Invariant subspaces and Rigidity

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Let $T :\to X$ be a linear operator. A subspace Y of X is called invariant if $T(Y) \subseteq Y$.
These subspaces can describe operator algebras. Indeed, let $\mathcal{A}$ be an operator algebra acting on a Hilbert space H. Define

$$\operatorname{Lat} \mathcal{A} = \{K \leq H : AK \subseteq K, \text{ for all } A \in \mathcal{A}\}.$$

We say that $\mathcal{A}$ is reflexive when

$$\mathcal{A} = \{A \in B(H) : AK \subseteq K, \text{ for all } K \in \operatorname{Lat}(\mathcal{A})\}.$$

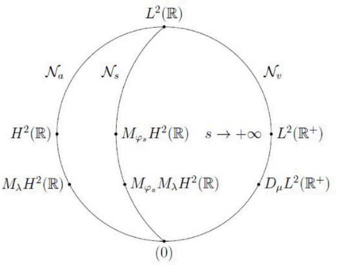


Figure: The lattice $Lat\mathcal{A}_p$ of the parabolic algebra (Katavolos - Power, 1997)

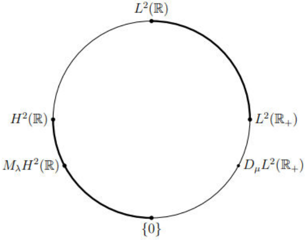


Figure: The lattice $Lat\mathcal{A}_{ph}$ of the triple semigroup algebra (K. - Power, 2015)

Figure: Footage from PhD


JOIN RIGIDITY
WE HAVE INVARIANT SUBSPACES

A bar-joint framework in $\mathbb{R}^d$ is a pair (G, p) where $G = (V, E)$ is a simple connected graph and $p : V \to \mathbb{R}^d$ is a placement of the vertices $v_1, v_2, \ldots$ as framework joints in $\mathbb{R}^d$. The framework edges are the line segments $[p(v_i), p(v_j)]$ associated with the edges E of the graph $G = (V, E)$.

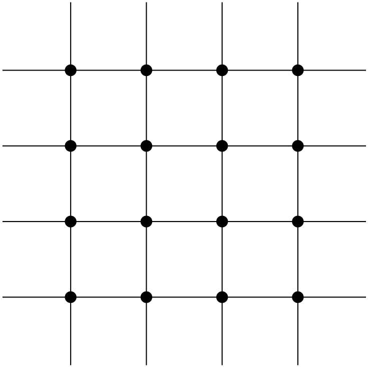
The rigidity matrix $R(G, p)$ of a framework (G, p) is an $|E| \times d|V|$ matrix, given by

$$vw \begin{bmatrix} \cdots & 0 & \overset{v}{p_v - p_w} & 0 & \cdots & 0 & \overset{w}{p_w - p_v} & 0 & \cdots \end{bmatrix}.$$

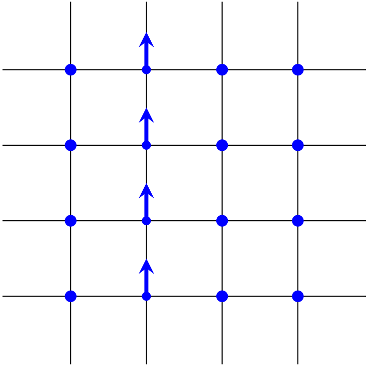
The complex infinitesimal flex space $\mathcal{F}(G, p; \mathbb{C})$ is the vector space $\ker R(G, p)$.

Spoiler alert: If (G, p) is "symmetric", then $\mathcal{F}(G, p; \mathbb{C})$ is "invariant" for the "shift" operators and their inverses.

The 2D grid framework $C_{\mathbb{Z}^2}$

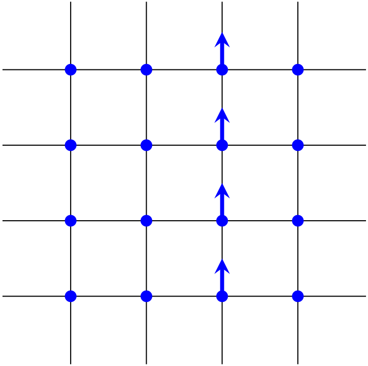


The 2D grid framework $C_{\mathbb{Z}^2}$



Check translation invariance of $\mathcal{F}(C_{\mathbb{Z}^2};\mathbb{C})$

The 2D grid framework $C_{\mathbb{Z}^2}$



Let Γ be a discrete abelian group. A Γ-symmetric graph is a pair (G, ∂) where $G = (V, E)$ is a simple undirected graph and

$$\partial : \Gamma \to \mathrm{Aut}(G) : v \mapsto \gamma v$$

is a group homomorphism. The orbit of a vertex $v \in V$ under ∂ is the set $[v] = \{\gamma v : \gamma \in \Gamma\}$.

We write V_0 the set of vertex orbits and similarly we get the set E_0 of all edge orbits. Assume that V_0, E_0 are finite, and that ∂ acts *freely* on the vertices and edges of G.

For each vertex orbit $[v] \in V_0$, choose a representative vertex $\tilde{v} \in [v]$ and denote the set of all such representatives by $\tilde{V}_0$.

The quotient graph $G_0 = (V_0, E_0)$ is the multigraph, that satisfies

$$[e] = [v][w] \text{ if there exists } \gamma \in \Gamma, \text{ such that } \tilde{v}(\gamma\tilde{w}) \in [e].$$

Fix an orientation on the edges of G_0, so that each edge in G_0 is an ordered pair $[e] = ([v], [w])$. Then for each $[e] = ([v], [w])$ there exists a *unique* $\gamma \in \Gamma$ such that $\tilde{v}(\gamma\tilde{w}) \in [e]$.

This group element is referred to as the gain on the directed edge $[e]$ and is denoted $\psi_{[e]}$. A gain graph for the Γ-symmetric graph (G, ∂) is any edge-labelled directed multigraph obtained from the quotient graph G_0 in this way.

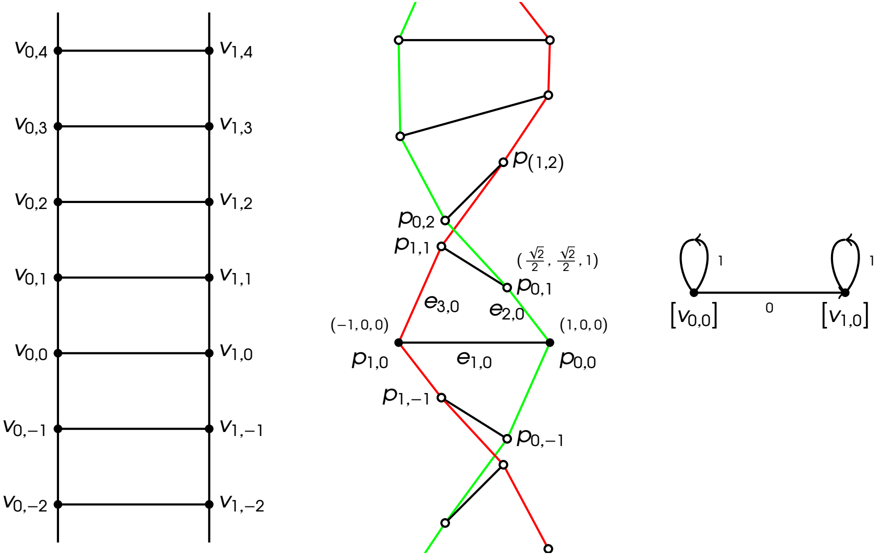


Figure: The double helix framework $\mathcal{G}_{dh}$ (center), underlying graph (left) and gain graph (right).

Denote by $\mathrm{Isom}(X)$ the group of affine isometries of X. A Γ-symmetric framework is a tuple $\mathcal{G} = (G, p, \partial, \tau)$ where $\tau : \Gamma \to \mathrm{Isom}(X)$ is a group homomorphism, (G, ∂) is a Γ-symmetric graph and (G, p) is a framework with the property that,

$$p(\tilde{v}, \gamma) = \tau(\gamma)p(\tilde{v}, 1_\Gamma), \qquad \text{for all } \gamma \in \Gamma \text{ and all } \tilde{v} \in \tilde{V}_0.$$

For each $\gamma \in \Gamma$, let $d\tau(\gamma)$ denote the linear isometry on X that is uniquely defined by the linear part of the affine isometry $\tau(\gamma)$. We denote by $\tilde{\tau}$ the unitary representation

$$\tilde{\tau}(\gamma)(x) = (d\tau(\gamma)x_{[v]})_{[v]\in V_0},$$

for all $x = (x_{[v]})_{[v]\in V_0} \in C^{d|V_0|}$.

The periodic case:

Suppose that $\Gamma = \mathbb{Z}^d$ and that τ is a group of translations. Then $d\tau$ is trivial and the rigidity matrix satisfies

$$R(G,p)U_z = W_z R(G,p),$$

the U_z, W_z are the appropriate shifts

$$(U_z f)(z') = f(z' - z), \quad \forall\, f \in \ell^2(\mathbb{Z}^d, C^{d|V_0|}),$$

$$(W_z g)(z') = g(z' - z), \quad \forall\, g \in \ell^2(\mathbb{Z}^d, \mathbb{C}^{|E_0|}).$$

Hence we can use tools from Fourier ananysis to get:

Theorem (Owen - Power, 2011)

Let $\mathcal{G} = (G, p)$ be a periodic framework with rigidity operator $R(G, p)$. Then there exists a function $\Phi \in L^\infty(\mathbb{T}^d, M_{|E_0|, d|V_0|})$ such that

$$\mathcal{F}_E R(G, p) \mathcal{F}_V = M_\Phi,$$

where $\mathcal{F}_*$ is the associated Fourier transform.

The *Rigid Unit Mode (RUM) spectrum* of $\mathcal{G}$ is the set

$$\Omega(\mathcal{G}) = \{\omega \in \mathbb{T}^d : \ker \Phi_{\mathcal{G}}(\overline{\omega}) \neq \{0\}\}.$$

Corollary (Owen-Power, 2011)

Let $a \in \mathbb{C}^{d|V_0|}$ be in the nullspace of $\Phi_{\mathcal{G}}(\overline{\omega})$. Then the function

$$u : \mathbb{Z}^n \to \mathbb{C}^{d|V_0|} : k \to \omega^k a$$

defines an ω-factor periodic infinitesimal flex.

In the general (abelian) symmetric case, the rigidity matrix satisfies

$$R(G,p)U_{\gamma,\tilde{\tau}} = W_{\gamma}R(G,p),$$

where $U_{\gamma,\tilde{\tau}}$ is the twisted shift

$$(U_{\gamma,\tilde{\tau}}f)(\gamma') = \tilde{\tau}(\gamma)f(\gamma'-\gamma), \quad \forall f \in \ell^2(\Gamma, \mathbb{C}^{d|V_0|}).$$

Check that the unitary representation $\tilde{\tau}$ gives rise to a unitary operator $T_{\tilde{\tau}} \in B(\ell^2(\Gamma, C^{d|V_0|}))$ where,

$$(T_{\tilde{\tau}} f)(\gamma) = \tilde{\tau}(-\gamma) f(\gamma), \quad \forall\, f \in \ell^2(\Gamma, \mathbb{C}^{d|V_0|}).$$

Theorem (K. - Kitson - McCarthy (2021))

Let Γ be a discrete abelian group and $\mathcal{G} = (G, \varphi, \partial, \tau)$ be a Γ-symmetric framework. Then

$$\mathcal{F}_E R(G,p) T_{\tilde{\tau}} \mathcal{F}_V = M_\Phi,$$

for some $\Phi \in L^\infty(\hat{\Gamma}, B(\mathbb{C}^{d|V_0|}, \mathbb{C}^{|E_0|}))$. Moreover, if $a \in \ker \Phi(\overline{\chi})$ then the function the function

$$u : \Gamma \to \mathbb{C}^{d\,|V_0|} : \chi \to \chi(\gamma) d\tau(\gamma) a.$$

defines a twisted phase periodic infinitesimal flex.

Back to the periodic case:

Let $\mathcal{G} = (G, p, \mathcal{T})$ be a periodic framework in $\mathbb{R}^d$. The **transfer function** $\Phi_{\mathcal{G}}(z)$ is the $|E_0| \times d\,|V_0|$ matrix over the Laurent polynomial ring.

Let

$$p(e) = p(v, k) - p(w, l)$$

be the vector associated with the edge $e = (v, k)(w, l)$.

The row for an edge $e = (v, k)(w, l)$ with $v \neq w$ takes the form

$$\begin{array}{cccccccccccc} & & & & v & & & & w & & & \\ e & [\;0 & \dots & 0 & p(e)z^{-k} & 0 & \dots & 0 & -p(e)z^{-l} & 0 & \dots & 0\;] \end{array}$$

while if $v = w$ it takes the form

$$\begin{array}{cccccccc} & & & & v & & & \\ e & [\;0 & \dots & 0 & p(e)(z^{-k} - z^{-l}) & 0 & \dots & 0\;]. \end{array}$$

The geometric flex spectrum of $\mathcal{G}$ is the set

$$\Gamma(\mathcal{G}) = \{\omega \in \mathbb{C}_*^d : \ker \Phi_{\mathcal{G}}(\overline{\omega}) \neq \{0\}\}.$$

Let $a \in \mathbb{C}^{d|V_0|}$ be in the nullspace of $\Phi_{\mathcal{G}}(\omega^{-1})$. Then the function

$$u : \mathbb{Z}^d \to \mathbb{C}^{d|V_0|} : k \to \omega^k a$$

defines a factor-periodic velocity field which is an infinitesimal flex.

Let $\mathcal{G} = (G, p, \mathcal{T})$ be a periodic framework in $\mathbb{R}^d$. A vectorial *pg*-sequence for $\mathcal{G}$, for this periodic structure, with geometric index $\omega \in \mathbb{C}_*^d$, is a velocity field $u_{\omega,h} : \mathbb{Z}^d \to \mathbb{C}^{d|V_0|}$ of the form

$$u_{\omega,h} : k \to \omega^k h(k)$$

where $h(z)$ is a vector-valued polynomial in $\mathbb{C}[z] \otimes \mathbb{C}^{d|V_0|}$.

Theorem (K. - Power, 2019)

Let $\mathcal{G} = (G, p, \mathcal{T})$ be a periodic framework in $\mathbb{R}^d$. Then $\mathcal{F}(\mathcal{G}; \mathbb{C})$ is the closed linear span of *pg*-sequences $u_{\omega,h}$ in $\mathcal{F}(\mathcal{G}; \mathbb{C})$.

The ``scalar case'' was proved by Lefranc in 1958.

Define the bilinear pairing

$$\langle \cdot, \cdot \rangle : \mathbb{C}(z) \times C(\mathbb{Z}^d) \to \mathbb{C} : \langle \sum_k a_k z^k, (u_k)_{k \in \mathbb{Z}^d} \rangle = \sum_k a_k u_k.$$

Similarly, for $p = (p_i) \in \mathbb{C}(z) \otimes \mathbb{C}^r$ and $u = (u_i) \in C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ we have the corresponding pairing

$$\langle \cdot, \cdot \rangle : (\mathbb{C}(z) \otimes \mathbb{C}^r) \times (C(\mathbb{Z}^d) \otimes \mathbb{C}^r) \to \mathbb{C} : \langle p, u \rangle = \langle (p_i), (u_i) \rangle = \sum_{i=1}^{r} \langle p_i, u_i \rangle.$$

For a subspace A of $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ we write

$$A^\perp = \{p \in \mathbb{C}(z) \otimes \mathbb{C}^r : \langle p, u \rangle = 0, \text{ for all } u \in A\}.$$

Similarly for a subspace B of $\mathbb{C}(z) \otimes \mathbb{C}^r$ we write $B^\perp$ for the annihilator in $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ with respect to the same pairing.

Lemma 1

Let A be a closed subspace of $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ and let M be a closed subspace of $\mathbb{C}(z) \otimes \mathbb{C}^r$. Then $A = (A^\perp)^\perp$ and $M = (M^\perp)^\perp$.

Lemma 2

A closed subspace A in $C(\mathbb{Z}^d) \otimes \mathbb{C}^r$ is shift invariant if and only if $A^\perp$ is a $\mathbb{C}(z)$-submodule of the module $\mathbb{C}(z) \otimes \mathbb{C}^r$.

An (interesting) example : $(M(\mathcal{G}))^\perp = \mathcal{F}(\mathcal{G}; \mathbb{C})$. where

$$M(\mathcal{G}) = \mathbb{C}(z)p_1(z) + \cdots + \mathbb{C}(z)p_m(z)$$

and $p_1(z), \ldots, p_m(z)$ are the vector-valued functions given by the rows of the transfer function.

...and then commutative algebra

Theorem (Lasker - Noether)

Every submodule of a finitely generated module over a Noetherian ring is a finite intersection of primary submodules.

Why do we need primary submodules?

Let Q be a finitely generated submodule of $\mathbb{C}[z] \otimes \mathbb{C}^r$ and $S[Q] = \mathbb{C}[[z]] \cdot Q$ be the corresponding $\mathbb{C}[[z]]$-module in $\mathbb{C}[[z]] \otimes \mathbb{C}^r$.

Proposition

Let Q be a primary submodule in $\mathbb{C}[z] \otimes \mathbb{C}^r$ with associated root 0. Then $Q = S[Q] \cap (\mathbb{C}[z] \otimes \mathbb{C}^r)$.

Theorem (K.-Power, 2021)

The following statements are equivalent for a periodic framework $\mathcal{G}$ in $\mathbb{R}^d$.

(i) $\mathcal{G}$ is first-order rigid.

(ii) $\mathcal{G}$ is exponentially rigid.

(iii) For a given periodic structure basis, there are no nontrivial factor-periodic flexes or nontrivial flexible lattice periodic flexes.

Theorem (K.-Power, 2021)

Let $\mathcal{G}$ be a periodic framework in $\mathbb{R}^d$ with a given periodic structure basis and associated geometric flex spectrum $\Gamma(\mathcal{G})$. Then the following statements are equivalent.

(i) $F(\mathcal{G}, \mathbb{C})$ is finite-dimensional.

(ii) $\Gamma(\mathcal{G})$ is a finite set.

Figure: Thank you!