Braced sphere triangulations and rigidity

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EPSRC
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Outline

Introduction to rigidity

Survey of recent work in normed spaces

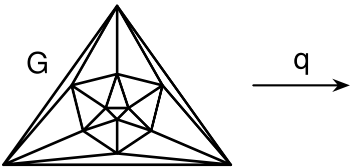
A new result for doubly braced triangulations

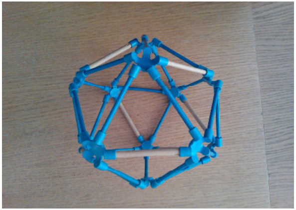
X finite dimensional real normed linear space.

$G = (V, E)$ simple undirected graph.

$q \in X^V$, $q = (q_v)_{v \in V}$.

The pair (G, q) is called a bar-joint framework in X.





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Target theorem: A graph is rigid in a given space for almost all placements iff it satisfies the following purely combinatorial (and easily verifiable) conditions...

An infinitesimal flex of (G, q) is a vector $u \in X^V$, $u = (u_v)_{v \in V}$, such that for every edge $vw \in E$,

$$\lim_{t \to 0} \frac{1}{t} \left(\|q_v + tu_v - (q_w + tu_w)\| - \|q_v - q_w\| \right) = 0.$$

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$\mathcal{T}(G,q)$ is the linear space of trivial infinitesimal flexes of (G,q).

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Theorem (Folklore)

G is minimally rigid in $\mathbb{R}$ iff it is a tree.

$G = (V, E)$ is (k, l)-sparse if $|E'| \leq k|V'| - l$ for "all" subgraphs (V', E').

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G is (k, l)-tight if it is (k, l)-sparse and $|E| = k|V| - l$.

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Theorem (Maxwell 1864)

If G is minimally rigid in ℓ_2^d then G is $\left(d,\frac{d(d+1)}{2}\right)$-tight.

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G is minimally rigid in ℓ_2^2 iff it is $(2, 3)$-tight.

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Longstanding open problem: Find a combinatorial characterisation of rigidity in ℓ_2^d, where $d \geq 3$.

Theorem (Gluck 1975, Whiteley 1990)

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Triangulations of compact surfaces without boundary are rigid in ℓ_2^3.

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Theorem (Cruickshank, K., Power 2019)

Triangulations of a torus with a single hole are minimally rigid in ℓ_2^3 iff they are $(3, 6)$-tight.

Workshop on Geometric Rigidity, Lancaster 2015

The rigidity matrix $R(G,q)$ takes the form,

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & v & & & & w & \end{array} \\
vw & \left(\begin{array}{ccccccccc}
 & & & \vdots & & & & \vdots & \\
0 & \cdots & 0 & \varphi(v,w) & 0 & \cdots & 0 & -\varphi(v,w) & 0 \ \cdots \\
 & & & \vdots & & & & \vdots &
\end{array}\right)
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where $\varphi(v,w)$ is the (unique) support functional for $q_v - q_w \in X$.

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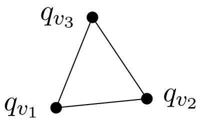
where $\varphi(v,w)$ is the (unique) support functional for $q_v - q_w \in X$.

- $\mathcal{F}(G,q) = \ker R(G,q)$.
- $\mathcal{T}(G,q)$ depends on $\mathrm{Isom}(X)$.

Example

Let $X = \ell_2^2$ and $G = K_3$.

Write $q_v = (q_v^x, q_v^y)$ for all $v \in V$.



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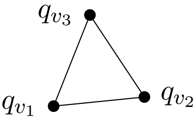

$R(G, q)$ is a $|E| \times 2|V|$-matrix:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \scriptstyle (v_1;x) & \scriptstyle (v_1;y) & \scriptstyle (v_2;x) & \scriptstyle (v_2;y) & \scriptstyle (v_3;x) & \scriptstyle (v_3;y) \end{array} \\
\begin{array}{c} v_1v_2 \\ v_1v_3 \\ v_2v_3 \end{array} &
\begin{pmatrix}
q_{v_1}^x - q_{v_2}^x & q_{v_1}^y - q_{v_2}^y & q_{v_2}^x - q_{v_1}^x & q_{v_2}^y - q_{v_1}^y & 0 & 0 \\
q_{v_1}^x - q_{v_3}^x & q_{v_1}^y - q_{v_3}^y & 0 & 0 & q_{v_3}^x - q_{v_1}^x & q_{v_3}^y - q_{v_1}^y \\
0 & 0 & q_{v_2}^x - q_{v_3}^x & q_{v_2}^y - q_{v_3}^y & q_{v_3}^x - q_{v_2}^x & q_{v_3}^y - q_{v_2}^y
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Theorem (K., Power 2014)

Let $p \in [1, \infty]$, $p \neq 2$. Then G is minimally rigid in ℓ_p^2 iff it is $(2, 2)$-tight.

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Conjecture

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Theorem (Dewar, K., Nixon 2021)

Triangulations of the projective plane are minimally rigid in ℓ_p^3, *for all* $p \in (1, \infty)$, $p \neq 2$.

Theorem (K. 2015)

Let X be a normed plane with a polygonal unit ball. Then G is minimally rigid in X iff it is $(2,2)$-tight.

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Theorem (K., Levene 2020)

Let $p \in [1,\infty]$, $p \neq 2$.

(i) *If G is minimally rigid in $(\mathcal{M}_n(\mathbb{R}), \|\cdot\|_{c_p})$ then it is $(n^2, 2n^2-n)$-tight.*

(ii) *If G is minimally rigid in $(\mathcal{H}_n(\mathbb{R}), \|\cdot\|_{c_p})$ then it is $(\frac{1}{2}n(n+1), n^2)$-tight.*

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Geometric constraint systems, Lancaster 2019

Joint work with James Cruickshank (NUI Galway), Eleftherios Kastis (Lancaster) and Bernd Schulze (Lancaster).

See our recent preprint:

Cruickshank, Kastis, Kitson, Schulze. Braced triangulations and rigidity. arxiv.org/abs/2107.03829

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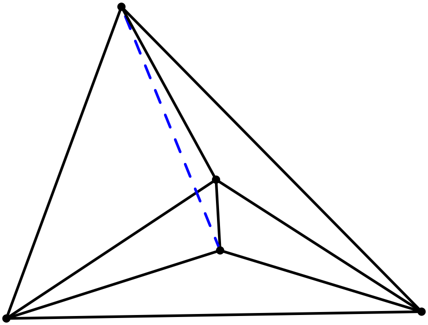
Theorem

An irreducible braced sphere triangulation with b braces has at most $11b - 4$ vertices.

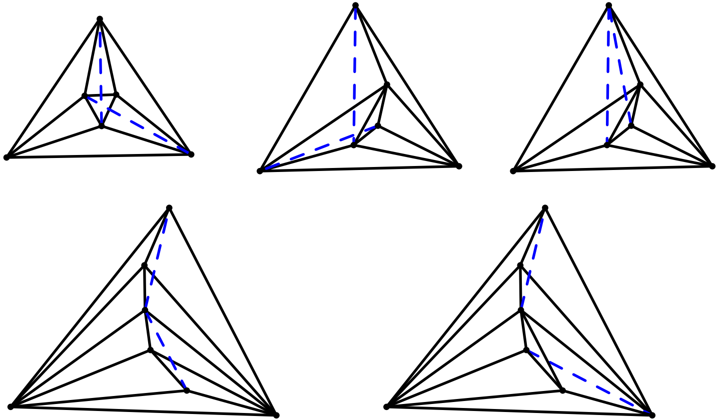
Theorem

There is exactly one irreducible unibraced sphere triangulation.



Theorem

There are exactly five irreducible doubly braced sphere triangulations.



For $p \in (1, \infty)$ define the mixed norm on $\mathbb{R}^3$ by,

$$\|(x, y, z)\|_{2,p} = ((x^2 + y^2)^{\frac{p}{2}} + |z|^p)^{\frac{1}{p}}.$$

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Conjecture

Doubly braced triangulations are minimally rigid in $(\mathcal{H}_2(\mathbb{R}), \|\cdot\|_{c_p})$, *for all* $p \in [1, \infty]$, $p \neq 2$.

Proof.

Every doubly braced sphere triangulation can be constructed from one of 5 irreducibles by “vertex splitting”.

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Vertex splitting preserves minimal rigidity in smooth and strictly convex normed spaces. $\square$

Thank you, Steve!

Congratulations and enjoy your retirement!