Construction of New Toric Quantum Codes

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Quantum Codes

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- **Toric Codes**: Kitaev proposes the class of *toric codes*, a subclass of the stabilizer quantum codes associated with the square lattice of torus $\mathbb{Z}^2$. 
Proposal

• **PROPOSAL:**
  It is possible to generate toric quantum codes by means of tessellations of square lattice of torus by translations of a determined fundamental region.
Stabilizer Codes
Stabilizer Codes

- Pauli group of \( n \) qubits:

\[
P_n = \pm \{I, \sigma_x, \sigma_y, \sigma_z\} \otimes n.
\]

\[
I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_x \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
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- **Properties:**
  - For each \( M \in \mathcal{P}_n \), we have \( M^2 = \pm I \);
  - If \( M^2 = I \), then \( M \) is Hermitian; if \( M^2 = -I \), then \( M \) is anti-Hermitian.
  - \( M, N \in \mathcal{P}_n \Rightarrow MN = NM \) or \( MN = -NM \).
Stabilizer Codes

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- A stabilizer code $C_S \subseteq \mathcal{H}$ associated to $S$ is the simultaneous eigenspace, with eigenvalue $+1$, comprising all elements of an Abelian subgroup $S$

$$C_S = \{|\psi\rangle; \quad M|\psi\rangle = |\psi\rangle \quad \forall M \in S\}.$$
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  - $E \in \mathcal{P}_n$ commute with every $M_i \in S \Rightarrow$ no error is detected.
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- The operators of $\mathcal{P}_n$ which commute with every $M_i \in S$ but do not belong to $S$, preserve the coding space $C_S$, by not acting trivially on it.

- The code distance is given by the least weight of $E \in \mathcal{P}_n$ such that $E$ commutes with every $M_i \in S$ but does not belong to $S$. 
Known Toric Quantum Codes
Kitaev’s Toric Codes

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  A_v = \sum_{j \in E_v} \sigma_x^j, \quad B_f = \sum_{j \in E_f} \sigma_z^j.
  \]
- \( C = \{ |\psi\rangle : A_v |\psi\rangle = |\psi\rangle, \quad B_f |\psi\rangle = |\psi\rangle \quad \forall \, v, f \} \).
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- $[[2m^2, 2, m]]$. 
Algebraic interpretation

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- The minimum distance of the code corresponds to the least number of edges in the dual lattice to be covered between the coset representatives, $d = m$. 
Bombin and Martin-Delgado’s Codes

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  \[
  X \quad \text{ } \quad X \\
  X \quad \text{ } \quad X \\
  X \quad \text{ } \quad X \\
  X \\
  X \\
  X \\
  X
  \]

- \( m \) two dimensional Lee spheres with radius \( r \) may be used to tessellate the torus \( \mathbb{Z}_m \times \mathbb{Z}_m \), where \( m = 2r^2 + 2r + 1 \) and \( r = 1, 2, \ldots \).
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  \[
  \begin{array}{c|c}
  \hline
  X & X \\
  \hline
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  \hline
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- The length of the code \(n\) is the number of edges of the Lee spheres and the minimum distance of the code is the radius \(r\) of Lee spheres.
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  ![Lattice Diagram]

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  - This system of lattices supplies codes with parameters \([d^2 + 1, 2, d]\) and keep the same properties of the original Kitaev’s code.
New Toric Quantum Codes
The lattice $\mathbb{Z}^2$ is generated by vectors $\nu_1 = (1, 0)$ and $\nu_2 = (0, 1)$, with fundamental region described by a square, with area equal to 1, and its corresponding quadratic form is $x^2 + y^2$. 
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- Lee spheres are a special type of polyominoes that was used to generate perfect classical codes and the class of $[[d^2 + 1, 2, d]]$ quantum codes.
Algebraic Approach

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- A systematic approach to tessellate the lattice \( \mathbb{Z}_m \times \mathbb{Z}_m \), is to determine the cosets, \( X \).
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- A systematic approach to tessellate the lattice $\mathbb{Z}_m \times \mathbb{Z}_m$, is to determine the cosets, $X$.
- $X \rightarrow (x, y) \in \mathbb{Z}_m \times \mathbb{Z}_m$ indicates the place of a polyomino.
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Algebraic Approach

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- To get a polyomino with area \( m \), the cardinality of \( \mathcal{A} \) must be \( m \), \(|\mathcal{A}| = m\).
Algebraic Approach

- The set of representatives of polyominoes, \( A \), corresponds to a classical lattice code.
- \( A \) is a subgroup of \((\mathbb{Z}_m \times \mathbb{Z}_m, +)\).
- To get a polyomino with area \( m \), the cardinality of \( A \) must be \( m, |A| = m \).
- The quadratic form of the lattice \( \mathbb{Z}_m \times \mathbb{Z}_m \) will be used to find the lattice vectors \((x, y) \in A\)

\[ x^2 + y^2 = m. \]
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\[
x^2 + y^2 = m.
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- Since we wish that \( |\mathcal{A}| = m \), we have:
  - \( \gcd(x, y) = 1 \Rightarrow \mathcal{A} = \langle (x, y) \rangle \).
  - \( \gcd(x, y) = \delta \neq 0, 1 \Rightarrow \mathcal{A} = \langle (x, y), (-y, x) \rangle \).
Parameters of the Codes

- Once the subspace given by the representatives is known, it is possible to choose the polyominoes that may tessellate the lattice.
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- Once the subspace given by the representatives is known, it is possible to choose the polyominoes that may tessellate the lattice.

- The quantum code associated to this tessellation is defined as:
  - \( n = \) number of edges of the polyomino = \( 2m \).
  - \( k = 2 \).
  - The minimum code distance is given by the shortest distance between two representatives of the polyominoes. Therefore

\[
d = d_M = |x| + |y|,
\]

where \( d_M \) is the *Mannhein distance*.

- \([2m, 2, d_M]\).
Shape of the Polyomino

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- It is possible to tessellate the same $\mathbb{Z}_m^2$ lattice by different polyominoes, without modifying the generated quantum code.

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- The optimum shape for the polyomino depends on the type of graph associated with the discrete channel without memory.

- This shape may generally be considered as the union of a square $x \times x$ with a square $y \times y$. 
When $m = 2t^2 + 2t + 1$, for $t = 1, 2, 3, \ldots$, Bombin and Martin-Delgado’s codes are reproduced.
\[ [d^2 + 1, 2, d] \] codes

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- \( m = (t + 1)^2 + t^2 \Rightarrow x = (t + 1) \) and \( y = t \Rightarrow A = \langle(x, y)\rangle \).
**$[[d^2 + 1, 2, d]]$ codes**

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- \( n = 2m = d^2 + 1 \)

- \([[d^2 + 1, 2, d]]\).

- **Example:** \( m = 5 \Rightarrow \mathcal{A} = \langle (2, 1) \rangle \). Code \([[10, 2, 3]]\) is obtained.
When $m$ is a perfect square, the Kitaev’s codes are reproduced.
[[2d^2, 2, d]] codes

- When $m$ is a perfect square, the Kitaev’s codes are reproduced.
  
  - $m = x^2 + y^2 \Rightarrow x = \pm \sqrt{m}, y = 0 \Rightarrow A = \langle (\sqrt{m}, 0), (0, \sqrt{m}) \rangle$. 
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  - \( d = |\sqrt{m}| \)
  - \( n = 2m = 2d^2 \)
  - \( [[2d^2, 2, d]]. \)
- Example: \( m = 4 \Rightarrow A = \{ (0, 0), (2, 0), (0, 2), (2, 2) \}. \) Code \( [[8, 2, 2]] \) is obtained.
New Class of Toric Codes

- When $m = x^2 + x^2$, we have $\mathcal{A} = \langle (x, x), (-x, x) \rangle$. 

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- \( k/n = 1/d \).

**Example:** \( m = 8 \Rightarrow \)
\( \mathcal{A} = \{(0, 0), (2, 2), (4, 4), (6, 6), (6, 2), (2, 6), (0, 4), (4, 0)\} \).
Code \([16, 2, 4]\) is obtained.
Conclusions
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- Besides reproducing known classes of codes, new classes of toric codes are determined. For instance, the class $[[d^2, 2, d]]$, the best known so far.
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