Exploring symmetry plane conditions in numerical Euler solutions

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**Abstract** The motion of the position of the maximum of vorticity $\|\zeta\|_\infty$ and growth of enstrophy are explored numerically using a recent calculation at moderately high resolution. We provide an exact analytic formula for the motion of $\|\zeta\|_\infty$ in the symmetry plane and use it to validate the numerical data. This motion drifts with respect to the local Lagrangian frame of reference due to the local gradient of vortex stretching, and can thus be associated with depletion of the circulation in the symmetry plane from the vicinity of $\|\zeta\|_\infty$. Despite this depletion, the numerical data is consistent with singular growth of the 3D enstrophy and symmetry-plane enstrophy that is bounded by inequalities analogous to the type known for the three-dimensional Navier–Stokes equations, using their respective enstrophies and with the viscosity being replaced by the circulation invariant.

### 10.1 Introduction

After the proof that the time integral of maximum vorticity controlled any possible singularities of the three-dimensional incompressible Euler equations (Beale, Kato, & Majda, 1984), it was realized that this was...
a quantity of low enough order to be accessible by numerical simulations, and the quest to determine numerically whether or not the Euler equations develop singularities ensued.

However, apparent singular growth in one quantity is insufficient for claiming consistency with singular growth as it is difficult to distinguish singular power-law growth from strong, but non-singular growth such as the exponential of an exponential. At a minimum, the growth in stretching near the position of maximum vorticity needs to be tracked independently and should grow in a manner consistent with the mathematical bounds, as first suggested by Pumir & Siggia (1990). Soon after that, the curvature of vortex lines and the maximum of velocity were added to the list of properties that might be required to become singular along with vorticity (Constantin, Fefferman, & Majda, 1996).

To date, only one calculation (Kerr, 1993) has attempted to address all of these possible conditions for a singularity of Euler. Evidence there, and in a later paper (Kerr, 2005), favoured the existence of a singularity. Most other calculations at that time (Pumir & Siggia, 1990) came down on the side against the existence of a singularity. The field lay dormant until a new calculation by Hou & Li (2006) suggested that the numerical experiment of Kerr (1993) had not been run long enough in time to support its conclusions in favour of a singularity. Recent work (Bustamante & Kerr, 2008) provides objective criticism of both works by considering the invariant circulations on the mirror-symmetric planes:

\[
\Gamma_y = \int_{z=0}^{\pi} \int_{x=0}^{4\pi} \zeta_y(x, 0, z, t) \, dx \, dz \quad \Gamma_z = \int_{y=0}^{2\pi} \int_{x=0}^{4\pi} \zeta_z(x, y, 0, t) \, dx \, dy,
\]

where \( \zeta = (\zeta_x, \zeta_y, \zeta_z) \) is the vorticity field. By reproducing the numerics of Hou & Li (2006) it is found that, starting at an early time, their numerics do not conserve these invariants, with \( \dot{\Gamma}_y < 0 \) and \( \dot{\Gamma}_z > 0 \) as in a viscous flow, implying these are not faithful solutions of the Euler equations. An online description is available (Kerr, 2010). Whilst the analysis of Bustamante & Kerr (2008) finds nothing contradicting the overall conclusion of Kerr (1993) in support of a singularity, it does find that the attempt to reach detailed scaling conclusions with the data set of Kerr (1993) was overly optimistic.

This paper presents analysis of the calculation of Bustamante & Kerr (2008) with the limited objective of demonstrating how calculations of this type can point to new types of mathematical analysis, even if there remain technical challenges to using these simulations to probe the details of any singular, or nearly singular, collapsing solutions. In particu-
lar, the focus is analysis that can be done in the $y = 0$ symmetry plane, the region where most of the action either promoting or suppressing singular behaviour would be. Two approaches are proposed. One is to investigate the motion of the position of the maximum vorticity with respect to the Lagrangian motion. The other is to look for how the enstrophy, or mean square vorticity, grows in this plane, and compare it to the overall growth of enstrophy in the full three-dimensional domain.

10.2 The calculation

The form of the Euler equations upon which most analysis is done, and which we integrate numerically, is the vorticity formulation:

$$\frac{\partial \zeta}{\partial t} + (u \cdot \nabla) \zeta = (\zeta \cdot \nabla) u,$$

where $\zeta = \nabla \times u$ is the vorticity.

As in many earlier papers (Kerr & Hussain, 1989, Pumir & Siggia, 1990, Kerr, 1993, Hou & Li, 2006) the initial condition in a fully periodic geometry consists of two anti-parallel vortices as shown in Figure 10.1, and the calculation will take advantage of the existence of two mirror-symmetric planes to reduce the computational domain, denoted by convention as the $z = 0$ dividing plane of coordinates $(x, y)$ between the vortices and the $y = 0$ symmetry plane of coordinates $(x, z)$ that runs through the initial perturbation of vortices (Kerr & Hussain, 1989). The details of how this particular “anti-parallel” calculation was set up are given in Bustamante & Kerr (2008).

An important additional condition, first required by Kerr (1993), is that the normal component of vorticity at the upper half of the symmetry plane ($y = 0, z > 0$) be of definite sign. This condition is preserved in time if the Euler equations (10.1) are satisfied. Moreover, the flux of vorticity through the upper half of the symmetry plane gives rise to a conserved circulation. In contrast to the calculation in Kerr (1993), the numerical method this time is strictly Fourier, in part to be able to compare directly with the numerics of Hou & Li (2006). Sine and cosine transforms are used to generate symmetries of the initial condition about the position of the perturbation and between the vortices, so only one quarter of the domain needs to be simulated. The adaptive method used in the direction of maximum collapse in Kerr (1993) is replaced by using...
significantly more mesh points in that direction than would have been feasible in the early 1990s.

Bustamante & Kerr (2008) emphasized why it was necessary to have an initial condition with only one sign of the vorticity if circulation, by virtue of Kelvin’s theorem, was to be conserved numerically. Two scaling conclusions were reached:

1) The growth of mean square vorticity, or enstrophy $Z = \frac{1}{2} \int |\zeta|^2 \, dV = \frac{1}{2} \|\zeta\|_2^2$ was far closer to a power law of the form $Z \sim (T_c - t)^\gamma$, where $T_c$ is the singular time and with $0.25 < \gamma < 0.5$, rather than the logarithmic growth reported earlier (Kerr, 1993).

Once this was realized, the Kerr (1993) data was refitted to earlier times and was found to agree with this conclusion. In both cases, the new analysis gave a more robust estimate of $T_c$.

2) Once more robust values for $T_c$ were obtained, the growth of the maximum of vorticity $\|\zeta(\cdot, t)\|_\infty = \sup_{x \in T^3} |\zeta(x, t)|$ could be compared to $\|\zeta\|_\infty \sim (T_c - t)^{-\gamma}$, without pre-supposing $\gamma \equiv 1$ as was done before (Kerr, 1993).

Such an assumption is unjustified because, if one assumes power law growth, the mathematics only tells us that $\gamma \geq 1$. The new analysis that was applied to both data sets (Bustamante & Kerr, 2008) concluded that if $\gamma = 1 + \delta_\gamma$, then $\delta_\gamma = 0$ is not supported by
the data and provided weak evidence, due to the limited resolution, that $\delta_\gamma \lesssim 0.25$.

10.3 Importance of the symmetry plane

The anti-parallel configuration was chosen for the calculations discussed above based upon vortex filament work in the 1980s which showed that attracting vortex configurations would invariably twist around into a locally anti-parallel configuration. The alternative grid point configurations with the best supporting evidence for singularities of Euler (Boratav, Pelz, & Zabusky, 1992, Orlandi & Carnevale, 2007, Grafke et al., 2008) show a similar tendency for the vortices to twist locally into similar configurations.

There are three advantages of starting a calculation in the anti-parallel configuration that are associated with the symmetry plane: 1) the maximum of the vorticity and other intense features are likely to be on or near this plane; 2) the circulation must be conserved on this plane (Bustamante & Kerr, 2008); and 3) the finest resolution can be allocated near it (Pumir & Siggia, 1990). One goal of this paper is to introduce further numerical analysis that can be applied on the symmetry plane which would be difficult to apply to an arbitrary, evolving vorticity configuration in three-dimensional space.

One of the advantages of working in this plane is that the number of variables is reduced. Let us recall that from the usual definition of enstrophy as a volume integral, $Z(t) = \frac{1}{2} \int |\zeta|^2 dV$, we can obtain its production, after manipulating equation (10.1), as

$$\dot{Z} = \int \zeta_i S_{ij} \zeta_j dV,$$

where $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the strain.

On the symmetry plane only one component of the vorticity, $\zeta_y$, and only one component of the out-of-plane strains, $S_{yy}$, are non-zero, while the velocity component perpendicular to the plane vanishes: $u_y = 0$. If $u_h \equiv (u_x, u_z)$ is the velocity in this plane, then the vorticity equation simplifies to

$$\frac{\partial \zeta_y}{\partial t} + (u_h \cdot \nabla)\zeta_y = -\nabla \cdot (u_h)\zeta_y \quad \text{or} \quad \frac{\partial \zeta_y}{\partial t} + \nabla \cdot (\zeta_y u_h) = 0. \quad (10.2)$$

The last form allows us to interpret the vorticity on the symmetry plane as a conserved density. From these equations we can define the “hori-
zontal enstrophy" $Z_h$ and its time evolution:

$$Z_h = \frac{1}{2} \int_{y=0}^{\infty} \zeta_y^2 \, dx \, dz \quad \text{and} \quad \dot{Z}_h = -\int_{y=0}^{\infty} \alpha \zeta_y^2 \, dx \, dz,$$

(10.3)

where $\alpha = S_{yy}$, and due to incompressibility of the 3D velocity field we have $\alpha = -\nabla \cdot \mathbf{u}_h$. Notice that the velocity field on the symmetry plane can be reconstructed from the knowledge of the scalars $\zeta_y$ and $\alpha$ on the symmetry plane. In addition, on the symmetry plane the curvature of the vortex lines $\kappa \neq 0$, their torsion $\tau = 0$, and the helicity density $h = \mathbf{u} \cdot \zeta = 0$.

There are enough simplifications here to suggest that the Euler dynamics on the symmetry plane might be simplified to a two-dimensional model with stretching analogous to that proposed by Gibbon, Fokas, & Doering (1999). This cannot be done unless ad hoc assumptions are made to close the out-of-plane pressure terms in the equation for $S_{yy}$. Some of the analysis here was originally designed to address this question.

The first new analysis is designed to address two tendencies that could suppress singular behaviour: (i) motion of the position of $\|\zeta\|_{\infty}$ against the Lagrangian flow, and (ii) flattening of the initial distribution of vorticity.

### 10.4 Motion in the symmetry plane

Here and in Bustamante & Kerr (2008), the periodic domain uses the coordinates $(x, y, z) \in [0, 4\pi] \times [-2\pi, 2\pi] \times [-\pi, \pi]$. Let us define the upper half of the symmetry plane as

$$\mathcal{U} = \{(x, y, z) : y = 0, z \in [0, \pi), x \in [0, 4\pi]\}.$$

Assume that the maximum of vorticity modulus $\|\zeta_y(\cdot, t)\|_{\infty}$ can be found at some point in $\mathcal{U}$ for a given time. Then we define the “position of vorticity maximum" $\mathbf{x}_M(t)$ such that $|\zeta_y(\mathbf{x}_M(t), t)| = \|\zeta_y(\cdot, t)\|_{\infty}$. Similarly we could define the centre of circulation $\mathbf{x}_c(t) = (x_c(t), z_c(t))$ by weighting the positions by the vorticity.

In vortex filament models, it is assumed that at all times the vortex cores are cylindrical and that, for every cross section of the core, the centre of circulation $\mathbf{x}_c(t)$ coincides with the position of maximum vorticity $\mathbf{x}_M(t)$. Once it was shown using simulations (Pumir & Kerr, 1987) that the cores are strongly deformed, being flattened not cylindrical, there is
no reason why $x_M$ and $x_c$ should be coincident, nor move with the Lagrangian motion of the fluid. Both can be determined numerically. The analysis presented here shows that $x_M$ drifts against the Lagrangian flow. Analysis of the motion of $x_c$ is in progress.

\[
\frac{dx_M}{dt} = u_h(x_M, t) + \mathbb{P} \cdot \nabla \alpha(x_M, t),
\]

where the $2 \times 2$ positive-definite matrix $\mathbb{P}$ is given by minus the matrix

\[
P_{ij} = \Pi_{ij} - \Pi_{\lambda} \delta_{ij}.
\]

Figure 10.2 Three velocities against the flattened elliptic contours of $|\zeta_y|$ around $x_M$, the position of $\|\zeta\|_\infty$ on the $(x, z)$ symmetry plane. The orthogonal major and minor axes defining length scales $\lambda_{\text{large}}, \lambda_{\text{small}}$ are shown. The three velocities are: the Lagrangian motion $u_h$; the drift defined as $P \cdot \nabla \alpha(x_M, t)$; and their sum, $\dot{x}_M$. The time of the snapshot corresponds to $t = 5.94$ (same as in top frame of Figure 10.4).
inverse of the Hessian of $\theta \equiv \log |\zeta_y|$, evaluated at the point $x_M(t)$:

$$P = - \begin{pmatrix} \theta_{xx} & \theta_{xz} \\ \theta_{zx} & \theta_{zz} \end{pmatrix}^{-1} \bigg|_{x=x_M(t)}.$$  

It is useful to define two natural length scales: $\lambda_{\text{Large}}$ and $\lambda_{\text{Small}}$, given by the square roots of the eigenvalues of $P$. The drift can be computed as long as these length scales remain finite. Figure 10.2 provides a picture of the drift of $x_M$ with respect to the Lagrangian motion, taken from simulation data at time $t = 5.94$. The absolute scale of the vectors has been chosen to improve presentation; the relative scales of the vectors correspond to their actual values. We see that the main contribution to the drift comes from the projection of the stretching gradient $\nabla \alpha$ along the large-length-scale eigenvector of $P$.

Figure 10.3 plots contours of $\zeta_y$ and $\alpha$ on the symmetry plane. Recall that there is a mirror image on the other side of the $z = 0$ line (as in Figure 10.1), so that together these vortices propagate to the left. However, the maximum of $\alpha$ is to the right, so the position $x_M(t)$ drifts to the right with respect to the Lagrangian flow. The long-term qualitative behaviour is demonstrated in Figure 10.4, where the following is shown: initially the innermost contour moves primarily to the left with the Lagrangian motion along the $z = 0$ plane at the bottom, ahead of the motion of $x_M$, as was qualitatively predicted. Then these inner points move up as the vorticity curls around at the leading edge of the vortical domain. The innermost contour expands due to the negative $\alpha$ shown in Figure 10.3 at the leading edge. During this period, the leading edge of the outer contour behaves in a similar manner, but the bulk of its points remain near $x_M$. This might explain why the total circulation seems to control the growth of enstrophy, as demonstrated in the next section. The final frame in Figure 10.4 compares the positions of $x_M$ and the points from the two $t = 5.94$ contours at the three times.

### 10.5 Growth of 3D enstrophy and symmetry-plane enstrophy

The most reliable scaling trend obtained in Bustamante & Kerr (2008) was that the growth of 3D enstrophy $Z$ follows the power-law

$$Z \sim (T_c - t)^{\gamma_\zeta} \quad \text{with} \quad 0.25 < \gamma_\zeta < 0.5,$$
where $T_c$ is the predicted singular time. As $t \to T_c$, the exponent saturates to $\gamma_\zeta \lesssim 0.47$. The corresponding trend from the Kerr (1993) data gives $\gamma_\zeta \lesssim 0.5$.

The trends obtained seem to suggest that $\gamma_\zeta \leq 1/2$ asymptotically, which would suggest that there might be an upper bound to the growth of enstrophy. What do we know about bounds for the growth of enstrophy, for either the Euler or Navier–Stokes equations or from other numerical simulations?

What is known for the Navier–Stokes equations is that

$$\dot{Z} \leq c Z^{3/\nu^3},$$

which would imply $Z \lesssim c_0 (T_c - t)^{-1/2}$, if this singular behaviour persisted. This possibility is what inspired new local analysis of a Navier–Stokes numerical simulation in the vicinity of strongly interacting vortex structures (Schumacher, Eckhardt, & Doering, 2010). Could there be any links between their analysis and any $Z \sim (T_c - t)^{-1/2}$ Euler results?

There are many differences between the simulations and analysis in Schumacher et al. (2010) and what is being suggested here, including the absence of viscosity in our case and their definition of locality. Since the mathematics predicting the upper bound for Navier–Stokes enstrophy growth is completely dependent upon the existence of a depletion mechanism, that is viscous dissipation, some replacement for this effect...
Figure 10.4 To demonstrate the qualitative behaviour of the motion of the position $x_M$ of the maximum of vorticity $\|\zeta\|_\infty$ on the symmetry plane, we consider two isocontours of vorticity modulus (denoted in and out) determined at $t = 5.94$, which are then particle-tracked to times $t = 7.42$ and $t = 8.98$. The position $x_M$ of maximum vorticity modulus (denoted by the crosses) has moved far outside the original innermost contour of $|\zeta|$. In other words, the position $x_M$ has a substantial drift (to the right, in this case) with respect to the Lagrangian frame of reference. The maximum vorticity (labelled $\|\omega\|_\infty$) at each time is given.

and its governing parameter (viscosity) would have to be found if we are to go further with any proposed analogous bound for enstrophy growth in the inviscid vorticity equation. For replacing viscosity, by dimensional arguments, there is only one choice, the conserved circulation of the vortices $\Gamma_y$ through the upper half of the symmetry plane. The importance
of initial circulation for allowing the simulations to achieve possibly singular behaviour has already been discussed. The question now is whether the conservation of circulation can have further importance in providing a constraint on singular growth.

The simulations using the initial conditions of Bustamante & Kerr (2008) and Kerr (1993) are uniquely qualified for testing this proposition because their vorticity fields have a definite sign at the upper half of the \( y = 0 \) symmetry plane. Based on dimensional analysis, two simple "model inequalities" can be proposed in terms of the positive constant \( \Gamma_y \):

(i) In analogy to (10.4), enstrophy \( Z \) might be controlled by:

\[
\dot{Z} \leq c \frac{Z^3}{\Gamma_y^3},
\]

where \( c \) is a dimensionless factor.

(ii) The growth of the symmetry-plane enstrophy \( Z_h \) (10.3) could be controlled by:

\[
\dot{Z}_h \leq c' \frac{Z_h^2}{\Gamma_y},
\]

where \( c' \) is a dimensionless factor. This can be partially justified using Navier–Stokes arguments similar to those used to derive (10.4) in two-dimensions, but neglecting that enstrophy in the two-dimensional Euler equations is strictly bounded and again by replacing viscosity with \( \Gamma_y \).

We do not know whether it is possible to derive these inequalities rigorously, but our numerical data suggests they might hold asymptotically. The saturated version of inequalities (10.5) and (10.6) would imply

\[
\frac{\Gamma_y^3}{Z^2} \approx 2c (T_c - t), \quad \frac{\Gamma_y}{Z_h} \approx c' (T_c - t),
\]

and our data verifies these late-time trends as shown in Figure 10.5. An empirically determined relative coefficient of 0.34 was chosen so that the two predictions converge asymptotically. In other words, we obtain \( c \approx 0.17c' \), which is the order of magnitude we hoped to find if our model inequalities were to be of significance.
Figure 10.5 Comparison of normalized enstrophy and \( y = 0 \) symmetry plane enstrophy. An extrapolation of the \( \Gamma_3^3/Z^2 \) curve from \( t = 9.375 \) to \( t = 10 \) is shown to demonstrate the trends towards a singularity.

10.6 Summary

Two possibly conflicting trends have been found by our new analysis. The first comes from the new diagnostic that looks at how the position of the maximum of vorticity \( x_M \) moves against the Lagrangian flow within the original region of strong circulation. This is closely tied to the flattening around \( x_M \) that has been noted in anti-parallel simulations starting with Pumir & Kerr (1987), who suggested that this flattening would generate a simple vortex sheet and suppress the singular trends.

The second new trend is that the enstrophy grows in a singular manner that could depend upon the original circulation playing a role for all times, suggesting that even if the circulation spreads locally, it is still important globally. Can these opposing trends be reconciled?

Kerr (1993) addressed the question of flattening by noting that as contours get stretched in the \( z \)-direction, the vortex sheet developed a kink. However, this alone cannot explain how singular trends can be maintained.

Perhaps the motion of the Lagrangian points illustrated in Figure 10.4 provides the answer. Especially for the last time, \( t = 8.98 \). By this time,
the region that contained $x_M$ (the point at which the vorticity modulus is equal to $\|\zeta\|_{\infty}$) at $t = 5.7$ has looped around so that it now sits over the new $x_M$, and therefore can still contribute the weight of its circulation to the Biot–Savart terms that would be driving any possible singularity.

Also note the white areas defined by $|\zeta| \lesssim \|\zeta\|_{\infty}$ at $t = 5.94$ and $t = 7.42$ around the position of maximum vorticity. These are within the highest vorticity contours and show that even through some of the circulation has been advected further ahead (to the left), it has largely been replaced by new circulation coming in from behind (from the right). A piece of analysis that we are working on and should address some of these issues is how the position of a centre of circulation moves with respect to $x_M$.

The new numerical analysis presented here is meant to inspire new mathematical analysis of the Euler equations that might be difficult, if not impossible, in the full three-dimensional domain. The critical parameter is the circulation. This is the integral of the vorticity, which is rarely considered. It can be used in this case because it is well-defined for the initial conditions and the domain considered, and it is conserved across all $(x, z)$ planes. There are many higher norms which could potentially be split into, or bounded by, the circulation and another higher-order norm. Perhaps some combination of these terms could be found that would explain the empirical scaling laws and the conjectures presented here.

Acknowledgments Support for this work was provided by the Leverhulme Foundation grant F/00 215/AC and UCD Seed Funding Projects SF304 and SF564. Computational support was provided by the Warwick Centre for Scientific Computing.

References


