Error Approximation for Backwards and Simple Continued Fractions

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Joint work with Cameron Bjorklund

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Outline of the Talk

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2. Introduce backwards continued fractions and their properties
3. Introduce the BCF and CF errors $\varepsilon_m(x)$ and $E_n(x)$
4. Construct bounds for $\varepsilon_m(x)$ and $E_n(x)$ on cylinder sets

What are Continued Fractions?

The *continued fraction (CF) expansion* for $x \in \mathbb{R}$ is the expression

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} =: [a_0, a_1, ...],$$

where $x_0 = x$, $a_0 = \lfloor x \rfloor$, $x_{i+1} = \frac{1}{x_i - a_i}$, and $a_i = \lfloor x_i \rfloor$. By this construction, we get that $a_0 \in \mathbb{Z}$ and $a_i \geq 1$ for each $i \geq 1$.

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Backwards Continued Fractions

The *backwards continued fraction* (BCF) expansion of $x \in \mathbb{R}$ is the expression

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where $x_0 = x$, $b_0 = \lfloor x \rfloor + 1$, $x_{i+1} = \frac{1}{b_i - x_i}$, and $b_i = \lfloor x_i \rfloor + 1$. By these calculations, we see that $b_0 \in \mathbb{Z}$ and $b_i \geq 2$ for each $i \geq 1$.

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$\frac{\pi}{4} = [0, 1, 3, 1, 1, 1, 15, 2, 72, ...]$.
Let's look at some convergents.

$$\frac{P_0}{Q_0} = [0] = 0 \Rightarrow \quad \frac{P_0}{Q_0} < \frac{\pi}{4}$$

$$\frac{P_1}{Q_1} = [0, 1] = 1 \Rightarrow \quad \frac{P_1}{Q_1} > \frac{\pi}{4}$$

$$\frac{P_2}{Q_2} = [0, 1, 3] = \frac{3}{4} \Rightarrow \quad \frac{P_2}{Q_2} < \frac{\pi}{4}$$

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$\frac{\pi}{4} = [[1, 5, 3, 17, 2, 74, 11, \ldots]]$.
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$$\frac{p_0}{q_0} = [[1]] = 1 \Rightarrow \qquad \frac{p_0}{q_0} > \frac{\pi}{4}$$
$$\frac{p_1}{q_1} = [[1, 5]] = \frac{4}{5} \Rightarrow \qquad \frac{p_1}{q_1} > \frac{\pi}{4}$$
$$\frac{p_2}{q_2} = [[1, 5, 3]] = \frac{11}{14} \Rightarrow \qquad \frac{p_2}{q_2} > \frac{\pi}{4}$$
$$\vdots$$

BCF convergents converge monotonically from above to the value they are estimating.

Why do we care about continued fractions?

- The denominators Q_m grow exponentially and

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However, their oscillatory nature makes them hard to work with.

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2. $[[b_0, ..., b_{n-1}, b_n]] < [[b_0, ..., b_{n-1}, b'_n, ..., b'_s]]$ where $b_n < b'_n$ and $s \geq n$, for any $b'_i \geq 2$.

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Continued Fractions

$$x = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$

$x \in \mathbb{R},\ a_0 \in \mathbb{Z}$ and $a_i \geq 1$.
The nth CF convergent of x is

$$\frac{P_n}{Q_n} = [a_0, a_1, a_2, \ldots, a_n]$$

The nth term CF error is

$$E_n(x) = \left| x - \frac{P_n}{Q_n} \right|$$

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Backwards Continued Fractions

$$y = b_0 - \cfrac{1}{b_1 - \frac{1}{b_2 - \ldots}}$$

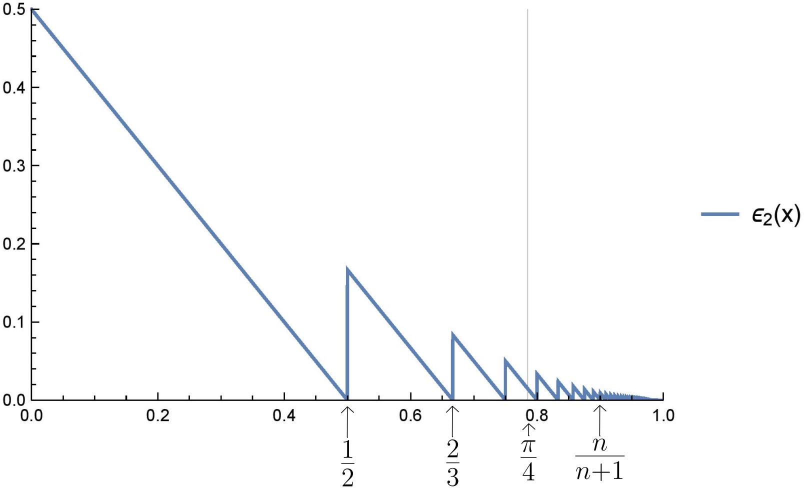
$y \in \mathbb{R},\ b_0 \in \mathbb{Z}$ and $b_i \geq 2$.
The nth BCF convergent of y is

$$\frac{p_n}{q_n} = [[b_0, b_1, b_2, \ldots, b_n]]$$

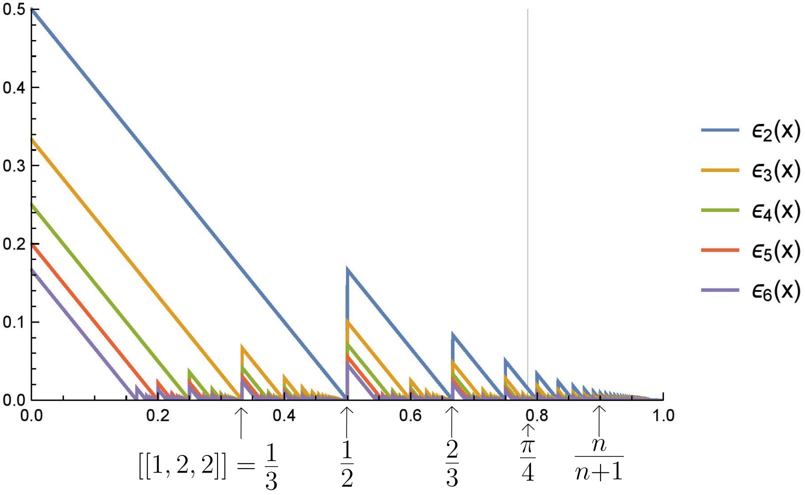
The nth term BCF error is

$$\varepsilon_n(x) = \left| x - \frac{p_n}{q_n} \right|$$

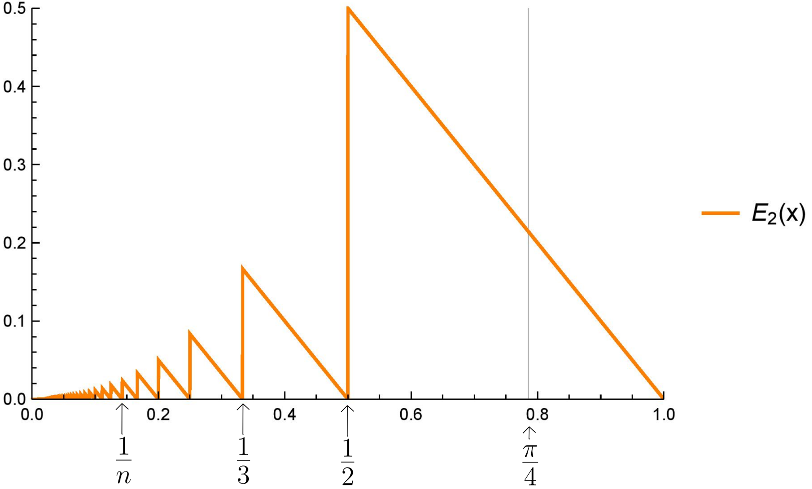
The BCF Error $\varepsilon_2(x)$



The BCF Error $\varepsilon_m(x)$



The CF Error $E_2(x)$



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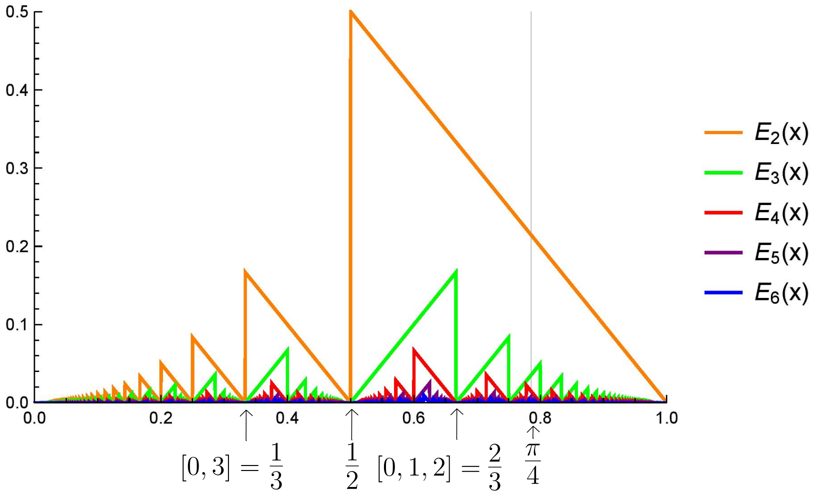


BCF Cylinder Sets and BCF Error Bounds $f_m^{[[1,b_2,\ldots,b_r]]}(x)$

The <u>rth-level BCF cylinder set</u> is

$$C_{[[b_1,\ldots,b_r]]} = \{x \in \mathbb{R} : r\text{th BCF convergent of } x = [[b_1, \ldots, b_r]]\}.$$

A <u>bounding curve of level r</u> is

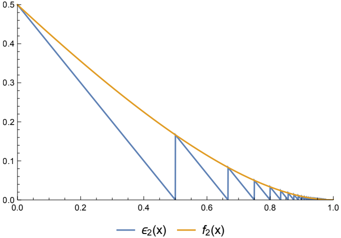
$f_m^{[[1,b_2,\ldots,b_r]]}(x) =$ limiting function for the BCF error of x given we are using a m-term BCF approximation and the r-term BCF expansion of x is $[[1, b_2 \ldots, b_r]]$.

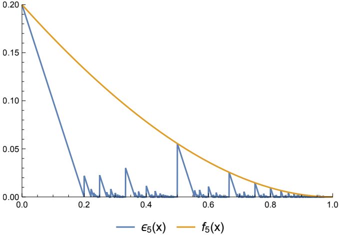
Overall Bound for the BCF Error, $f_m(x)$

Theorem (L-Bjorklund)

For any $x \in [0,1]$*, we have* $\varepsilon_m(x) \le f_m(x)$ *where*

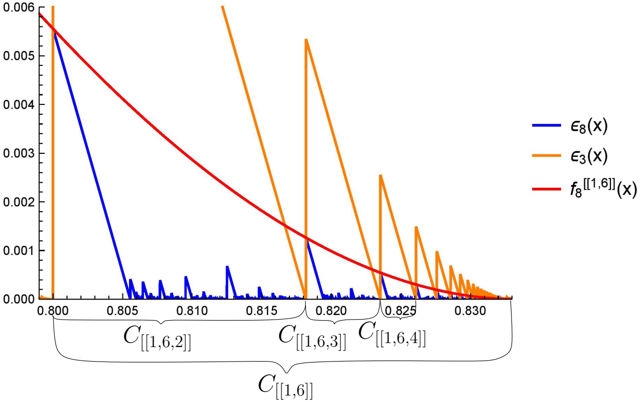
$$f_m(x) = \frac{1}{\left(\frac{1}{1-x}\right)\left((m-1)\frac{1}{1-x}+1\right)} = \frac{(1-x)^2}{m-x}.$$





Key Idea for Bounding the Error

To obtain a bound for the error on $C_{[[1,b_2,\ldots,b_r]]}$, we must find the maximum of $\varepsilon_m(x)$ on $C_{[[1,b_2,\ldots,b_r,n]]}$ for $n = 2, 3, \ldots$ and interpolate along these values.



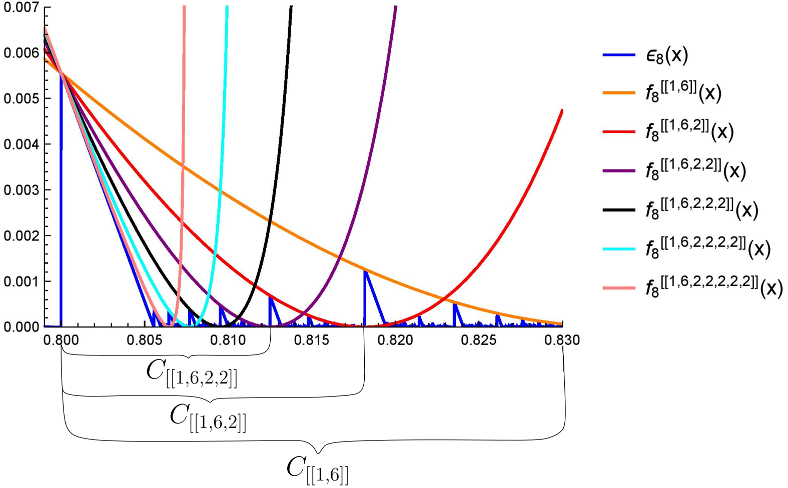
The General Case $f_m^{[[1,b_2,\ldots,b_r]]}(x)$

Theorem (L-Bjorklund)

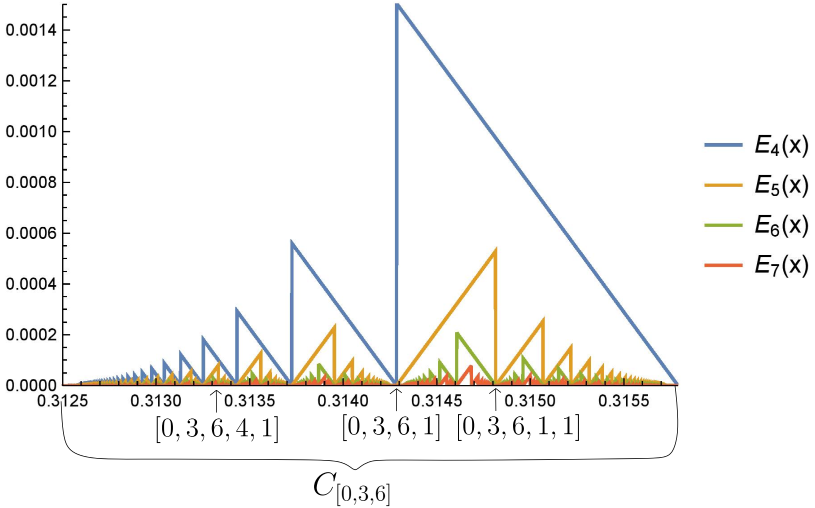
For $x \in C_{[[1,b_2,\ldots,b_r]]}$, i.e. the rth BCF convergent for x is $[[1, b_2, \ldots, b_r]] = \frac{p_r}{q_r}$, we have the mth BCF error $\varepsilon_m(x)$ is bounded by

$$f_m^{[[1,b_2,\ldots,b_r]]}(x) = \frac{(p_r - q_r x)^2}{(m-r) + q_r(p_r - q_r x)} = \frac{\varepsilon_r(x)^2}{\frac{m-r}{q_r^2} + \varepsilon_r(x)}.$$

Some Examples of $f_m^{[[1,b_2,\ldots,b_r]]}(x)$



The Maximum of $E_m(x)$ on $C_{[0,c_2,\ldots,c_r]}$



The General Case for CF Bounds $g_m^{[0,c_2,\dots,c_r]}(x)$

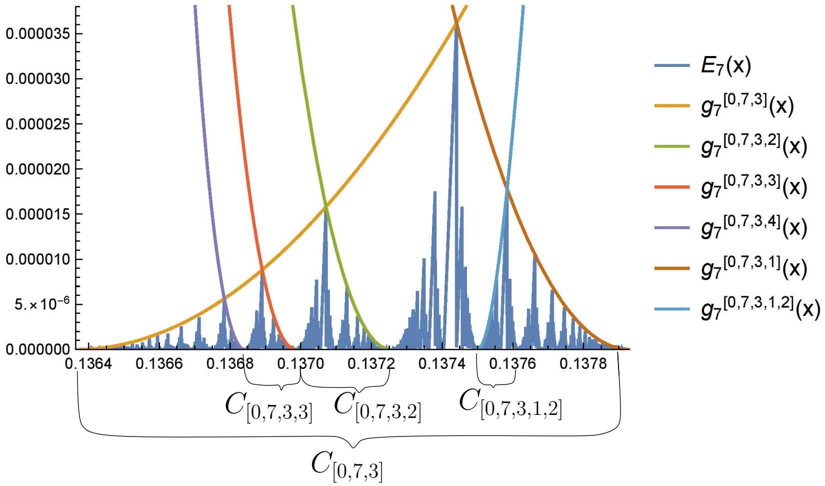
Theorem (L-Bjorklund)

We have the following bounding function for the CF error $E_m(x)$ on $C_{[0,c_2,\dots,c_r]}$:

$$g_m^{[0,c_2,\dots,c_r]}(x) = \begin{cases} \dfrac{(P_r - Q_r x)^2}{F_{m-r+1}F_{m-r} + Q_r(P_r - Q_r x)} & \text{if } m-r \text{ is even} \\[2ex] \dfrac{(P_r - Q_r x)^2}{F_{m-r+1}F_{m-r} - Q_r(P_r - Q_r x)} & \text{if } m-r \text{ is odd,} \end{cases}$$

where F_n is the nth Fibonacci number, and we have $F_0 = 0$, $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Some Examples of $g_m^{[0,c_2,\ldots,c_r]}(x)$



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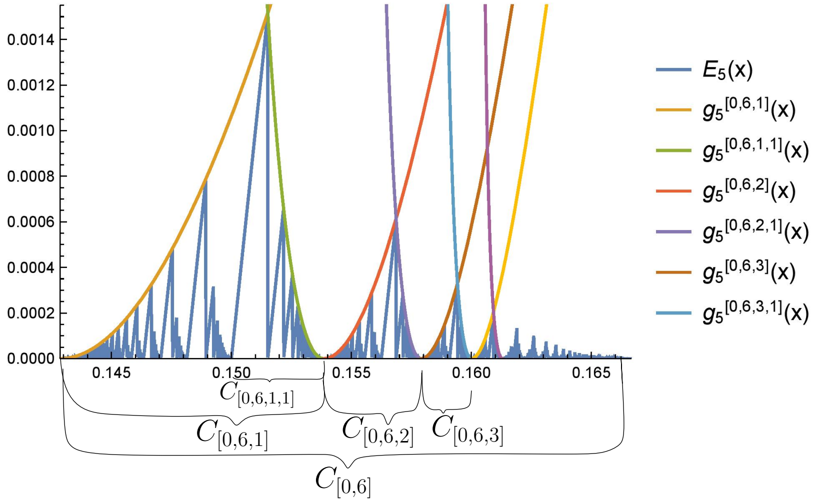

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- Reprove old results and prove new results on CFs using bounding curves.

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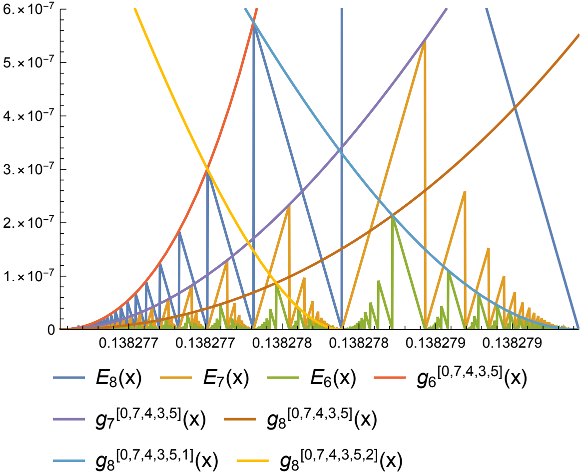
- Reprove old results and prove new results on CFs using bounding curves.
- Study expansions of transcendental numbers.

Thank You!



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$E_6(x)$
$g_6^{[0,7,4,3,5]}(x)$
$g_7^{[0,7,4,3,5]}(x)$
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