Measures on quantum logics of idempotent matrices over finite fields

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We consider the real Hilbert space $H$ with the scalar product $(\cdot, \cdot)$ and the quantum logic of all projections (i.e., self-adjoint idempotents) in the set of all linear operators on $H$. Two projections, $P$ and $Q$, are said to be orthogonal iff $PQ = QP = 0$. A function $\mu$ on the set of all projections with non-negative real values is said to be a measure iff

$$\mu\left(\sum_n P_n\right) = \sum_n \mu(P_n) \quad (*)$$

for any sequence or finite set $(P_n)$ of pairwise orthogonal projections.
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**Theorem 0** (A.M. Gleason, 1957). *Any measure $(\dim(H) \geq 3)$ admits the representation*

$$\mu(P) = tr(TP) \quad (1)$$

*where $T$ is a unique positive nuclear operator in $H$.***
The representation (1) fails if $\mu$ is a signed measure (with values in $(-\infty, +\infty)$) and $H$ is finite-dimensional. A construction of counterexamples uses existence of a function on the real line which is additive but not linear.
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The crucial case is $\dim(H) = 3$. The proof uses triples $P_1, P_2, P_3$ of pairwise orthogonal one-dimensional projections. Any such triple gives an equation

$$\mu(P_1) + \mu(P_2) + \mu(P_3) = \mu(Id)$$

connecting the values of $\mu$. 
Theorem 1 (DM, 1989). Let us consider the quantum logic $\mathcal{P}(H)$ of all continuous linear idempotents on $H$ (dim$(H) = \infty$). Then any finitely additive signed measure which is $\sigma$-additive on every $\sigma$-subalgebra of $\mathcal{P}(H)$ admits the representation
\[ \mu(P) = \text{tr}(TP) \] (1)
where $T$ is a unique nuclear operator in $H$. 
One of ideas of the proof is to use the classical Gleason theorem.
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Namely, we present the set $\mathcal{P}(H)$ of all idempotents as the union

$$\mathcal{P}(H) = \bigcup A \Pi_A(H)$$

where $\Pi_A(H)$ is the set of all idempotents which are self-adjoint w.r.t. a scalar product $A$. 

Thus, for any scalar product $A$, we have (1) with some $T = T_A$. So, the idea of the proof is to glue all the $T_A$ by using $\Pi_A(H) \cap \Pi_B(H) \neq \emptyset$. 

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Why we need extensions of the field of rationals?

The idempotent version of the Gleason theorem may be reformulated for Banach spaces (which do not have scalar products) and even for linear topological spaces.
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Unfortunately, in such general setting we cannot use the classical Gleason theorem. In fact, in the finite-dimensional case this theorem is not true for signed measures and in the infinite-dimensional case the topology is not defined by a scalar product.
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So, we need some analogs of the Gleason theorem which would be proved independently from the Gleason theorem for Hilbert spaces.

This brings us to the case of linear spaces over the field $\mathbb{Q}$ of rationals.
Theorem 2 (DM, 1995). Any $\mathbb{Q}$-valued measure on the set of all rational idempotent $n \times n$-matrices ($\dim(H) \geq 3$) admits the representation

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The first variant of the proof used computer calculations. Now I proved this theorem without computing by using some symmetrization construction.
How to prove without computing?

Let $I_0, I_1, I_2, \ldots, I_s$ be commuting involutions on the set of idempotents and $\nu = \nu(\epsilon_1, \ldots, \epsilon_s)$ defined by
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$$\nu(P) = \mu(P) + \sum_i \varepsilon_i \mu(I_i(P)) + \sum_{i<j} \varepsilon_i \varepsilon_j \mu(I_iI_j(P)) + \sum_{i<j<k} \varepsilon_i \varepsilon_j \varepsilon_k \mu(I_iI_jI_k(P)) + \ldots$$

where $\varepsilon_i = \pm 1$. 
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$$\nu(P) = \mu(P) + \sum_{i \leq s} \varepsilon_i \mu(I_i(P)) + \sum_{i < j \leq s} \varepsilon_i \varepsilon_j \mu(I_i I_j(P)) + \sum_{i < j < k \leq s} \varepsilon_i \varepsilon_j \varepsilon_k \mu(I_i I_j I_k(P)) + \ldots$$

where $\varepsilon_i = \pm 1$.

Every $\nu$ is either invariant or changes the sign of $I_i(P)$, and

$$\mu = 1/2^s \cdot \sum_{\varepsilon_i} \nu(\varepsilon_1, \ldots, \varepsilon_s).$$
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In this direction we have no success.
The Gleason theorem and a problem for finite fields

**Problem.** Let $\mathbb{F}_{p^k}$ be a finite field and $\mathcal{P}(\mathbb{F}_{p^k})$ the set of all idempotent $\mathbb{F}_{p^k}$-valued $n \times n$-matrices. Consider a function $\mu$ on $\mathcal{P}(\mathbb{F}_{p^k})$ with values in the prime field $\mathbb{F}_p$ satisfying

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\mu\left(\sum_{i \leq n} P_i\right) = \sum_{i \leq n} \mu(P_i)
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for any finite set of pairwise orthogonal projections $P_i$. Does $\mu$ admit an additive extension to the set of all $\mathbb{F}_{p^k}$-valued $n \times n$-matrices ($n \geq 3$) with trace belonging to $\mathbb{F}_p$?
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**Remark.** The case $n > 3$ reduces to the case $n = 3$. 
Theorem 3 (DM, 1995). The problem has the affirmative solution in the case when $F_{p^k}$ is prime (i.e., $k = 1$).
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The first proof used computer calculations. Now I proved this theorem without computing by using some symmetrization construction. (The construction of the proof of Theorem 2 is not convenient in this case since we cannot divide by 2.)
My students (Khomutova, Skvortsov) constructed programs for the cases $p^k = 8$, $p^k = 9$. The calculation gives the affirmative answer in these cases, too. Unfortunately, I cannot verify the computing, so I am not sure.
References


