Weighted shift directed graph operator algebras

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Joint work with David Kribs and Stephen Power

IWOTA, August 2021

Mathematics Genealogy Project

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MathSciNet

Ph.D. University of Edinburgh 1976

Dissertation: *Intertwining Operators*

Mathematics Subject Classification: 47—Operator theory

Advisor: Frank Featherstone Bonsall

Students:
Click here to see the students ordered by family name.

Name	School	Year	Descendants
Heffernan, David	Lancaster University	1995	
Haworth, Paul	Lancaster University	2001	
Levene, Rupert	Lancaster University	2004	
Nixon, Tony	Lancaster University	2011	
Kastis, Eleftherios		2017	

Outline

1. Single variable shifts, unweighted and weighted, and their operator algebras
2. Shifts on directed graphs, unweighted and weighted, and their operator algebras
3. Commutants and bicommutants

Single variable shifts

Single variable unweighted shift and the Hardy algebra

- $H = \ell^2(\mathbb{N}_0)$, o.n.b. $(\xi_n)_{n\geq 0}$
- $S \in B(H)$, $S = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{bmatrix} : \xi_n \mapsto \xi_{n+1}$
- $\mathcal{S} := \overline{\text{alg}}^{\text{WOT}}\{I, S\} \cong H^\infty(\mathbb{D})$, Hardy algebra
- $\mathbb{N}_0 \to \mathcal{S}$, $k \mapsto S^k$ is a semigroup homorphism ($S^k S^l = S^{k+l}$)
- $S^k \xi_n = \xi_{n+k}$

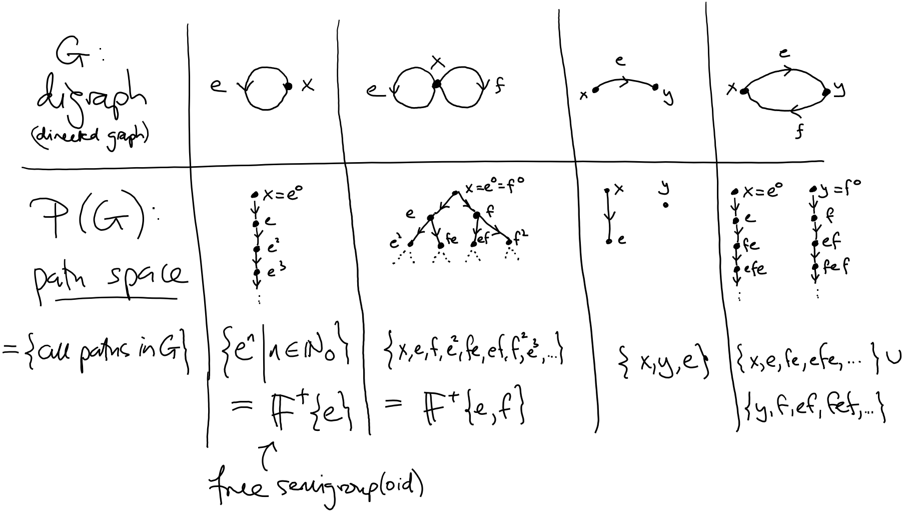
Single variable weighted shifts and their algebras

[Shields 1974, survey]

- $H = \ell^2(\mathbb{N}_0)$, o.n.b. $(\xi_n)_{n\geq 0}$, and fix weights $\lambda_0, \lambda_1, \lambda_2, \dots > 0$
- $S_\lambda \in B(H)$, $S_\lambda = \begin{bmatrix} 0 & & & \\ \lambda_0 & 0 & & \\ & \lambda_1 & 0 & \\ & & \ddots & \ddots \end{bmatrix} : \xi_n \mapsto \lambda_n \xi_{n+1}$
- $\mathcal{S}_\lambda := \overline{\text{alg}}^{\text{WOT}}\{I, S_\lambda\}$, "weighted Hardy algebra"
- $\mathbb{N}_0 \to \mathcal{S}_\lambda, k \mapsto S_\lambda^k$ is a semigroup homorphism ($S_\lambda^k S_\lambda^l = S_\lambda^{k+l}$)
- $S_\lambda^k \xi_n = \lambda(n,k)\xi_{n+k}$ where $\lambda(n,k) = \prod_{n\leq i<n+k} \lambda_i$
- "Cocycle condition": $\lambda(n, k_1 + k_2) = \lambda(n + k_1, k_2)\lambda(n, k_1)$

Shifts on directed graphs

Digraphs and paths



Remark: $\mathcal{P}(G)$ is finite $\iff$ G is acyclic

Unweighted shifts on digraphs and free semigroupoid algebras

- G: digraph, $H = \ell^2(\mathcal{P}(G))$, o.n.b. $(\xi_v)_{v \in \mathcal{P}(G)}$
- For $w, u \in \mathcal{P}(G)$, define partial isometries $L_w, R_u \in B(H)$:

$$L_w : \xi_v \mapsto \begin{cases} \xi_{wv} & wv \in \mathcal{P}(G), \\ 0 & \text{else} \end{cases} \qquad R_u : \xi_v \mapsto \begin{cases} \xi_{vu} & vu \in \mathcal{P}(G), \\ 0 & \text{else} \end{cases}$$

- Free semigroupoid operator algebras:

$$\mathcal{L} := \overline{\text{alg}}^{\text{WOT}}\{I, L_w \mid w \in \mathcal{P}(G)\}, \qquad \mathcal{R} := \overline{\text{alg}}^{\text{WOT}}\{I, R_u \mid u \in \mathcal{P}(G)\}$$

- '90s$\rightarrow$: Davidson, Pitts, Popescu, Kribs, Katsoulis, Solel, Power, Kennedy, Kakariadis, ... commutants, invariant subspaces, (hyper)reflexivity, wandering vectors, ...

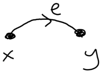
Theorem (Kribs-Power, 2004) $\mathcal{L}'' = \mathcal{L}$

$\mathcal{L}' = \mathcal{R}$ and $\mathcal{R}' = \mathcal{L}$; hence, $\mathcal{L}'' = \mathcal{L}$.

Examples

1. $G =$ 

   $\mathcal{P}(G) = \{e^k \mid k \in \mathbb{N}_0\} \cong \mathbb{N}_0$, $L_e = R_e = S$, $\mathcal{L} = \mathcal{R} = \mathcal{S}$, the Hardy algebra.

2. $G =$ 

   $\mathcal{P}(G) = \{x, y, e\}$, $\mathcal{L} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & 0 & b \end{bmatrix} \right\}$, $\mathcal{R} = \mathcal{L}' = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & c & a \end{bmatrix} \right\}$.

Weighted left shifts on digraphs, and their algebras

- Left weight (LW): $\lambda : \mathcal{P}(G) \times \mathcal{P}(G) \to (0, \infty)$ obeying the "left cocycle relation" $\lambda(v, w_2 w_1) = \lambda(w_1 v, w_2)\lambda(v, w_1)$
- For $w \in \mathcal{P}(G)$, define

$$L_{\lambda,w} : \xi_v \mapsto \begin{cases} \lambda(v, w)\xi_{wv} & wv \in \mathcal{P}(G), \\ 0 & \text{else}. \end{cases}$$

- $\|L_{\lambda,w}\|_{B(H)} = \sup_v \lambda(v, w)$. Could be infinite!
- Say λ is *left bounded* (LB) if $\|L_{\lambda,w}\|_{B(H)} < \infty$ for all $w \in \mathcal{P}(G)$.
- cocycle relation $\implies$ semigroupoid homomorphism: $L_{\lambda,w_2 w_1} = L_{\lambda,w_2} L_{\lambda,w_1}$
- There is a unique $\alpha : \mathcal{P}(G) \to (0, \infty)$ with $\alpha|_{V(G)} \equiv 1$ so that $\lambda(v, w) = \frac{\alpha(wv)}{\alpha(v)}$. (Namely, $\alpha(v) := \lambda(v, s(v))$.)
- For LBLW λ, consider $\mathcal{L}_\lambda := \overline{\text{alg}}^{\text{WOT}}\{I, L_w \mid w \in \mathcal{P}(G)\}$.

Weighted right shifts on digraphs, and their algebras

- Right weight (RW): $\rho : \mathcal{P}(G) \times \mathcal{P}(G) \to (0, \infty)$ obeying the "right cocycle relation" $\rho(v, u_1u_2) = \rho(vu_1, u_2)\rho(v, u_1)$
- For $u \in \mathcal{P}(G)$, define

$$R_{\rho,u} : \xi_v \mapsto \begin{cases} \rho(v,u)\xi_{vu} & vu \in \mathcal{P}(G), \\ 0 & \text{else}. \end{cases}$$

- Say ρ is *right bounded* (RB) if $\|R_{\rho,u}\|_{B(H)} < \infty$ for all $u \in \mathcal{P}(G)$.
- cocycle relation $\implies$ semigroupoid homomorphism: $R_{\rho,u_1u_2} = R_{\rho,u_1}R_{\rho,u_2}$
- There is a unique $\alpha : \mathcal{P}(G) \to (0, \infty)$ with $\alpha|_{V(G)} \equiv 1$ so that $\rho(v,u) = \frac{\alpha(vu)}{\alpha(v)}$. (Namely, $\alpha(v) := \rho(v, r(v))$.)
- For RBRW ρ, consider $\mathcal{R}_\rho := \overline{\text{alg}}^{\text{WOT}}\{I, R_u \mid u \in \mathcal{P}(G)\}$.

Commutants and bicommutants

Commutants and bicommutants

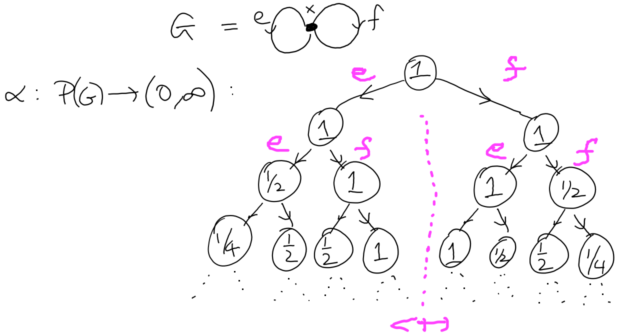
Say a LW λ and a RW ρ are *companions* if they come from the same map $\alpha : \mathcal{P}(G) \to (0, \infty)$.

Theorem (KLP, 2017)

If λ is a LBLW and ρ is a RBRW, and λ, ρ are companions, then $\mathcal{L}'_\lambda = \mathcal{R}_\rho$ and $\mathcal{R}'_\rho = \mathcal{L}_\lambda$.
In particular, if λ is a LBLW with a RB companion, then $\mathcal{L}''_\lambda = \mathcal{L}_\lambda$.

- Taking $\lambda = \rho \equiv 1$ recovers the theorem of [Kribs-Power 2004].
- If the companion of λ is not RB, the situation is less clear…

Example



- $\alpha \rightsquigarrow \lambda$, LBLW
- $\alpha \rightsquigarrow \rho$, RW, not RB
- $\mathcal{L}'_\lambda = \mathbb{C}I$, so $\mathcal{L}''_\lambda = B(H) \neq \mathcal{L}_\lambda$.

Congratulations on your retirement,
and thank you, Steve!