Error Approximation for Backwards and Simple Continued Fractions

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In this paper, we provide a new framework for studying continued fractions by means of the *backwards* continued fraction (BCF). We develop an approximation theory for BCFs, show the correspondence between continued fractions and their backwards continued fractions counterpart, and illustrate a rich approximation theory for continued fractions based off the methods of the approximation theory for the backwards case. In particular, we construct explicit functions that are sharp bounds for the BCF or CF error infinitely often over any BCF or CF cylinder set, and along the way work out the details to pass seamlessly between the BCF and CF expansion of any real number.

Contents

1 Introduction

In modern language, approximating real numbers via their continued fraction expansions dates to 1737 with Euler's *De fractionibus continuis dissertatio*. These expansions have very strong properties with respect to accuracy, precision, and the rates of convergence. For example, a classical theorem of Lagrange states that if $|x - \frac{p}{q}| < \frac{1}{2q^2}$, then $\frac{p}{q}$ is a convergent for the continued fraction expansion of x, i.e.

$\frac{p}{q} = \frac{P_n}{Q_n}$ for some n. Moreover, if $\frac{p}{q}$ is a best approximation of the second kind, i.e. if $|q'x - p'| < |qx - p|$ then $q' > q$, then $\frac{p}{q}$ is also a convergent. In this paper, we aim to tackle similar questions of "how good is the approximation of x by y", where first we take y to be a convergent of the *backwards continued fraction* of x, then take y to be a convergent of the continued fraction of x, and then compare the approximation theories we've constructed.

Informally, a *backwards continued fraction expansion* of $x \in \mathbf{R}$ is an expansion $x = [[b_0, b_1, ...]]$ where

$$[[b_0, b_1, ...]] := b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots}}}$$

with $b_0 \in \mathbf{Z}$ and $b_n \geq 2$ for $n \geq 1$ (the construction of the b_i's is discussed in Section 2). Such strings are in 1-1 correspondence with the reals giving us a unique infinite expansion for each real number. Truncations of this infinite expansion lead to rational numbers $\frac{p_n}{q_n} = [[b_0, ..., b_n]]$ called *convergents* which converge monotonically from above to x as $n \to \infty$. The *mth term BCF approximation to x* is the function $\varepsilon_m(x) = |x - \frac{p_m}{q_m}|$ where $\frac{p_m}{q_m}$ is the mth convergent to x. Our goal is to study the strength and rate of convergence of these convergents for $x \in [0, 1]$.

In this paper, our aim is to construct bounds for $\varepsilon_m(x)$ when we take x to be in the *rth level BCF cylinder set* $C_{[[b_0,...,b_r]]} = \{x \in [0, 1] : \text{ the } r\text{th convergent of } x \text{ is } [[b_0, ..., b_r]]\}$. We are successful in constructing *BCF error bound functions* $f_m^{[[b_0,...,b_r]]}(x)$ defined on $C_{[[b_0,...,b_r]]}$ such that $\varepsilon_m(x) \leq f_m^{[[b_0,...,b_r]]}(x)$ for all $x \in C_{[[b_0,...,b_r]]}$, where $f_m^{[[b_0,...,b_r]]}$ are rational functions in $\varepsilon_r(x)$. The full description and construction of these error bounds is contained in Section 5. After constructing the bounds for the BCF error, we define the above terms in the language of *continued fractions* and perform a similar analysis. Continued fractions have been shown to yield the best approximations with respect to many metrics, however bounding the CF error over CF cylinder sets as done here does not seem to have been treated before. As in the case of the BCFs, we construct *CF error bound functions* $g_m^{[c_0,...,c_r]}(x)$ defined on $C_{[c_0,...,c_r]}$ such that $E_m(x) \leq g_m^{[c_0,...,c_r]}(x)$ for all $x \in C_{[c_0,...,c_r]}$, where $g_m^{[c_0,...,c_r]}$ are rational functions in $E_r(x)$ (see Section 6 for full details on $g_m^{[c_0,...,c_r]}(x)$, the *CF cylinder set* $C_{[c_0,...,c_r]}$, and the *CF error* $E_m(x)$). Bounds on the error are constructed by looking at the structure of the expansions of the unit interval, of which the BCF expansions are easier to work with than CF expansions (see Theorem 1).

Backwards continued fractions have made their appearance in many other fields of mathematics beyond number theory. For example, in dynamics BCFs have been studied from the angle of measure preserving properties of the associated map $g(x) = \frac{1}{1-x}$ on the unit interval and for classifying geodesic flows in the upper half-plane [AF84; Kat96]. In geometry, Duke et. al used BCF expansions to classify geometric invariants for real quadratic fields [DIT16], and in topology Hatcher and Thurston utilized BCF expansions to characterize surfaces in knots [HT85]. This list is by no means all encompassing and aims to serve merely as an indicator of their different uses. As for rational approximations, beyond the simple calculation $|x - \frac{p_n}{q_n}| \leq \frac{1}{q_n}$ for the nth BCF approximation to x, the approximation theory for BCFs does not seem to have its place yet in the literature so we aim to fill this gap. In the process of constructing a rigorous foundation for the approximation theory of real numbers by BCFs, an approximation scheme for how to approximate numbers by CFs is also constructed, yielding classical results in a new framework. The success in the extension of methods for BCFs to CFs is an indication that BCFs provide a natural framework to construct rational approximations.

By first investigating the BCF error which has a rather nice structure to work with (Theorem 1), an approach to bounding the CF error comes to light. The essence of the method is quite classical,

boiling down to partitioning an interval and interpolating a function at maximums of an error over each sub-interval. In particular, we partition the interval into *cylinder sets* for a fixed number of digits in the BCF and CF expansions (i.e. into sets of the form $C_{[[b_0,\ldots,b_r]]}$ or $C_{[c_0,\ldots,c_r]}$). Next we find the maximum of the error (either $\varepsilon_m(x)$ or $E_m(x)$) within each cylinder set as well as its x-value, and then construct a function which assigns to each calculated x-value the corresponding maximum (either $f_m^{[[b_0,\ldots,b_r]]}(x)$ or $g_m^{[c_0,\ldots,c_r]}(x)$). Such examples can be seen in the following picture, and all the pieces at play will be defined during the course of the paper.

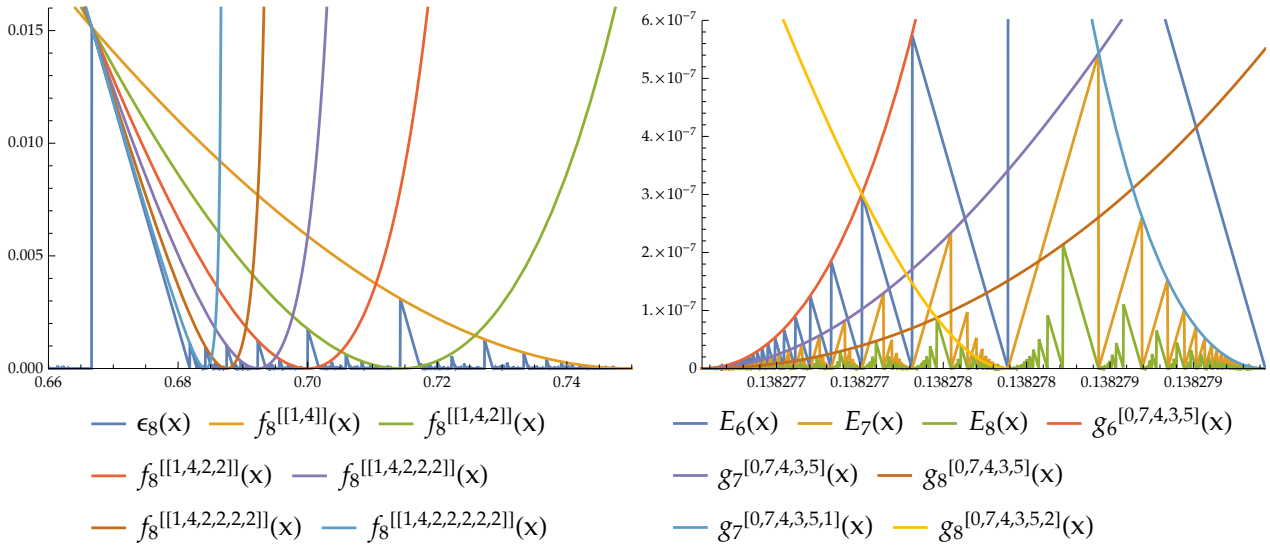

Figure 1: (a) 8-term BCF error and some bounding curves, (b) 6-, 7-, and 8-term CF errors with some bounding curves.

In the course of this paper, we answer the following questions in the affirmative:

(Q1) Does there exist a strictly decreasing continuous function that bounds the BCF error sharply infinitely often for the BCF expansion of any $x \in [0,1]$ given that it has m terms in its backwards continued fraction expansion? (Theorem 5)

(Q2) Does there exist a strictly decreasing continuous function that sharply bounds the BCF error infinitely often for the BCF expansion of any $x \in [0,1]$ given that its BCF expansion starts with $[[1, b_2, ..., b_r]]$ and it has $m \geq r+1$ terms in its backwards continued fraction expansion? (Theorem 7)

(Q3) Does there exist a strictly decreasing continuous function that sharply bounds the error in the CF expansion of any $x \in [0,1]$ infinitely often given that its CF expansion starts with $[0, c_2, ..., c_r]$ and it has $m \geq r+1$ terms in its continued fraction expansion? (Theorem 8)

(Q4) Can we describe explicitly how to go from a CF expansion to a BCF expansion and vice-versa? (Theorem 4)

(Q5) Can we describe the set of numbers that are better approximated by their m-term BCFs than their m-term CFs? (Corollary 2)

The structure of the paper is as follows: Section 2 provides the necessary background information on BCFs as well as introducing computational tools to deal with BCF expansions, Section 3 introduces

continued fractions and establishes the connection between BCFs and CFs, Section 4 introduces the errors $\varepsilon_m(x)$ and $E_m(x)$ and their construction on the unit interval, Section 5 contains the construction of the BCF error bounding functions $f_m^{[[1,b_2,\ldots,b_r]]}(x)$ as well as a few examples building up to the general case of $f_m^{[[1,b_2,\ldots,b_r]]}(x)$, Section 6 contains the construction of the CF error bounding functions $g_m^{[0,c_2,\ldots,c_r]}(x)$, and the Appendix contains some necessary matrix calculations.

2 Background on Backwards Continued Fractions

We begin this section by introducing one of the main players of this paper, the backwards continued fraction. Formally, the *backwards continued fraction* (BCF) expansion of $x \in \mathbf{R}$, as introduced by Hirzebruch in [Hir73], is the expression

$$x = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots}}} =: [[b_0, b_1, \ldots]].$$

where $x_0 = x$, $b_0 = \lfloor x \rfloor + 1$, $x_{i+1} = \frac{1}{b_i - x_i}$, and $b_i = \lfloor x_i \rfloor + 1$. In this definition, we see that $b_0 \in \mathbf{Z}$ and $b_i \geq 2$ for each $i \geq 1$. Similarly to the case of standard continued fractions, we define the n^{th} *convergent* $\frac{p_n}{q_n}$ of x to be

$$\frac{p_n}{q_n} = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_n}}}} = [[b_0, \ldots, b_n]].$$

The above construction of associating an infinite string $(b_0, b_1, \ldots)$ (with $b_0 \in \mathbf{Z}$, $b_i \geq 2$ for $i \geq 1$) to a real number gives us a one-to-one correspondence of infinite BCF expansions and the reals. Following in the footsteps of Zagier and Katok [Kat96; Kat03; Zag81], we make note of the following collection of properties associated to BCFs:

(2.1) x is rational if and only if from some point on all the b_i's are equal to 2;

(2.2) x is a quadratic irrational, i.e. a root of an irreducible quadratic polynomial with coefficients in $\mathbf{Z}$, if and only if its BCF expansion is eventually periodic, i.e. there exists a $k \in \mathbf{N}$ such that $b_{i+rk} = b_{i+k}$ for any $r \in \mathbf{N}$, $i \in \{0, 1, \ldots, m-1\}$, so $x = [[b_0, \ldots, b_{k-1}, \overline{b_k, \ldots, b_{k+m}}]]$ (the line over $b_k, \ldots, b_{k+m}$ signifies that those numbers repeat periodically and that m is the least period), except in the case where $b_{k+i} = 2$ for all $i \geq 0$ and x is rational by (2.1);

(2.3) a quadratic irrational x has a purely periodic BCF expansion if and only if $0 < x' < 1 < x$, where x' is conjugate to x, i.e. it is the other root of the same quadratic polynomial as x;

(2.4) if $x = [[\overline{b_0, \ldots, b_m}]]$, then $\frac{1}{x'} = [[\overline{b_m, \ldots, b_0}]]$.

(2.5) two numbers $x = [[b_0, b_1, \ldots]]$ and $y = [[c_0, c_1, \ldots]]$ have the same tails, i.e. $b_{n+i} = c_{m+i}$ for some n, m and all i, if and only if there exists $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ such that $\gamma(x) = y$ where γ acts by Möbius transformations,

$$\gamma(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az + b}{cz + d}.$$

Example: Consider the expansion and the 6th convergent of $x = \frac{-1+2\sqrt{23}}{5}$, which by (2.2) has a periodic BCF expansion.

$$\frac{-1+2\sqrt{23}}{5} = [[2, \overline{22, 5, 2, 3, 2, 2, 2, 3, 2, 5}]], \quad \frac{p_6}{q_6} = [[2, 22, 5, 2, 3, 2, 2]] = \frac{2042}{1045}.$$

Backwards continued fractions were introduced in number theory by Hirzebruch and Zagier as a tool in calculating the class number for imaginary quadratic fields $K = \mathbf{Q}[\sqrt{-D}]$ given information on the corresponding real quadratic field $\mathbf{Q}[\sqrt{D}]$ [Hir73; Zag81]. They were successful in this endeavor given that $D = p$ is a prime congruent to 3 (mod 4). Additionally, the language of backwards continued fractions provided the proper landscape for Gauss reduction theory of matrices in $\mathrm{SL}_2(\mathbf{Z})$. More information on the comprehensive reduction theory of $\mathrm{SL}_2(\mathbf{Z})$ can be found in [Kat96; Zag81]. For the sake of completeness, we include all necessary information we need below. Most information below can be found in [Kat96; Kat03; Zag81], and is cited as such.

In Section 4, we will use the following pieces of information regarding convergents of BCF expansions to construct error estimates.

(B1) For $[[b_0, ..., b_m]] = \frac{p_m}{q_m}$, the sequences p_m and q_m satisfy the recurrences $p_r = b_r p_{r-1} - p_{r-2}$ and $q_r = b_r q_{r-1} - q_{r-2}$, subject to the initial conditions $p_0 = b_0$, $p_1 = b_1 b_0 - 1$, $q_0 = 1$, and $q_1 = b_1$.

(B2) The matrix $\begin{pmatrix} p_r & q_r \\ -p_{r-1} & -q_{r-1} \end{pmatrix}$, which we call *the BCF matrix of convergents*, is calculated in $\mathrm{SL}_2(\mathbf{Z})$ by

$$\begin{pmatrix} p_r & q_r \\ -p_{r-1} & -q_{r-1} \end{pmatrix} = \begin{pmatrix} b_r & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_{r-1} & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(B3) The convergents satisfy $p_{r-1} q_r - p_r q_{r-1} = 1$ for all r.

Properties (B1)−(B3) will come back into play when we construct explicit functions bounding the error for a BCF expansion with a given number of terms.

A crucial property that we require for our analysis that standard continued fractions do not satisfy is that of "well-ordering at each level", which we make precise in the statement of the following theorem.

Theorem 1. *Let $[[b_0, b_1, ...]]$ and $[[c_0, c_1, ...]]$ be two infinite backwards continued fractions. We have $[[b_0, b_1, ...]] < [[c_0, c_1, ...]]$ if and only if the first index n for which $b_n \neq c_n$ satisfies $b_n < c_n$. The same statement holds for finite backwards continued fractions provided we also make an allowance for the equalities $[[b_0, ..., b_n]] := [[b_0, ..., b_n + 1, 2, 2, ...]]$. In particular, the set of BCF expansions satisfy the following well-ordering properties:*

(i) $[[b_0, ..., b_n]] < [[b_0, ..., b_n + 1]]$;

(ii) $[[b_0, ..., b_{n-1}, b_n]] < [[b_0, ..., b_{n-1}, b'_n, ..., b'_s]]$ where $b_n < b'_n$ and $s \geq n$, for any $b'_i \geq 2$.

Proof. We proceed by induction on n. If $b_0 < c_0$, then we clearly $[[b_0]] = b_0 < c_0 = [[c_0]]$ and the base case is done. If $b_0, ..., b_n, z, w$ and are positive numbers with $z < w$, then

$$[[b_0, ..., b_n, z]] = [[b_0, [[b_1, ..., b_n, z]]]] < [[b_0, [[b_1, ..., b_n, w]]]] = [[b_0, ..., b_n, w]]$$

and we get the result by the induction hypothesis. The two well-ordering properties follow directly from the inequality applied to the appropriate finite BCF expansions. □

There does exist a similar theorem for CF expansions noted in [Bom07] that depends on the parity of the length of the expansion, which we state here for completeness.

Theorem 2. [Bom07] *Let* $[a_0, a_1, ...]$ *and* $[b_0, b_1, ...]$ *be two simple infinite continued fractions. We have* $[a_0, a_1, ...] < [b_0, b_1, ...]$ *if and only if the first index* n *for which* $a_n \neq b_n$ *satisfies* $(-1)^n a_n < (-1)^n b_n$. *The same statement holds for simple finite continued fractions provided we also make an allowance for the equality* $[a_0, ..., a_n - 1, 1] = [a_0, ..., a_n]$.

In light of this result, we will make use of writing $[a_0, ..., a_n - 1, 1] = [a_0, ..., a_n]$ for the remainder of the paper, and we'll mention when this situation arises. One major take-away of Theorem 1 is that given a finite BCF expansion $[[b_0, ..., b_m]]$, we can make sense of the next m-term BCF expansion, i.e. given a m-term BCF $[[b_0, ..., b_m]]$, the BCF $[[b_0, ..., b_m + 1]]$ is the smallest of all m-term BCF expansions which are larger than $[[b_0, ..., b_m]]$. Thus Theorem 1 is one sense in which BCF expansions are easier to work with than CF expansions, for which the inequalities depend on the parity of n in the notation of Theorem 2.

3 Connecting Backwards Continued Fractions and Continued Fractions

The main goal of this section is to be able to convert backwards continued fractions to simple continued fractions (CFs) and back again. This goal is met in Theorem 4 which states for a BCF expansion $x = [[b_0, \widehat{B_1}, b_1, ..., b_{n-1}, \widehat{B_n}]]$, with B_i as defined in Definition 1, its CF expansion is $[b_0 - 1, B_1 + 1, b_1 - 2, B_2 + 1, \ldots, B_n + 1]$ (as defined below). Later we will use this relationship between the two types of continued fractions to compare their approximation strengths. Since the theory of continued fractions is quite classical, we direct the interested reader to the book of Hensley [Hen06] for a thorough treatment of continued fractions. Furthermore, this source can be referred to for any detail on continued fractions that is omitted.

A *simple continued fraction* (or *continued fraction*) expansion for $x \in \mathbf{R}$ is the expression

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} =: [a_0, a_1, ...],$$

where $x_0 = x$, $a_0 = \lfloor x \rfloor$, $x_{i+1} = \frac{1}{x_i - a_i}$, and $a_i = \lfloor x_i \rfloor$. In this definition, we see that $a_0 \in \mathbf{Z}$ and $a_i \geq 1$ for each $i \geq 1$. We define the n^{th} *convergent* $\frac{P_n}{Q_n}$ of x to be

$$\frac{P_n}{Q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}} = [a_0, ..., a_n].$$

Similarly to the case of BCFs, we make note of the following properties of CF convergents.

(C1) For $[a_0, ..., a_m] = \frac{P_m}{Q_m}$, the sequences P_m and Q_m satisfy the recurrences $P_r = a_r P_{r-1} + P_{r-2}$ and $Q_r = a_r Q_{r-1} + Q_{r-2}$ subject to the initial conditions $P_0 = a_0$, $P_1 = a_1 a_0 + 1$, $Q_0 = 1$, and $Q_1 = a_1$.

(C2) The matrix $\begin{pmatrix} P_r & Q_r \\ P_{r-1} & Q_{r-1} \end{pmatrix}$, which we call *the CF matrix of convergents*, is calculated in $\mathrm{GL}_2(\mathbf{Z})$ by

$$\begin{pmatrix} P_r & Q_r \\ P_{r-1} & Q_{r-1} \end{pmatrix} = \begin{pmatrix} a_r & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{r-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(C3) The convergents satisfy $P_rQ_{r-1} - P_{r-1}Q_r = (-1)^{r-1}$ for all r. (Compare to (B3) which does not have the sign dependence on r.)

The first step in transferring information from one language to another is already seen in some literature. Hirzebruch offers a way to turn a continued fraction with three terms into a backwards continued fraction in [Hir73]:

$$[a_0, a_1, z] = [[a_0 + 1, \underbrace{2, 2, \ldots, 2}_{a_1 - 1 \ 2's}, z + 1]], \tag{1}$$

and later Katok noted how to do this in general in [Kat03] (which we include as Corollary 1). The first step in the direction of turning a BCF expansion into its CF expansion can be worked out to be

$$[[b_0, b_1]] = [b_0 - 1, 1, b_1 - 1]. \tag{2}$$

This is equivalent to the equality

$$a - \frac{1}{b} = (a - 1) + \frac{1}{1 + \frac{1}{(b-1)}}.$$

Inducting on this idea immediately gives us a nice theorem.

Theorem 3. *For a BCF $x = [[b_0, b_1, \ldots, b_n]]$ with at least two terms, we have the (perhaps irregular) continued fraction expansion is*

$$[b_0 - 1, 1, b_1 - 2, 1, b_2 - 2, 1, \ldots, b_{n-1} - 2, 1, b_n - 1].$$

Proof. We proceed by induction on the number of terms in a BCF. For a BCF with two terms, we have $[[b_0, b_1]] = [b_0 - 1, 1, b_1 - 1]$ by (2) above. Assume for induction that $[[b_0, b_1, \ldots, b_n]]$ has the continued fractions expansion we assert. Consider $x = [[b_0, b_1, \ldots, b_{n+1}]]$. Let $z = [[b_n, b_{n+1}]] = [b_n - 1, 1, b_{n+1} - 1]$ so we can write $x = [[b_0, b_1, \ldots, b_{n-1}, z]]$. By the induction hypothesis, we have that

$$\begin{aligned} x &= [b_0 - 1, 1, b_1 - 2, 1, \ldots, b_{n-1} - 2, 1, z - 1] \\ &= [b_0 - 1, 1, b_1 - 2, 1, \ldots, b_{n-1} - 2, 1, [b_n - 1, 1, b_{n+1} - 1] - 1] \\ &= [b_0 - 1, 1, b_1 - 2, 1, \ldots, b_n - 2, 1, b_{n+1} - 1] \end{aligned}$$

□

One issue we have with the previous theorem is that it does not always produce a *regular continued fraction*, where a continued fraction is regular if after the first term, all terms are at least 1. This irregularity occurs when a digit of 2 is encountered in the BCF expansion. When looking at backwards continued fraction expansions, it is impossible to ignore the vast abundance of long strings of the integer 2 (especially with property (2.1) in mind). For example,

$$\begin{aligned} \frac{1 + \sqrt{19}}{2} &= [[3, \overline{4, 2, 2, 2, 2, 2, 2, 2, 4, 5}]], \\ \frac{8 + \sqrt{57}}{3} &= [[\overline{6, 2, 2, 2, 2, 4, 2, 2, 2, 2, 6, 7, 3, 46, 3, 7}]], \text{ and} \\ e &= [[3, 4, 3, 2, 2, 2, 3, 8, 3, 2, 2, 2, 2, 2, 2, 2, 3, 12, ...]]. \end{aligned}$$

In fact, with the noticeable abundance of twos in a typical backwards continued fraction, there are usually quite a lot of zeros in the associated continued fraction! In light of this, our next goal is to understand the behaviour of how a sequence of alternating ones and zeros acts in a continued fraction.

Proposition 1. *Given the continued fraction* $x = [\underbrace{1, 0, 1, 0, ..., 0, 1}_{m \text{ ones}}, z]$ *for* z *an indeterminate, the regular continued fraction expansion is* $x = [m, z]$.

Proof. We induct on the number of ones. We start with the cases of $m = 0$ and $m = 1$ ones, which are $[0, z]$ and $[1, z]$ respectively, and we're done with the base cases. Assume for induction that $[1, 0, 1, ..., 0, 1, z] = [m, z]$ for m 1's, let's look at $[\underbrace{1, 0, ..., 0, 1}_{m+1\ 1's}, z]$. Define $w = [\underbrace{1, 0, ..., 0, 1}_{m\ 1's}, z]$ which equals $[m, z]$ by the induction hypothesis, so

$$[\underbrace{1, 0, ..., 0, 1}_{m+1\ 1's}, z] = [1, 0, w] = 1 + \frac{1}{0 + \frac{1}{w}} = 1 + w = 1 + [m, z] = [m + 1, z].$$

□

In relation to Theorem 3, Proposition 1 gives us the proper way of turning an irregular continued fraction obtained from a BCF into a simple continued fraction. It can be deduced from the statement of Theorem 3 that the strings of 2's in a BCF are what give rise to these irregular continued fractions which we must simplify. It turns out that these strings of 2's are integral to our analysis, so much so that we make following definition:

Definition 1. *The natural form of a BCF expansion* x *is*

$$x = [[b_0; \widehat{B}_1, b_1, \widehat{B}_2, b_2, \dots]] = [[b_0, \underbrace{2, 2, \dots, 2}_{B_1}, b_1, \underbrace{2, 2, \dots, 2}_{B_2}, b_2, \dots]],$$

where $b_i \in \mathbf{Z}$ *with* $b_i \geq 3$ *for* $i \geq 1$ *and* $\widehat{B}_i \in \mathbf{Z}_{\geq 0}$. *We call the* B_i*'s the block sizes of* x. *If a BCF is finite, the natural form must end in a (perhaps empty) string of twos* $\widehat{B}_n$. *It is clear every BCF, infinite or finite, has a natural form.*

Remark: To get the natural form of a BCF given its standard BCF expansion, block all strings of 2's together (except the first digit) and in between every two digits which are strictly greater than 2, insert an empty block of 2's. For example, we have the equivalences of standard and natural forms $[[3, 2, 4, 5, 3]] = [[3, \hat{1}, 4, \hat{0}, 5, \hat{0}, 3, \hat{0}]]$ and $[[2, 7, 2, 2, 2, 3, 2, 2, 19, 2, 2, 2, 2]] = [[2, \hat{0}, 7, \hat{3}, 3, \hat{2}, 19, \hat{4}]]$.

With this definition at hand, we can now state the main theorem of this section.

Theorem 4. *For* $x = [[b_0; \widehat{B}_1, b_1, \widehat{B}_2, b_2, \dots, \widehat{B}_n]]$ *in natural form, the simple continued fraction expansion of* x *is*

$$x = [b_0 - 1, B_1 + 1, b_1 - 2, B_2 + 1, \dots, B_n + 1].$$

Lemma 1. $[[b_0; \widehat{B}_1, z]] = [b_0 - 1, B_1 + 1, z - 1]$.

Proof. We will induct on B_1. If $B_1 = 0$, then

$$[[b_0; \widehat{0}, z]] = [[b_0, z]] = b_0 - \frac{1}{z} = b_0 - 1 + \frac{z - 1}{z} = b_0 - 1 + \frac{1}{\frac{z}{z-1}} = b_0 - 1 + \frac{1}{1 + \frac{1}{z-1}} = [b_0 - 1, 0 + 1, z - 1].$$

Now assume that $[[b_0; \widehat{B_1}, z]] = [b_0 - 1, B_1 + 1, z - 1]$.

$$[[b_0; \widehat{B_1 + 1}, z]] = [[b_0; \widehat{B_1}, 2 - \frac{1}{z}]] = [b_0 - 1, B_1 + 1, 1 - \frac{1}{z}] = b_0 - 1 + \frac{1}{B_1 + 1 + \frac{1}{1 - \frac{1}{z}}} = b_0 - 1 + \frac{1}{B_1 + 1 + \frac{z}{z-1}}$$

$$= b_0 - 1 + \frac{1}{B_1 + 1 + 1 + \frac{1}{z-1}} = [b_0 - 1, B_1 + 2, z - 1].$$

□

We now proceed with a proof of the main theorem.

Proof of Theorem 4: We first note that the CF expansion $[b_0 - 1, B_1 + 1, b_1 - 2, B_2 + 1, \ldots, B_n + 1]$ where $[[b_0; \widehat{B}_1, b_1, \widehat{B}_2, b_2, \ldots, \widehat{B}_n]]$ is a natural form BCF is in fact a regular continued fraction. Since $b_i \geq 3$ outside of b_0 and $B_i \geq 0$, we have $b_i - 2$ and $B_i + 1$ are each at least 1 where needed.

To prove the theorem, we induct on the number of blocks $\widehat{B}_i$. A BCF with one block, $[[b_0, \widehat{B}_1, b_1]]$, is a straight application of lemma 2.

Assume that any BCF with n blocks has the CF representation we claim. Consider the BCF $x = [[b_0; \widehat{B}_1, b_1, \widehat{B}_2, b_2, \ldots, b_n, \widehat{B}_{n+1}]]$. Let $y = [[b_1, \widehat{B_2}, b_2, \ldots, b_n, \widehat{B}_{n+1}]]$, the tail of x. We have

$$y = [b_1 - 1, B_2 + 1, b_2 - 2, \ldots, B_{n+1} + 1]$$
$$x = [[b_0, \widehat{B}_1, y]] = [b_0 - 1, B_1 + 1, y - 1] = [b_0 - 1, B_1 + 1, [b_1 - 1, B_1 + 1, b_2 - 2, \ldots, B_{n+1} + 1] - 1]$$
$$= [b_0 - 1, B_1 + 1, [b_1 - 2, B_2 + 1, \ldots, B_{n+1} + 1]\,] = [b_0 - 1, B_1 + 1, b_1 - 2, \ldots, B_{n+1} + 1].$$

□

In light of the previous theorem, the proper way to transfer from CFs to BCFs becomes clear, which we demonstrate in the following corollary. We note that the the following corollary can be found as the last of the exercises of in the BCF section of Katok's book [Kat03]. A variant of Theorem 4 is shown by Dajani and Kraaikamp in [DK00] by means of an algorithm using insertion and singularization, and was previously noted by Zagier (Aufgabe 3, pp.131) [Zag81], but the use of the natural form is new to the literature.

Corollary 1. *Given a standard continued fraction $x = [a_0, ..., a_n]$, its BCF expansion is*

$$[[a_0 + 1, \widehat{a_1 - 1}, a_2 + 2, ..., a_{n-1} + 2, \widehat{a_n - 1}]] \qquad \text{if } n \text{ odd,}$$
$$[[a_0 + 1, \widehat{a_1 - 1}, a_2 + 2, ..., a_n + 2, \widehat{0}]] \qquad \text{if } n \text{ even.}$$

Proof. Clearly by Theorem 4 the proposed BCF expansions are equal to x. □

Theorem 4 and Corollary 1 provide us with the proper way of transferring back and forth between BCF and CF expansions. With this effective way of equating the two expansions at hand, we can look at the length of the expansions to investigate the strength of approximation.

Definition 2. *The natural length, $\eta(x)$, of a BCF $x = [[b_0; \widehat{B}_1, b_1, \ldots, b_{n-1}, \widehat{B}_n]]$ is $\eta(x) = n$. The CF-length of a number $y = [a_1, a_2, \ldots, a_n]$ is n. The BCF-length of a number $z = [[b_1, b_2, \ldots, b_n]]$ (in standard form, not natural form) is n.*

Corollary 2. *The CF-length of x is equal to $2\eta(x)$. The BCF-length of $x = [[b_0; \widehat{B}_1, b_1, \ldots, b_{n-1}, \widehat{B}_n]]$ is $\eta(x) + \sum_{i=1}^{n} B_i$. Moreover, we have that a rational number x can be approximated in less steps by BCFs than CFs if $\eta(x) > \sum_{i=1}^{\eta(x)} B_i$.*

Proof. By Theorem 4, since the maps between CFs and BCFs in natural form yield valid CF/BCF expansions, the result is immediate. As for the case of writing the BCF-length in terms of the natural length, this follows directly from expressing x in its natural form and comparing lengths. □

Example: The natural forms of $[[1, 4, 2, 2, 2, 3, 2, 2, 4, 2, 2, 2, 5, 2]]$ and $[[2, 5, 7]]$ are $[[1; \widehat{0}, 4, \widehat{3}, 3, \widehat{2}, 4, \widehat{3}, 5, \widehat{1}]]$ and $[[2; \widehat{0}, 5, \widehat{0}, 7, \widehat{0}]]$, respectively. Theorem 4 tells us that these BCFs are equal to the CFs $[0, 1, 2, 4, 1, 3, 2, 4, 3, 2]$ and $[1, 1, 3, 1, 5, 1]$, respectively. We can see Corollary 2 at play by noting that

$$[[1, 4, 2, 2, 2, 3, 2, 2, 4, 2, 2, 2, 5, 2]] = [0, 1, 2, 4, 1, 3, 2, 4, 3, 2]$$

is approximated in more terms by its BCF expansion than its CF expansion and $[[2,5,7]] = [1,1,3,1,5,1]$ is approximated in less terms by its BCF expansion than its CF expansion since $\eta([[2,5,7]]) = 3 > \sum_{i=1}^{3} 0 = 0$. In standard notation, the above equality can be read as

$$1 - \cfrac{1}{4 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{3 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{4 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{5 - \frac{1}{2}}}}}}}}}}}}} = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{3 + \frac{1}{2}}}}}}}}}.$$

4 The Nature of the Errors $\varepsilon_m(x)$ and $E_n(x)$

The primary goal of this paper is to construct effective bounds for approximations of real numbers by their m-term BCF/CF expansion. To this end, it is necessary to understand what the approximation looks like on all of [0,1] and then construct approximations on properly chosen subsets.

Definition 3. *The mth term BCF approximation error $\varepsilon_m(x)$ of $x \in [0,1]$ is given by*

$$\varepsilon_m(x) = \left| x - \frac{p_m}{q_m} \right|,$$

where $\frac{p_m}{q_m}$ is the m-term BCF expansion of x. The nth term CF approximation error $E_n(x)$ of $x \in [0,1]$ is given by

$$E_n(x) = \left| x - \frac{P_n}{Q_n} \right|,$$

where $\frac{P_n}{Q_n}$ is the n-term CF expansion of x.

By Theorem 1, the BCF convergents $\frac{p_m}{q_m}$ of x are always over-estimates and it follows that $\varepsilon_m(x) = \frac{p_m}{q_m} - x$ for every m. With the definition of $\varepsilon_m(x)$ and $E_n(x)$, the task at hand is to walk through the calculation of the errors starting with $\varepsilon_2(x)$ and $E_2(x)$. This recursive procedure starts at the index 2 since the one step approximations $\varepsilon_1(x) = 1 - x$ and $E_1(x) = x$ are defined on all of the unit interval and do not have any further dependence on where x sits in $[0,1]$.

4.1 The BCF Error $\varepsilon_m(x)$

We will investigate the BCF error $\varepsilon_m(x)$ by looking at an irrational number in the unit interval and construct the mth term approximation for it. This procedure will highlight the structure of BCF expansions.

Consider the irrational number $\frac{\pi}{4} \approx 0.7853982$, which by the calculation for BCF expansions is $\frac{\pi}{4} = [[1,5,3,17,2,74,11,...]]$. The first BCF convergent for $\frac{\pi}{4}$ is $\frac{p_1}{q_1} = 1$, which gives us the first error $\varepsilon_1(\frac{\pi}{4}) = |\frac{\pi}{4} - \frac{p_1}{q_1}| = 1 - \frac{\pi}{4} \approx 0.2146018$. This case is not terribly interesting as every number in the unit interval has first digit 1. When we arrive at index 2, we have $\frac{p_2}{q_2} = [[1,5]] = \frac{4}{5}$ and $\varepsilon_2(\frac{\pi}{4}) = |\frac{\pi}{4} - \frac{4}{5}| \approx 0.0146018$. In fact, one can work out all the numbers in the interval $(\frac{3}{4}, \frac{4}{5}]$ have digit 5 as their second BCF digit. More generally, the interval $([[1,n-1]],[[1,n]]] = (\frac{n-2}{n-1}, \frac{n-1}{n}]$ is comprised of all numbers in $[0,1]$ with second BCF digit n. The slope of the BCF error on each of these sub-intervals is -1 since the error is explicitly given by $\varepsilon_2(x) = \frac{n-1}{n} - x$ on $(\frac{n-2}{n-1}, \frac{n-1}{n}]$. Combining this observation

with $\varepsilon_2([[1,n]]) = \varepsilon_2(\frac{n-1}{n}) = 0$, we obtain that the limit of the BCF error as x approaches $\frac{n-2}{n-1}$ from the right to be

$$\frac{n-1}{n} - \frac{n-2}{n-1} = \frac{n^2-2n+1-(n^2-2n)}{n(n-1)} = \frac{1}{n(n-1)}.$$

Putting all this information together, we get the graph of $\varepsilon_2(x)$ in Figure 2.

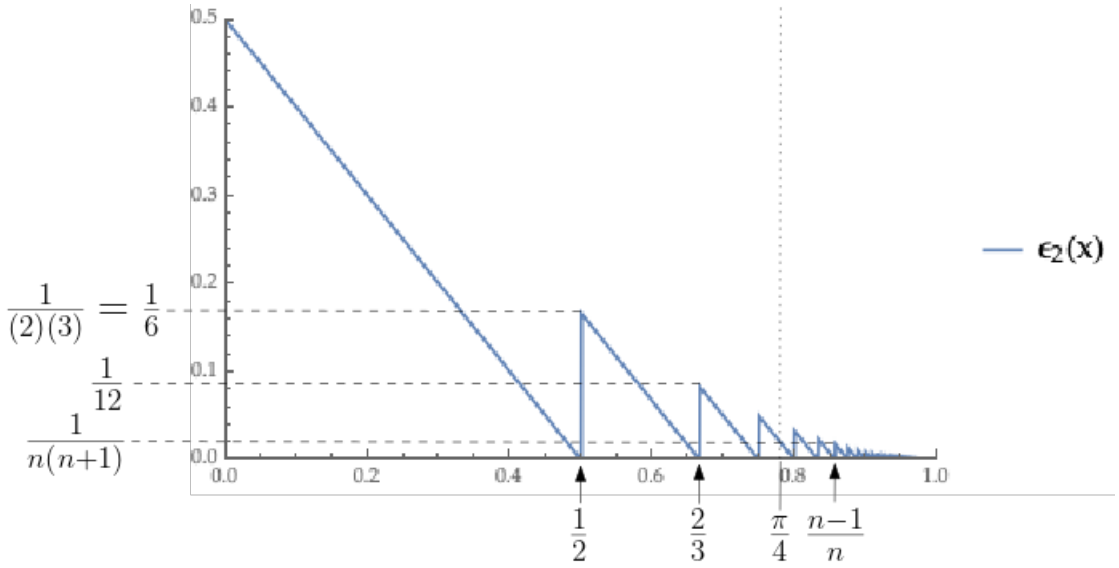

Figure 2: The graph of $\varepsilon_2(x)$ and the highlighted value of ε_2 at $\frac{\pi}{4}$.

The third BCF convergent of $\frac{\pi}{4}$ is $\frac{p_3}{q_3} = [[1,5,3]] = \frac{11}{14}$, giving us the error $\varepsilon_3(\frac{\pi}{4}) = |\frac{\pi}{4} - \frac{11}{14}| \approx 0.00031608571$. The elements in $[0,1]$ that share the second digit 5 and third digit 3 with $\frac{\pi}{4}$ are those in the interval $([[1,5,2]],[[1,5,3]]] = (\frac{7}{9}, \frac{11}{14}]$ and in general, the sets on which the third term error $\varepsilon_3(x)$ is continuous and has constant slope -1 are of the form $([[1,k,n]],[[1,k,n+1]]]$ for $k,n \geq 2$. The BCF error is 0 at $[[1,k,n+1]]$ and approaches

$$\left(1 - \frac{1}{k - \frac{1}{n+1}}\right) - \left(1 - \frac{1}{k - \frac{1}{n}}\right) = \frac{1}{(kn-1)(k(n+1)-1)}$$

as x approaches $[[1,k,n]]$ from the right. In general, Theorem 1 tells us $[[1,b_2,...,b_n]] < [[1,b_2,...,b_n+1]]$ for any n and combined with the fact that the mth BCF error will always have slope -1 on the subinterval $([[1,b_2,...,b_{m-1},n]],[[1,b_2,...,b_{m-1},n+1]]]$, we can generate the following plots in Figure 3 of $\varepsilon_m(x)$ for various values of m. In particular, $\varepsilon_2(x)$, $\varepsilon_3(x)$, $\varepsilon_4(x)$,... are continuous with slope -1 on the intervals $([[1,n]],[[1,n+1]]]$, $([[1,k,n]],[[1,k,n+1]]]$, $([[1,\ell,k,n]],[[1,\ell,k,n+1]]]$,... for $n,k,\ell \geq 2$ respectively.

Observe that in constructing $\varepsilon_m(x)$, we can see a scaled down version of $\varepsilon_2(x)$ in each interval $([[1,b_2,...,b_{m-1}]],[[1,b_2,...,b_{m-1}+1]]]$, which is related to the fact that the interval can be partitioned by the mth digit in the m-term BCF expansion

$$([[1,b_2,...,b_{m-1}]],[[1,b_2,...,b_{m-1}+1]]] = \bigcup_{n\geq 2} ([[1,b_2,...,b_{m-1}+1,n]],[[1,b_2,...,b_{m-1}+1,n+1]]].$$

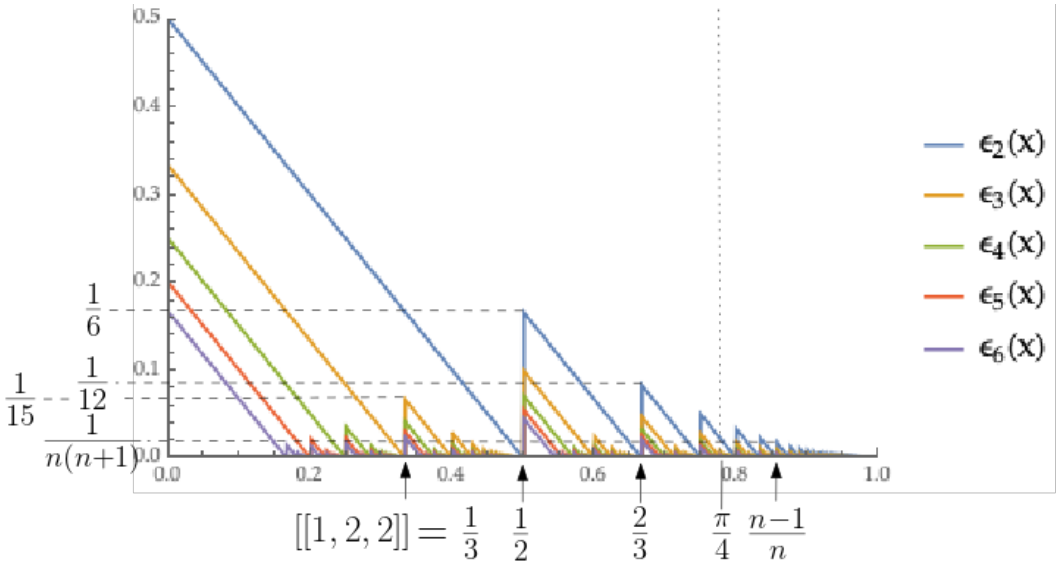

Figure 3: The $m = 2$-, 3-, 4-, 5-, and 6- term BCF error curves $\varepsilon_m(x)$.

4.2 The CF Error $E_m(x)$

The nature of $E_m(x)$ is similar to that of $\varepsilon_m(x)$ but its key difference follows from Theorem 2 which tells us $[a_1, a_2, ..., a_r, n] < [a_1, a_2, ..., a_r, n+1]$ with $n \geq 1$ if and only if r is even. In a manner akin to the above, we will look at the step-by-step building up of the CF error by looking at the CF expansion of $\frac{\pi}{4}$, which by the calculation for CF expansions or by Theorem 4 applied to the BCF expansion has CF expansions $\frac{\pi}{4} = [0, 1, 3, 1, 1, 1, 15, 2, 72, 1, 9, ...]$. The first CF convergent for $\frac{\pi}{4}$ is $\frac{P_1}{Q_1} = 0$, which gives us the first CF error to be $E_1(\frac{\pi}{4}) = |\frac{\pi}{4} - 0| = \frac{\pi}{4}$. Just as before, this case is not terribly interesting as every number in the unit interval has first digit 0. As for the second CF convergent, we have $\frac{P_2}{Q_2} = [0, 1] = 1$ and $E_2(\frac{\pi}{4}) = |\frac{\pi}{4} - 1| \approx 0.2146018$. In fact, all the numbers in the interval $(\frac{1}{2}, 1]$ have second digit 1 in their CF expansion. More generally, we have that the interval $(\frac{1}{n+1}, \frac{1}{n}]$ is the collection of elements in $[0, 1]$ with second digit n. The slope of the CF error on each of these sub-intervals is -1 since Theorem 2 tells us that $\frac{P_2}{Q_2}$ is an over-estimate and the CF error is given explicitly by $E_2(x) = \frac{1}{n} - x$. By knowing the slope of the error is -1 on $(\frac{1}{n+1}, \frac{1}{n}]$ and $E_2(\frac{1}{n}) = 0$, we have that the limit of the error as x approaches $\frac{1}{n+1}$ from the right to be

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

Using this information, we can now graph $E_2(x)$.

The third CF convergent of $\frac{\pi}{4}$ is $\frac{P_3}{Q_3} = [0, 1, 3] = \frac{3}{4}$, which has error $E_3(\frac{\pi}{4}) = |\frac{\pi}{4} - \frac{3}{4}| \approx 0.0353982$. In contrast with the case of BCF approximation where we had $\frac{p_3}{q_3} > \frac{\pi}{4}$, Theorem 2 tells us that $\frac{P_3}{Q_3} < \frac{\pi}{4}$. This inequality holds for each $x \in [0, 1]$, i.e. if we denote the third CF convergent to x by $\frac{P_3}{Q_3}(x)$, we have $\frac{P_3}{Q_3}(x) \leq x$ for all x in the unit interval. Furthermore, all elements in the sub-interval $[[0, 1, 3], [0, 1, 4])$ share second digit 1 and third digit 3 with $\frac{\pi}{4}$ and the sets on which the 3rd term CF error $E_3(x)$ is continuous with constant slope 1 are of the form $[[0, k, n], [0, k, n+1])$ for $k, n \geq 1$. Here the slope is 1 instead of -1 since the error is given by $E_3(x) = x - \frac{P_3}{Q_3}$. The third term CF error is 0 at $[0, k, n]$ and approaches

$$0 + \frac{1}{k + \frac{1}{n+1}} - 0 + \frac{1}{k + \frac{1}{n}} = \frac{(n+1)(kn+1) - n(k(n+1)+1)}{(kn+1)(k(n+1)+1)} = \frac{1}{(kn+1)(k(n+1)+1)}$$

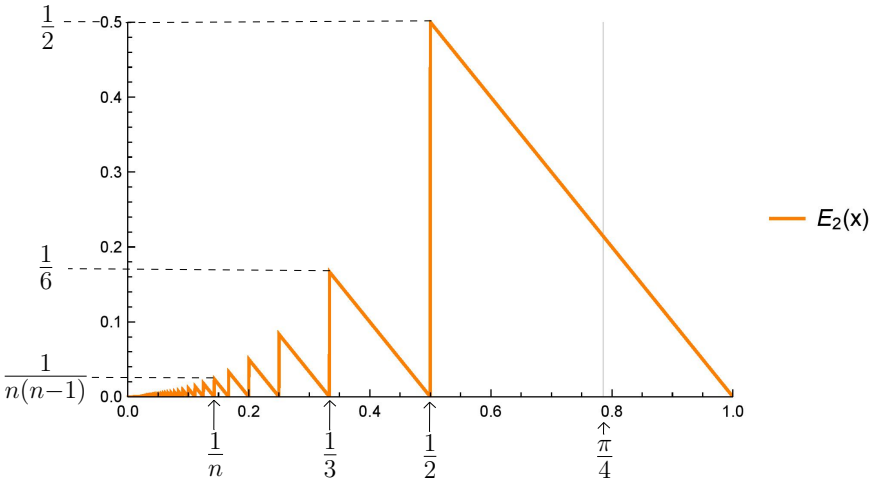

Figure 4: The graph of $E_2(x)$ and the value of E_2 at $\frac{\pi}{4}$.

as x approaches $[0, k, n+1]$ from the left. Continuing in this fashion, Theorem 2 tells us $[a_1, a_2, ..., a_r, n] < [a_1, a_2, ..., a_r, n+1]$ with $n \geq 1$ if and only if r is even, and combining this with the fact that $E_m(x)$ has slope 1 for m odd and slope -1 for m even on the sets $[[0, c_2, ..., c_m], [0, c_2, ..., c_m+1])$ and $([0, c_2, ..., c_m+1], [0, c_2, ..., c_m]]$ respectively, we can create the following graphs of $E_m(x)$ for various values of m.

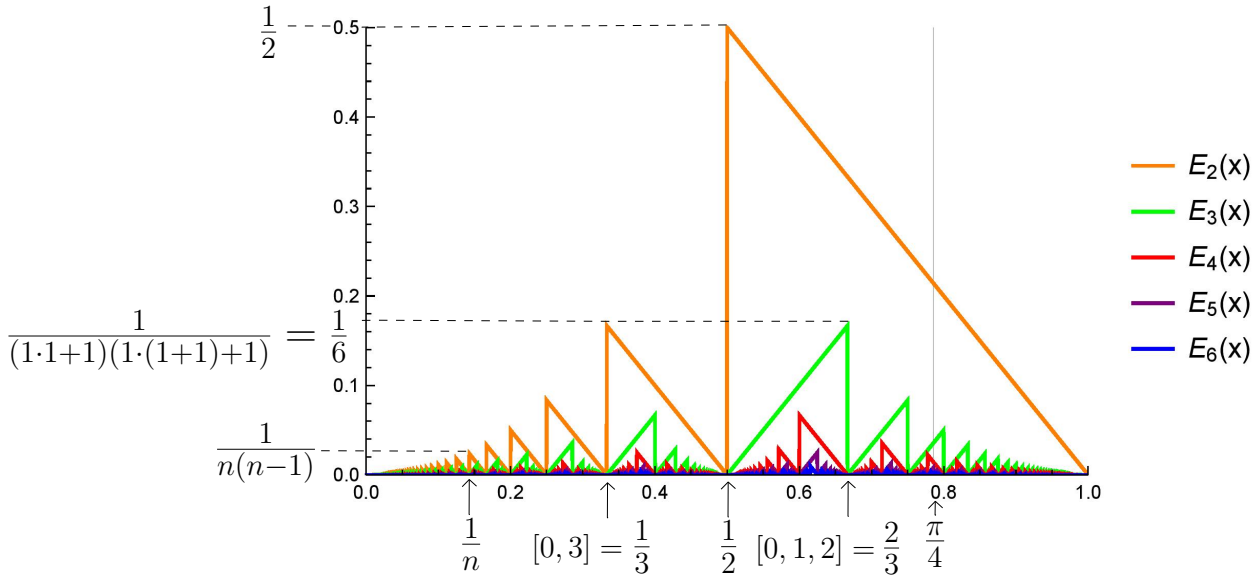

Figure 5: The $m = 2$-, 3-, 4-, 5-, and 6- term CF error curves $E_m(x)$.

Observe that in constructing $E_m(x)$, we can see a scaled down version of $E_2(x)$ in each interval $([0, c_2, ..., c_{m-1}], [0, c_2, ..., c_{m-1}+1]]$ for m even and a scaled down version of $E_2(1-x)$ in each interval $[[0, c_2, ..., c_m+1], [0, c_2, ..., c_m])$ for m odd. This is related to the fact that the interior of the interval can be partitioned by the mth digit in the m-term CF expansion

$$([0, c_2, ..., c_{m-1}]], [0, c_2, ..., c_{m-1}+1]) = \bigcup_{n\geq 1} ([0, c_2, ..., c_{m-1}+1, n+1], [0, c_2, ..., c_{m-1}+1, n]] \quad (m \text{ even})$$

$$([0, c_2, ..., c_{m-1}+1], [0, c_2, ..., c_{m-1}]] = \bigcup_{n\geq 1} [[0, c_2, ..., c_{m-1}+1, n], [0, c_2, ..., c_{m-1}+1, n+1]) \quad (m \text{ odd}).$$

5 Error Approximation for BCFs

In this section, we will exploit the myriad of nice qualities of BCFs introduced in Section 2 to obtain explicit information on the approximation behaviour of BCFs. For the remainder of this paper, we are going to be talking about how many terms we need in an expansion of x to get the error within the bounds of some value. Hence it is natural to think of b_0 as the first term, b_1 as the second term, and so on. Thus we will start our expansions indexed at 1 (compared to 0) and will use the convention of writing x and $\frac{p_m}{q_m}$ as the following:

$$x = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ddots}}} = [[b_1, b_2, ...]], \qquad \frac{p_m}{q_m} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ddots - \frac{1}{b_m}}}} = [[b_1, ..., b_m]].$$

Definition 4. *The mth term BCF approximation error $\varepsilon_m(x)$ of $x \in [0,1]$ is given by*

$$\varepsilon_m(x) = \left| x - \frac{p_m}{q_m} \right|,$$

where $\frac{p_m}{q_m}$ is the m-term BCF expansion of x.

With this notation at hand, our goal will be to find a continuous function which bounds the mth term error $\varepsilon_m(x)$, given some *initial information* on the expansion of x. Such initial information is given in the form that x lies in the rth-level BCF cylinder set

$$C_{[[b_1,...,b_r]]} = \{x \in \mathbf{R} : r\text{th BCF convergent of } x = [[b_1, ..., b_r]]\}.$$

For the remainder of this paper, we use the notation

$$f_m^{[[1,b_2,...,b_r]]}(x) = \text{limiting function for the error of } x \text{ given we are using a } m\text{-term BCF approximation and the } r \text{ term BCF expansion of } x \text{ is } [[1, b_2, ..., b_r]].$$

If the value of x has no information given, we denote the error by $f_m(x)$ without any superscript. We call such curves $f_m^{[[1,b_2,...,b_r]]}(x)$ bounding curves of level r.

The remark in the middle of Section 1 translates to $\varepsilon_m(x) \leq \frac{1}{q_m}$ for any x, so here we show how much better of an approximation one gets given which r-level cylinder set $C_{[[b_1,...,b_r]]}$ contains x. Using the bounds below, one can find much tighter bounds on $\varepsilon_m(x)$ than $\frac{1}{q_m}$.

Remark: $\mathrm{SL}_2(\mathbf{Z})$ acts transitively on the set of unimodular intervals $\mathcal{U} := \{(\frac{a}{b}, \frac{c}{d}] : ad - bc = 1\}$. To see this, note that each unimodular interval $(\frac{a}{b}, \frac{c}{d}] \in \mathcal{U}$ corresponds to the element in $\mathrm{SL}_2(\mathbf{Z})$ which has a and b in one column and c and d in the other, so the group action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathcal{U}$ is left multiplication. Since any group acts transitively on itself, it follows that $\mathrm{SL}_2(\mathbf{Z})$ acts transitively on $\mathcal{U}$. Additionally, the set of all BCF cylinder sets $\{C_{[[1,b_2,...,b_r]]} : r \geq 1, b_i \geq 2 \text{ for } i = 2, ..., r\} \subseteq \mathcal{U}$ (see calculations in Theorem 7). In fact, one can go even further and show that $\{C_{[[1,b_2,...,b_r]]} : r \geq 1, b_i \geq 2 \text{ for } i = 2, ..., r\} = \mathcal{U}$. Therefore, $\mathrm{SL}_2(\mathbf{Z})$ acts transitively on the set of all BCF cylinder sets. On a related note, every BCF convergent matrix can be written uniquely as a sequence of the generators of $\mathrm{SL}_2(\mathbf{Z})$ coded by the expansion of the corresponding BCF [Kat96; Zag81].

Before attacking the general case of bounding $\varepsilon_m(x)$ given the first r-terms in x's BCF expansion, we investigate a few special cases to illuminate our approach. The included figures highlight the structure of the set of BCF expansions which we aim to characterize through our investigation of the error.

5.1 Overall bound on [0,1]

We start our analysis with the case where we have no information on x other than that it lies in the unit interval. It serves as an introduction to the methods that will be used throughout Sections 5 and 6.

Theorem 5. *For any $x \in [0,1]$ and its mth BCF convergent $\frac{p_m}{q_m}$, we have $\varepsilon_m(x) \leq f_m(x)$, where*

$$f_m(x) = \frac{1}{(\frac{1}{1-x})((m-1)\frac{1}{1-x}+1)} = \frac{(1-x)^2}{m-x}.$$

Proof. The numbers $\frac{n-1}{n} = 1 - \frac{1}{n} = [[1,n]]$ for $n \geq 2$ are the best approximations for 2 step estimates, and are therefore approximated perfectly regardless of how many terms we take. Thus, our overall bound for the m-term approximation depends only on the error given at the x-values $[[1,n]]$. The error $\varepsilon_m(x)$ of the m-term approximation as x approaches $\frac{n-1}{n}$ from the right is given by the difference between $\frac{n-1}{n}$ and the next m-term convergent; this is because there are no intermediate m-term convergents on the interval from $[[1,n]]$ to the next m-term convergent, so $\varepsilon_m(x)$ is decreasing with a slope of -1. Since we want $f_m(x)$ to be continuous on $[0,1]$, we must have that $f_m([[1,n]])$ equals the right hand limit of $\varepsilon_m(x)$ as $x \to [[1,n]]$. By well ordering, the next m-term BCF of $[[1,n]]$ is $[[1,n+1,\underbrace{2,...,2}_{m-2}]] = [[1,n+1,\widehat{m-2}]]$.

Utilizing the easily verifiable identity $\frac{1}{r} = [[1,\widehat{r-1}]]$, we have

$$\lim_{x \to \frac{n-1}{n}^+} \varepsilon_m(x) = \left(1 - \cfrac{1}{n+1-\cfrac{1}{2-\cfrac{1}{\ddots_{-\frac{1}{2}}}}}\right) - \left(1-\frac{1}{n}\right) = -\frac{1}{n+\frac{1}{m-1}} + \frac{1}{n}$$
$$= \frac{\frac{1}{m-1}}{n(n+\frac{1}{m-1})} = \frac{1}{n(n(m-1)+1)}.$$

Our limiting function $f_m(x)$ should take the value $\frac{1}{n(n(m-1)+1)}$ at $x = [[1,n]]$, i.e. $f_m(\frac{n-1}{n}) = \frac{1}{n(n(m-1)+1)}$. Furthermore, this is the only arithmetic information f_m necessarily must satisfy. By solving $x = \frac{n-1}{n}$ for n, one obtains

$$n = \frac{1}{1-x}.$$

Plugging in this value of n into $f_m(\frac{n-1}{n}) = \frac{1}{n(n(m-1)+1)}$ gives us the following bound for the m-term BCF approximation of x:

$$f_m(x) = \frac{1}{(\frac{1}{1-x})((m-1)\frac{1}{1-x}+1)} = \frac{(1-x)^2}{m-x}.$$

□

5.2 Beginning with all 2's

In Section 3, we saw that the abundance of 2's that appear in a BCF are integral to their connection to standard CF expansions. With this in mind, in this subsection we aim to bound $\varepsilon_m(x)$ for values of $x \in [0, \frac{1}{m-1}]$, i.e. values of x that have the mth convergents of the form $\frac{p_m}{q_m} = [[1,\widehat{m-2},n]]$. Since the interval

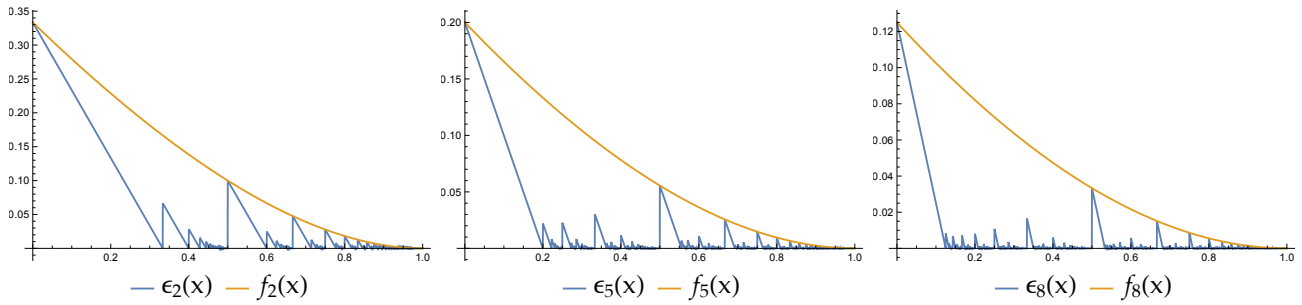

Figure 6: The $m = 2$-, 5-, and 8-term BCF errors $\varepsilon_m(x)$ on [0,1] with the overall bounding curve $f_m(x)$ above it.

$[0, \frac{1}{m-1}]$ is the left-most sub-interval of $[0,1]$, we call $f_m^{[[1,2,\ldots,2]]}(x)$ the first section bounding curve (in the above terminology, this is a bounding curve of level $m-1$). The x-values that lie in this interval have the nice property that their BCF starts with a string of 2's, and are therefore integrally related to rational numbers by properties (2.1) and (2.5). To deal with this case, we need a certain calculation.

Lemma 2. *For the mth term convergents, we have*

$$[[\ell, \underbrace{2, \ldots, 2}_{m-2}, n]] = \ell - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{\ddots - \frac{1}{n}}}} = \ell - 1 + \frac{n-1}{(m-1)(n-1)+1}.$$

Proof. Using the matrix product representation (B2) with Lemma 4 from the calculation section below, we see that the mth convergent for $[[\ell, \widehat{m-2}, n]]$ is $\ell - 1 + \frac{n-1}{(m-1)(n-1)+1}$, where we obtain this by dividing p_m by q_m calculated in the below matrix:

$$\begin{aligned}
\begin{pmatrix} p_m & q_m \\ -p_{m-1} & -q_{m-1} \end{pmatrix} &= \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^{m-2} \begin{pmatrix} \ell & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} m-1 & m-2 \\ -(m-2) & -(m-3) \end{pmatrix} \begin{pmatrix} \ell & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} n((\ell-1)(m-1)+1) - ((\ell-1)(m-2)+1) & (m-1)(n-1)+1 \\ -((\ell-1)(m-1)+1) & -(m-1) \end{pmatrix} \\
&= \begin{pmatrix} (\ell-1)((m-1)(n-1)+1) + n - 1 & (m-1)(n-1)+1 \\ -((\ell-1)(m-1)+1) & -(m-1) \end{pmatrix}.
\end{aligned}$$

□

For determining $\varepsilon_m(x)$ on $C_{[[1,\widehat{m-2}]]}$, we take $\ell = 1$ in the above lemma. Just as before, we determine the limiting function $f_m^{[[1,\widehat{m-2}]]}(x)$ by analyzing its behaviour at specific values of x. The mth level error for two consecutive convergents in this region of the interval is given by the difference $[[1, \underbrace{2, \ldots, 2}_{m-2}, n]]$ and $[[1, \underbrace{2, \ldots, 2}_{m-2}, n+1]]$. Using the lemma from above, we obtain

$$\begin{aligned}
\lim_{x \to \frac{n-1}{(m-1)(n-1)+1}^+} \varepsilon_m(x) &= [[1, \underbrace{2, \ldots, 2}_{m-2}, n+1]] - [[1, \underbrace{2, \ldots, 2}_{m-2}, n]] = \frac{n}{(m-1)(n)+1} - \frac{n-1}{(m-1)(n-1)+1} \\
&= \frac{1}{((m-1)n+1)((m-1)(n-1)+1)}.
\end{aligned}$$

Again, solving for x at the specific value where we calculated the value of the error in the above line, in this case $\frac{n-1}{(m-1)(n-1)+1}$, we get

$$x = \frac{n-1}{(m-1)(n-1)+1} \Rightarrow x((m-1)(n-1)+1) = (n-1) \Rightarrow n = \frac{(2-m)x+1}{1-(m-1)x} = \frac{x}{1-(m-1)x} + 1.$$

Combining this value of n with the bound on the error above gives us the following theorem.

Theorem 6. *For $x \in [0, \frac{1}{m-1}]$, the mth degree error $\varepsilon_m(x)$ is bounded from above by $f_m^{[[1,2,\ldots,2]]}(x)$, where*

$$\begin{aligned} f_m^{[[1,2,\ldots,2]]}(x) &= \frac{1}{\left((m-1)\left(\frac{x}{1-(m-1)x}+1\right)+1\right)\left((m-1)\left(\frac{x}{1-(m-1)x}\right)+1\right)} \\ &= \frac{(1-(m-1)x)^2}{m-(m-1)^2x}. \end{aligned}$$

Note: This exactly agrees with the above case for $m = 2$,

$$f_2^{[[1]]}(x) = \frac{(1-(1)x)^2}{2-(1)^2x} = \frac{(1-x)^2}{2-x} = f_2(x)$$

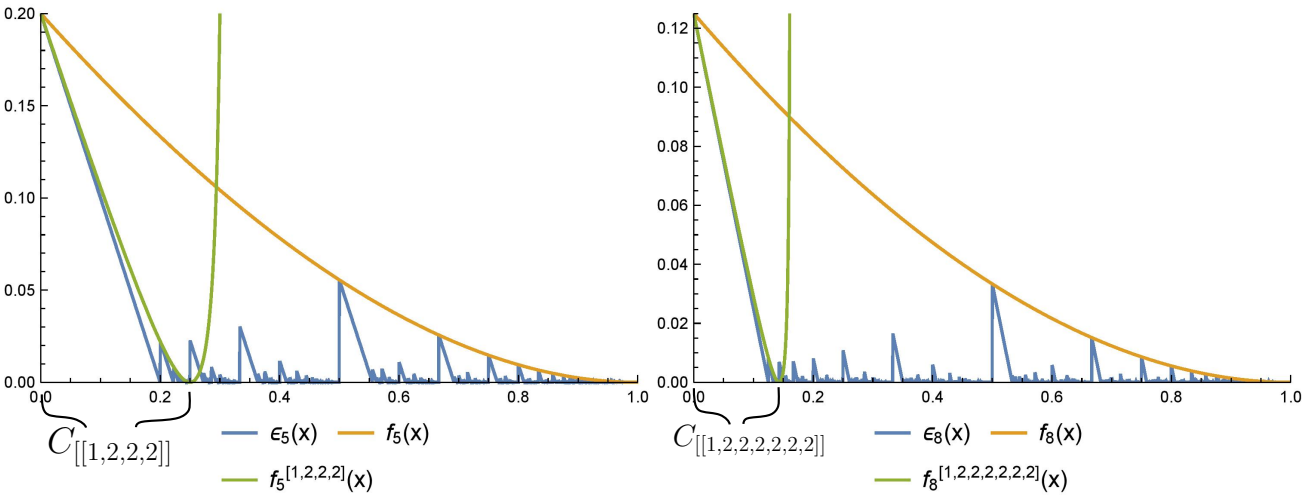

Figure 7: The $m = 5$- and 8-term BCF errors $\varepsilon_m(x)$ on [0,1] with the overall bounding curve $f_m(x)$ and first section bounding curve.

5.3 General case

We take our approach in two steps, the first to illuminate the central idea we are aiming for and the second to generalize the first.

5.3.1 The case of bounding the mth term error given $m-1$ terms of the BCF

Suppose that we know the first $m-1$ terms in the BCF expansion of x, i.e. we are given that

$$x \in ([[1, b_2, \ldots, b_{m-1}]], [[1, b_2, \ldots, b_{m-1}+1]]) \quad \Longleftrightarrow \quad x \in C_{[[1,b_2,\ldots,b_{m-1}]]}.$$

This translates to x having the $m-1$-term BCF expansion $[[1, b_2, ..., b_{m-1}]] = \frac{p_{m-1}}{q_{m-1}}$. We partition our cylinder set $C_{[[1,b_2,...,b_{m-1}]]}$ into the disjoint union $\bigcup_{n\geq 2} C_{[[1,b_2,...,b_{m-1},n]]}$ and find the maximum value of the error on each more refined cylinder set. An important thing to note is that the error approaches its supremum on $C_{[[1,b_2,...,b_{m-1},n]]}$ as x approaches $[[1, b_2, ..., b_{m-1}, n]]$ from the right, so these are the x-values where we need the value of the error $\varepsilon_m(x)$. We will then interpolate across these x-values to get the bounding function $f_m^{[[1,b_2,...,b_{m-1}]]}(x)$.

Applying (B2) to the BCF $[[1, b_2, ..., b_{m-1}, n]]$, we get $[[1, b_2, ..., b_{m-1}, n]] = \frac{np_{m-1}-p_{m-2}}{nq_{m-1}-q_{m-2}}$. The mth term error for the BCF expansion at $[[1, b_2, ..., b_{m-1}, n]]$ is therefore given by

$$\begin{aligned}
\lim_{x\to[[1,b_2,...,b_{m-1},n]]^+} \varepsilon_m(x) &= [[1, b_2, ..., b_{m-1}, n+1]] - [[1, b_2, ..., b_{m-1}, n]] \\
&= \frac{(n+1)p_{m-1} - p_{m-2}}{(n+1)q_{m-1} - q_{m-2}} - \frac{np_{m-1} - p_{m-2}}{nq_{m-1} - q_{m-2}} \\
&= \frac{-p_{m-1}q_{m-2} + p_{m-2}q_{m-1}}{((n+1)q_{m-1} - q_{m-2})(nq_{m-1} - q_{m-2})} \qquad \text{(exploiting recurrence relation (B3))} \\
&= \frac{1}{((n+1)q_{m-1} - q_{m-2})(nq_{m-1} - q_{m-2})}.
\end{aligned}$$

Solving for x when $x = [[1, b_2, ..., b_{m-1}, n]]$ yields

$$x = [[1, b_2, ..., b_{m-1}, n]] = \frac{np_{m-1} - p_{m-2}}{nq_{m-1} - q_{m-2}} \Rightarrow n = \frac{p_{m-2} - q_{m-2}x}{p_{m-1} - q_{m-1}x}.$$

Combining the last two blocks of equations gives us our limiting function

$$\begin{aligned}
f_m^{[[1,b_2,...,b_{m-1}]]}(x) &= \frac{1}{\left(\left(\frac{p_{m-2}-q_{m-2}x}{p_{m-1}-q_{m-1}x} + 1\right) q_{m-1} - q_{m-2}\right)\left(\frac{p_{m-2}-q_{m-2}x}{p_{m-1}-q_{m-1}x} q_{m-1} - q_{m-2}\right)} \\
&= \frac{(p_{m-1} - q_{m-1}x)^2}{1 + q_{m-1}(p_{m-1} - q_{m-1}x)}.
\end{aligned}$$

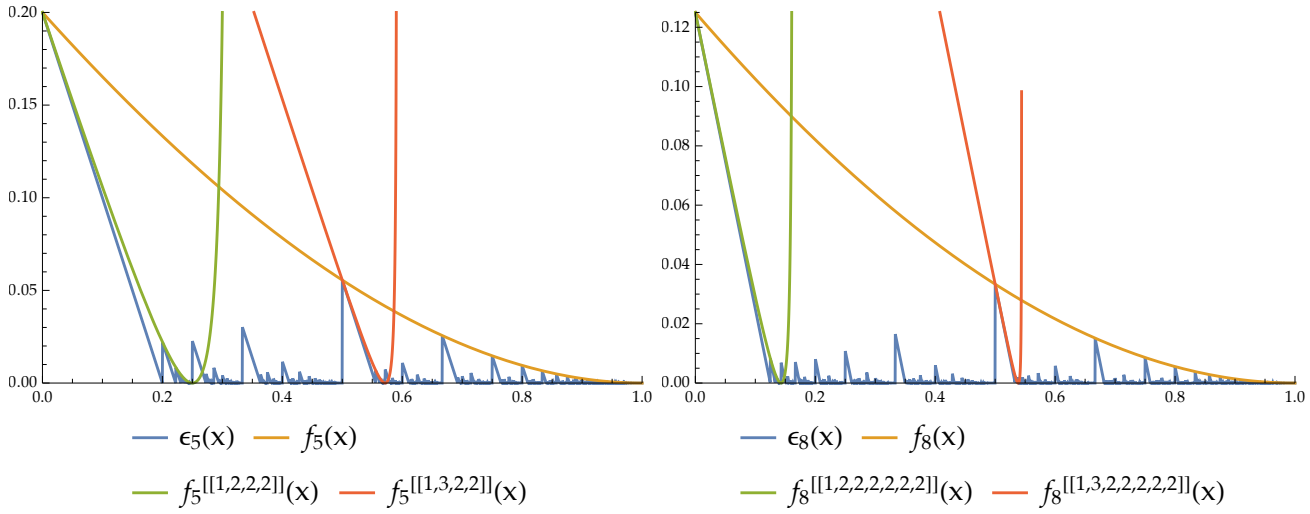

Figure 8: The $m = 5$- and 8-term BCF errors $\varepsilon_m(x)$ on [0,1] with the overall bounding curve $f_m(x)$, the first section bounding curve $f_m^{[[1,\widehat{m-2}]]}(x)$, and a bounding curve of level $m-1$.

5.3.2 The case of bounding the mth term error given $r \leq m-1$ terms of the BCF

We now consider the situation where we know the first r digits in the BCF expansion $[[1, b_2, ..., b_r]]$, where $r \leq m-1$, and wish to approximate the mth term error for x lying in this cylinder set.

Theorem 7. *For $x \in C_{[[1,b_2,...,b_r]]}$, i.e. the rth convergent for x is $[[1, b_2, ..., b_r]] = \frac{p_r}{q_r}$, we have the mth BCF error $\varepsilon_m(x)$ is bounded by*

$$f_m^{[[1,b_2,...,b_r]]}(x) = \frac{(p_r - q_r x)^2}{(m-r) + q_r(p_r - q_r x)} = \frac{\varepsilon_r(x)^2}{\frac{m-r}{q_r^2} + \varepsilon_r(x)}.$$

Proof. As before, we will determine the error at a specific set of points in the cylinder set and by interpolating along them, we obtain the overall error we wish to calculate. In this case, the set of points we need to determine the error at are $y_n = [[1, b_2, ..., b_r, n]]$. The mth term error at the $r+1$-term BCF $[[1, b_2, ..., b_r, n]]$ is given by the difference between it and the next m-term BCF expansion, which is $[[1, b_2, ..., b_r, n+1, \widehat{m-(r+1)}]]$. Utilizing property (B2) to calculate $[[1, b_2, ..., b_r, n+1, \widehat{m-(r+1)}]]$, and noting that by continuity we must have $f_m^{[[1,b_2,...,b_r]]}(y_n) = \lim_{x \to y_n^+} \varepsilon_m(x)$, we arrive at

$$\begin{aligned}
f_m^{[[1,b_2,...,b_r]]}(y_n) &= [[1, b_2, ..., b_r, n+1, \widehat{m-(r+1)}]] - [[1, b_2, ..., b_r, n]] \\
&= \frac{((m-r)n+1)p_r - (m-r)p_{r-1}}{((m-r)n+1)q_r - (m-r)q_{r-1}} - \frac{np_r - p_{r-1}}{nq_r - q_{r-1}} \\
&= \frac{((m-r)n+1)(p_{r-1}q_r - p_r q_{r-1}) + (m-r)n(p_r q_{r-1} - p_{r-1}q_r)}{(((m-r)n+1)q_r - (m-r)q_{r-1})(nq_r - q_{r-1})} \qquad \text{(by (B3))} \\
&= \frac{1}{(((m-r)n+1)q_r - (m-r)q_{r-1})(nq_r - q_{r-1})}.
\end{aligned}$$

Akin to the other analyses, we set $x = y_n = \frac{np_r - p_{r-1}}{nq_r - q_{r-1}}$ and solve for $n = \frac{p_{r-1} - q_{r-1}x}{p_r - q_r x}$. Plugging this in as the value for n in $f_m^{[[1,b_2,...,b_r]]}(y_n)$ above and simplifying again using the fundamental recurrence, we get

$$\begin{aligned}
f_m^{[[1,b_2,...,b_r]]}(x) &= \frac{1}{(((m-r)\frac{p_{r-1} - q_{r-1}x}{p_r - q_r x} + 1)q_r - (m-r)q_{r-1})(\frac{p_{r-1} - q_{r-1}x}{p_r - q_r x}q_r - q_{r-1})} \\
&= \frac{(p_r - q_r x)^2}{(m-r) + q_r(p_r - q_r x)}.
\end{aligned}$$

For the specific value of $r = m-1$, this equation agrees with the final result obtained in the previous section showing us this is the proper generalization of the previously worked out example. □

Remark: One would want that if we were given no information on x other than that $x \in \mathbf{R}$ (i.e. $x \in C_{[[]]}$), then we should have $f_1^{[[]]}(x) = 1$ to bound the classic sawtooth function $\varepsilon_1(x) = 1 - x$ on $(0, 1]$ and extended periodically to every interval of the form $(n, n+1]$ for $n \in \mathbf{Z}$. By taking the natural choice of initial BCF convergent matrix

$$\begin{pmatrix} p_0 & q_1 \\ -p_{-1} & -q_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we see that

$$f_1^{[[]]}(x) = \frac{(p_0 - q_0 x)^2}{(1-0) + q_0(p_0 - q_0 x)} = \frac{1}{1+0} = 1.$$

Additionally, to take a degree r bounding curve from $[0,1]$ to any interval of the form $[\ell, \ell+1]$ for $\ell \in \mathbf{Z}$, note that the 1-periodic nature of $\varepsilon_m(x)$ on $\mathbf{R}$ implies

$$f_m^{[[\ell+1,b_2,\ldots,b_r]]}(x) = f_m^{[[1,b_2,\ldots,b_r]]}(x-\ell).$$

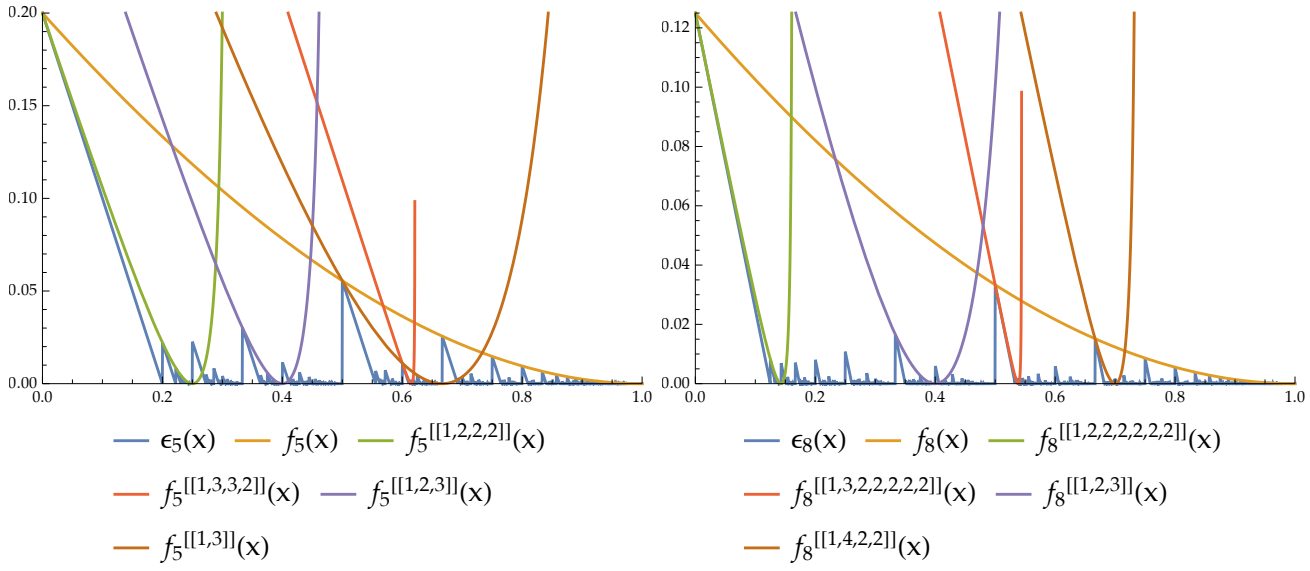

Figure 9: The $m = 5$- and 8-term BCF errors $\varepsilon_m(x)$ on [0,1] with the overall bounding curve $f_m(x)$, the first section bounding curve $f_m^{[[1,\widehat{m-2}]]}(x)$, a bounding curve of degree $m-1$, and some bounding curves of level $r < m-1$.

6 CF Approximations

The approximation scheme introduced for BCFs in the previous section illuminated a novel approach for approximation error: given a cylinder set of depth $r \leq m-1$ and the mth degree error, by looking at the x-values yielding the maximum values for the error in each nested cylinder set of depth $r+1$ and interpolating the error at these maximums, we get a upper bound for the error on all of the r-term cylinder set. In order to try to do this in the case of CF's, we need to locate such x values and the values of the error at them. Similarly to above, we need the following definitions.

Definition 5. *The mth term CF approximation error $E_m(x)$ of $x \in [0,1]$ is given by*

$$E_m(x) = \left| x - \frac{P_m}{Q_m} \right|,$$

where $\frac{P_m}{Q_m}$ is the m-term CF expansion of x. The rth level CF cylinder set $C_{[c_1,\ldots,c_r]}$ is

$$C_{[c_1,\ldots,c_r]} = \{x \in \mathbf{R} : r\text{th CF convergent of } x = [c_1,\ldots,c_r]\}.$$

Unlike the case of BCFs where the cylinder sets were nested in such a way that the maximum of the error on each cylinder set was obtained at the left endpoint, the x-value where $E_m(x)$ is maximized on $C_{[0,c_2,\ldots,c_r,n]}$ is not simply the left endpoint of the cylinder set. To handle this problem, we have the following lemma.

Lemma 3. *The mth term CF error $E_m(x)$ is maximized on $C_{[0,c_2,...,c_r,n]}$ as x approaches*

$$y_{n,m-r} = [0, c_2, ..., c_r, n, \underbrace{1, ..., 1}_{m-r \ 1's}] = \frac{(nF_{m-r+1} + F_{m-r})P_r + F_{m-r+1}P_{r-1}}{(nF_{m-r+1} + F_{m-r})Q_r + F_{m-r+1}Q_{r-1}}$$

from the left/right (depending on if m is odd/even), where $[0, c_2, ..., c_r] = \frac{P_r}{Q_r}$.

Proof. We begin by noting that for a general cylinder set $C_{[a_1,...,a_n]}$, its size is given by $|[a_1, ..., a_n] - [a_1, ..., a_n - 1]|$. This definition makes sense even in the case of $a_n = 1$ since by Theorem 2, $[a_1, ..., a_{n-1}, 1] = [a_1, ..., a_{n-1} + 1]$ and we can take the difference in CFs taken as expansions in $n - 1$ terms. If $[a_1, ..., a_{n-1}] = \frac{P_{n-1}}{Q_{n-1}}$, then

$$\begin{aligned} |[a_1, ..., a_n] - [a_1, ..., a_n - 1]| &= \left| \frac{a_n P_{n-1} + P_{n-2}}{a_n Q_{n-1} + Q_{n-2}} - \frac{(a_n - 1)P_{n-1} + P_{n-2}}{(a_n - 1)Q_{n-1} + Q_{n-2}} \right| \\ &= \frac{1}{(a_n Q_{n-1} + Q_{n-2})((a_n - 1)Q_{n-1} + Q_{n-2})}. \end{aligned}$$

This number gets smaller as we take our $a_i's$ to be larger, hence this quotient is the largest when all $a_i = 1$ for $i \leq n - 1$ and $a_n = 2$.

The above line of reasoning tells us that $E_m(x)$ is maximized on $C_{[0,c_2,...,c_r,n]}$ at the m-term CF whose remaining expansion after the digit n is all 1's except the last digit which is a 2 (this verifies that it is indeed the smallest number approximated using m-terms since if the last term was a 1, we could truncate the fraction by one step). Explicitly, this value is

$$x = [\underbrace{0, c_2, ..., c_r, n, 1, ..., 1, 2}_{m}] = [0, c_2, ..., c_r, n, \underbrace{1, ..., 1}_{m-r \ 1's}] =: y_{n,m-r}.$$

Utilizing the matrix representation for CF convergents, Lemma 4 from the calculations section, and a similar calculation to Lemma 2 above, we have

$$\begin{aligned} \begin{pmatrix} P_m & P_{m-1} \\ Q_m & Q_{m-1} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_3 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_r & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{m-r} \\ &= \begin{pmatrix} P_r & P_{r-1} \\ Q_r & Q_{r-1} \end{pmatrix} \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{m-r+1} & F_{m-r} \\ F_{m-r} & F_{m-r-1} \end{pmatrix} \\ &= \begin{pmatrix} P_r & P_{r-1} \\ Q_r & Q_{r-1} \end{pmatrix} \begin{pmatrix} nF_{m-r+1} + F_{m-r} & nF_{m-r} + F_{m-r-1} \\ F_{m-r+1} & F_{m-r} \end{pmatrix} \\ &= \begin{pmatrix} (nF_{m-r+1} + F_{m-r})P_r + F_{m-r+1}P_{r-1} & (nF_{m-r} + F_{m-r-1})P_r + F_{m-r}P_{r-1} \\ (nF_{m-r+1} + F_{m-r})Q_r + F_{m-r+1}Q_{r-1} & (nF_{m-r} + F_{m-r-1})Q_r + F_{m-r}Q_{r-1} \end{pmatrix} \end{aligned}$$

Taking the appropriate quotients of matrix entries yields

$$y_{n,m-r} := [0, c_2, ..., c_r, n, \underbrace{1, ..., 1}_{m-r}] = \frac{(nF_{m-r+1} + F_{m-r})P_r + F_{m-r+1}P_{r-1}}{(nF_{m-r+1} + F_{m-r})Q_r + F_{m-r+1}Q_{r-1}}$$

□

Remark: Lemma 3 can also be proven using geometric methods. Since the supremum of the error over a cylinder set is equal to the diameter of the cylinder set, finding the cylinder set with the highest error is the same as finding the widest cylinder set. From pictures, it is clear that you must pick the widest cylinder set at each level which corresponds to adding a 1 to the string. To be more precise, one

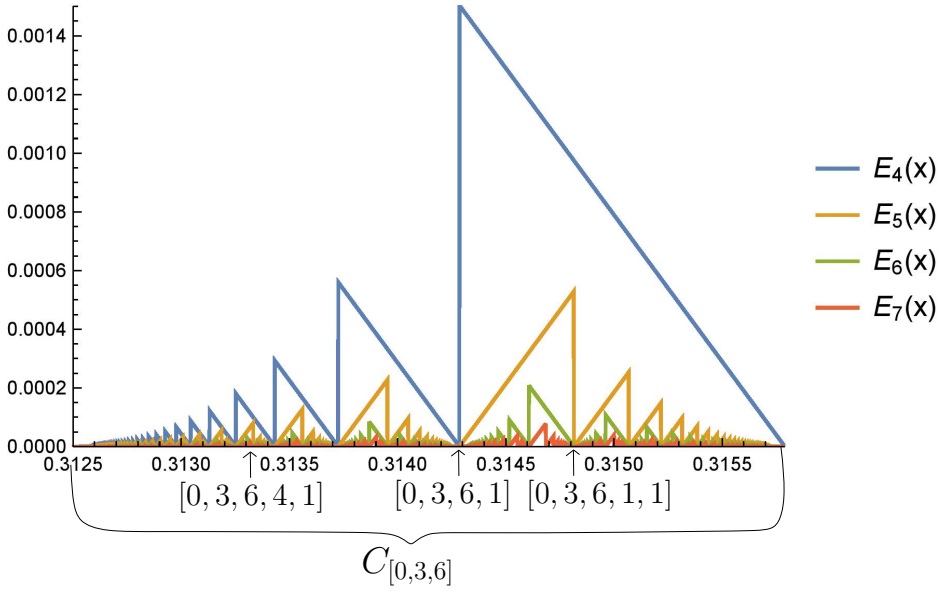

Figure 10: The nested errors of $E_m(x)$ for $m = 4, 5, 6, 7$ on the cylinder set $C_{[0,3,6]}$. The largest orange peak within each blue triangle corresponds to $y_{n,4-3}$ where n is the enumeration of the blue triangles counting from the right starting with 1. In general, the largest peak of the mth error within the nth blue triangle occurs at $y_{n,m-3}$.

can work out that the Lebesgue measure of $C_{[s]}$ (assuming s is an admissable CF string of length k and can not be shortened, so its last digit is not a 1) is $1/Q_k(Q_k + Q_{k-1})$. Therefore the widest cylinder set of a certain length is the one that minimizes Q_k and Q_{k-1}, which just by looking at the recurrence relation $Q_n = c_n Q_{n-1} + Q_{n-2}$ we see that it must be where all the c_i's are each 1 except the last digit which is $c_k = 2$.

Similarly to the case of BCF error, we use the notation

$$g_m^{[0,c_2,\ldots,c_r]}(x) = \text{limiting function for the CF error of } x \text{ given we are using a } m\text{-term CF approximation and the } r \text{ term CF expansion of } x \text{ is } [0, c_2 \ldots, c_r].$$

So we have $E_m(x) \leq g_m^{[0,c_2,\ldots,c_r]}(x)$ for any choice of parameters $c_2, \ldots, c_r \in \mathbf{Z}_{\geq 1}$. Using the g_m terminology, Lemma 3 tells us that $g_m^{[0,c_2,\ldots,c_r]}(x)$ is maximized on $C_{[0,c_2,\ldots,c_r,n]}$ at $y_{n,m-r}$ (as can be seen in Figure 6). This is also the only piece of arithmetic information we have defining our bounding function, so we now proceed with explicitly working out an expression for $g_m^{[0,c_2,\ldots,c_r]}(x)$.

6.1 Bounding the m-term CF Error on $C_{[0,c_2,\ldots,c_r]}$

Now that we know where $E_m(x)$ approaches its maximal value in each nested cylinder set $C_{[0,c_2,\ldots,c_r,n]}$, we can handle the case of maximizing $E_m(x)$ on $C_{[0,c_2,\ldots,c_r]}$ to obtain $g_m^{[0,c_2,\ldots,c_r]}(x)$.

Theorem 8. *For $x \in C_{[0,c_2,\ldots,c_r]}$, i.e. the rth CF convergent for x is $[0, c_2, \ldots, c_r] = \frac{P_r}{Q_r}$, we have the mth CF error $E_m(x)$ is bounded by*

$$g_m^{[0,c_2,\ldots,c_r]}(x) = \frac{(P_r - Q_r x)^2}{F_{m-r+1}F_{m-r} + (-1)^m Q_r(P_r - Q_r x)} = \frac{E_r(x)^2}{\frac{F_{m-r+1}F_{m-r}}{Q_r^2} + (-1)^{m-r}E_r(x)}$$

where F_n is the nth Fibonacci number satisfying $F_0 = 0$, $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Proof. From the above lemma, we know that $g_m^{[0,c_2,\ldots,c_r]}(x)$ is maximized at $y_{n,m-r}$ for $x \in C_{[0,c_2,\ldots,c_r,n]}$, and we know its value at $y_{n,m-r}$ is given by the difference of $y_{n,m-r}$ and the previous m term CF convergent, which by subtracting 1 from the final digit of 2 in $y_{n,m-r}$ when written as an m-term estimate yields $y_{n,m-r-1}$. Combining this with calculation work taken care of in Lemma 4, we obtain

$$\begin{aligned} g_m^{[0,c_2,\ldots,c_r]}(y_{n,m-r}) &= |y_{n,m-r} - y_{n,m-r-1}| \\ &= \left| \frac{(nF_{m-r+1} + F_{m-r})P_r + F_{m-r+1}P_{r-1}}{(nF_{m-r+1} + F_{m-r})Q_r + F_{m-r+1}Q_{r-1}} - \frac{(nF_{m-r} + F_{m-r-1})P_r + F_{m-r}P_{r-1}}{(nF_{m-r} + F_{m-r-1})Q_r + F_{m-r}Q_{r-1}} \right| \\ &= \frac{|F_{m-r}^2 - F_{m-r+1}F_{m-r-1}|}{((nF_{m-r+1} + F_{m-r})Q_r + F_{m-r+1}Q_{r-1})((nF_{m-r} + F_{m-r-1})Q_r + F_{m-r}Q_{r-1})} \\ &= \frac{1}{((nF_{m-r+1} + F_{m-r})Q_r + F_{m-r+1}Q_{r-1})((nF_{m-r} + F_{m-r-1})Q_r + F_{m-r}Q_{r-1})}. \end{aligned}$$

By setting $x = y_{n,m-r}$ and solving for n in terms of x, we have

$$x = \frac{(nF_{m-r+1} + F_{m-r})P_r + F_{m-r+1}P_{r-1}}{(nF_{m-r+1} + F_{m-r})Q_r + F_{m-r+1}Q_{r-1}} \Rightarrow n = \frac{-(P_{r-1} - Q_{r-1}x)}{P_r - Q_r x} - \frac{F_{m-r}}{F_{m-r+1}}.$$

Plugging this value of n into $\frac{1}{(nF_{m-r+1}+F_{m-r})Q_r+F_{m-r+1}Q_{r-1}}$ yields

$$\begin{aligned} &\frac{1}{\left(\left(\frac{-(P_{r-1}-Q_{r-1}x)}{P_r-Q_rx} - \frac{F_{m-r}}{F_{m-r+1}}\right)F_{m-r+1} + F_{m-r}\right)Q_r + F_{m-r+1}Q_{r-1}} \\ = &\frac{P_r - Q_r x}{-F_{m-r+1}Q_r(P_{r-1} - Q_{r-1}x) - F_{m-r}Q_r(P_r - Q_r x) + F_{m-r}Q_r(P_r - Q_r x) + F_{m-r+1}Q_{r-1}(P_r - Q_r x)} \\ = &\frac{P_r - Q_r x}{F_{m-r+1}(-1)^r} \end{aligned}$$

and when we plug this value of n into the right term we get

$$\begin{aligned} &\frac{1}{\left(\left(\frac{-(P_{r-1}-Q_{r-1}x)}{P_r-Q_rx} - \frac{F_{m-r}}{F_{m-r+1}}\right)F_{m-r} + F_{m-r-1}\right)Q_r + F_{m-r}Q_{r-1}} \\ = &\frac{F_{m-r+1}(P_r - Q_r x)}{F_{m-r+1}F_{m-r}((-Q_r(P_{r-1} - Q_{r-1}x) + Q_{r-1}(P_r - Q_r x)) + (-F_{m-r}^2 + F_{m-r+1}F_{m-r-1})Q_r(P_r - Q_r x)} \\ = &\frac{F_{m-r+1}(P_r - Q_r x)}{F_{m-r+1}F_{m-r}(-1)^r + (-1)^{m-r}Q_r(P_r - Q_r x)} \end{aligned}$$

where we used the identity $F_{m-r+1}F_{m-r-1} - F_{m-r}^2 = (-1)^{m-r}$ which can be deduced by taking determinants of the matrix in Lemma 4. Multiplying these two expressions together clears the $F_{m-r+1}(-1)^r$ factor and gives us the formula for $g_m^{[0,c_2,\ldots,c_r]}(x)$ as

$$g_m^{[0,c_2,\ldots,c_r]}(x) = \frac{(P_r - Q_r x)^2}{F_{m-r+1}F_{m-r} + (-1)^m Q_r(P_r - Q_r x)}$$

□

7 Appendix

In the above arguments, there were a few instances where we needed some matrix computations to calculate explicit convergents. However, these computations do not add much value to the exposition of the proofs where they were used, so their proofs were postponed until now.

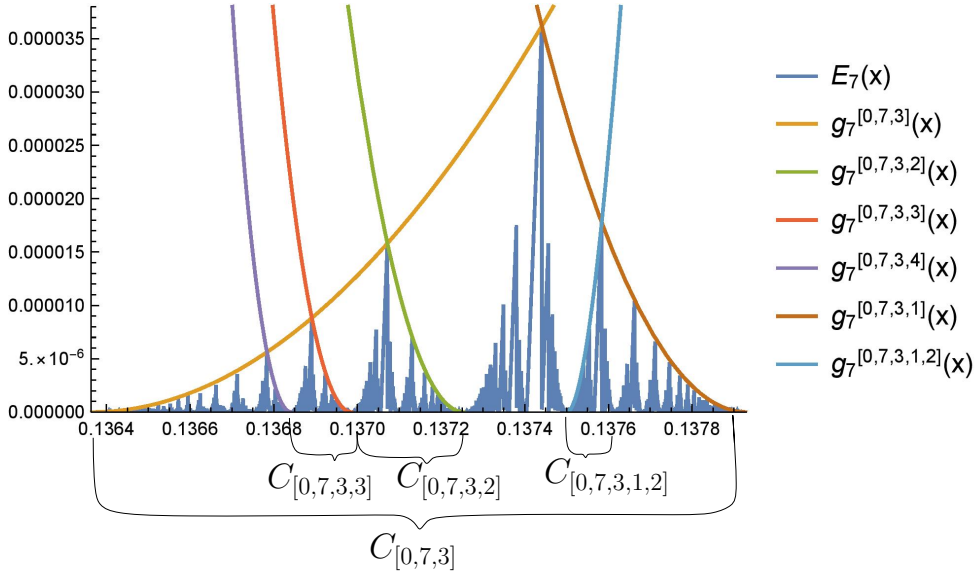

Figure 11: The $m = 7$-term CF error $E_7(x)$ on $C_{[0,7,3]}$ with the main bounding curve $g_m^{[0,7,3]}(x)$, and some bounding curves of level $r \leq m-1$.

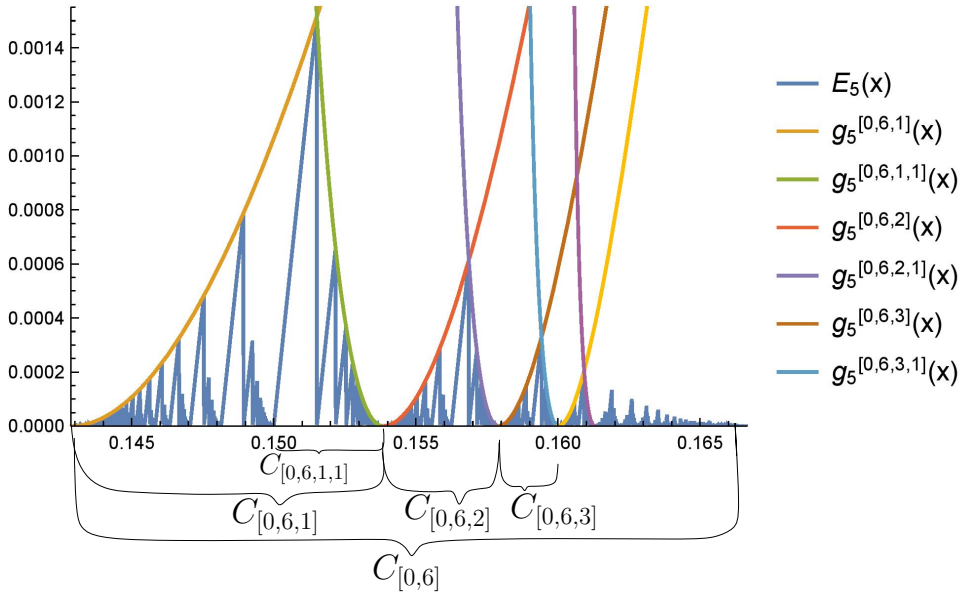

Figure 12: The $m = 5$-term CF error $E_5(x)$ on $C_{[0,6]}$ with the pairs of bounding curves $g_5^{[0,6,n]}(x)$ and $g_5^{[0,6,n,1]}(x)$ for $n = 1, 2, 3, 4$ ($n = 4$ case not included on plot legend).

Lemma 4. *For $k \in \mathbf{N}$, we have*

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^k = \begin{pmatrix} k+1 & k \\ -k & -(k-1) \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix},$$

where F_n is the nth Fibonacci number, satisfying the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial condition $F_0 = 0$, $F_1 = 1$.

Proof. These are both done by induction. The base cases of $k = 1$ for both matrix calculations hold by simply plugging in 1 and comparing. Assume for induction that the expressions above for k hold, we

now show it for $k+1$.

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^{k} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} k+1 & k \\ -k & -(k-1) \end{pmatrix} = \begin{pmatrix} k+2 & k+1 \\ -(k+1) & -k \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} = \begin{pmatrix} F_{k+1}+F_k & F_k+F_{k-1} \\ F_k+F_{k-1} & F_k \end{pmatrix}$$
$$= \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}.$$

□

References

[AF84] Roy L. Adler and Leopold Flatto. "The Backward Continued Fraction map and Geodesic Flow." In: *Ergodic Theory and Dynamical Systems* 4 (1984), pp. 487–492.

[Bom07] Enrico Bombieri. "Continued fractions and the Markoff tree." In: *Expositiones Mathematicae* 25 (2007), pp. 187–213.

[DK00] Karma Dajani and Cor Kraaikamp. "The Mother of All Continued Fractions ." In: *Colloquim Mathematicum* 84/85 (2000), pp. 109–123.

[DIT16] William Duke, Özlem Imamoglu, and Árád Tóth. "Geometric Invariants for Real Quadratic Fields." In: *Annals of Mathematics* 184 (2016), pp. 949–990.

[HT85] Allen Hatcher and William Thurston. "Incompressible Surfaces in 2-Bridge Knot Complements ." In: *Inventiones Mathematicae* 79 (1985), pp. 225–246.

[Hen06] Doug Hensley. *Continued Fractions*. World Scientific Publishing Company, 2006.

[Hir73] Friedrich Hirzebruch. "Hilbert Modular Surfaces." In: *L'Enseignement Mathématique* 19 (1973), pp. 183–281.

[Kat96] Svetlana Katok. "Coding of Closed Geodesics after Gauss and Morse." In: *Geometriae Dedicata* 63.10 (1996), pp. 123–145.

[Kat03] Svetlana Katok. *MASS Selecta: Continued fractions, hyperbolic geometry and quadratic forms*. American Mathematical Society, 2003.

[Zag81] Don Zagier. *Zetafunktionen und quadratische Körper: eine Einfiihrung in die höhere Zahlentheorie*. Springer-Berlin, 1981.

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