Interpolation and Duality on RKHS spaces

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Setting

A Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H}$ on a set X is a Hilbert space of functions on X s.t. point evaluations are cnts. So

$$\forall z \in X \ \exists k_z \in \mathcal{H} \ s.t. \quad h(z) = \langle h, k_z \rangle.$$

The function $K(w,z) = \langle k_w, k_z \rangle$ on $X \times X$ is the kernel.

An RKHS on a ball $\mathbb{B}_d \subset \mathbb{C}^d$ is unitarily invariant if

$$K(w,z) = \sum_{n \geq 0} a_n \langle w, z \rangle^n \quad a_0 = 1, \ a_n > 0.$$

It is regular if $\displaystyle\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1.$ Regularity forces $\sum_{n\geq 0} a_n z^n$ to have radius of convergence 1.

Moreover the norm is invariant under unitaries acting on $\mathbb{B}_d$.
The monomials $\{z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d} : \alpha \in \mathbb{N}_0^d\}$ form orthogonal basis.
So $\mathcal{H} \subset \mathrm{Hol}(\mathbb{B}_d)$ consists of analytic functions.

Examples

- Hardy space H^2 on $\mathbb{D}$. $K(w,z) = \frac{1}{1-w\bar{z}}$.
  $H^2 = \{f = \sum_{n\geq 0} a_n z^n : \|f\|_2^2 = \sum_{n\geq 0} |a_n|^2 < \infty\}$.
- Drury-Arveson space H^2_d for $2 \leq d \leq \infty$. $K(w,z) = \frac{1}{1-\langle w,z\rangle}$.
- Dirichlet space
  $\mathcal{D} = \{f = \sum_{n\geq 0} a_n z^n : \|f\|_{\mathcal{D}}^2 = \sum_{n\geq 0}(n+1)|a_n|^2 < \infty\}$.
- Bergman space
  $L^2_a(\mathbb{B}_d) = \{f \in \mathrm{Hol}(\mathbb{B}_d) : \int_{\mathbb{B}_d} |f(z)|^2\, dA < \infty\}$.
- Hardy space on balls:
  $H^2(\mathbb{B}_d) = \{f \in \mathrm{Hol}(\mathbb{B}_d) : \int_{\partial\mathbb{B}_d} |f(z)|^2\, d\sigma < \infty\}$.
- $H_s(\mathbb{B}_d)$, $s \in \mathbb{R}$, has reproducing kernel
  $K(w,z) = \sum_{n=0}^{\infty}(n+1)^s\langle w,z\rangle^n$.
  Here, $s = 0$ corresponds to the Drury–Arveson space and $s = -1$ yields the Dirichlet space.

Multipliers: $\mathrm{Mult}(\mathcal{H}) = \{f : f\mathcal{H} \subset \mathcal{H}\}$.
Then M_f is bounded on $\mathcal{H}$, and

$$\|f\|_\infty \leq \|f\| := \|M_f\|.$$

Here, the coordinates: z_i, $1 \leq i \leq d$, are multipliers.
$A(\mathcal{H})$ denotes the norm closure of the polynomials in $\mathrm{Mult}(\mathcal{H})$.
Note that $A(\mathcal{H}) \subset C(\overline{\mathbb{B}_d})$.

Examples

- Hardy space $A(H^2) = A(\mathbb{D})$ and $\mathrm{Mult}(H^2) = H^\infty(\mathbb{D})$.
- DA-space $M_{z_1}, \dots, M_{z_d}$ is the universal model for a commuting row contraction. It plays the role in the unilateral shift for this multivariable context.
- Bergman spaces and Hardy spaces on balls: $A(H^2(\mathbb{B}_d)) = A(L^2_a(\mathbb{B}_d)) = A(\mathbb{B}_d)$ and $\mathrm{Mult}(H^2(\mathbb{B}_d)) = \mathrm{Mult}(L^2_a(\mathbb{B}_d)) = H^\infty(\mathbb{B}_d)$.

Interpolation Problems

1. Peak Interpolation.

Given a compact $E \subset \partial\mathbb{B}_d$ and $h \in C(E)$ with $\|h\|_\infty = 1$,
is there an $f \in A(\mathcal{H})$ s.t. $f|E = h$?
(a) Can one obtain $\|f\| < 1 + \varepsilon$ for any $\varepsilon > 0$?
(b) Can one obtain $\|f\| = 1$?
(c) Can one obtain $|f(z)| < 1$ for $z \in \overline{\mathbb{B}_d} \setminus E$?

E must be "small".
For $A(\mathbb{D})$, Rudin-Carleson yields (b), (c) if $|E| = 0$.
For $A(\mathbb{B}_d)$, $d \geq 2$, Bishop and others get (b), (c) if E is totally null.
For $A(H^2_d)$, Clouâtre-D get (a), (c) if E is Mult(H^2_d)-totally null.

The multiplier result is more delicate than for uniform algebras because the norm is greater than the sup norm. So $(c) \not\Rightarrow (b)$.

2. Pick or Nevanlinna-Pick Interpolation.

Given $F = \{z_1, \dots, z_n\} \subset \mathbb{B}_d$, $w_1, \dots, w_n \in \mathbb{C}$,
is there an $f \in A(\mathcal{H})$ s.t.

$$f(z_i) = w_i, \quad 1 \le i \le n \quad \text{and} \quad \|f\| \le 1?$$

In certain spaces (Pick spaces), this holds iff

$$\big[k(z_i, z_j)(1 - w_i\overline{w_j})\big] \ge 0.$$

In all RKHS, this is a necessary condition.

Many of our examples are complete Pick spaces (same works for matrix valued interpolation), but some, like all Bergmann spaces and Hardy spaces for $d \ge 2$, are not.
This problem has been extensively studied in RKHSs.

3. Pick and Peak Interpolation.

Given $F = \{z_1, \dots, z_n\} \subset \mathbb{B}_d$, $E \subset \partial\mathbb{B}_d$,
$w_1, \dots, w_n \in \mathbb{C}$, and $h \in C(E)$ with $\|h\|_\infty = 1$,
is there an $f \in A(\mathcal{H})$ s.t. $f(z_i) = w_i$, $1 \le i \le n$ and $f|E = h$
and $\|f\| < 1 + \varepsilon$ for any $\varepsilon > 0$?

Izzo: yes for uniform algebras if the Pick and Peak problems can be solved separately.
The $\varepsilon > 0$ is necessary since some Pick data has unique solution.

Duality

$A(\mathbb{D})^* = H^\infty_* \oplus_1 \mathrm{Sing}(\mathbb{T})$

$A(\mathbb{B}_d)^* = H^\infty(\mathbb{B}_d)_* \oplus_1 \mathrm{TS}(\partial\mathbb{B}_d)$ Henkin, Cole, Range, Valskii

$A(H^2_d)^* = \mathrm{Mult}(H^2_d)_* \oplus_1 \mathrm{TS}(H^2_d)$ Clouâtre-D

Definition

A measure μ on $\partial\mathbb{B}_d$ is *Mult*(H)-Henkin (Hen(H)) if it extends to a weak-$*$ cnts. functional on Mult(H).

A measure ν is *Mult*(H)-totally singular (TS(H)) if

$$\nu \perp \mu \ \forall \mu \in \mathrm{Hen}(H).$$

A set E is *Mult*(H)-totally null (TN(H)) if $\mu(E) = 0 \ \forall \mu \in \mathrm{Hen}(H)$.

Hen(H) and TS(H) are complementary closed bands.

Theorem

$$A(H)^* \simeq \mathrm{Mult}(H)_* \oplus_1 \mathrm{TS}(H) \quad \textit{completely isometrically.}$$

Corollary

$$A(H)^{**} \simeq \mathrm{Mult}(H) \oplus \mathcal{W}_s \quad \text{completely isometrically}$$

where $\mathcal{W}_s = \mathrm{TS}(H)^*$ is an abelian von Neumann algebra.

Idea: $0 \to K(\mathcal{H}) \to C^*(A(H)) \to C(\partial\mathbb{B}_d) \to 0.$
$\varphi \in A(H)^*$. Extend to $\tilde{\varphi} \in C^*(A(H))^*$ by HB. $\|\varphi\| = \|\tilde{\varphi}\|_{cb}$.
Wittstock's theorem: $\exists V, W$ isometries, $*$-reps π_a and π_s s.t.

$$\tilde{\varphi} = V^*(\underbrace{\mathrm{id}^{(\alpha)} \oplus \pi_a}_{\text{Henkin}} \oplus \underbrace{\pi_s}_{\text{TS}})W.$$

Back to Interpolation

If $F \subset \mathbb{B}_d$, let $\mathrm{Mult}(H)|_F = \{f|_F : \varphi \in \mathrm{Mult}(H)\}$ with
$$\|f|_F\| = \inf\{\|g\| : g|_F = f|_F\}.$$

Theorem (Pick-peak)

H regular unitarily invariant.
$F \subset \mathbb{B}_d$ finite; $E \subset \partial\mathbb{B}_d$ closed, $\mathrm{Mult}(H)$ totally null.
Then the restriction map

$$\Phi_{F\cup E} : A(H) \to \mathrm{Mult}(H)|_F \oplus C(E), \quad \Phi_{F\cup E}(f) = f|_F \oplus f|_E$$

is a complete quotient map.

Idea: show that

$$\Phi^* : (\mathrm{Mult}(H)|_F)^* \oplus_1 M(E) \to \mathrm{Mult}(H)_* \oplus_1 TS(\mathrm{Mult}(H))$$

is a complete isometry.

Corollary

If H is a complete NP space,
$F = \{z_1, \dots, z_k\} \subset \mathbb{B}_d$; $E \subset \partial\mathbb{B}_d$ *closed,* $\mathrm{Mult}(H)$ *totally null*
$W_1, \dots W_k \in \mathcal{M}_n$, $h \in \mathcal{M}_n(C(E))$ *with* $\|h\|_\infty \le 1$, $\varepsilon > 0$ *and*

$$\left[K(z_i, z_j)(I_n - W_i W_j^*)\right] \ge 0$$

then $\exists f \in \mathcal{M}_n(A(H))$ *such that*

$$f(z_i) = W_i, \quad f|_E = h \quad \textit{and} \quad \|f\|_{\mathcal{M}_n(\mathrm{Mult}(H))} < 1 + \varepsilon.$$

Remark

The $\varepsilon > 0$ is necessary even in the scalar case for $A(\mathbb{D})$. When the matrix $\left[\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j}\right]$ is singular, the interpolating function on F of norm 1 is unique. So for most choices of h, the ε is required.

Theorem (peak)

H regular unitarily invariant. $E \subset \partial\mathbb{B}_d$ closed, $\mathrm{Mult}(H)$ totally null. Let $g \in \mathcal{M}_n(C(E))$. Then $\exists f \in \mathcal{M}_n(A(H))$ s.t.

$$f|_E = g, \quad \|f\| = \|g\| \quad \text{and} \quad \|f(z)\| < \|g\| \text{ for } z \in \overline{\mathbb{B}_d} \setminus E.$$

Sketch. $I(E) = \{f \in A(H) : f|_E = 0\} \triangleleft A(H)$.
Show that $I(E)^\perp = M(E) = \mathrm{TS}(E)$. Thus

$$A(H)^* \simeq \big(\mathrm{Mult}(H)_* \oplus_1 \mathrm{TS}(E^c)\big) \oplus_1 \mathrm{TS}(E).$$

$\therefore I(E)$ is an M-ideal. $\Rightarrow \Phi_E$ takes $\overline{b_1(A(H))}$ onto $\overline{b_1(C(E))}$.

Build $h \in A(H)$ with $h|_E = 1$, $\|h\| = 1$ and $|h(z)| < 1$ for $z \in \overline{\mathbb{B}_d} \setminus E$. Multiply interpolant by h.

★: linear selection.

Interpolating sets

Closed $E \subset \partial\mathbb{B}_d$ is interpolating (I) if Φ_E is surjective (no norm).

Theorem

Suppose that there are non-empty TN sets.
If E is an interpolating set, then E is $\mathrm{Mult}(H)$ totally null.

- then points are TN, so can apply peak theorem to $E \cup \{z\}$.
- restriction is automatically a complete surjection (uniformly).

Theorem

Suppose that there are no non-empty TN sets.
If E is an interpolating set, then E is finite.

Theorem

H regular unitarily invariant. $E \subset \partial\mathbb{B}_d$ closed. TFAE

1. E is TN
2. E is a PI set
3. E is a PPI set
4. E is a P set ($\exists h \in A(H)$, $\|h\| = 1 = h|_E$, $|h(z)| < 1\ \forall z \notin E$)

If there are non-empty TN sets, then

5. E is an interpolating set.

Existence of TN sets

- Most classical examples have non-trivial TN sets.
- If $\sum a_n < \infty$, then $A(H) \subseteq \mathrm{Mult}(H) \subset C(\overline{\mathbb{B}_d})$. There are no TN sets.
- There is an unbounded kernel with no TN sets.
- if TN sets exist, then interpolating sequences exist. $F = \{z_i : i \geq 1\} \subset \mathbb{B}_d$ is interpolating if the map $\mathrm{Mult}(H) \to l^\infty$, $f \to (f(z_i))$ is surjective.

Zero sets

In $A(\mathbb{D})$ and $A(\mathbb{B}_d)$, zero sets coincide with PI sets.

Proposition

If E is a TN set, then $\exists h \in A(H)$ s.t. $h^{-1}(0) = E$.

Proof. Let h peak exactly on E, and set $f = 1 - h$.

Theorem

In Drury-Arveson space, there exist zero sets which are not TN.

Based on a construction by Hartz showing that

$$\mathrm{Hen}(H^2_d) \supsetneq \mathrm{Hen}(H^2(\mathbb{B}_d)).$$

Happy Birthday, Steve!!