Operator Algebras, Graph Theory, and Distinguishing Quantum States with LOCC

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IWOTA Lancaster 2021
Operator Algebras to Geometric Rigidity
Session to Honour Prof. Stephen Power's Retirement

UNIVERSITY of GUELPH
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Outline

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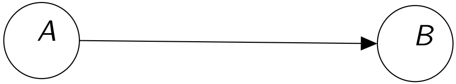
Quantum Local Operations and Classical Communication (LOCC)

- **Local Operations and Classical Communication (LOCC)** basic idea: multiple parties share set of quantum states spread across their subsystems, on which each party can perform local *quantum* operations, and then transmit their results only using *classical* information.
- The LOCC paradigm is fundamental in QIT; e.g., hybrid classical-quantum communication networks, probing locality vs quantum entanglement, power of distributed quantum algorithms, quantum teleportation protocols, etc.
- The key general LOCC problem we focus on here: **distinguishing quantum states** amongst a set of known states, where two parties (A, B) can perform quantum measurements on their individual subsystems, and then communicate classically.

Quantum LOCC

- Further, as arbitrary LOCC operations are notoriously difficult to characterize mathematically, we restrict ourselves to the case of **one-way LOCC**, where the communication is limited to one predetermined direction. This still captures many key examples and settings (though not all).
- One-way local operations and classical communication as a simple visual:



Bipartite Case

- Two parties A, B are separated at distance labs.
- They control their (finite dimensional) subsystem Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$.
- $\mathcal{H}_A \otimes \mathcal{H}_B$ is prepared in a pure state from a known set of states (unit vectors) $\mathcal{S} = \{|\psi_j\rangle\}_j$.
- Identify the particular j using one-way LOCC.

One-way LOCC Measurements

Mathematical description of one-way LOCC

A **one-way LOCC measurement** (with A going first) is a set of positive operators $\mathbb{M} = \{A_k \otimes B_{k,j}\}_{k,j}$ such that

$$\sum_k A_k = I_A \quad \text{and} \quad \sum_j B_{k,j} = I_B \quad \forall k.$$

$\Rightarrow$ If outcome $A_k \otimes B_{k,j}$ is obtained for some k and j, the conclusion is the prepared state was the one identified with the pair k, j.

One-way LOCC

Example (Easy one – Bell basis states)

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right) \\ |\psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\right) \end{aligned}$$

Here:

- Alice: $A_1 = |0\rangle\langle 0|$ and $A_2 = |1\rangle\langle 1|$.
- Bob: $\{B_{1,1}, B_{1,2}\} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\} = \{B_{2,1}, B_{2,2}\}$.
- If Alice gets outcome 0, then tells Bob, who after measurement gets outcome 0, then the state is $|\psi_0\rangle$. Similarly it would be $|\psi_1\rangle$ if Bob measured a 1.

Operator relations for one-way LOCC

$\Rightarrow$ Let $|\Phi\rangle$ be the standard *maximally entangled state* on two-qudit space $\mathbb{C}^d \otimes \mathbb{C}^d$; $|\Phi\rangle = \frac{1}{\sqrt{d}}\big(|0\rangle|0\rangle + \ldots + |d-1\rangle|d-1\rangle\big)$. (A state $(I \otimes U)|\Phi\rangle$ is maximally entangled precisely when U is unitary.)

Theorem (Nathanson '13)

Let $\mathcal{S} = \{|\psi_i\rangle = (I \otimes U_i)|\Phi\rangle\} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d$ be a set of orthogonal states (with U_i operators on $\mathbb{C}^d$). Then TFAE:

- *(i) The elements of $\mathcal{S}$ can be distinguished with one-way LOCC.*
- *(ii) There exists a set of states $\{|\phi_k\rangle\}_{k=1}^{r} \subseteq \mathbb{C}^d$ and positive numbers $\{m_k\}$ such that $\sum_k m_k|\phi_k\rangle\langle\phi_k| = I$ and for all k and $i \neq j$,*

$$\langle \phi_\mathbf{k} | \mathbf{U_j^* U_i} | \phi_\mathbf{k} \rangle = \mathbf{0}.$$

Operator relations for one-way LOCC

In the example above...

Example

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) = (I_2 \otimes I_2)|\Phi\rangle$$
$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B) = (I_2 \otimes X)|\Phi\rangle,$$

where X is the single-qubit Pauli bit flip operator. Here we have $U_1 = I_2$, $U_2 = X$, and the diagonal entries are zero;

$$U_j^* U_i = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Outline

Some Important Operator Structures

⇒ Recall the basic structure theory for *finite-dimensional C^*-algebras*: every such algebra $\mathfrak{A}$ is unitarily equivalent to an orthogonal direct sum of the form $\bigoplus_k (M_{m_k}(\mathbb{C}) \otimes I_{n_k})$.

Definition

Let $\mathfrak{A}$ be a unital C^*-algebra. Any linear subspace $\mathfrak{S}$ contained in $\mathfrak{A}$ which contains the identity and is closed under taking adjoints is called an **operator system**.

Definition

Let $\mathcal{H}$ be a Hilbert space and let $\mathfrak{S} \subseteq B(\mathcal{H})$ be a set of operators on $\mathcal{H}$ that form an operator system. A vector $|\psi\rangle \in \mathcal{H}$ is said to be a **separating vector** *of $\mathfrak{S}$ if $A|\psi\rangle \neq 0$ whenever A is a nonzero element of $\mathfrak{S}$.*

Some Important Operator Structures

$\Rightarrow$ If $\mathcal{H}$ is finite-dimensional and $\mathfrak{S}$ is in fact a C^*-subalgebra, then we may use the decomposition above for such algebras to determine the existence of a separating vector as follows.

Theorem (Pereira '03)

The C^*-algebra $\bigoplus_k (M_{m_k}(\mathbb{C}) \otimes I_{n_k})$ has a separating vector if and only if $n_k \geqslant m_k$ for all k.

E.g., In the case of the diagonal algebra $\mathfrak{A}_\Delta \cong \mathbb{C}^d$, $m_k = n_k = 1$ for all $1 \leqslant k \leqslant d$, and hence $\mathfrak{A}_\Delta$ has a separating vector, for example: $|\psi\rangle = \frac{1}{\sqrt{d}}(|0\rangle + \ldots + |d-1\rangle)$ does the job.

Taken together, these notions and our early results led to...

Operator Structure – LOCC Result

Theorem

Suppose the operator system $\mathfrak{S}_0 = \operatorname{span}\{U_i^* U_j, I\}_{i\neq j}$ *is closed under multiplication and hence is a* C^**-algebra. Then* $\mathcal{S} = \{(I\otimes U_i)|\Phi\rangle\}$ *is distinguishable by one-way LOCC if and only if* $\mathfrak{S}_0$ *has a separating vector.*

E.g., Road Map for Examples from Operator Structures:

- If we have $\mathfrak{A} = \operatorname{Alg}(\mathfrak{S}_0) \cong \oplus_k (M_{m_k} \otimes I_{n_k})$, then $\mathfrak{A}$ has a separating vector if and only if $n_k \geqslant m_k$ for all k.

- $\Rightarrow$ To find sets of *indistinguishable states*, we can look for sets $\{U_i\}$ such that $\mathfrak{S}_0 = \mathfrak{A}$ and $m_k > n_k$ for some k.

- $\Rightarrow$ We are led to consider sets of unitaries $\{U_i\}$, such that the set is (up to scalar multiples) closed under multiplication, taking *'s, and taking inverses.

Application: States from the Stabilizer Formalism

...Sets of such unitaries are plentiful in **Quantum Error Correction**...

Theorem

Let $\{U_i\}$ be a complete set of 4^k 'logical Pauli operators for a stabilizer k-qubit code' on $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$. Then $\mathcal{S} = \{(I \otimes U_i)|\Phi\rangle\}_i$ is distinguishable by one-way LOCC if and only if $k \leqslant \frac{n}{2}$.

Remark

The upper bound $(2k = n)$ gives sets that saturate a known bound for one-way distinguishable states on $\mathbb{C}^d \otimes \mathbb{C}^d$ $(d = 2^n)$. For $2k < n$, this produces (non-trivial) distinguishable sets; significant as it is known that many sets defined from $\mathcal{P}_n$ with $< 2^n$ operators (here $4^k < 2^n$) are *not* one-way distinguishable.

Outline

Graph Theory Nomenclature

- We write $G = (V, E)$ for a **simple graph** with vertex set V and edge set E. For $v, w \in V$, we write $v \sim w$ if the edge $\{v, w\} \in E$.
- The **complement** of G is the graph $\overline{G} = (V, \overline{E})$, where the edge set $\overline{E}$ consists of all two-element sets from V that are not in E.
- Another graph G' is a **subgraph** of G, written $G' \leqslant G$, if $V' \subseteq V$ and $E' \subseteq E$ with $v, w \in V'$ whenever $\{v, w\} \in E'$.
- A set of graphs $\{G_i = (V_i, E_i)\}$ **covers** G if $V = \cup_i V_i$ and $E = \cup_i E_i$, and $\{G_i\}$ is a **clique cover** for G if $\{G_i\}$ covers G and if each of the G_i is a complete graph (i.e., a clique).
- The **clique cover number** $\mathrm{cc}(G)$ is the smallest possible number of subgraphs contained in a clique cover of G.

Product States and Orthogonal Graph Representations

- Given a graph $G = (V, E)$, a function $\phi : V \to \mathbb{C}^d \backslash \{0\}$ is an **orthogonal representation** of G if for all vertices $v_i \neq v_j \in V$,

$$v_i \not\sim v_j \iff \langle \phi(v_i), \phi(v_j) \rangle = 0.$$

- Given a set of product states $\{|\psi_k^A\rangle \otimes |\psi_k^B\rangle\}_{k=1}^r$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, Alice's graph is the unique graph $\mathbf{G_A}$ with $V = \{1, 2, \dots, r\}$ such that $k \mapsto |\psi_k^A\rangle$ is an orthogonal representation of G_A. Similarly, Bob's graph $\mathbf{G_B}$ is defined by orthogonality (or not) of the $|\psi_k^B\rangle$.

- By construction, the states are mutually orthogonal iff Alice's graph is a subgraph of the complement of Bob's graph; $G_A \leqslant \overline{G_B}$.

- E.g., $\mathcal{S} = \{|\psi_1\rangle = |00\rangle, |\psi_2\rangle = |01\rangle, |\psi_3\rangle = |10\rangle, |\psi_4\rangle = |11\rangle\}$. Then $V = \{1, 2, 3, 4\}$, $G_A = \{(1, 2), (3, 4)\}$, $G_B = \{(1, 3), (2, 4)\}$, and $\overline{G_B} = \{(1, 2), (3, 4), (2, 3), (1, 4)\}$.

One-Way LOCC Distinguishability of Product States via Graph Theory

Theorem

Given a set of product states in $\mathcal{H}_A \otimes \mathcal{H}_B$, let G_A and G_B be the graphs of the states from Alice and Bob's perspectives, respectively. Let $\phi : V_A \to \mathcal{H}_A$ be the association of vertices with Alice's states and assume that the set $\{\phi(v) : v \in V\}$ spans $\mathcal{H}_A$. Then the states are distinguishable with one-way LOCC with Alice measuring first if and only if there exists

(1) *a graph G satisfying $G_A \leqslant G \leqslant \overline{G_B}$,*

(2) *a clique cover $\{V_j\}_{j=1}^{k}$ of G, and,*

(3) *a POVM $\{Q_j\}$ on $\mathcal{H}_A$ such that for all $v \in V_A$, $Q_j\phi(v) \neq 0$ implies that $v \in V_j$.*

The Single Qubit Sender Case: Full Graph Characterisation

Theorem

A set of orthonormal product states in $\mathbb{C}^2 \otimes \mathbb{C}^d$, for $d \geqslant 2$, is distinguishable via one-way LOCC with Alice going first if and only if there is some graph between the two graphs G_A and $\overline{G_B}$ with clique cover number at most two; that is, there is a graph G such that

$$G_A \leqslant G \leqslant \overline{G_B} \quad \textit{and} \quad \mathrm{cc}\,(G) \leqslant 2. \tag{1}$$

- E.g., $\mathcal{S} = \{|\psi_1\rangle = |00\rangle, |\psi_2\rangle = |01\rangle, |\psi_3\rangle = |10\rangle, |\psi_4\rangle = |11\rangle\}$.

- $\Rightarrow$ This set is distinguishable with Alice going first, and can easily be seen to satisfy the theorem condition with:

$$G_A = G = \{(1,2),(3,4)\} \lneq \overline{G_B} = \{(1,2),(3,4),(2,3),(1,4)\},$$

and

$$\mathrm{cc}\,(G) = 2.$$

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Conclusion & Outlook

- The use of techniques from **operator theory** and **operator algebras** in the subject of **LOCC** is relatively new, and our results on state distinguishability so far suggest some avenues worth pursuing further (e.g. connections with **quantum error correction**). We have obtained new results and examples, and new versions/realizations of previous results and examples.
- More recently, we have introduced **graph theory** techniques into our state distinguishability analysis, and we are continuing this work ...see references on the next slide.
- (Shameless extra plug) Generalization of Nielsen's Theorem on **LOCC state convertibility** ($\Leftrightarrow$ spectral **majorization**) to the **von Neumann algebra** framework: J.Crann, D.K., R.Levene, I.Todorov, Comm. Math. Phys., 378 (2020), 1123-1156.

References for our state distinguishability work

(also see references therein for other entrance points into literature)

- D.K., Comfort Mintah, Michael Nathanson, Rajesh Pereira, *Vector representations of graphs and distinguishing quantum product states with one-way LOCC*, Linear Algebra and its Applications, 602 (2020), 223-239.
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(...and more to come)

Have A Nice Day

Thank you!