Matrices for analysing rigidity and global rigidity of periodic frameworks

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Outline

Rigidity and global rigidity

- (Bar-joint) framework: (G, p), where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a map.

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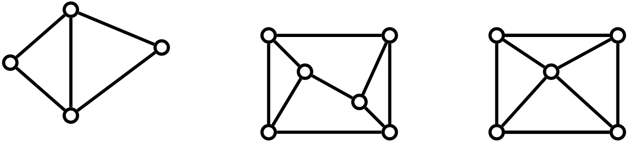
- (G, p) is called (locally) rigid if there exists a neighborhood of p in which every framework (G, q) that is equivalent to (G, p) is congruent to (G, p).

Examples in 2D

Figure: A flexible and a rigid framework in $\mathbb{R}^2$.



(a) (b) (c)

Figure: (a), (b) are rigid but not globally rigid in $\mathbb{R}^2$. (c) is globally rigid in $\mathbb{R}^2$.

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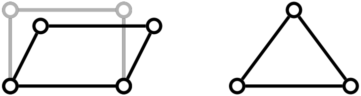

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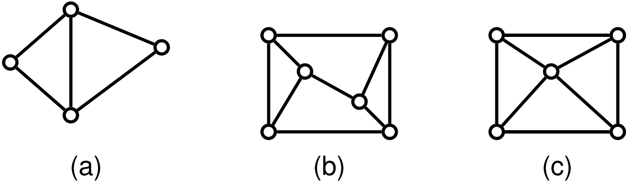

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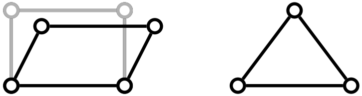

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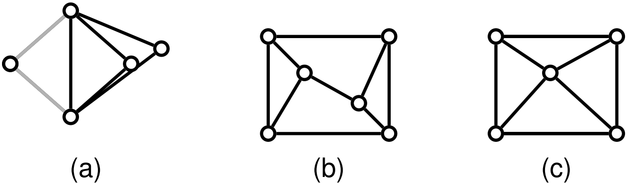

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Rigidity and the rigidity matrix

The rigidity matrix

- Rigidity map of graph G:

$$f_G : \mathbb{R}^{d|V|} \ni p \mapsto \left(\dots, \|p_i - p_j\|^2, \dots \right)^T \in \mathbb{R}^{|E|}$$

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$$R(G,p) = \begin{array}{c} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} 1 & & & i & & & & & j & & & & |V| \end{array} \\ \left(\begin{array}{ccccccccccccc} & & & & & & \vdots & & & & & & \\ 0 & \dots & 0 & p_i - p_j & 0 & \dots & & \dots & 0 & p_j - p_i & 0 & \dots & 0 \\ & & & & & & \vdots & & & & & & \end{array} \right) \end{array} \begin{array}{c} \\ \\ \text{edge } \{i, j\} \end{array}$$

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- An element in the kernel of $R(G,p)$ is an infinitesimal motion. A trivial infinitesimal motion is the derivative of a motion of congruent frameworks.
- (G,p) is called infinitesimally rigid if every infinitesimal motion of (G,p) is trivial.

Checking for local rigidity

- **Thm**: (G, p) (with p affinely spanning $\mathbb{R}^d$) is infinitesimally rigid in $\mathbb{R}^d$ if

$$\operatorname{rank}\big(R(G,p)\big) = d|V| - \binom{d+1}{2}.$$

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- If (G, p) is infinitesimally rigid, then it is rigid. The converse is also true if p is generic (or regular):
- **Theorem (Asimov-Roth, 78):** Let p be generic (or regular). Then (G, p) is rigid in $\mathbb{R}^d$ iff

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- Thus rigidity is a generic property of the graph.
- Pollaczek-Geiringer (1927) and Laman (1970) found a characterisation of generic rigidity in 2D which can be checked in poly. time. Open problem in higher dimensions.

Global rigidity and the stress matrix

The stress matrix

- A self-stress of a framework (G, p) is a vector ω in $\mathbb{R}^{|E|}$ satisfying

$$\sum_{j \in N_G(i)} \omega_{ij}(p_i - p_j) = 0 \qquad \text{for all } i \in V.$$

Equivalently ω is a self-stress of (G, p) if $\omega^\top R(G, p) = 0$.

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- The stress matrix Ω (of ω) is the Laplacian of G weighted by ω, i.e.,

$$\Omega = \begin{pmatrix} \sum_{i \neq 1} \omega_{1i} & -\omega_{12} & \cdots & -\omega_{1|V|} \\ -\omega_{12} & \sum_{i \neq 2} \omega_{2i} & \cdots & -\omega_{2|V|} \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{1|V|} & -\omega_{2|V|} & \cdots & \sum_{i \neq |V|} \omega_{i|V|} \end{pmatrix}$$

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- Note that $\Omega := B(G)^\top \mathrm{diag}(\omega) B(G)$, where $\mathrm{diag}(\omega)$ denotes the diagonal matrix whose diagonal vector is equal to ω, and $B(G)$ is the oriented incidence matrix of G.

Checking for global rigidity

- It is NP-hard to decide if a given framework (G, p) is globally rigid (Saxe 1979).
- The problem becomes tractable if p is generic, i.e., coordinates of p are algebraically independent over $\mathbb{Q}$.
- **Thm. (Connelly, 06 and Gortler, Healy, Thurston, 10):** If p is generic, then (G, p) is globally rigid in $\mathbb{R}^d$ if and only if

$$\exists \text{ self-stress } \omega \quad \text{such that} \quad \dim \ker \Omega = d + 1.$$

- Note that this implies that global rigidity is a generic property.
- Hendrickson (1992) established necessary conditions for generic global rigidity in all dimensions.
- Jackson and Jordán (2005) showed that Hendrickson's conditions are also sufficient in 2D (and these can be checked in poly. time). This is not the case in higher dimensions (Connelly, 1992).

Periodic frameworks

Periodic graphs

- A $\mathbb{Z}^d$-labeled graph is a pair (G, z) of a finite directed (multi-) graph G and a map $z : E(G) \to \mathbb{Z}^d$ such that
  - $z((u, u)) \neq 0$
  - $z((u, v)) \neq z((w, x))$ if $u = w$ and $v = x$
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- The covering graph of (G, z) is the (undirected, simple) graph $\tilde{G}$ with vertex set $V(\tilde{G}) = V(G) \times \mathbb{Z}^d$ and edge set $E(\tilde{G}) = E(G) \times \mathbb{Z}^d$, where an edge (e, γ) with $e = (u, v)$ joins the vertices (u, γ) and $(v, \gamma + z(e))$.

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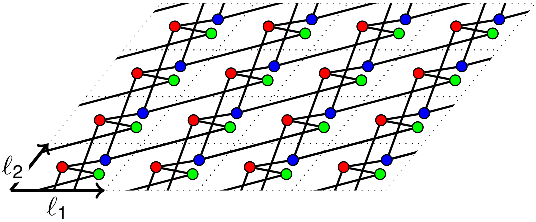
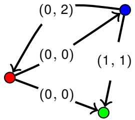
- (G, z) is also called the quotient $\mathbb{Z}^d$-labeled graph of $\tilde{G}$, and $\tilde{G}$ is called a d-periodic graph.

Periodic frameworks



L-periodic framework $(\tilde{G}, \tilde{p})$



$\mathbb{Z}^d$-labeled framework $((G, z), p, L)$

- Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a d-periodic graph and $L : \mathbb{Z}^d \to \mathbb{R}^d$ be a non-singular homomorphism (i.e., $L(\mathbb{R}^d)$ has rank d). L is called the lattice.

  A pair $(\tilde{G}, \tilde{p})$ with $\tilde{p} : \tilde{V} \to \mathbb{R}^d$ is an L-periodic framework in $\mathbb{R}^d$ if

  $$\tilde{p}((v, \alpha)) + L(\gamma) = \tilde{p}((v, \alpha + \gamma)) \qquad \text{for all } \gamma \in \mathbb{Z}^d \text{ and all } (v, \alpha) \in \tilde{V},$$

- $\mathbb{Z}^d$-labeled framework: $((G, z), p, L)$, where (G, z) is $\mathbb{Z}^d$-labeled graph, $p : V(G) \to \mathbb{R}^d$ and L is a non-singular lattice.

Periodic rigidity and global rigidity

- Let $(\tilde{G}, \tilde{p})$ be an L-periodic framework and $(\tilde{G}, \tilde{q})$ be an L'-periodic framework in $\mathbb{R}^d$
- $(\tilde{G}, \tilde{p})$ and $(\tilde{G}, \tilde{q})$ are equivalent if

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- L-periodic framework $(\tilde{G}, \tilde{p})$ is called (locally) rigid if in some open neighborhood of $(\tilde{p}, L)$ every L'-periodic framework $(\tilde{G}, \tilde{q})$ in $\mathbb{R}^d$ which is equivalent to $(\tilde{G}, \tilde{p})$ is congruent to $(\tilde{G}, \tilde{p})$.

Rigidity in terms of the $\mathbb{Z}^d$-labeled framework

- The rigidity map of G is the function $f_G : \mathcal{R}(G) \to \mathbb{R}^{|E|}$ defined by

$$f_G(p, L) = (\|v_{e_1}\|^2, \|v_{e_2}\|^2, \ldots, \|v_{e_{|E|}}\|^2).$$

where

$$v_e := p(j) + Lz(e) - p(i).$$

is the edge vector of $e = (i, j) \in E(G)$.

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- **Prop:** An L-periodic framework $(\tilde{G}, \tilde{p})$ is globally rigid if and only if its quotient framework $((G, z), p, L)$ is globally rigid. (Analogously for local rigidity.)

Periodic rigidity matrix

The periodic rigidity matrix

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- Rigidity matrix of $((G, z), p, L)$: The Jacobian of f_G at (p, L).
- $R(G, p, L)$ is the matrix of size $|E| \times (d|V| + d^2)$ whose rows are of the form

$$\begin{array}{cc} & \begin{array}{ccccccccccccc} & & & i & & & & j & & & & & \end{array} \\ ((i,j),z_e) & \begin{bmatrix} 0 & \dots & 0 & v_e^\top & 0 & \dots & 0 & -v_e^\top & 0 & \dots & 0 & (z_e \otimes v_e)^\top \end{bmatrix} \end{array}$$

where $z_e = z(e) = \begin{pmatrix} z_e^1 \\ \vdots \\ z_e^d \end{pmatrix} \in \mathbb{Z}^d$ and $z_e \otimes v_e = \begin{pmatrix} z_e^1 v_e \\ \vdots \\ z_e^d v_e \end{pmatrix} \in \mathbb{R}^{d^2}$.

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- This matrix was first established by Borcea and Streinu in 2010.

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- The kernel of $R(G, p, L)$ is the space of infinitesimal motions of $((G, z), p, L)$. The trivial infinitesimal motions are always in the kernel, so $\text{nullity } R(G, p, L) \geq \binom{d+1}{2}$.

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$$\text{rank } R(G,p,L) = d|V| + d^2 - \binom{d+1}{2} = d|V| + \binom{d}{2}$$

- If $((G,z),p,L)$ is infinitesimally rigid, then it is rigid. The converse is also true if (p,L) is generic (or regular). Thus periodic rigidity is a generic property.

- Malestein and Theran (2013) found a characterisation of generic periodic rigidity in 2D which can be checked in poly. time. Open problem in higher dimensions.

Periodic stress matrix

The periodic stress matrix

- A self-stress of a framework $((G, z), p, L)$ is a vector ω in $\mathbb{R}^{|E|}$ satisfying

$$\sum_{e \in \delta_G(i)} \omega(e) v_e = 0 \quad \text{for all } i \in V(G),$$

where $\delta_G(i)$ denotes the set of edges of G incident with i, and

$$\sum_{e \in E(G)} \omega(e) v_e z_e^\top = 0.$$

Equivalently ω is a self-stress of $((G, z), p, L)$ if $\omega^\top R(G, p, L) = 0$.

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- The stress matrix Ω (of ω) is the Laplacian of (G,z) weighted by ω, i.e.,

$$\Omega := I(G)^\top \mathrm{diag}(\omega) I(G),$$

where $\mathrm{diag}(\omega)$ denotes the diagonal matrix whose diagonal vector is equal to ω, and $I(G)$ is the $|E(G)| \times (|V(G)| + d)$ matrix of the form

$$\begin{array}{cc} & \begin{array}{ccccccccccccc} & & & i & & & & j & & & & & \end{array} \\ ((i,j);z(e)) & \left[\begin{array}{ccccccccccccc} 0 & \dots & 0 & 1 & 0 & \dots & 0 & -1 & 0 & \dots & 0 & z_e^1 & \dots & z_e^d \end{array}\right]. \end{array}$$

Application of the stress matrix

- **Thm. (**2021+**)** (G, p, L) is globally rigid if it has a self-stress ω such that
  - $\dim \ker \Omega = d + 1$ and
  - $\Omega \succeq 0$
  - $\nexists$ non-zero symmetric $d \times d$ matrix A with $v_e^\top A v_e = 0$ for all $e \in E(G)$.

- **Proof sketch:**
  - **Prop. 1:** Let
  $$[P\,L] := \begin{bmatrix} p_1 & p_2 & \dots & p_{|V|} & \ell_1 & \dots \ell_d \end{bmatrix} \in \mathbb{R}^{d \times (|V|+d)}.$$
  Then ω is an equilibrium stress of (G, p, L) if and only if $[P\,L]\Omega = 0$.
  - **Prop. 2:** Define the energy function (quadratic form)
  $$\mathcal{E}_{G,\omega}(p, L) := \begin{bmatrix} p^\top & \ell^\top \end{bmatrix} (\Omega \otimes I_d) \begin{bmatrix} p \\ \ell \end{bmatrix},$$
  Then we have
  $$\mathcal{E}_{G,\omega}(p, L) = \sum_{e \in E(G)} \omega(e) \|v_e\|^2$$

Proof (cont.)

- **Proof sketch (cont.):**
  - $\Omega \succeq 0$ implies that $\mathcal{E}_{G,\omega}$ is psd and hence convex.
  - Since ω is a self-stress of (G, p, L), (p, L) is a minimizer of $\mathcal{E}_{G,\omega}$.
  - By Prop. 1, this is equivalent to $[P\,L]\Omega = 0$.
  - Since $\dim\ker\Omega = d + 1$ and L is non-singular, the row vectors of $\begin{bmatrix} P & L \\ \mathbf{1}^\top & \mathbf{0}^\top \end{bmatrix}$ span the co-kernel of Ω.
  - Let (G, p', L') be equivalent to (G, p, L). By Prop. 2,
  $$\mathcal{E}_{G,\omega}(p', L') = \mathcal{E}_{G,\omega}(p, L).$$
  So (p', L') is also a minimizer of $\mathcal{E}_{G,\omega}$. Hence ω is self-stress of (G, p', L').
  - By Prop. 1, $[P'\,L']\Omega = 0$. So there exists a $d \times d$ matrix S and a vector $t \in \mathbb{R}^d$ such that $p_i' = Sp_i + t$ for all $i = 1, \ldots |V|$ and $\ell_j' = S\ell_j$ for all $j = 1, \ldots d$.
  - Thus, for each $e = uv \in E(G)$ we have
  $$v_e^\top (SS^\top - I_d) v_e = \langle v_e, v_e \rangle - \langle v_e', v_e' \rangle = 0.$$
  - Thus $SS^\top = I_d$, and (G, p, L) is congruent to (G, p', L').

Thank you!

Questions?