Classification of Rosenbloom-Tsfasman Block Codes

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Rosenbloom-Tsfasman Block Spaces and Codes
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  Packing Radius and Perfect Codes
  Covering Radius and Quasi-Perfect Codes
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Poset Codes

[Brualdi, Graves, Lawrence, 1995]

Notation and Definitions

- $P$ a poset (in general, on $[n] = \{1, 2, \ldots, n\}$).
Poset Codes

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- $P$-weight on $\mathbb{F}^n$: 

$$\omega_P(v) = |\langle \text{supp}(v) \rangle|$$
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Poset Codes

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  \]
- where \( supp(v) = \{i; v_i \neq 0\} \)
- \( \langle supp(v) \rangle = \) ideal generated by \( supp(v) \)
Poset Codes

Particular Cases

- Particular case 1: Hamming metric
Poset Codes

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- $i \leq j \iff i = j$
Poset Codes

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Poset Codes

Particular Cases

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- Particular case 2: **Rosenbloom - Tsfasman**
  
  \[ 1 < 2 < \cdots < n \]
Poset Codes

**Particular Cases**

- **Particular case 1:** Hamming metric
- \( i \leq j \iff i = j \)
- **Particular case 2:** Rosenbloom - Tsfasman
- \( 1 < 2 < \cdots < n \)

\[
\omega_P(v) = \max\{i; v_i \neq 0\}
\]
Block Codes

[Feng, Xu and Hickernell, 2006]
Notation and Definitions

$\pi = (k_1, k_2, \ldots, k_n)$ a partition of $N \in \mathbb{N}$
Block Codes

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- $\pi = (k_1, k_2, \ldots, k_n)$ a partition of $N \in \mathbb{N}$
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- the $\pi$-metric on $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$:

$$
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  $$\omega_\pi(v) = |supp_\pi(v)|$$
- Where $v = v_1 + \cdots + v_n$, $v_i \in V_i$, and $supp_\pi(v) = \{ i; v_i \neq 0 \}$
Rosenbloom-Tsfasman Block Codes

Definition

- The **Rosenbloom-Tsfasman block weight** $w_{\pi}$ (**\(\pi\)-weight)** of a vector $0 \neq v = v_1 + v_2 + \ldots + v_n \in V$ is

$$w_{\pi}(v) = \max \{i : v_i \neq 0\};$$
Definition

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- The **Rosenbloom-Tsfasman block metric** (or simply $\pi$-metric):

  $$d_{\pi} (u, v) := w_{\pi} (u - v).$$
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  \[
  d_\pi (u, v) := w_\pi (u - v) .
  \]

- **The \( \pi\text{-Rosenbloom-Tsfasman space}** (or simply a \( \pi\text{-space} \)):
  \( (V, d_\pi) \)
Definition

The π-minimal distance of a linear code $C \subseteq V$:

$$d_{\pi} = d_{\pi}(C) := \min \{ d_{\pi}(c, c') : c \neq c' \in C \}$$
Minimal Weight

- **Definition**
  The \( \pi \)-minimal distance of a linear code \( C \subseteq V \):

\[
d_{\pi} = d_{\pi}(C) := \min \{ d_{\pi}(c, c') : c \neq c' \in C \}
\]

- **Definition**
  The \( \pi \)-minimal weight of \( C \):

\[
w_{\pi}(C) := \min \{ w_{\pi}(c) : c \neq 0 \in C \}.
\]
The generalized Rosenbloom-Tsfasman block weight (or $\pi$-weight) of $D \subseteq V$:

$$||D|| := \max \{ w_\pi(x) : x \in D \};$$
Generalized Weights - Wei

Definition
The generalized Rosenbloom-Tsfasman block weight (or $\pi$-weight) of $D \subseteq V$:

$$\|D\| := \max \{ w_\pi(x) : x \in D \} ;$$

Definition
The $r$-th Rosenbloom-Tsfasman block weight (or $r$-th $\pi$-weight) of a linear code $C \subseteq V$:

$$d_r = d_r(C) := \min \{ \|D\| : D \subseteq C, \dim(D) = r \}$$
Definition

The \( \pi \)-generalized weight hierarchy \((d_1, d_2, \ldots, d_k)\) of an \([N; k; d_1, \ldots, d_k]\) linear code.
Generalized Weights - Wei

Definition

The \( \pi \)-generalized weight hierarchy \((d_1, d_2, \ldots, d_k)\) of an \([N; k; d_1, \ldots, d_k]\) linear code.

Remark: The generalized weight hierarchy is increasing but not necessarily strict.
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Best looking generating matrix

$$ \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & (\text{Id}_{s_m \times s_m} | 0) \\ 0 & \cdots & 0 & 0 & \cdots & (\text{Id}_{s_{m-1} \times s_{m-1}} | 0) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & (\text{Id}_{s_1 \times s_1} | 0) & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} $$
Theorem (Extension of Ozen and Siap - 2004)

Let $C$ be an $[N; k]$ linear code. Then $C$ admits a generating matrix of the form

$$
\begin{pmatrix}
G_{sm}\pi_1 & \cdots & G_{sm}\pi_{t_1} & G_{sm}\pi_{t_1+1} & \cdots & G_{sm}\pi_{t_m-1} & G_{sm}\pi_{t_m-1+1} & \cdots \\
G_{sm-1}\pi_1 & \cdots & G_{sm-1}\pi_{t_1} & G_{sm-1}\pi_{t_1+1} & \cdots & G_{sm-1}\pi_{t_m-1} & 0 & \cdots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
G_{s1}\pi_1 & \cdots & G_{s1}\pi_{t_1} & 0 & \cdots & 0 & 0 & \cdots \\
\end{pmatrix}
$$

where $G_{st}$ is an $s \times t$ matrix with $s \leq t$, for each $1 \leq i \leq m$ the rank of $G_{si}\pi_{ti}$ is $s_i$ and $s_1 + s_2 + \ldots + s_m = k$. 
Definition
A matrix like the one in Theorem 7 is called a matrix of type \(((s_1, t_1), \ldots, (s_m, t_m))\).
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A matrix like the one in Theorem 7 is called a matrix of type 
\(((s_1, t_1), \ldots, (s_m, t_m)).\]

Definition
A matrix of type \(((s_1, t_1), \ldots, (s_m, t_m))\) is said to be in a canonical form if \(G_{s_i \pi_j} = 0\) for \(1 \leq j \leq t_i - 1\) and
\[
G_{s_i \pi_{t_i}} = \left( \text{Id}_{s_i \times s_i} \mid 0_{s_i \times (\pi_{t_i} - s_i)} \right)
\]
for every \(i = 1, \ldots, m.\)
Best looking form generating matrix = Canonical Form

\[
\begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & (\text{Id}_{s_m \times s_m} \mid \mathbf{0}) \\
0 & \cdots & 0 & 0 & \cdots & (\text{Id}_{s_{m-1} \times s_{m-1}} \mid \mathbf{0}) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & (\text{Id}_{s_1 \times s_1} \mid \mathbf{0}) & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
Definition

A linear isometry $T$ of the $\pi$-metric space $(V, d_\pi)$ is a linear transformation $T : V \rightarrow V$ that preserves $\pi$-metric: $d_\pi(T(u), T(v)) = d_\pi(u, v)$ for every $u, v \in V$. 

Firer, Panek, Muniz Classification of Rosenbloom-Tsfasman Block Codes
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$$d_\pi(T(u), T(v)) = d_\pi(u, v)$$ for every $u, v \in V$.

Definition
Two linear codes $C, C' \subseteq V$ are considered to be equivalent if there is a linear isometry $T : V \rightarrow V$ such that $T(C) = C'$. 
Theorem (Canonical Form)

Let $C$ be $[N; k; d_1, \ldots, d_k]$ linear code with generating matrix $G$ of type $((s_1, t_1), \ldots, (s_m, t_m))$. Then there is a linear isometry $T$ of $(V, d_{\pi})$ such that the linear code $T(C)$ has a generating matrix in a canonical form of type $((s_1, t_1), \ldots, (s_m, t_m))$.

Proof: We start with a generating matrix $G = (g_{ij})$ of type $((s_1, t_1), \ldots, (s_m, t_m))$ (existence ensured by previous Theorem).
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**Proof:** We start with a generating matrix $G = (g_{ij})$ of type $((s_1, t_1), \ldots, (s_m, t_m))$ (existence ensured by previous Theorem). $v_i$ is the vector defined by the $i$-th row of $G$ (counted from bottom to top). $\delta = \{v_1, \ldots, v_k\}$ is a base of $C$. 


Consider the \( \pi \)-decomposition

\[
\nu_i = \nu_{i1} + \ldots + \nu_{in},
\]

with \( \nu_{ij} \in V_j \)
Consider the $\pi$-decomposition

$$v_i = v_{i1} + \ldots + v_{in},$$

with $v_{ij} \in V_j$

Define

$$w_i = v_{il} = v_i - p_{(1,2,\ldots,t_l-1)}(v_i).$$

with $p_{(1,2,\ldots,t_l-1)}$ projection in the coordinate space.

**Notation:**
Consider the \( \pi \)-decomposition

\[
v_i = v_{i1} + \ldots + v_{in},
\]

with \( v_{ij} \in V_j \)

Define

\[
w_i = v_{it_l} = v_i - p_{(1,2,\ldots,t_l-1)}(v_i).
\]

with \( p_{(1,2,\ldots,t_l-1)} \) projection in the coordinate space.

**Notation:**

\[
v_L^i = v_{s_1+\ldots+s_L-1+i} \quad \text{and} \quad w_L^i = w_{s_1+\ldots+s_L-1+i}
\]
\[ \alpha_l = \{ v_l^1, v_l^2, \ldots, v_l^{s_l} \} \quad \text{and} \quad \beta_l = \{ w_l^1, w_l^2, \ldots, w_l^{s_l} \} \]

are both linearly independent.
\[ \alpha_l = \{ v^1_l, v^2_l, \ldots, v^{s_l}_l \} \quad \text{and} \quad \beta_l = \{ w^1_l, w^2_l, \ldots, w^{s_l}_l \} \]

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Let \( \tilde{\beta}_l = \{ u^1_l, \ldots, u^{\pi t_l - s_l}_l \} \) be such that \( \beta_l \cup \tilde{\beta}_l \) is a base for \( V_{t_l} \).
\[ \alpha_l = \{v^1_l, v^2_l, \ldots, v^{s_l}_l\} \text{ and } \beta_l = \{w^1_l, w^2_l, \ldots, w^{s_l}_l\} \]

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It follows that \( \alpha_l \cup \widetilde{\beta}_l \) is also a base for \( V_{t_l} \).
Classification of Rosenbloom-Tsfasman Block Codes

Fundamental Properties of $\pi$-Codes

\[ \alpha_l = \{ v_1^l, v_2^l, \ldots, v_s^l \} \] and \[ \beta_l = \{ w_1^l, w_2^l, \ldots, w_s^l \} \] are both linearly independent.

Let \( \tilde{\beta}_l = \{ u_1^l, \ldots, u_{\pi t_l - s_l}^l \} \) be such that \( \beta_l \cup \tilde{\beta}_l \) is a base for \( V_{t_l} \).

It follows that \( \alpha_l \cup \tilde{\beta}_l \) is also a base for \( V_{t_l} \).

\[ \beta = \bigcup_{l=1}^{m} (\beta_l \cup \tilde{\beta}_l) \quad \text{and} \quad \alpha = \bigcup_{l=1}^{m} (\alpha_l \cup \tilde{\beta}_l) \]

are bases of \( V_{t_1} \oplus V_{t_2} \oplus \ldots \oplus V_{t_m} \).
Let $\Lambda = \{t_1, \ldots, t_m\}$, $\gamma_i = \{e^1_i, \ldots, e^{\pi_i}_i\}$ is the canonical base for the block space $V_i$ and

$$\gamma = \bigcup_{i \notin \Lambda} \gamma_i,$$

we conclude that

$$\alpha \cup \gamma \text{ and } \beta \cup \gamma$$

are bases of $V$. 
\( L : V \rightarrow V \) the L.T. defined by

\[
L(v^i) = w^i, \quad L(u^i) = u^i \quad \text{and} \quad L(e^i) = e^i \quad \text{if} \ i \notin \Lambda.
\]

and

\( S(L(G)) \) is a matrix in canonical form.

[ -, Muniz, Panek, 2007 ensures both \( L \) and \( S \) are isometries.
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and

\[ L(G) = H \] is a matrix such that \( H_{s;\pi_j} = 0 \) for \( 1 \leq j \leq t_i - 1. \)
\( L : V \rightarrow V \) the L.T. defined by

\[
L(v_i^j) = w_i^j, \quad L(u_i^j) = u_i^j \quad \text{and} \quad L(e_i^j) = e_i^j \quad \text{if} \ i \notin \Lambda.
\]

and

\( L(G) = H \) is a matrix such that \( H_{s_i \pi_j} = 0 \) for \( 1 \leq j \leq t_i - 1 \).

\( S : V \rightarrow V \) the L.T. defined by

\[
S(w_i^j) = e_i^{t_j}, \quad S(u_i^j) = e_i^{s_j + j} \quad \text{and} \quad S(e_i^j) = e_i^j \quad \text{if} \ i \notin \Lambda.
\]
L : V → V the L.T. defined by

\[ L(v^i) = w^i, \quad L(u^i) = u^i \quad \text{and} \quad L(e^i) = e^i \text{ if } i \notin \Lambda. \]

and

\[ L(G) = H \text{ is a matrix such that } H_{s_i \pi_j} = 0 \text{ for } 1 \leq j \leq t_i - 1. \]

S : V → V the L.T. defined by

\[ S(w^i) = e^i_{t_i}, \quad S(u^i) = e^i_{s_t+j} \quad \text{and} \quad S(e^i) = e^i \text{ if } i \notin \Lambda. \]

\[ S(L(G)) \text{ is a matrix in canonical form.} \]
\( L : V \rightarrow V \) the L.T. defined by
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\[
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\]

\( S(L(G)) \) is a matrix in canonical form.

[ , Muniz, Panek, 2007 ensures both \( L \) and \( S \) are isometries.
**Theorem (Classification Theorem)**

*The canonical form of generating matrices classifies π-codes, in the sense that any equivalence class of codes contains a unique code that has a generating matrix in canonical form.*

**Proof.**

The existence of a code that has a generating matrix in canonical form was proved in the Canonical Form Theorem. The uniqueness of such a code follows from the fact that different matrices in canonical form generate codes with different π-weight hierarchy.
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Classification of Rosenbloom-Tsfasman Block Codes
Theorem (Generalized Singleton Bound)

Let $C$ be an $[N; k]$ linear code of type $((s_1, t_1), \ldots, (s_m, t_m))$. Then for each $1 \leq j \leq m$,

$$d_{s_1+s_2+\ldots+s_j}(C) \leq n - m + j$$

In particular, we have the Singleton Bound

$$d_1(C) \leq n - m + 1.$$
Theorem

Let $C$ be a code of type $((s_1, t_1), \ldots, (s_m, t_m))$. If $j = t_i$ for some $i$, then

$$A^{(r)}_j = \begin{cases} \min\{s_i, r\} \sum_{s=1}^{\min\{s_i, r\}} \frac{(q)^{\sigma_{i-1}}(q)s_i}{(q)^{r-s}(q)^{\sigma_{i-1}-r+s}(q)s(q)s_{i-s}q^{(r-s)s(\sigma_{i-1}-r-s)(s_i-s)}} \\ 0 \end{cases}$$

and $A^{(r)}_j = 0$ otherwise.
Packing Radius

Definition
The packing radius $R = R(C)$ of a linear code $C$ is

$$R := \max \left\{ r : B_{\pi} (c; r) \cap B_{\pi} (c'; r) = \emptyset \text{ for every } c \neq c' \in C \right\}.$$ 

Theorem
The packing radius of an $[N; k; d_{\pi}]$ linear $\pi$-code is

$$R(C) = d_{\pi} (C) - 1.$$
MDS Codes

Theorem

Let $(V, d_{\pi})$ be the $\pi$-space with $\pi_i = 1$ for every $1 \leq i \leq n$ and let $C$ be an $[N; k]$ $\pi$-code. Then $C$ is MDS iff $C$ is $\pi$-perfect.
Covering Radius

**Theorem**

An \([N; k; d_1]\) \(\pi\)-code \(C\) is quasi-perfect iff

\[
p(d_1+1, \ldots, n) (C) = V_{d_1+1} \oplus V_{d_1+2} \oplus \ldots \oplus V_n
\]

and \(p_{d_1} (C) \neq V_{d_1}\).
Theorem

Let $C$ be a $\pi$-code and $R(C) = d_\pi(C) - 1$ its packing radius. If $u \in V$ is a $\pi$-coset leader of a coset $C_v$ and $w_\pi(u) \leq R(C)$, then $u$ is the unique $\pi$-coset leader of $C_v$. 


L. Panek, M. Firer, H.K. Kim and J.Y. Hyun, Groups of