From Positive Completions to Operator System Quotients

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The semester that Steve visited UH he taught a course on several complex variables. Most examples of course focused on the 2 variable case and he always used the variables “zed” $\mathcal{Z}$ and “ zee” $\mathfrak{z}$.

Overview

In this talk I want to show how some work with Stephen Power and Roger Smith in the 1980's on matrix completions leads naturally to the concept of operator system quotients and provided some of the earliest concrete examples.

Then I will outline how operator system quotients can simplify some results in C*-algebras. In fact they have led to alternative proofs of Kirchberg's characterization of WEP C*-algebras, to a new proof that the Cuntz algebras $\mathcal{O}_n$ are nuclear, and to Kavruk's construction of a finite dimensional operator system that is a nuclearity detector.

A matrix is **paritally defined** if only some of its entries are specified and the remaining entries are viewed as variables. Here's an example of a partially defined matrix over $\mathbb{C}$:

$$P = \begin{pmatrix} 1 & 1 & ? & -1 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ -1 & ? & 1 & 1 \end{pmatrix}$$

In this case there are only two distinct **fully defined principal submatrices**:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

which are both positive semidefinite. Such a matrix is called **partially positive**.

It is natural to wonder if given a partially positive matrix, can one always choose values for the ?'s that makes the resulting matrix positive? Such a matrix is called a **positive completion**.

The above matrix is an example of a partially positive matrix with no positive completion.

We only consider partially defined matrices such that $p_{i,j}$ defined implies that $p_{j,i}$ is defined. In this case one can associate a graph to the *pattern of defined entries* by saying that (i,j) is an edge iff $p_{i,j}$ is defined. For the above matrix the graph is the 4-cycle:

$$1 \to 2 \to 3 \to 4 \to 1.$$

Grone, Johnson, Sa, and Wolkowicz(1984) proved:
Theorem(GJSW): Fix a graph G on n vertices. Then every partially positive matrix with pattern G has a positive completion if and only if G is a chordal graph.
Later this theorem was extended to the case that the entries of the partially defined matrix are all elements of $B(\mathcal{H})$, with identical statements.

For a fixed graph G on n vertices we introduced the subspace

$$\mathcal{S}_G := span\{E_{i,i}, E_{i,j} : (i,j) \text{ is an edge } \} \subseteq M_n.$$

There is a one-to-one correspondence between linear maps $\Phi : \mathcal{S}_G \to B(\mathcal{H})$ and partially defined operator matrices with pattern G by setting

$$P = \big(\Phi(E_{i,j})\big), \forall E_{i,j} \in \mathcal{S}_G.$$

Theorem(PPS): (1) A partially defined operator matrix with pattern G has a positive completion iff the map $\Phi_P : \mathcal{S}_G \to B(\mathcal{H})$ is completely positive.
(2) P is partially positive iff Φ_P is positive on the convex hull of the rank one positives in $\mathcal{S}_G$.
Combining with GJSW:
(3) A graph is chordal iff $M_k(\mathcal{S}_G)^+$ is the convex hull of the its rank one positives, for all k.
This last fact also yields a shorter proof of a result from graph theory that says that a graph is chordal iff it has a *perfect vertex elimination scheme*.

Ordered Quotients

Another way to view a partially defined matrix P with pattern G is as coset:

$$P \to P + \mathcal{S}_G^{\perp} \in M_n/\mathcal{S}_G^{\perp}.$$

In this case P has a positive completion iff the coset has a positive lifting of the quotient map.
Thus, it is natural to make the quotient $M_n/\mathcal{S}_G^{\perp}$ a matrix ordered space via

$$(P_{i,j} + \mathcal{S}_G^{\perp}) \geq 0 \iff (P_{i,j} + K_{i,j}) \geq 0, \exists K_{i,j} \in \mathcal{S}_G^{\perp}.$$

With this family of matrix cones $M_n/\mathcal{S}_G^{\perp}$ becomes an abstract operator system in the sense of Choi-Effros.

With these identifications, PPS (1) becomes a statement about the dual space $(\mathcal{S}_G)^d$. Namely,

$$\mathcal{S}_G^d \simeq M_n/\mathcal{S}_G^{\perp},$$

is a complete order isomorphism.
This leads one to wonder what operator systems can one obtain as quotients of matrix algebras and as quotients of subspaces of matrix algebras, i.e., *subquotients* of the matrices.

Let $\mathbb{F}_n$ denote the free group on n generators, $e = g_1, ..., g_n$, let $C^*(\mathbb{F}_{n-1})$ denote the full group C*-algebra and let

$$\mathcal{S}_n = span\{g_i g_j^* : 1 \leq i, j \leq n\} \subseteq C^*(\mathbb{F}_{n-1}).$$

Farenick-P proved that if $D_0 \subseteq M_n$ denotes the diagonal matrices of trace 0, then

$$M_n / D_0 \simeq \mathcal{S}_n,$$

via the map $E_{i,j} \to \frac{g_i g_j^*}{n}$ is a unital complete order isomorphism of operator systems.
This was then used to give a simpler proof of Kirchberg's result that

$$C^*(\mathbb{F}_n) \otimes_{min} B(\mathcal{H}) = C^*(\mathbb{F}_n) \otimes_{max} B(\mathcal{H}),$$

and this lead to new characterisations of WEP C*-algebras.

If we let $v_1, ..., v_n$ denote a set of Cuntz isometries and let

$$\mathcal{V}_n = span\{1, v_1, ..., v_n, v_1^*, ..., v_n^*\} \subseteq \mathcal{O}_n,$$

then Da Zhang proved that $\mathcal{V}_n$ is unitally completely order isomorphic to the quotient of the operator system

$$\mathcal{S}_n = \{\begin{pmatrix} a & v^* \\ w & bI_n \end{pmatrix} : v, w \in \mathbb{C}^n\} \subseteq M_{n+1},$$

by the subspace

$$\mathcal{K} = \{\begin{pmatrix} a & 0 \\ 0 & -aI_n \end{pmatrix} : a \in \mathbb{C}\},$$

i.e. $\mathcal{S}_n/\mathcal{K} \simeq \mathcal{V}_n$, later P-Zhang used this quotient representation to give a new proof that $\mathcal{O}_n$ is nuclear.

To see $\mathcal{O}_n$ nuclear, let $\mathcal{A}$ be a unital C*-algebra. Prove that the fact that

$$\mathcal{S}_n \to \mathcal{V}_n \to 0,$$

is a complete quotient map implies

$$\mathcal{S}_n \otimes_\tau \mathcal{A} \to \mathcal{V}_n \otimes_\tau \mathcal{A} \to 0,$$

is a complete quotient map for $\tau = max$(easy) and $\tau = min$(hard, but matrix theory). Next show

$$\mathcal{S}_n \otimes_{min} \mathcal{A} = \mathcal{S}_n \otimes_{max} \mathcal{A}.$$

Hence, by a diagram chase, basically the 5 lemma,

$$\mathcal{V}_n \otimes_{min} \mathcal{A} = \mathcal{V}_n \otimes_{max} \mathcal{A}.$$

Now using Choi's theory of multiplicative domains, this implies,

$$\mathcal{O}_n \otimes_{min} \mathcal{A} = \mathcal{O}_n \otimes_{max} \mathcal{A}.$$

Finally, these quotient techniques helped Kavruk to prove that if we consider the 5 dimensional operator system

$$\mathcal{W}_{3,2} = \{(a_1, ..., a_6) \in \ell_6^\infty : a_1 + a_2 + a_3 = a_4 + a_5 + a_6\}$$
$$\text{with } \mathcal{W}_{3,2}^d \simeq (\ell_3^\infty \oplus \ell_3^\infty)/\{(a, a, a, -a, -a, -a)\},$$

then a C*-algebra $\mathcal{A}$ is nuclear iff

$$\mathcal{A} \otimes_{min} \mathcal{W}_{3,2} = \mathcal{A} \otimes_{max} \mathcal{W}_{3,2}.$$

A 5 dimensional nuclearity detector!

Thanks!