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From Fourier binests to Harmonic operators, via crossed products

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Plan

• Revisit the Fourier binest algebra and the parabolic algebra studied with S.C. Power in the 90's from a new perspective.

• Revisit the space of operators which are 'jointly harmonic' with respect to a family of complex measures on a group G.

• Represent both pairs of spaces inside suitable crossed products.

Act One: The parabolic and Fourier binest algebras

The *parabolic algebra* $\mathcal{A}_p \subseteq \mathcal{B}(L^2\mathbb{R})$ is

$$\mathcal{A}_p := \overline{\text{span}}^{\mathrm{w}^*}\{M_{e_t}\lambda_s : e_\lambda(x) = e^{itx}, t \in \mathbb{R}_+, s \in \mathbb{R}^+\}$$

$$= \overline{\text{span}}^{\mathrm{w}^*}\{M_h\lambda_s : h \in H^\infty(\mathbb{R}), s \in \mathbb{R}^+\}$$

where $M_h \in \mathcal{B}(L^2\mathbb{R})$ is the multiplication operator $f \mapsto hf$ and $(\lambda_s f)(x) = f(x-s)$ on $L^2(\mathbb{R})$.

These operators leave invariant both the *Volterra nest* $\mathcal{N}_v$ (consisting of all projections onto the subspaces $L^2[t,\infty),\ t \in \mathbb{R}$), and the *analytic nest* $\mathcal{N}_a$ which consists of all projections onto the subspaces $e_t H^2(\mathbb{R}),\ t \in \mathbb{R}$, (here $e_t(x) := e^{itx}$).
We called the union $\mathcal{N}_v \cup \mathcal{N}_a$ together with the trivial subspaces $\{0\}$ and $L^2(\mathbb{R})$, the *Fourier binest* $\mathcal{N}_{fb}$.

The parabolic and Fourier binest algebras are equal

So $\mathcal{A}_p$ leaves each $P \in \mathcal{N}_{fb}$ invariant. Hence, if $\Theta_P : X \mapsto P^\perp XP$, a w*-continuous complete contraction on $\mathcal{B}(L^2\mathbb{R})$, then $\mathcal{A}_p \subseteq \ker \Theta_P$. Thus considering the common kernel of the Θ_p's,

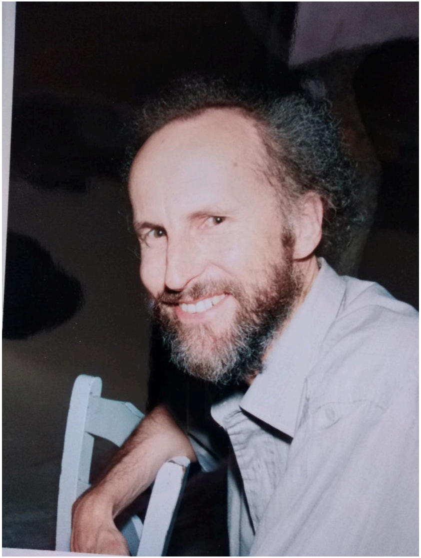
$$\bigcap \{\ker \Theta_P : P \in \mathcal{N}_{fb}\} = \{T \in \mathcal{B}(L^2\mathbb{R}) : P^\perp TP = 0 \ \forall P \in \mathcal{N}_{fb}\} := \mathcal{A}_{fb}$$

(the Fourier binest algebra) we have $\mathcal{A}_p \subseteq \mathcal{A}_{fb}$.

In [**?**] we proved with Steve that $\mathcal{A}_p = \mathcal{A}_{fb}$.

Idea of proof: Identify Hilbert-Schmidt operators in both algebras as being linearly isomorphic to the tensor product $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$ (*R. Levene*: $H^2(\mathbb{R}) \otimes L^2(\mathbb{R}^+)$) and there is an approximate identity of such Hilbert-Schmidt operators.

The parabolic and Fourier binest algebras are equal!



Act Two: Harmonic functions, Harmonic operators, and All That

Let $\mu \in M(G)$ be a probability measure on a loc. compact group G.

• Say $\phi : G \to \mathbb{C}$ is a μ-harmonic function ($\phi \in \mathcal{H}(\mu)$) if ϕ is a fixed point of the map P_μ given by

$$(P_\mu \phi)(s) = \int_G \phi(st) d\mu(t).$$

• Quantisation: Say $T \in \mathcal{B}(L^2(G))$ is a μ-harmonic operator if

$$\int_G \rho_t T \rho_t^{-1} d\mu(t) = T. \quad \text{Write} \quad T \in \widetilde{\mathcal{H}}(\mu).$$

(ρ: right-regular repr. of G)

Harmonic functions, Harmonic operators, and All That

So $\widetilde{\mathcal{H}}(\mu)$ is the fixed point set of the map

$$\Theta(\mu) : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^2(G)) : T \to \int_G \rho_t T \rho_t^{-1} d\mu(t)$$

which is weak-* continuous (completely contractive)
and extends P_μ.

Harmonic functions, Harmonic operators

Clearly multiplication operators M_ϕ by μ-harmonic functions are in $\widetilde{\mathcal{H}}(\mu)$ and so are left translation operators $\lambda_s,\ s \in G$.
Thus, if

$$\mathrm{Bim}(\mathcal{H}(\mu)) := \overline{\mathrm{span}}^{w^*}\{M_\phi\lambda_s : \phi \in \mathcal{H}(\mu), s \in G\} \subseteq \mathcal{B}(L^2G)$$

(the weak* closed $vN(G)$-bimodule generated by $\mathcal{H}(\mu)$), we have

$$\mathrm{Bim}(\mathcal{H}(\mu)) \subseteq \widetilde{\mathcal{H}}(\mu)\,.$$

More generally: consider jointly harmonic functions (resp. operators) i.e. harmonic under a family Λ of complex measures, we have

$$\mathrm{Bim}(\mathcal{H}(\Lambda)) \subseteq \widetilde{\mathcal{H}}(\Lambda)\,.$$

Harmonic functions generate Harmonic operators under the AP

As a consequence of a more general result [see Act Three below], we have

Theorem

For any $\Lambda \subseteq M(G)$,

$$\mathrm{Bim}(\mathcal{H}(\Lambda)) = \widetilde{\mathcal{H}}(\Lambda)$$

provided G has the Approximation Property AP of Haagerup-Kraus.

(AP: existence of a very weak form of approximate identity for the cb multipliers of the Fourier algebra of G.)

Harmonic functions generate Harmonic operators under the AP

This was first proved in case G is abelian, or compact, or weakly amenable discrete with M. Anoussis and I.G. Todorov [**?**]. Then generalised as above by J. Crann and M. Neufang [**?**].

• When $\Lambda = \{\mu\}$ for a probability measure, the result holds for any G (Izumi, Jaworski-Neufang). Reason: $\mathcal{H}(\mu)$ is linearly and covariantly completely isometrically isomorphic to a Von Neumann algebra (the 'Poisson boundary').

• For general Λ, the result generalises the idea of the 'non-commutative Poisson boundary'.

• Not known if AP necessary.

To view both under a different perspective: $\rightsquigarrow$

Act Three: The operator space crossed products

Let $\mathcal{V} \subseteq \mathcal{B}(H)$ be a w*-closed operator space (no algebra structure), with predual $\mathcal{V}_*$ and let $s \mapsto \alpha_s$ be an action of G on $\mathcal{V}$ by weak-* continuous complete isometries.

To represent both G and $\mathcal{V}$ simultaneously and covariantly:

Define, for each $v \in \mathcal{V}$, an element $\pi_\alpha(v) \in \mathcal{V}\bar{\otimes}L^\infty(G)$ by duality:

$$\langle \pi_\alpha(v), \omega \otimes h \rangle := \int_G \langle \alpha_s^{-1}(v), \omega \rangle h(s) ds, \qquad \omega \in \mathcal{V}_*, h \in L^1(G).$$

Considering $L^\infty(G) \subseteq \mathcal{B}(L^2 G)$ we have a map

$$\pi_\alpha : \mathcal{V} \to \mathcal{V}\bar{\otimes}L^\infty(G) \subseteq \mathcal{V}\bar{\otimes}\mathcal{B}(L^2 G) \subseteq \mathcal{B}(H \otimes L^2 G).$$

Also let $\tilde{\lambda} : G \to \mathcal{B}(H \otimes L^2 G) : s \mapsto \tilde{\lambda}_s := \mathrm{Id}_H \otimes \lambda_s$.

Covariance:

$$\pi_\alpha(\alpha_s(v)) = \tilde{\lambda}_s \pi_\alpha(v) \tilde{\lambda}_s^{-1}.$$

The spatial crossed product

$$\pi_\alpha : \mathcal{V} \to \mathcal{V}\bar{\otimes}\mathcal{B}(L^2G) \subseteq \mathcal{B}(H \otimes L^2G)$$
$$\tilde{\lambda} : G \to \mathcal{B}(H \otimes L^2G)$$
$$\pi_\alpha(\alpha_s(v)) = \tilde{\lambda}_s \pi_\alpha(v) \tilde{\lambda}_s^{-1}$$

The *spatial crossed product $\mathcal{V} \rtimes_\alpha G$* is defined to be the $vN(G)$-bimodule generated in $\mathcal{V}\bar{\otimes}\mathcal{B}(L^2G)$ by $\pi_\alpha(\mathcal{V})$: it is the weak* closed space

$$\mathcal{V} \rtimes_\alpha G := \overline{\mathrm{span}}^{\mathrm{w}^*}\{\pi_\alpha(v)\tilde{\lambda}_s,\ v \in \mathcal{V}, s \in G\} \subseteq \mathcal{V}\bar{\otimes}\mathcal{B}(L^2G).$$

It is not hard to see that $\mathcal{V} \rtimes_\alpha G$ consists of fixed points for the G-action

$$s \in G \mapsto \alpha_s \otimes \mathrm{Ad}\rho_s : \quad G \curvearrowright \mathcal{V}\overline{\otimes}\mathcal{B}(L^2G)$$

The Fubini crossed product

It is not hard to see that $\mathcal{V} \rtimes_\alpha G$ consists of fixed points for the G-action

$$s \in G \mapsto \alpha_s \otimes \mathrm{Ad}\rho_s : \quad G \curvearrowright \mathcal{V}\overline{\otimes}\mathcal{B}(L^2G)$$

on the tensor product $\mathcal{V}\overline{\otimes}\mathcal{B}(L^2G)$, where ρ is the right regular representation of G.

The (apriori larger) space of fixed points of $\alpha \otimes \mathrm{Ad}\rho$,

$$\{y \in \mathcal{V}\overline{\otimes}\mathcal{B}(L^2G) : \ (\alpha_s \otimes \mathrm{Ad}\rho_s)(y) = y \ \forall s \in G\} := \mathcal{V} \rtimes_\alpha^{\mathcal{F}} G$$

is called the Fubini crossed product of $\mathcal{V}$ by α.Thus

$$\mathcal{V} \rtimes_\alpha G \subseteq \mathcal{V} \rtimes_\alpha^{F} G\,.$$

Comparing the Crossed products

$$\mathcal{V} \rtimes_\alpha G \subseteq \mathcal{V} \rtimes_\alpha^F G\,.$$

In special cases, for example when $\mathcal{V}$ is a von Neumann algebra, the equality $\mathcal{V} \rtimes_\alpha G = \mathcal{V} \rtimes_\alpha^F G$ holds for all G, but not in general.

Crann - Neufang [**?**] proved that we have coincidence for all $\mathcal{V}$ when G has the AP.

Theorem (D. Andreou, [?])

The spatial and Fubini crossed products coincide for all dual operator spaces $\iff$ G has the AP.

Bimodules and Crossed products

Now specialise to the case of the action $G \overset{\alpha_G}{\curvearrowright} L^\infty(G)$ by left translation. Then both crossed products can be represented on $L^2(G)$:

Proposition (D. Andreou)

For any translation invariant weak closed space $\mathcal{V} \subseteq L^\infty(G)$,*

$$\mathcal{V} \rtimes_{\alpha_G} G \overset{\Psi}{\simeq} \mathrm{Bim}(\mathcal{V}) \quad \textit{and} \quad \mathcal{V} \rtimes^F_{\alpha_G} G \overset{\Psi}{\simeq} \ker \Theta(\mathcal{V}_\perp).$$

*where $\Psi : \mathcal{B}(L^2 G) \to \mathcal{B}(L^2(G \times G))$ is an isometric normal *-morphism.*

Here

$$\ker \Theta(\mathcal{V}_\perp) := \bigcap \{ T \in \mathcal{B}(L^2 G) : \int_G \rho_t T \rho_t^{-1} f(t) dt = 0 \, \forall f \in \mathcal{V}_\perp \}.$$

Note $\Psi : \lambda_s \to 1 \otimes \lambda_S$ and $\Psi : M_f \to M_{\alpha_G(f)}$ where $\alpha_G(f)(s,t) = f(ts)$.

Resolution: Back to the Examples

- In particular, for the jointly Λ-harmonic case, $\Lambda \subseteq M(G)$, we get

$$\mathcal{H}(\Lambda) \rtimes_{\alpha_G} G \overset{\Psi}{\simeq} \mathrm{Bim}(\mathcal{H}(\Lambda)) \quad \text{and} \quad \mathcal{H}(\Lambda) \rtimes^{F}_{\alpha_G} G \overset{\Psi}{\simeq} \ker \Theta(\mathcal{H}(\Lambda)_\perp)\,.$$

But $\ker \Theta(\mathcal{H}(\Lambda)_\perp) = \widetilde{\mathcal{H}}(\Lambda)$.
So we obtain the equality

$$\mathrm{Bim}(\mathcal{H}(\Lambda)) = \ker \Theta(\mathcal{H}(\Lambda)_\perp)$$

under the AP.

Resolution: Back to the Examples

- Also, for the holomorphic case, $\mathcal{V} = H^\infty(\mathbb{R})$,

$$H^\infty(\mathbb{R}) \rtimes_{\alpha_\mathbb{R}} \mathbb{R} \overset{\Psi}{\simeq} \mathrm{Bim}(H^\infty(\mathbb{R})) \quad \text{and} \quad H^\infty(\mathbb{R}) \rtimes^F_{\alpha_\mathbb{R}} \mathbb{R} \overset{\Psi}{\simeq} \ker \Theta(H^\infty(\mathbb{R})_\perp)$$

(which are of course equal). But

$$\mathrm{Bim}(H^\infty(\mathbb{R})) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{M_h\lambda_s : h \in H^\infty(\mathbb{R}),\ s \in \mathbb{R}\}$$

so, restricting to $s \in \mathbb{R}^+$,

$$\mathcal{A}_p = \overline{\mathrm{span}}^{\mathrm{w}^*}\{M_h\lambda_s : h \in H^\infty(\mathbb{R}),\ s \in \mathbb{R}^+\} \overset{\Psi}{\simeq} H^\infty(\mathbb{R}) \rtimes_{\alpha_\mathbb{R}} \mathbb{R}^+$$

where the 'semi-crossed product' is defined to be the weak* closed subspace of $H^\infty(\mathbb{R}) \rtimes_{\alpha_\mathbb{R}} \mathbb{R}$ generated by $\pi_\alpha(h)\tilde{\lambda}_s$ with $h \in H^\infty(\mathbb{R})$ and $s \in \mathbb{R}^+$.

$$H^\infty(\mathbb{R}) \rtimes_{\alpha_\mathbb{R}} \mathbb{R}^+ := \overline{\mathrm{span}}^{\mathrm{w}^*}\{\pi_\alpha(h)\tilde{\lambda}_s : h \in H^\infty(\mathbb{R}),\ s \in \mathbb{R}^+\}.$$

The Fourier binest algebra is a semi-crossed product

Thus, the equality $\mathcal{A}_p = \mathcal{A}_{fb}$ justifies the point of view that the Fourier binest algebra is (isomorphic to) a 'continuous' semi-crossed product of $H^\infty(\mathbb{R})$, by an action of $\mathbb{R}^+$.

$$\mathcal{A}_{fb} \overset{\Psi}{\simeq} H^\infty(\mathbb{R}) \rtimes_{\alpha_{\mathbb{R}}} \mathbb{R}^+.$$

This is implicit (in a discrete way) in some ideas of S.C. Power and in some recent results of E. Kastis.

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Στην Υγειά σου **Steve**! За здоровье!