MATH30090 Metric Spaces 2019–2020

Exercise sheet 2

1. For a metric space \((X, d)\) and \(x \in X\), we say \(x\) is an isolated point in \((X, d)\) if there is some \(r > 0\) so that \(B(x, r) = \{x\}\). We write \(\text{Iso}(X, d)\) for the set of all isolated points of \((X, d)\).

(a) Show that \(x\) is an isolated point if and only if \(\{x\}\) is open.
(b) Deduce that every subset of \(X\) is open in \((X, d)\) if and only if \(\text{Iso}(X, d) = X\).

2. Compute the sets:

(a) \(\text{Iso}(X, d_\delta)\) where \(X\) is any non-empty set and \(d_\delta\) is the discrete metric on \(X\);
(b) \(\text{Iso}(\mathbb{R}, d_E)\);
(c) \(\text{Iso}(\mathbb{N}, d(m, n) = |m - n|)\);
(d) \(\text{Iso}(\mathbb{R}^2, d_E)\);
(e) \(\text{Iso}(B([0, 1]), d_\infty)\).

3. Find an example of a metric space \((X, d)\) with infinitely many isolated points, and infinitely many non-isolated points (in other words, so that both \(\text{Iso}(X, d)\) and \(X \setminus \text{Iso}(X, d)\) are infinite sets).

4. (a) Show that if \(F\) is a finite subset of \(X\), then \(\overline{F} = F\), and \(F^o = F \cap \text{Iso}(X, d)\), and \(\partial F = F \setminus \text{Iso}(X, d)\).
(b) Suppose \(F\) is a finite set which contains no isolated point of \((X, d)\). Compute \(\overline{F}, F^o\) and \(\partial F\) in this case.

5. Let \((X, d)\) be a metric space. A set \(A \subseteq X\) is said to be clopen if \(A\) is both open and closed.

(a) Show that \(A\) is clopen if and only if \(A^c\) is clopen.
(b) Show that if \(x\) is an isolated point of \((X, d)\), then \(\{x\}\) and \(\{x\}^c\) are both clopen.
(c) What is the boundary of a clopen set?
(d) Show that every metric space has at least two clopen subsets.
(e) Show that \((\mathbb{R}, d_E)\) has only two clopen subsets.
(f) Which subsets of \((X, d_\delta)\) are clopen?
(g) For \(X = [1, 2) \cup (3, 4]\) and \(d(x, y) = |x - y|\), which subsets of \(X\) are clopen in \((X, d)\)?

6. Let \((X, d)\) be a metric space and let \(A \subseteq X\).

(a) Is it always true that \(\partial A = \partial(\overline{A})\)? Is it ever true?
(b) Is it always true that \(\partial A = \partial(A^o)\)? Is it ever true?
(c) Show that \(\overline{A} = A^o \cup \partial A\) and \(A^o = \overline{A} \setminus \partial A\).
7. Let \((X, d)\) be a metric space and let \(D \subseteq X\). We say that \(D \subseteq X\) is \emph{dense in} \((X, d)\) if \(\overline{D} = X\).

(a) Give an example of a set \(D \subseteq \mathbb{R}\) so that both \(D\) and \(D^c\) are dense in \((\mathbb{R}, d_E)\) (and explain why your answer is correct).

(b) Repeat (a) for \((\mathbb{R}^2, d_E)\) instead of \((\mathbb{R}, d_E)\).

(c) Let \(d_\delta\) be the discrete metric on a set \(X\). Determine the dense subsets of \((X, d_\delta)\).

8. Let \(D \subseteq X\). Show that the following four conditions are equivalent:

(a) \(D\) is dense in \((X, d)\);

(b) \((D^c)^o = \emptyset\);

(c) Every \(A \subseteq X\) with \(A^o \neq \emptyset\) has \(A^o \cap D \neq \emptyset\);

(d) Every ball \(B \subseteq X\) has \(B \cap D \neq \emptyset\).

9. (a) Let \((X, d)\) be a metric space. Prove that the only subset of \(X\) which is both closed and dense in \((X, d)\) is \(X\) itself.

(b) Find an example of an open and dense set in \((\mathbb{R}, d_E)\). Then find uncountably many such examples.

(c) Find an example of a non-open and dense set in \((\mathbb{R}, d_E)\). Then find uncountably many such examples.

10. The following statements about an arbitrary metric space \((X, d)\) are false. In each case, find a counterexample. [Hint: you might want to take \((X, d) = (\mathbb{R}, d_E)\).]

(a) If \(A_1\) and \(A_2\) are any two subsets of \(X\), then \(\overline{A_1 \cap A_2} = \overline{A_1} \cap \overline{A_2}\).

(b) If \(B_1, B_2, B_3, \ldots\) are all subsets of \(X\), then \(\left(\bigcap_{i \in \mathbb{N}} B_i\right)^o = \bigcap_{i \in \mathbb{N}} B_i^o\).

11. (a) Let \(U = \{(x, y) \in \mathbb{R}^2: x > 0\}\). Find open balls \(B_1, B_2, B_3, \ldots\) in \((\mathbb{R}^2, d_E)\) with \(U = \bigcup_{i \in \mathbb{N}} B_i\).

(b) Explain why it is not possible to do this for \(\bar{U} = \{(x, y) \in \mathbb{R}^2: x \geq 0\}\).

12. Let \((X, d)\) be a metric space and let \(A \subseteq X\).

(a) Show that \(A^o\) is the largest open set in \((X, d)\) which is a subset of \(A\).

(b) Show that \(\overline{A}\) is the smallest closed set in \((X, d)\) which is a superset of \(A\).