilon}{2} \qquad \text{on } M \times [\tau_0, \tau_1] \tag{4.14}$$

if $0 < \delta \ll 1$ is sufficiently small (depending on ε).

By similar logic using instead the supersolution from Lemma 4.7 and the interior estimate Lemma 4.8, one can show the remaining C^2 bound. The argument is left as an exercise

Exercise 4.11. *For parameters ε, μ, μ_0, and τ_0 sufficiently large or small as in the statement of the theorem, show that*

$$\sum_{k=0}^{2} |\overline{\nabla}^k h|_{\overline{g}} \leq e^{\lambda_* \tau} f^{|\lambda_*|} \qquad \text{on } M \times [\tau_0, \tau_1].$$

(Hint: Note that $Ae^{\lambda_* \tau} f^{|\lambda_*|} > \varepsilon$ when $f > A^{-\frac{1}{|\lambda_*|}} \varepsilon^{\frac{1}{|\lambda_*|}} e^{\tau}$.)

With the exercise and estimate (4.14), the proof is complete. □

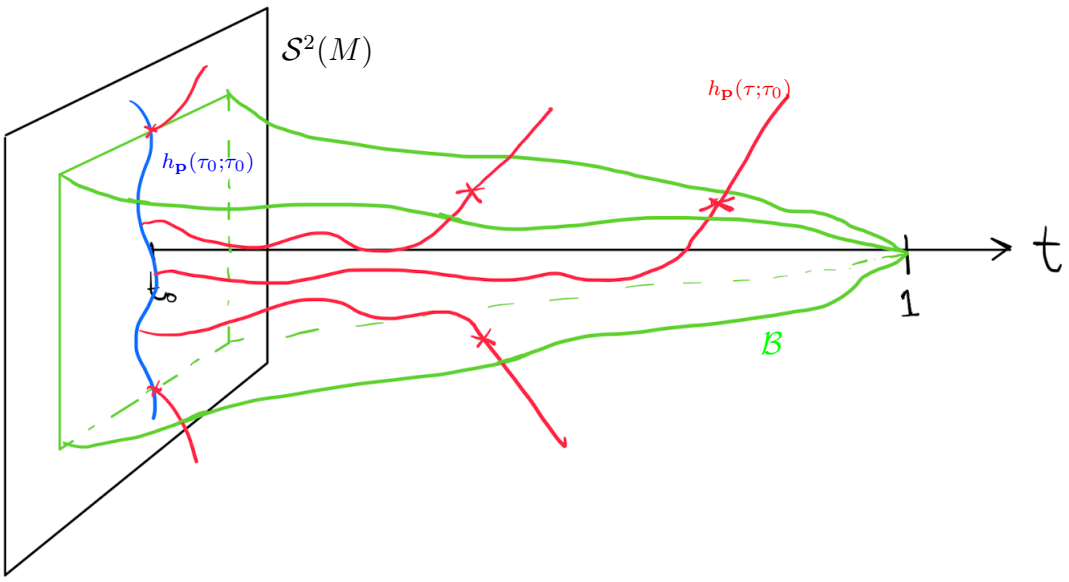
Chapter 5

Ricci Flows Forming AC Shrinker Singularities

In this chapter, we combine the analysis of the previous chapters to show that, given any AC shrinker, there exists a Ricci flow starting from certain perturbations of the shrinker metric for which the flow develops a finite-time singularity modeled on the shrinker (Corollary 5.8). The overall proof strategy aims to perturb the approximate Ricci flow solutions from Remark 2.13 to an actual Ricci flow with similar dynamics. To obtain the suitable perturbation, we appeal to a Ważewski Box argument [Waż47].

5.1 Overview of the Ważewski Box Argument

Ważewski Box arguments are one method to perturb approximate solutions of a differential equation into exact solutions of the equation [Waż47]. We outline the strategy of this argument.



First, construct a "box" $\mathcal{B} = \bigcup_{\tau_0 \leq \tau < \infty} (\mathcal{B}(\tau) \times \{\tau\})$ such that

- $\mathcal{B}(\tau)$ shrinks down to 0 as $\tau \to \infty$ $(t \to 1)$ and

- each time-slice $\mathcal{B}(\tau) \subseteq \mathcal{S}^2(M)$ splits off a finite-dimensional factor

$$\mathcal{B}(\tau) = \mathcal{B}^K(\tau) \times \mathcal{B}^{\infty-K}(\tau)$$

such that each factor is a closed ball.

Consider a *collection* of initial data $h_{\mathbf{p}}(\tau_0)$ parametrized by $\mathbf{p}$ (and initial time τ_0) as in the picture. Let $h_{\mathbf{p}}(\tau) = h_{\mathbf{p}}(\tau; \tau_0)$ denote the corresponding solution of

$$\begin{cases} \partial_\tau h_{\mathbf{p}} = \overline{\Delta}_f h_{\mathbf{p}} + 2\overline{Rm}[h_{\mathbf{p}}] + \mathcal{E}(h_{\mathbf{p}}) \\ h_{\mathbf{p}}(\tau_0; \tau_0) = h_{\mathbf{p}}(\tau_0). \end{cases} \tag{5.1}$$

Suppose for the sake of contradiction that all $h_{\mathbf{p}}(\tau)$ exit the box $\mathcal{B}$ at some finite time $\tau_{exit}(\mathbf{p}) < \infty$. Develop an argument to show that $h_{\mathbf{p}}(\tau)$ can only exit through the finite-dimensional portion

$$\left(\partial \mathcal{B}^K(\tau)\right) \times \mathcal{B}^{\infty-K}(\tau) \subsetneq \partial \mathcal{B}(\tau).$$

$\left(\partial \mathcal{B}^K(\tau)\right) \times \mathcal{B}^{\infty-K}(\tau) \simeq S^{K-1}$ is homotopic to a $K-1$ sphere. If the parameters $\mathbf{p}$ vary over a closed K-dimensional ball B^K, then the function

$$\mathbf{p} \mapsto h_{\mathbf{p}}(\tau_{exit}(\mathbf{p})) \in (\partial \mathcal{B}^K) \times \mathcal{B}^{\infty-K} \simeq S^{K-1}$$

can be used to construct a continuous retraction $B^K \to S^{K-1}$. However, no such retraction exists by homology considerations. This contradiction therefore indicates that there exists $\mathbf{p}_*$ such that $h_{\mathbf{p}_*}(\tau; \tau_0)$ never exits the box $\mathcal{B}$. Since $\mathcal{B}(\tau)$ shrinks down to 0 as $\tau \to \infty$, this shows that $h_{\mathbf{p}_*}(\tau; \tau_0)$ is a solution to (5.1) that limits to 0 as $\tau \to \infty$.

5.2 Definition of the Box

Definition 5.1. Assume Assumption 3.1.

For $h \in L^2_f(S^2(M))$, define

$$\pi_u h \doteqdot \sum_{j=1}^{K} (h, h_k)_{L^2_f} h_k \qquad \text{and} \qquad \pi_s h \doteqdot \sum_{j=K+1}^{\infty} (h, h_k)_{L^2_f} h_k$$

to be the L^2_f-projections onto the part of the spectrum above and below λ_*.

Definition 5.2 (The Box)**.** Assume Assumption 3.1. For parameters $\varepsilon, \mu > 0$, define

$$\mathcal{B}[\tau] = \mathcal{B}[\tau; \mu, \varepsilon]$$

to be the set of $h \in \mathcal{S}^2(M)$ such that

$$\|\pi_u h\|_{L^2_f(M)} \leq \mu e^{\lambda_* \tau} \tag{5.2}$$

$$\|\pi_s h\|_{L^2_f(M)} \leq \mu e^{\lambda_* \tau} \tag{5.3}$$

$$\text{and } \|h\|_{C^2(M)} \leq \varepsilon. \tag{5.4}$$

5.3 Evolution of $L^2_f(M)$ Norms

Lemma 5.3. *If $h \in \mathcal{S}^2(M)$ satisfies*

$$\sum_{k=0}^{2} |\overline{\nabla}^k h|_{\overline{g}} \leq \min\{\varepsilon, e^{\lambda_* \tau} f^{|\lambda_*|}\},$$

then

$$\|\mathcal{E}(h)\|_{L^2_f(M)} \leq C e^{2\lambda_* \tau}$$

for some $C = C(M^n, \overline{g}, f, \lambda_*) > 0$ so long as $0 < \varepsilon \ll 1$ and $\tau \geq \tau_0 \gg 1$ (both depending only on the soliton $(M^n, \overline{g}, f)$ and λ_*).

Proof. By the definition of the error term $\mathcal{E}(h)$ (2.3), there holds the pointwise estimate

$$|\mathcal{E}(h)|_{\overline{g}} \lesssim \left(|h|_{\overline{g}} + |\overline{\nabla} h|_{\overline{g}} + |\overline{\nabla}^2 h|_{\overline{g}}\right)^2$$

when $0 < \varepsilon \ll 1$. Therefore,

$$\begin{aligned}
\|\mathcal{E}\|^2_{L^2_f} &= \int_{\{f \leq e^\tau\}} |\mathcal{E}|^2_{\overline{g}} e^{-f} dV_{\overline{g}} + \int_{\{f > e^\tau\}} |\mathcal{E}|^2_{\overline{g}} e^{-f} dV_{\overline{g}} \\
&\lesssim \int_{\{f \leq e^\tau\}} e^{4\lambda_* \tau} f^{4|\lambda_*|} e^{-f} dV_{\overline{g}} + \int_{\{f > e^\tau\}} \varepsilon^4 e^{-f} dV_{\overline{g}} \\
&\leq e^{4\lambda_* \tau} \int_M f^{4|\lambda_*|} e^{-f} dV_{\overline{g}} + \int_{\{f > e^\tau\}} e^{-f} dV_{\overline{g}}.
\end{aligned}$$

Since f grows quadratically (Proposition 1.21) and the shrinker is asymptotically conical, it can be shown that

$$\int_M f^{4|\lambda_*|} e^{-f} dV_{\overline{g}} < \infty \qquad \text{and} \qquad \int_{\{f > e^\tau\}} e^{-f} dV_{\overline{g}} \lesssim e^{-\frac{1}{100}(e^\tau)} \leq e^{4\lambda_* \tau}$$

when $\tau \geq \tau_0 \gg 1$. The claimed estimate for $\|\mathcal{E}\|_{L^2_f(M)}$ follows. □

Lemma 5.4. *Assume Assumption 3.1. Let $\tau_0 < \tau_1$ and let h be a smooth solution of*

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E}(h) \qquad \text{on } M \times [\tau_0, \tau_1]$$

such that

$$\sum_{k=0}^{2} |\overline{\nabla}^k h|_{\overline{g}} \leq \min\{\varepsilon, e^{\lambda_* \tau} f^{|\lambda_*|}\} \qquad \text{on } M \times [\tau_0, \tau_1].$$

Then, for all $\tau \in [\tau_0, \tau_1]$, the stable and unsta ble projections satisfy

$$\begin{aligned}
&\frac{d}{d\tau} \|e^{-\lambda_* \tau} \pi_u h\|_{L^2_f(M)} \geq (\lambda_K - \lambda_*) \|e^{-\lambda_* \tau} \pi_u h\|_{L^2_f(M)} - C e^{\lambda_* \tau_0} \qquad \text{and} \\
&\frac{d}{d\tau} \|e^{-\lambda_* \tau} \pi_s h\|_{L^2_f(M)} \leq (\lambda_{K+1} - \lambda_*) \|e^{-\lambda_* \tau} \pi_s h\|_{L^2_f(M)} + C e^{\lambda_* \tau_0}
\end{aligned}$$

for some $C = C(M^n, \overline{g}, f, \lambda_*) > 0$ so long as $0 < \varepsilon \ll 1$ and $\tau_0 \gg 1$ (both depending only on the soliton $(M^n, \overline{g}, f)$ and λ_*).

Proof. We compute

$$\begin{aligned}
&\frac{d}{d\tau}\|e^{-\lambda_*\tau}\pi_u h\|^2_{L^2_f(M)}\\
=\ &-2\lambda_* e^{-2\lambda_*\tau}\|\pi_u h\|^2_{L^2_f} + 2e^{-2\lambda_*\tau}(\pi_u h, \partial_\tau \pi_u h)_{L^2_f}\\
=\ &-2\lambda_* e^{-2\lambda_*\tau}\|\pi_u h\|^2_{L^2_f} + 2e^{-2\lambda_*\tau}(\pi_u h, (\overline{\Delta}_f + \overline{Rm})\pi_u h)_{L^2_f} + 2e^{-2\lambda_*\tau}(\pi_u h, \cancel{\pi_u}\mathcal{E}(h))_{L^2_f}\\
\geq\ &-2\lambda_* e^{-2\lambda_*\tau}\|\pi_u h\|^2_{L^2_f} + 2e^{-2\lambda_*\tau}\lambda_K\|\pi_u h\|^2_{L^2_f} - 2e^{-2\lambda_*\tau}\|\pi_u h\|_{L^2_f}\|\mathcal{E}\|_{L^2_f}\\
\geq\ &2(\lambda_K - \lambda_*)\|e^{-\lambda_*\tau}\pi_u h\|^2_{L^2_f} - 2Ce^{\lambda_*\tau}\|e^{-\lambda_*\tau}\pi_u h\|_{L^2_f}
\end{aligned}$$

where the last inequality follows from Lemma 5.3 when $0 < \varepsilon \ll 1$ and $\tau_0 \gg 1$. Thus,

$$\frac{d}{d\tau}\|e^{-\lambda_*\tau}\pi_u h\|_{L^2_f(M)} \geq (\lambda_K - \lambda_*)\|e^{-\lambda_*\tau}\pi_u h\|_{L^2_f(M)} - Ce^{\lambda_*\tau_0}.$$

An analogous argument applies to show the claimed estimate for $\frac{d}{d\tau}\|e^{-\lambda_*\tau}\pi_s h\|_{L^2_f(M)}$. □

Corollary 5.5. *Assume Assumption 3.1. Let $\tau_0 < \tau_1$ and let h be a smooth solution of*

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E}(h) \qquad \text{on } M \times [\tau_0, \tau_1]$$

such that

$$\sum_{k=0}^{2} |\overline{\nabla}^k h|_{\overline{g}} \leq \min\{\varepsilon, e^{\lambda_*\tau} f^{|\lambda_*|}\} \qquad \text{on } M \times [\tau_0, \tau_1].$$

Let $\mu > 0$. If $0 < \varepsilon \ll 1$ (depending only on the soliton and λ_*) and $\tau_0 \gg 1$ (depending on the soliton, λ_*, and μ), then the following hold:

1. *(Cannot Exit through Stable Side)*

$$\|\pi_s h(\tau_0)\|_{L^2_f} < \mu e^{\lambda_*\tau_0} \implies \|\pi_s h\|_{L^2_f} < \mu e^{\lambda_*\tau} \qquad \forall \tau \in [\tau_0, \tau_1].$$

2. *(Hits Unstable Side Transversely) If $\|\pi_u h(\tau)\|_{L^2_f} = \mu e^{\lambda_*\tau}$ for some $\tau \in (\tau_0, \tau_1)$, then, at that time τ,*

$$\frac{d}{d\tau}\|e^{-\lambda_*\tau}\pi_u h(\tau)\|_{L^2_f(M)} > 0.$$

Proof. (Cannot Exit through Stable Side)

Suppose for the sake of contradiction that there is a time such that $\|\pi_s h(\tau)\|_{L^2_f} = \mu e^{\lambda_*\tau}$ and consider the first such time. Then, at this time,

$$\frac{d}{d\tau}\|e^{-\lambda_*\tau}\pi_s h(\tau)\|_{L^2_f} \geq 0. \tag{5.5}$$

On the other hand, Lemma 5.4 implies that at this time

$$\frac{d}{d\tau}\|e^{-\lambda_*\tau}\pi_s h(\tau)\|_{L^2_f} \leq (\lambda_{K+1} - \lambda_*)\|e^{-\lambda_*\tau}\pi_s h(\tau)\|_{L^2_f} + Ce^{\lambda_*\tau_0} = (\lambda_{K+1} - \lambda_*)\mu + Ce^{\lambda_*\tau_0}.$$

Since $\lambda_{K+1} < \lambda_* < 0$, this quantity will be negative for $\tau_0 \gg 1$ (depending on μ), which contradicts (5.5).

(Hits Unstable Side Transversely)

If $\|\pi_u h(\tau)\|_{L^2_f} = \mu e^{\lambda_*\tau}$, then Lemma 5.4 implies, at this time,

$$\frac{d}{d\tau}\|e^{-\lambda_*\tau}\pi_u h(\tau)\|_{L^2_f} \geq (\lambda_K - \lambda_*)\|e^{-\lambda_*\tau}\pi_u h(\tau)\|_{L^2_f} - Ce^{\lambda_*\tau_0} = (\lambda_K - \lambda_*)\mu - Ce^{\lambda_*\tau_0}.$$

Since $\lambda_* < 0$ and $\lambda_K > \lambda_*$, this quantity will be positive if $\tau_0 \gg 1$ (depending on μ). □

5.4 The Main Result and Its Proof

Lemma 5.6 ([Sto26, Proposition 2.33])**.** *Assume Assumption 3.1. Then for any $j, m \in \mathbb{N}$ with $j \leq K$, there exists a constant $C_m = C_m(M^n, \overline{g}, f, \lambda_*, m) > 0$ such that the eigenfunction h_j satisfies the pointwise derivative estimate*

$$|\overline{\nabla}^m h_j|_{\overline{g}} \leq C_m f^{|\lambda_*|} \qquad \text{on } M.$$

Theorem 5.7. *Assume Assumption 3.1. Fix a cutoff function*[1] *$\eta : M \to [0,1]$ such that*

$$\eta(x) \equiv 1 \quad \text{on } \left\{ f < \frac{1}{2} e^{\tau_0} \right\}, \qquad \operatorname{supp} \eta \subseteq \left\{ f < e^{\tau_0} \right\},$$
$$\text{and } \sup_M |\overline{\nabla}^k \eta|_{\overline{g}} \leq C_k = C_k(M^n, \overline{g}, f, \lambda_*, k) \qquad \forall k \in \mathbb{N}.$$

So long as

- *$0 < \varepsilon \ll 1$ (depending only on the soliton and λ_*),*
- *$0 < r_0 \ll 1$ (depending only on the soliton, λ_*, and ε),*
- *$0 < \mu \ll 1$ (depending only on the soliton, λ_*, ε, and r_0),*
- *$0 < \mu_0 \ll 1$ (depending only on the soliton, λ_*, ε, and r_0), and*
- *$\tau_0 \gg 1$ (depending only on the soliton, $\lambda_*, \varepsilon, r_0, \mu, \mu_0$),*

the following holds:
If the initial data $h(\tau_0) \in \mathcal{S}^2(M)$ is such that

$$\|h(\tau_0)\|_{L^2_f(M)} \leq \mu_0 e^{\lambda_* \tau_0}, \qquad \|\pi_s h(\tau_0)\|_{L^2_f(M)} \leq \frac{\mu}{3} e^{\lambda_* \tau_0}, \text{ and}$$
$$\sum_{j=0}^{4n+2} |\overline{\nabla}^j h(\tau_0)|_{\overline{g}} \leq \mu_0 \min\left\{1, e^{\lambda_* \tau_0} f^{|\lambda_*|}\right\},$$

then there exists a perturbation of the form

$$h_{\mathbf{p}}(\tau_0) = h(\tau_0) + \eta e^{\lambda_* \tau_0} \sum_{j=1}^{K} p_j h_j$$

for some $\mathbf{p} = (p_1, \ldots, p_K) \in \overline{B}_{r_0} \subseteq \mathbb{R}^K$ such that the solution to

$$\partial_\tau h = \overline{\Delta}_f h + 2\overline{Rm}[h] + \mathcal{E}(h) \tag{5.6}$$

starting from $h_{\mathbf{p}}(\tau_0)$ exists for all $\tau \geq \tau_0$ and $h_{\mathbf{p}}(\tau) \in \mathcal{B}[\tau; \mu, \varepsilon]$ for all $\tau \in [\tau_0, \infty)$.

Proof. Consider the collection of initial data

$$h_{\mathbf{p}}(\tau_0) = h(\tau_0) + \eta e^{\lambda_* \tau_0} \sum_{j=1}^{K} p_j h_j$$

parametrized by $\mathbf{p} \in \overline{B}_{r_0} \subseteq \mathbb{R}^K$ where $\overline{B}_{r_0}$ denotes the closed Euclidean ball of radius r_0 centered at the origin.

By exploiting the correspondence between (rrRF) and (RF) (Proposition 2.6), it is possible to show that the corresponding solution $h_{\mathbf{p}}(\tau)$ to (5.6) starting from $h_{\mathbf{p}}(\tau_0)$ exists

[1] Such cutoff functions always exist when $\tau_0 \gg 1$ depending on the soliton and λ_*.

for $\tau \in [\tau_0, \tau_*(\mathbf{p}))$. Without loss of generality, take $\tau_*(\mathbf{p}) \in [\tau_0, \infty]$ to be the maximal existence time.

Suppose for the sake of contradiction that, for each $\mathbf{p} \in \overline{B}(\mathbf{0}, r_0)$, the solution $h_{\mathbf{p}}(\tau) \notin \mathcal{B}[\tau; \mu, \varepsilon]$ for some $\tau < \tau_*(\mathbf{p})$. Define $\tau_{exit}(\mathbf{p})$ to be the infimum of such times.

Consider the function

$$\Phi : \overline{B}_{r_0} \to \mathbb{R}^K$$

$$\Phi(\mathbf{p}) = \left(\langle h_{\mathbf{p}}(\tau_{exit}(\mathbf{p})), h_k \rangle_{L^2_f(M)} \right)_{1 \le k \le K}.$$

Claim 1. *The image satisfies*

$$\Phi(\overline{B}_{r_0}) \subseteq \mathbb{R}^K \setminus \{\mathbf{0}\}$$

so long as $0 < \varepsilon \ll 1$, $0 < \mu_0, r_0 \ll 1$ *(depending on* ε*),* $0 < \mu \ll 1$*, and* $\tau_0 \gg 1$ *(depending on* $\varepsilon, \mu, \mu_0, r_0$*).*

Proof of Claim. Let $\mathbf{p} \in \overline{B}_{r_0}$. Then $\mathbf{h}_{\mathbf{p}}(\tau) \in \mathcal{B}[\tau; \mu, \varepsilon]$ for all $\tau \in [\tau_0, \tau_{exit}(\mathbf{p}))$.

By the pointwise estimates for h_j (Lemma 5.6), the initial data satisfies

$$\begin{aligned}
\sum_{j=0}^{4n+2} |\overline{\nabla}^j h_{\mathbf{p}}(\tau_0)|_{\overline{g}} &\le \sum_{j=0}^{4n+2} |\overline{\nabla}^j h(\tau_0)|_{\overline{g}} + e^{\lambda_* \tau_0} \sum_{j=1}^{K} |p_j| \sum_{k=0}^{4n+2} |\overline{\nabla}^k (\eta h_j)|_{\overline{g}} \\
&\le \mu_0 e^{\lambda_* \tau_0} f^{|\lambda_*|} + C e^{\lambda_* \tau_0} r_0 f^{|\lambda_*|} \\
&= (\mu_0 + C r_0) e^{\lambda_* \tau_0} f^{|\lambda_*|}
\end{aligned}$$

for some $C = C(M^n, \overline{g}, f, \lambda_*) > 0$ which may vary from line to line. Similarly,

$$\begin{aligned}
&\sum_{j=0}^{4n+2} |\overline{\nabla}^j h_{\mathbf{p}}(\tau_0)|_{\overline{g}} \\
&\le \sum_{j=0}^{4n+2} |\overline{\nabla}^j h(\tau_0)|_{\overline{g}} + e^{\lambda_* \tau_0} \sum_{j=1}^{K} |p_j| \sum_{k=0}^{4n+2} |\overline{\nabla}^k (\eta h_j)|_{\overline{g}} \\
&\le \mu_0 + C e^{\lambda_* \tau_0} r_0 (e^{\tau_0})^{|\lambda_*|} && (\operatorname{supp} \eta \subseteq \{f < e^{\tau_0}\}) \\
&\le (\mu_0 + C r_0) && (\lambda_* < 0)
\end{aligned}$$

for some $C = C(M^n, \overline{g}, f, \lambda_*) > 0$.

Combined with the estimates provided by $\mathcal{B}[\tau; \mu, \varepsilon]$, the global C^2 estimate (Theorem 4.9) gives

$$\sum_{j=0}^{2} |\overline{\nabla}^j h_{\mathbf{p}}(\tau)|_{\overline{g}} \le \min \left\{ \frac{\varepsilon}{2}, e^{\lambda_* \tau} f^{|\lambda_*|} \right\} \qquad \forall \tau \in [\tau_0, \tau_{exit}(\mathbf{p})]$$

so long as $0 < \varepsilon \ll 1$, $0 < \mu_0, r_0 \ll 1$ (depending on ε), $0 < \mu \ll 1$, and $\tau_0 \gg 1$ (depending on $\varepsilon, \mu, \mu_0, r_0$). In particular, $h_{\mathbf{p}}$ cannot exit the box $\mathcal{B}$ through the side defined by $\|h\|_{C^0(M)} = \varepsilon$.

We also have

$$\begin{aligned}
\|\pi_s h_{\mathbf{p}}(\tau_0)\|_{L^2_f} &\le \|\pi_s h(\tau_0)\|_{L^2_f} + e^{\lambda_* \tau_0} \sum_{j=1}^{K} |p_j| \|\pi_s (\eta - 1) h_j\|_{L^2_f} \\
&\le \frac{\mu}{3} e^{\lambda_* \tau_0} + C e^{\lambda_* \tau_0} r_0 \sum_{j=1}^{K} \|h_j\|_{L^2_f(\{f > \frac{1}{2} e^{\tau}\})} \qquad (5.7)
\end{aligned}$$

By the h_j pointwise estimates (Lemmma 5.6), for any $1 \leq j \leq K$,

$$\int_{\{f>\frac{1}{2}e^{\tau_0}\}} |h_j|^2 e^{-f} dV \lesssim \int_{\{f>\frac{1}{2}e^{\tau_0}\}} f^{|\lambda_*|} e^{-f} dV \lesssim e^{-\frac{1}{100}(\frac{1}{2}e^{\tau_0})} \tag{5.8}$$

if $\tau_0 \gg 1$. Indeed, the last inequality follows from approximating

$$\int_{\{f>\frac{1}{2}e^{\tau_0}\}} f^{|\lambda_*|} e^{-f} dV \approx \int_{\{r>\sqrt{2}e^{\tau_0/2}\}} r^{2|\lambda_*|} e^{-\frac{1}{4}r^2} r^{n-1} dr dA.$$

Inserting this estimate into (5.7), it follows that

$$\|\pi_s h_{\mathbf{p}}(\tau_0)\|_{L^2_f} < \frac{\mu}{2} e^{\lambda_* \tau_0}$$

if $\tau_0 \gg 1$ (depending on μ, r_0). Corollary 5.5 then implies

$$\|\pi_s h_{\mathbf{p}}(\tau)\|_{L^2_f} < \frac{\mu}{2} e^{\lambda_* \tau} \qquad \forall \tau \in [\tau_0, \tau_{exit}(\mathbf{p})].$$

That is, $h_{\mathbf{p}}$ cannot exit the box $\mathcal{B}$ through the side defined by $\|\pi_s h(\tau)\|_{L^2_f} = \mu e^{\lambda_* \tau}$.

Therefore, $h_{\mathbf{p}}$ can only exit the box $\mathcal{B}$ through the side defined by $\|\pi_u h\|_{L^2_f} = \mu e^{\lambda_* \tau}$. Hence,

$$\Phi(\mathbf{p}) = \Big(\langle h_{\mathbf{p}}(\tau_{exit}(\mathbf{p})), h_k \rangle_{L^2_f(M)} \Big)_{1 \leq k \leq K} \neq \mathbf{0}.$$

□

Claim 2. Φ *is continuous.*

Proof Sketch of Claim. It suffices to show that $\mathbf{p} \mapsto \tau_{exit}(\mathbf{p})$ is continuous.

The proof of the previous claim shows that $h_{\mathbf{p}}$ can only exit the box $\mathcal{B}$ by saturating the inequality $\|\pi_u h_{\mathbf{p}}\|_{L^2_f(M)} \leq \mu e^{\lambda_* \tau}$. By Corollary 5.5, this implies $\|\pi_u h_{\mathbf{p}}(\tau)\|_{L^2_f} > \mu e^{\lambda_* \tau}$ for τ slightly larger than $\tau_{exit}(\mathbf{p})$. By continuous dependence of solutions on their initial data, $h_{\mathbf{p}}$ then "pulls" nearby $h_{\mathbf{p}'}$ out of the box for nearby times. Hence, $\tau_{exit}(\mathbf{p}')$ cannot be much larger than $\tau_{exit}(\mathbf{p})$ for $\mathbf{p}'$ near $\mathbf{p}$.

Corollary 5.5 implies $\|\pi_u h_{\mathbf{p}}(\tau)\|_{L^2_f} < \mu e^{\lambda_* \tau}$ for $\tau \in [\tau_0, \tau_{exit}(\mathbf{p}))$. Combined with the proof of the previous claim, this implies $h_{\mathbf{p}}(\tau)$ lies in the interior of $\mathcal{B}(\tau)$ for $\tau_0 \leq \tau < \tau_{exit}(\mathbf{p})$. By continuous dependence of solutions on their initial data, $h_{\mathbf{p}}$ "pulls" nearby $h_{\mathbf{p}'}$ into the box for nearby times, and so $\tau_{exit}(\mathbf{p}')$ cannot be much smaller than $\tau_{exit}(\mathbf{p})$ for $\mathbf{p}'$ near $\mathbf{p}$.

This proves $\mathbf{p} \mapsto \tau_{exit}(\mathbf{p})$ is continuous. □

Claim 3. *If $\overline{A} = \overline{B}_{r_0} \setminus B_{r_0/2}$ denotes the closed annulus, then the restriction $\Phi|_{\overline{A}} : \overline{A} \to \mathbb{R}^K \setminus \{\mathbf{0}\}$ is homotopic to the identity $Id_{\overline{A}} : \overline{A} \to \overline{A} \subseteq \mathbb{R}^K \setminus \{\mathbf{0}\}$ so long as $\mu_0 < \frac{r_0}{100K}$, $\mu < \frac{r_0}{4}$, and $\tau_0 \gg 1$ (depending on r_0).*

Proof of Claim. We compute

$$\begin{aligned}
\langle h_{\mathbf{p}}(\tau_0), h_j \rangle_{L^2_f(M)} &= \langle e^{\lambda_* \tau_0} \sum_{i=1}^K p_i h_i, h_j \rangle_{L^2_f} + \langle h(\tau_0), h_j \rangle_{L^2_f} + \langle (1-\eta) e^{\lambda_* \tau_0} \sum_{i=1}^K p_i h_i, h_j \rangle_{L^2_f} \\
&= e^{\lambda_* \tau_0} p_j + \langle h(\tau_0), h_j \rangle_{L^2_f} + \langle (1-\eta) e^{\lambda_* \tau_0} \sum_{i=1}^K p_i h_i, h_j \rangle_{L^2_f}.
\end{aligned}$$

We have

$$\left|\langle h(\tau_0), h_j\rangle_{L^2_f}\right| \leq \mu_0 e^{\lambda_*\tau_0},$$

and, using (5.8),

$$\left|\langle (1-\eta)e^{\lambda_*\tau_0}\sum_{i=1}^K p_i h_i, h_j\rangle_{L^2_f}\right| \lesssim e^{\lambda_*\tau_0}\sum_{i=1}^K |p_i|\|h_i\|_{L^2_f(\{f>\frac{1}{2}e^{\tau_0}\})} \lesssim e^{\lambda_*\tau_0} r_0 e^{-\frac{1}{100}(\frac{1}{2}e^{\tau_0})}$$
$$\leq e^{\lambda_*\tau_0}\frac{r_0}{100K}$$

if $\tau_0 \gg 1$ (depending on r_0). So

$$\langle h_{\mathbf{p}}(\tau_0), h_j\rangle_{L^2_f} = e^{\lambda_*\tau_0}p_j + e^{\lambda_*\tau_0}\tilde{E} \qquad \text{where } |\tilde{E}| \leq \mu_0 + \frac{r_0}{100K}.$$

If also $\mu_0 \leq \frac{r_0}{100K}$ and $\frac{r_0}{4} > \mu$, then we see $\tau_{exit}(\mathbf{p}) = \tau_0$ for all $\mathbf{p} \in \overline{A}$. Additionally, it follows that

$$\Phi_s(\mathbf{p}) = e^{\lambda_*\tau_0}\mathbf{p} + s(\Phi(\mathbf{p}) - e^{\lambda_*\tau_0}\mathbf{p})$$

is a homotopy from $\Phi|_{\overline{A}}$ to $e^{\lambda_*\tau_0}\mathrm{Id}_{\overline{A}}$ (through maps $\overline{A} \to \mathbb{R}^K \setminus \{\mathbf{0}\}$). $e^{\lambda_*\tau_0}\mathrm{Id}_{\overline{A}}$ is clearly homotopic to $\mathrm{Id}_{\overline{A}}$ so the claim is proved. □

In summary, we have a continuous map $\Phi : \overline{B}_{r_0} \to \mathbb{R}^K\setminus\{\mathbf{0}\}$ such that $\Phi|_{\overline{A}}$ is homotopic to the identity $\mathrm{Id}_{\overline{A}} : \overline{A} \to \mathbb{R}^K \setminus \{\mathbf{0}\}$. The latter property implies $\Phi|_{\overline{A}}$ induces the identity map map on $(K-1)^{th}$-homology groups

$$\mathrm{Id} : H_{K-1}(\overline{A}) \cong \mathbb{Z} \to H_{K-1}(\mathbb{R}^K \setminus \{\mathbf{0}\}) \cong \mathbb{Z}.$$

On the other hand, the former property implies

$$(s, \mathbf{p}) \mapsto \Phi(s\mathbf{p}) \qquad (s, \mathbf{p}) \in [0,1] \times \overline{A}$$

defines a homotopy from $\Phi|_{\overline{A}}$ to a constant map, which induces the zero map on $(K-1)^{th}$-homology groups

$$0 : H_{K-1}(\overline{A}) \cong \mathbb{Z} \to H_{K-1}(\mathbb{R}^K \setminus \{\mathbf{0}\}) \cong \mathbb{Z}.$$

This contradiction proves there exists some $\mathbf{p}_* \in \overline{B}_{r_0}$ such that $h_{\mathbf{p}_*}(\tau) \in \mathcal{B}[\tau; \mu, \varepsilon]$ for all $\tau \in [\tau_0, \tau_*(\mathbf{p}))$.

Finally, if $\tau_*(\mathbf{p}_*) < \infty$, then the fact that $h_{\mathbf{p}_*}(\tau) \in \mathcal{B}[\tau]$ for all $\tau \in [\tau_0, \tau_*(\mathbf{p}_*))$ would give sufficient estimates to extend the solution $h_{\mathbf{p}_*}$ to times beyond $\tau_*(\mathbf{p}_*)$, contradicting the maximality of $\tau_*(\mathbf{p}_*)$. Hence, $\tau_*(\mathbf{p}_*) = \infty$ and $h_{\mathbf{p}_*}(\tau) \in \mathcal{B}[\tau]$ for all $\tau \in [\tau_0, \infty)$. □

By Proposition 2.6, a solution $h(\tau) \in \mathcal{B}[\tau]$ for all $\tau \geq \tau_0$ gives sufficient estimates to transform the rescaled reparametrized Ricci flow solution $\overline{g} + h(\tau)$ to a Ricci flow solution $G(t)$ with time t defined by $\tau = -\ln(1-t)$. The above theorem (with suitably chosen ε, μ) therefore yields the following corollary.

Corollary 5.8. *Assume Assumption 3.1. Fix a cutoff function as in $\eta : M \to [0,1]$ as in Theorem 5.7 (and with $e^{-\tau_0} = 1 - t_0$). Let $\phi_t : M \to M$ denote the family of diffeomorphisms associated to the soliton as in Theorem 1.14.*

Consider $t_0 < 1$ and initial data

$$G(t_0) = (1-t_0)\phi_{t_0}^*(\overline{g} + h)$$

such that

$$\|h\|_{L^2_f(M)} \leq \mu_0(1-t_0)^{|\lambda_*|}, \qquad \|\pi_s h\|_{L^2_f(M)} \leq \frac{\mu_0}{3}(1-t_0)^{|\lambda_*|}, \text{ and}$$
$$\sum_{j=0}^{4n+2} |\overline{\nabla}^j h|_{\overline{g}} \leq \mu_0 \min\left\{1, (1-t_0)^{|\lambda_*|} f^{|\lambda_*|}\right\}.$$

If

- $0 < r_0 \ll 1$ *(depending only on the soliton,* λ_**),*
- $0 < \mu_0 \ll 1$ *(depending only on the soliton,* λ_*, r_0*), and*
- $0 < 1 - t_0 \ll 1$ *(depending only on the soliton,* λ_*, r_0, μ_0*),*

then there exists a perturbation of the form

$$G(t_0) = (1-t_0)\phi^*_{t_0}\left(\overline{g} + h + (1-t_0)^{|\lambda_*|}\eta \sum_{j=1}^{K} p_j h_j\right) \qquad \text{with } |(p_1, \ldots, p_K)| \leq r_0$$

such that the Ricci flow $G(t)$ *starting from* $G(t_0)$ *exists for* $t \in [t_0, 1)$*, develops a finite-time singularity at time* 1*, and the singularity is modeled on the AC shrinker* $(M^n, \overline{g}, f)$ *in the following sense: there exists a family of diffeomorhpisms* $\psi_t : M \to M$ $(t \in [t_0, 1))$ *such that the rescaled and reparametrized solution satisfies*

$$\frac{1}{1-t}(\phi_t^{-1})^* \psi_t^* G(t) \xrightarrow[t \nearrow 1]{C^\infty_{loc}(M,\overline{g})} \overline{g}.$$

Remark 5.9. By the definition of the box $\mathcal{B}$, the convergence $\frac{1}{1-t}(\phi_t^{-1})^*\psi_t^* G(t) \xrightarrow[t\nearrow 1]{} \overline{g}$ happens at a rate of at most $(1-t)^{|\lambda_*|}$. A priori, the box $\mathcal{B}$ only gives L^2_f convergence, but the local C^2 estimate (Theorem 3.10) and local interior C^m estimate (Theorem 3.9) improve the convergence to a C^∞_{loc} convergence.

Using similar techniques as presented in this mini-course, one can even "compactify" the Ricci flows in Corollary 5.8 to find Ricci flows on *closed* manifolds that form finite-time singularities modeled on any given AC shrinker. In particular, any AC shrinker arises as a finite-time singularity of some Ricci flow on a closed manifold.

Theorem 5.10 ([Sto26])**.** *Let* $(M^n, \overline{g}, f)$ *be an AC shrinker. Then there exists a* ***closed*** *manifold* $\mathcal{M}^n$ *and a Ricci flow* $(G(t))_{t\in[0,T)}$ *on* $\mathcal{M}$ *such that the flow develops a finite-time Type I singularity at* $T < \infty$ *which, informally speaking, is modeled on the given shrinker* $(M, \overline{g}, f)$*.*

Bibliography

[Bam21] Richard H Bamler. Structure theory of non-collapsed limits of Ricci flows. *arXiv e-prints*, page arXiv:2009.03243, September 2021.

[BC25] Richard H. Bamler and Eric Chen. Degree theory for 4-dimensional asymptotically conical gradient expanding solitons. *Communications in Pure and Applied Mathematics*, 79(5):1151–1298, 2025.

[Che09] Bing-Long Chen. Strong uniqueness of the Ricci flow. *Journal of Differential Geometry*, 82(2):363 – 382, 2009.

[CZ10] Huai-Dong Cao and Detang Zhou. On complete gradient shrinking Ricci solitons. *J. Differential Geom.*, 85(2):175–186, 06 2010.

[FIK03] Mikhail Feldman, Tom Ilmanen, and Dan Knopf. Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons. *Journal of Differential Geometry*, 65(2):169–209, 2003.

[Ham95] Richard S. Hamilton. The formation of singularities in the ricci flow. *Surveys In Differential Geometry*, 2, 1995.

[KW15] Brett Kotschwar and Lu Wang. Rigidity of asymptotically conical shrinking gradient Ricci solitons. *Journal of Differential Geometry*, 100(1):55 – 108, 2015.

[MW17] Ovidiu Munteanu and Jiaping Wang. Conical structure for shrinking Ricci solitons. *Journal of the European Mathematical Society*, 19(11):3377–3390, 2017.

[Shi89] Wan-Xiong Shi. Deforming the metric on complete riemannian manifolds. *Journal of Differential Geometry*, 30(1):223–301, 1989.

[Sto26] Maxwell Stolarski. Closed ricci flows with singularities modeled on asymptotically conical shrinkers. *Geometry & Topology*, 30(2026), 2026.

[Waż47] Tadeusz Ważewski. Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires. *Annales de la Société Polonaise de Mathématique*, 20:279–313, 1947.