THE GEL'FAND PROBLEM FOR THE BIHARMONIC OPERATOR

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Abstract. We study stable and finite Morse index solutions of the equation $\Delta^2 u = e^u$. If the equation is posed in $\mathbb{R}^N$, we classify radial stable solutions. We then construct non radial stable solutions and we prove that unlike the corresponding second order problem, no Liouville-type theorem holds, unless additional information is available on the asymptotics of solutions at infinity. Thanks to this analysis, we prove that stable solutions of the equation on a smoothly bounded domain (supplemented with Navier boundary conditions) are smooth if and only if $N \leq 12$. We find an upper bound for the Hausdorff dimension of their singular set in higher dimensions and conclude with an *a priori* estimate for solutions of bounded Morse index, provided they are controlled in a suitable Morrey norm.

1. Introduction

A classical result attributed[1] to G.I. Barenblatt asserts that there exists infinitely many solutions to the equation

$$-\Delta u = 2e^u \qquad \text{in } \Omega, \tag{1.1}$$

whenever Ω is the unit ball of $\mathbb{R}^3$ and the equation is supplemented with a homogeneous Dirichlet boundary condition. The result appeared in a volume edited by I.M. Gel'fand [18], whose name the problem now bears. We refer the interested reader to the book [12] for some of the developments of this equation in the more than sixty years that separate us from Barenblatt's discovery. Let us simply mention that K. Nagasaki and T. Suzuki [25] completly classified the solutions found by Barenblatt according to their Morse index. Much of what can be said of the equation posed in a general domain Ω rests, through a blow-up analysis, upon Liouville-type theorems for finite Morse index solutions of the equation posed in entire space. This lead N. Dancer and A. Farina [9] to proving that in dimension $3 \leq N \leq 9$, any solution to (1.1) in $\Omega = \mathbb{R}^N$ is unstable outside every compact set and so, it has infinite Morse index.

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[1] see H. Brezis, [5].

In the present work, we consider the fourth-order analogue of the Gel'fand problem. Motivated by the aforementioned results and by the expanding literature on fourth-order equations, see in particular the books [17] by F. Gazzola, H.-Ch. Grunau, and G. Sweers, and [15] by P. Esposito, N. Ghoussoub, and Y. Guo, we want to classify solutions of

$$\Delta^2 u = e^u \qquad \text{in } \mathbb{R}^N, \tag{1.2}$$

which are stable (resp. stable outside a compact set), that is, solutions such that

$$\int_\Omega e^u \varphi^2 \, dx \leq \int_\Omega |\Delta\varphi|^2 \, dx \quad \text{for all } \varphi \in H^2(\Omega) \cap H^1_0(\Omega), \tag{1.3}$$

where Ω is $\mathbb{R}^N$(resp. the complement of some compact subset of $\mathbb{R}^N$). Consider first radial solutions. Noting that the equation is invariant under the scaling transformation

$$u_\lambda(x) = u(\lambda x) + 4\ln\lambda, \quad x \in \mathbb{R}^N, \lambda > 0, \tag{1.4}$$

we may always assume that $u(0) = 0$.

Proposition 1.1. *Let $5 \leq N \leq 12$. Assume u is a radial solution of* (1.2). *Let $v = -\Delta u$, $0 = u(0)$, and $\beta = v(0)$. There exist $\beta_1 > \beta_0 > 0$ depending on N only such that*

(i) *$\beta \geq \beta_0$.*
(ii) *If $\beta = \beta_0$, u is unstable outside every compact set.*
(iii) *If $\beta \in (\beta_0, \beta_1)$, u is unstable, but u is stable outside a compact set.*
(iv) *If $\beta \geq \beta_1$, u is stable.*

Remark 1.2. The fact that no radial solution exists for $\beta < \beta_0$ is due to G. Arioli, F. Gazzola and H.-C. Grunau [1]. In addition, E. Berchio, A. Farina, A. Ferrero, and F. Gazzola first proved in [3] that for $5 \leq N \leq 12$, u is stable outside a compact set[2] if and only if $\beta > \beta_0$. The novelty here is the number β_1: our result characterizes stable radial solutions when $5 \leq N \leq 12$. See [3], [33] for the remaining cases $1 \leq N \leq 4$ and $N \geq 13$.

In particular, there is no hope of proving a result similar to that of Dancer and Farina in our context, without further restrictions on the solution. One might ask whether all stable solutions are radial, at least in dimension $5 \leq N \leq 12$. This is still not the case.

Theorem 1.3. *Assume $N \geq 5$. Take a point $x^0 = (x^0_1, \ldots, x^0_N) \in \mathbb{R}^N$, parameters $\alpha_1, \ldots, \alpha_N > 1 + N/2$, and let*

$$p(x) = \sum_{i=1}^N \alpha_i (x_i - x^0_i)^2. \tag{1.5}$$

[2]the notion of stability outside a compact set that we use here is stronger than the one given in [3]. One can easily check that the results of [3] remain true in our setting.

Then, there exists a solution u of (1.2) such that

$$u(x) = -p(x) + \mathcal{O}(|x|^{4-N}) \quad \text{as } |x| \to \infty. \tag{1.6}$$

In particular, u is stable outside a compact set (resp. stable, if $\min_{i=1,\dots,N} \alpha_i$ is large enough) and u is not radial about any point if the coefficients α_i are not all equal.

Remark 1.4. Using the scaling (1.4), one immediately obtains a solution u of (1.2) such that $u(x) = -p(x) + C + \mathcal{O}(|x|^{4-N})$ as $|x| \to \infty$, under the sole assumption that $\alpha_i > 0$ for all $i = 1, \dots, N$.

Remark 1.5. In fact, any solution satisfying (1.6) has finite Morse index, thanks to the Cwikel-Lieb-Rozenbljum formula (see G.V. Rozenbljum [26] and D. Levin [22] for the formula, as well as A. Farina [16] for its application to semilinear elliptic equations). We thank A. Farina for this observation.

Remark 1.6. In dimension $N \geq 13$, the radial solution $u_{\beta_0}(x) = -4\ln|x| + \mathcal{O}(1)$ is stable, see [3]. Observe that u_{β_0} does not satisfy (1.6). In dimension $5 \leq N \leq 12$, it would be interesting to determine whether, up to rescaling and rotation, all stable solutions do satisfy (1.6).

All stable solutions that we have encountered so far have quadratic behavior at infinity. In particular, letting

$$v = -\Delta u \quad \text{and} \quad \overline{v}(r) = \fint_{\partial B_r} v \, d\sigma,$$

these solutions satisfy $\overline{v}(\infty) > 0$, where [3]

$$\overline{v}(\infty) := \lim_{r \to +\infty} \overline{v}(r).$$

This motivates the following Liouville-type result.

Theorem 1.7. *Assume* $5 \leq N \leq 12$. *Let u be a solution of* (1.2) *such that* $\overline{v}(\infty) = 0$. *Then, u is unstable outside every compact set.*

Remark 1.8. As observed in [3], in dimension $N = 4$, applying inequality (1.3) with a standard cut-off function, we easily see that if u is stable outside a compact set, then $e^u \in L^1(\mathbb{R}^4)$. Thanks to the work of C.-S. Lin [23], up to a rotation of space, u must satisfy

$$u(x) = -p(x) - 4\gamma \ln|x| + c_0 + \mathcal{O}(|x|^{-\tau}) \quad \text{as } |x| \to \infty, \tag{1.7}$$

where $p(x)$ is of the form (1.5) with $\alpha_i \geq 0$, $\gamma = \frac{1}{32\pi^2}\int_{\mathbb{R}^4} e^u \, dx \leq 2$, c_0 and $\tau > 0$ are constants. Conversely, there exist solutions of the form (1.7), as proved by J. Wei and D. Ye in [36]. If in addition $\overline{v}(\infty) = 0$, then $\gamma = 2$ and up to translation and the scaling (1.4),

$$u(x) = -4\ln\left(1 + |x|^2\right) + \ln 384, \qquad \text{for all } x \in \mathbb{R}^4.$$

[3]For any solution of (1.2), v is superharmonic. In particular, its spherical average $\overline{v}$ is a decreasing function of r. We will prove that $v > 0$, so that $\overline{v}(\infty) \geq 0$ is always well-defined.

Now, let us turn to bounded domains. We begin by recalling a few known results on the Gel'fand problem for the biharmonic operator, when the domain is the unit ball and the equation is supplemented with a homogeneous Dirichlet boundary condition, i.e. for $\lambda > 0$, we consider the equation

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B, \\ u = |\nabla u| = 0 & \text{on } \partial B. \end{cases}$$

It is known that there exists an extremal parameter $\lambda^* = \lambda^*(N) > 0$ such that the problem has at least one solution (which is stable) for $\lambda < \lambda^*$, a weak stable solution u^* for $\lambda = \lambda^*$ and no solution for $\lambda > \lambda^*$ (see G. Arioli, F. Gazzola, F.-C. Grunau and E. Mitidieri [2]). The unique weak stable solution u^* associated to $\lambda = \lambda^*$ is classical if and only if $1 \le N \le 12$ (see the work by J. Dávila, I. Guerra, M. Montenegro and one of the authors [10], as well as a simplification due to A. Moradifam [24]). It is also known that if $N \ge 5$, the problem has a unique singular radial solution for some $\lambda = \lambda_S$ (as follows from the analysis in [2] and [10]) and, as in Barenblatt's result, that it has infinitely many regular radial solutions for the same value of the parameter (see the delicate work due to J. Dávila, I. Flores and I. Guerra [11]). The case of general domains with Dirichlet boundary conditions is essentially unexplored, due to the lack of a comparison principle. The case of homogeneous Navier boundary conditions, namely the equation

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \ge 1$, seems, for now, easier to deal with. Our results can be summarized as follows.

Theorem 1.9. *Let $N \ge 1$ and let Ω be a smoothly bounded domain of $\mathbb{R}^N$. Let u^* be the extremal solution of (1.8).*

(i) *If $1 \le N \le 12$, then $u^* \in C^\infty(\overline{\Omega})$.*
(ii) *If $N \ge 13$, then $u \in C^\infty(\Omega \setminus \Sigma)$, where Σ is a closed set whose Hausdorff dimension is bounded above by*

$$\mathcal{H}_{dim}(\Sigma) \le N - 4p^*$$

and $p^* > 3$ is the largest root of the polynomial $(X-\frac{1}{2})^3 - 8(X-\frac{1}{2}) + 4$.

Remark 1.10. Theorem 1.9 was first proved for $1 \le N \le 8$ by C. Cowan, P. Esposito, and N. Ghoussoub, see [7]. As we were completing this work, we learnt that C. Cowan and N. Ghoussoub just improved their result to $1 \le N \le 10$, see [8]. The question of partial regularity in large dimension was recently settled by K. Wang for the classical Gel'fand problem, see [31] and [32]. Our approach is somewhat different from his. We expect that the computed exponent p^* is not optimal. Similarly, as observed by P. Esposito [14], the methods that we develop here will not yield the expected critical curve (resp. dimension) for the Lane-Emden system (resp. for the MEMS problem), for which new ideas must be found. To the authors knowledge, the state of the art on these equations is contained in the following references: [7], [8], the works of J. Wei and D. Ye [37], of J. Wei, X. Xu and W. Yang [35], of C. Cowan [6] and of H. Hajlaoui, A. Harrabi and D. Ye [?hhy].

Finally, our Liouville-type result, Theorem 1.7, will be used to prove the following.

Theorem 1.11. *Let $5 \leq N \leq 12$ and let Ω be a smoothly bounded domain of $\mathbb{R}^N$. Assume in addition that Ω is convex. Let $u \in C^4(\overline{\Omega})$ be any classical solution of (1.8) and let $v = -\Delta u$. There exists a compact subdomain $\omega \subset \Omega$ such that if*

$$\int_{B_r(x_0)} v\, dx \leq K r^{N-2}, \tag{1.9}$$

for every ball $B_r(x_0) \subset \omega$ and for some constant $K > 0$, then, there exists a number C depending only on λ, Ω, N, K and the Morse index of u, such that

$$||u||_{L^\infty(\Omega)} \leq C.$$

Definition 1.12. In the above, the Morse index of a solution u is the number of negative eigenvalues of the linearized operator $L = \Delta^2 - \lambda e^u$ with domain $D(L) = \{u \;:\; u, \Delta u \in H^2(\Omega) \cap H_0^1(\Omega)\}$. According to standard spectral theory, this number is finite.

Remark 1.13. If u is stable, then (1.9) automatically holds for some constant K depending only on Ω, N, and ω. See Lemma 5.2. We do not know whether this remains valid for solutions of bounded Morse index. In addition, it would be interesting to know how the number C depends on the Morse index of u. See the work of X.-F. Yang [38] for a result in this direction, in a subcritical setting.

Notation. For any given function f, $\overline{f}$ connotes the spherical average of f. We write $f \lesssim g$ (resp. $f \gtrsim g$), when there exists a numerical constant C such that $f \leq Cg$ (resp. $f \geq Cg$). $B_R(x)$ denotes the open ball centered at x and of radius R. For shorthand, $B_R = B_R(0)$ and $A_R = B_{2R} \setminus \overline{B_R}$ is the open annulus of radii R and $2R$.

2. Classification of stable radial solutions

We prove here Proposition 1.1. Take $\beta \geq \beta_0$ and let $u = u_\beta$ be the radial solution such that $u(0) = 0$ and $v(0) = \beta$. We claim that if β is large enough, then u_β is stable. We shall use the following inequality, found in [1]:

$$u_\beta \leq -\frac{\beta - \beta_0}{2N} r^2 \quad \text{for all } r > 0. \tag{2.1}$$

Simply choose $\overline{\beta} > \beta_0$ such that

$$e^{-\frac{\overline{\beta} - \beta_0}{2N} r^2} \leq \frac{N^2(N-4)^2}{16 r^4} \quad \text{for all } r > 0. \tag{2.2}$$

Combining (2.1), (2.2), and the Hardy-Rellich inequality, we deduce that u_β is stable for $\beta \geq \overline{\beta}$. So, we may define $\Lambda := \{\beta > \beta_0 \ : \ u_\beta \text{ is stable}\}$ and $\beta_1 = \inf \Lambda$. By standard ODE theory, one easily proves that $\beta_1 = \min \Lambda$. According to [3], u_{β_0} is unstable. So, $\beta_1 > \beta_0$. Also, by a result in [1], solutions are ordered : if $\tilde{\beta} > \beta$, then $u_{\tilde{\beta}} \leq u_\beta$. So, Λ is the interval $[\beta_1, +\infty)$. $\square$

3. Construction of nonradial solutions

We present here the proof of Theorem 1.3. Take a polynomial p of the form (1.5). Without loss of generality, we may assume that $x^0 = 0$. We look for a solution u of the form $u = -p(x) + z(x)$, so that z and $w = -\Delta z$ satisfy

$$\begin{cases} -\Delta z = w & \text{in } \mathbb{R}^N, \\ -\Delta w = e^{-p(x)} e^z & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

For $x \in \mathbb{R}^N$, let

$$Z(x) = (1 + |x|^2)^{2-N/2}, \quad W(x) = (1 + |x|^2)^{1-N/2}.$$

We claim that (Z, W) is a super-solution of (3.1). Indeed, straightforward calculations yield

$$\begin{aligned} -\Delta Z &\geq 2(N-4)W & &\text{in } \mathbb{R}^N, \\ -\Delta W &= N(N-2)(1 + |x|^2)^{-1-N/2} & &\text{in } \mathbb{R}^N. \end{aligned}$$

Since $Z \leq 1$ and since we assumed that $p(x) \geq (1 + N/2)|x|^2$ in $\mathbb{R}^N$, we have

$$-\Delta W \geq (1 + |x|^2)^{-1-N/2} e^Z \geq e^{-(1+N/2)|x|^2} e^Z \geq e^{-p(x)} e^Z \quad \text{in } \mathbb{R}^N,$$

which proves our claim.

Since the system is cooperative, and $(0, 0)$ and (Z, W) form a pair of ordered sub- and super-solutions, we obtain the existence of a solution of (3.1) which further satisfies $0 < z \leq Z$ and $0 < w \leq W$ in $\mathbb{R}^N$. Hence, $u(x) := -p(x) + z(x)$ is a solution of (1.2).

To prove that u is stable outside a compact set, let us observe again that $Z \leq 1$ in $\mathbb{R}^N$. So, we can find $\rho > 0$ large such that

$$e^{u(x)} \leq e^{Z(x)} e^{-p(x)} \leq e^{1-p(x)} \leq \frac{N^2(N-4)^2}{16|x|^4}, \quad \text{for all } |x| > \rho. \tag{3.2}$$

By the Hardy-Rellich inequality, u is stable outside $\overline{B_\rho}$.

Remark now that if $\min_{i=1,\dots,N} \alpha_i > 0$ is large enough then (3.2) is valid for all $x \in \mathbb{R}^N$, so u is stable in $\mathbb{R}^N$. This ends the proof of Theorem 1.3. $\square$

4. Regularity of the extremal solution in dimension $1 \leq N \leq 12$

In this section, we prove the first part of Theorem 1.9. Let u denote the minimal solution to (1.8) associated to a parameter $\lambda \in (\lambda^*/2, \lambda^*)$. Up to rescaling, we may assume that $\lambda = 1$. The first ingredient in our proof is the following consequence of the stability inequality (1.3):

$$\int_\Omega e^{\frac{u}{2}}\varphi^2\,dx \leq \int_\Omega |\nabla\varphi|^2\,dx \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{4.1}$$

(4.1) was proved independently[4] by C. Cowan and N. Ghoussoub in [8] and A. Farina, B. Sirakov, and one of the authors in [13]. See e.g. Lemma 6.1 below for its proof. Now, Let us write the problem (1.8) as a system in the following way:

$$\begin{cases} -\Delta u = v & \text{in } \Omega, \\ -\Delta v = e^u & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Fix $\alpha > \frac{1}{2}$ and multiply the first equation in (4.2) by $e^{\alpha u} - 1$. Integrating over Ω, we obtain

$$\int_\Omega (e^{\alpha u} - 1)\,v\,dx = \alpha\int_\Omega e^{\alpha u}|\nabla u|^2\,dx = \frac{4}{\alpha}\int_\Omega \left|\nabla\left(e^{\frac{\alpha u}{2}} - 1\right)\right|^2\,dx.$$

By (4.1),

$$\int_\Omega e^{\frac{u}{2}}\left(e^{\frac{\alpha u}{2}} - 1\right)^2\,dx \leq \int_\Omega \left|\nabla\left(e^{\frac{\alpha u}{2}} - 1\right)\right|^2\,dx.$$

Combining these two inequalities, we deduce that

$$\int_\Omega e^{\frac{u}{2}}\left(e^{\frac{\alpha u}{2}} - 1\right)^2\,dx \leq \frac{\alpha}{4}\int_\Omega (e^{\alpha u} - 1)\,v\,dx. \tag{4.3}$$

Similarly, multiply the second equation in (4.2) by $v^{2\alpha-1}$ and use (4.1) to deduce that

$$\int_\Omega e^{\frac{u}{2}}v^{2\alpha}\,dx \leq \frac{\alpha^2}{2\alpha - 1}\int_\Omega e^u v^{2\alpha-1}\,dx. \tag{4.4}$$

By Hölder's inequality,

$$\int_\Omega e^u v^{2\alpha-1}\,dx \leq \left(\int_\Omega e^{\frac{u}{2}}v^{2\alpha}\,dx\right)^{\frac{2\alpha-1}{2\alpha}}\left(\int_\Omega e^{(\alpha+\frac{1}{2})u}\,dx\right)^{\frac{1}{2\alpha}} \quad \text{and}$$

$$\int_\Omega e^{\alpha u}v\,dx \leq \left(\int_\Omega e^{\frac{u}{2}}v^{2\alpha}\,dx\right)^{\frac{1}{2\alpha}}\left(\int_\Omega e^{(\alpha+\frac{1}{2})u}\,dx\right)^{\frac{2\alpha-1}{2\alpha}}.$$

[4]N. Ghoussoub informed us that the result was first made available online by C. Cowan and himself. See also the Appendix for another result found by several authors.

Plugging these inequalities in (4.4) and (4.3) respectively, we deduce that

$$\left(\int_\Omega e^{\frac{u}{2}} v^{2\alpha}\,dx\right)^{\frac{1}{2\alpha}} \le \frac{\alpha^2}{2\alpha-1}\left(\int_\Omega e^{(\alpha+\frac12)u}\,dx\right)^{\frac{1}{2\alpha}} \quad \text{and}$$

$$\int_\Omega e^{\frac{u}{2}}\left(e^{\frac{\alpha u}{2}}-1\right)^2 dx \le \frac{\alpha}{4}\left(\int_\Omega e^{\frac{u}{2}} v^{2\alpha}\,dx\right)^{\frac{1}{2\alpha}}\left(\int_\Omega e^{(\alpha+\frac12)u}\,dx\right)^{\frac{2\alpha-1}{2\alpha}}.$$

Multiplying these inequalities, it follows that

$$\int_\Omega e^{\frac{u}{2}}\left(e^{\frac{\alpha u}{2}}-1\right)^2 dx \le \frac{\alpha^3}{8\alpha-4}\int_\Omega e^{(\alpha+\frac12)u}\,dx$$

so that

$$\left(1-\frac{\alpha^3}{8\alpha-4}\right)\int_\Omega e^{(\alpha+\frac12)u}\,dx \le 2\int_\Omega e^{\frac{\alpha+1}{2}u}\,dx.$$

Apply Hölder's inequality to the right-hand side. Then,

$$\int_\Omega e^{\frac{\alpha+1}{2}u}\,dx \le \left(\int_\Omega e^{\frac{2\alpha+1}{2}u}\,dx\right)^{\frac{\alpha+1}{2\alpha+1}} |\Omega|^{\frac{\alpha}{2\alpha+1}}$$

and so

$$\left(1-\frac{\alpha^3}{8\alpha-4}\right)\left(\int_\Omega e^{\frac{2\alpha+1}{2}u}\,dx\right)^{\frac{\alpha}{2\alpha+1}} \le 2\,|\Omega|^{\frac{\alpha}{2\alpha+1}}. \tag{4.5}$$

Let $\alpha^* > 5/2$ denote the largest root of the polynomial X^3-8X+4. We have just proved that e^u is uniformly bounded in $L^p(\Omega)$ for every $p < p^* = \alpha^*+\frac12$. In particular, e^u is bounded in $L^p(\Omega)$ for some $p>3$. Using elliptic regularity applied to (1.8), this implies that u^* is bounded, hence smooth, whenever $N \le 12$. □

5. Growth of the L^1 norm

This section and the next provide preparatory results that will be used both for the proof of the Liouville-type theorem and the partial regularity result. We begin with the case of $\mathbb{R}^N$.

Lemma 5.1. *Assume $N \ge 5$ and let u be a solution of (1.2) which is stable (resp. stable outside the ball $\overline{B_{R_0}}$). Let $v=-\Delta u$, $\overline{v}$ its spherical average, and assume that $\overline{v}(\infty)=0$. Lrt B_R denote the open ball of radius R (resp. the open annulus of radii R and $2R$). Then, there exists a positive constant C depending only on N (resp. on N, u, R_0) such that*

$$\int_{B_R} v\,dx \le CR^{N-2} \quad \textit{for every } R>0 \textit{ (resp. } R>2R_0\textit{)}. \tag{5.1}$$

Proof. We claim that $v > 0$. Since the equation is invariant under translation, it suffices to prove that $v(0) > 0$. Let $\overline{v}$ be the spherical average of v and assume by contradiction that $v(0) = \overline{v}(0) \leq 0$. We have

$$\begin{cases} -\Delta \overline{v} = \overline{e^u} & \text{in } \mathbb{R}^N, \\ -\Delta \overline{u} = \overline{v} & \text{in } \mathbb{R}^N. \end{cases} \tag{5.2}$$

In particular, $-\Delta \overline{v} = -r^{1-N}(r^{N-1}v')' > 0$, and so $\overline{v}$ is a decreasing function of $r > 0$. Since we assumed that $\overline{v}(0) \leq 0$, it follows that $\overline{v}(r) < 0$ for all $r > 0$. So, $\overline{u}$ is subharmonic. It follows that $\overline{u}$ is an increasing function of $r > 0$. In particular, $\overline{u}$ is bounded below and so, given $R > 2R_0$,

$$\int_{B_{2R}\setminus B_R} e^{\overline{u}}\, dx \geq e^{u(0)} \int_{B_{2R}\setminus B_R} dx \gtrsim R^N. \tag{5.3}$$

Take a cut-off function $\varphi \in C^2_c(B_4 \setminus B_{\frac{1}{2}})$, with $0 \leq \varphi \leq 1$ in $\mathbb{R}^N$ and $\varphi \equiv 1$ in $B_2 \setminus B_1$. Applying Jensen's inequality and the stability inequality (1.3) with test function $\varphi(x/R)$, we find

$$\int_{B_{2R}\setminus B_R} e^{\overline{u}} dx \leq \int_{B_{2R}\setminus B_R} \overline{e^u} dx \lesssim R^{N-4}$$

which contradicts (5.3), if R is chosen large enough. Hence, $v(0) > 0$. Now, if u is stable, take $r > 0$, take a standard cut-off function $\varphi \in C^2_c(B_2)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ in B_1. Apply the stability inequality (1.3) with test function $\varphi(x/r)$ to get

$$\int_{B_r} e^u\, dx \lesssim r^{N-4}. \tag{5.4}$$

If u is only stable outside $\overline{B_{R_0}}$, fix $r > 2R_0$ and take a cut-off function $\varphi \in C^2_c(\mathbb{R}^N)$ such that $\varphi \equiv 0$ in B_{R_0}, $\varphi \equiv 1$ in $B_r \setminus B_{R_0+1}$, $\varphi \equiv 0$ outside B_{2r}, and $|\Delta\varphi| \lesssim r^{-2}$ in $B_{2r} \setminus B_r$. Using the stability inequality (1.3) with this test function we find that

$$\int_{B_r} e^u\, dx = \int_{B_{R_0+1}} e^u\, dx + \int_{B_r\setminus B_{R_0+1}} e^u\, dx \lesssim 1 + r^{N-4} \lesssim r^{N-4}. \tag{5.5}$$

Now, we rewrite the first equation in (5.2) as

$$-(r^{N-1}\overline{v}')' = r^{N-1}\overline{e^u}.$$

Integrate on $(0, r)$. By (5.4), which holds for every $r > 0$ if u is stable (resp. by (5.5), which holds for $r > 2R_0$ if u is stable outside $\overline{B_{R_0}}$),

$$-r^{N-1}\overline{v}'(r) = \int_0^r t^{N-1}\overline{e^u} dt \lesssim r^{N-4}.$$

We integrate once more between R and $+\infty$. Since $\overline{v}(\infty) = 0$, we obtain

$$\overline{v}(R) \lesssim R^{-2}.$$

Clearly, (5.1) follows. □

Now, we turn to an analogous result on bounded domains. Let us recall that the extremal solution is stable.

Lemma 5.2. *Let $N \geq 3$ and let u be the extremal solution of (1.8). Set $v = -\Delta u$. Fix $x_0 \in \Omega$ and let $R_0 = \mathrm{dist}(x_0, \partial\Omega)/2$. Then, there exists a constant $C > 0$ depending only on N, Ω, and R_0 such that for all $r < R_0$,*

$$\int_{B_r(x_0)} v \, dx \leq C r^{N-2}. \tag{5.6}$$

Proof. Without loss of generality, we may assume that $x_0 = 0$ and $\lambda^* = 1$.

Step 1. There exists a constant $C = C(N, R_0) > 0$ such that for all $r \in (0, R_0)$,

$$\int_{[r<|x|<R_0]} e^u |x|^{2-N} \, dx \leq C r^{-2}. \tag{5.7}$$

To see this, consider the function $\psi : B_{2R_0} \to \mathbb{R}^N$ given by

$$\psi(x) = \begin{cases} a - b|x|^2 & \text{if } |x| < r, \\ |x|^{1-N/2} & \text{if } r \leq |x| < 2R_0, \end{cases} \tag{5.8}$$

where the constants $a = \frac{N+2}{4} r^{1-N/2}$ and $b = \frac{N-2}{4} r^{-1-N/2}$ are chosen so that ψ is C^1 and piecewise C^2. Take a standard cut-off function

$$\zeta \in C_c^2(B_2) \quad \text{such that } 0 \leq \zeta \leq 1 \text{ and } \zeta = 1 \text{ on } B_1, \tag{5.9}$$

so that $\varphi(x) = \psi(x)\zeta(x/R_0)$ belongs to $H_0^1(\Omega) \cap H^2(\Omega)$.

Since u is stable, we have

$$\int_{[r<|x|<R_0]} e^u |x|^{2-N} \, dx \leq \int_\Omega e^u \varphi^2 \, dx \leq \int_\Omega |\Delta\varphi|^2 \, dx. \tag{5.10}$$

By (5.8),

$$\int_{[|x|<r]} |\Delta\varphi|^2 \, dx \leq 4N^2 b^2 \int_{[|x|<r]} dx \leq C b^2 r^N \leq C' r^{-2}. \tag{5.11}$$

Similarly,

$$\int_{[r<|x|<R_0]} |\Delta\varphi|^2 \, dx \leq C \int_{[r<|x|]} |x|^{-2-N} \, dx \leq C' r^{-2}. \tag{5.12}$$

Finally,

$$\int_{[R_0<|x|]\cap\Omega} |\Delta\varphi|^2 \, dx \leq C(N, R_0) \leq C' r^{-2}. \tag{5.13}$$

Collecting (5.10), (5.11), (5.12), and (5.13), the estimate (5.7) follows.

Step 2. Take as before a standard cut-off ζ satisfying (5.9) and let

$$\psi(x) = \int_{\mathbb{R}^N} |x-y|^{2-N} \zeta(y) \, dy, \qquad \text{for } x \in \mathbb{R}^N.$$

Then, there exists a constant C, depending on N only such that

$$\psi(x) \le C \min\{1, |x|^{2-N}\}, \qquad \text{for } x \in \mathbb{R}^N. \tag{5.14}$$

Indeed, if $|x| > 4$ and $|y| < 2$,

$$|x-y| \ge |x| - |y| \ge \frac{1}{2}|x|$$

so that

$$\psi(x) \le 2^{N-2}|x|^{2-N} \int_{\mathbb{R}^N} \zeta(y)\, dy \le C|x|^{2-N},$$

while if $|x| \le 4$ and $|y| < 2$,

$$|x-y| \le |x| + |y| \le 6$$

and so

$$\psi(x) \le ||\zeta||_{L^\infty(\mathbb{R}^N)} \int_{|x-y|<6} |x-y|^{2-N}\, dy \le C.$$

(5.14) follows.

Step 3. Take $r \in (0, R_0)$, ζ a standard cut-off function satisfying (5.9), and let φ_r be the solution to

$$\begin{cases} -\Delta\varphi_r = \zeta(x/r) & \text{in } \Omega, \\ \varphi_r = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.15}$$

Then, there exists a constant C depending on N only such that

$$\varphi_r(x) \le C r^2 \min\left\{1, r^{N-2}|x|^{2-N}\right\}, \quad \text{for all } x \in \Omega. \tag{5.16}$$

This easily follows from the maximum principle, observing that a constant multiple of $r^2\psi(x/r)$ is a supersolution to the above equation.

Step 4. There exists a constant C depending on N and Ω only, such that

$$\int_\Omega e^u\, dx \le C. \tag{5.17}$$

This is an obvious consequence of Equation (4.5) (which holds for any $\alpha \in (\frac{1}{2}, \frac{5}{2}]$) and Hölder's inequality.

Step 5. Multiply (5.15) by $v = -\Delta u$ and integrate by parts. Then,

$$\int_{[|x|<r]} v\, dx \le \int_\Omega v\zeta(x/r)\, dx = \int_\Omega e^u \varphi_r\, dx.$$

Using Step 3, we have

$$\int_{[|x|<r]} e^u \varphi_r\, dx \le Cr^2 \int_{[|x|<r]} e^u\, dx.$$

Using stability with test function $\zeta(x/r)$, we also have

$$\int_{[|x|<r]} e^u\, dx \le Cr^{N-4}$$

and so

$$\int_{[|x|<r]} e^u \varphi_r \, dx \le C r^{N-2}.$$

By Step 1 and Step 3,

$$\int_{[r<|x|<R_0]} e^u \varphi_r \, dx \le C r^N \int_{[r<|x|<R_0]} e^u |x|^{2-N} \, dx \le C r^{N-2}.$$

Finally, using (5.17),

$$\int_{[R_0<|x|]\cap\Omega} e^u \varphi_r \, dx \le C R_0^{2-N} r^N \int_\Omega e^u \, dx \le C' r^{N-2}.$$

(5.6) follows. □

6. A bootstrap argument

Our next task consists in improving the L^1-estimates of the previous section to L^p-estimates for larger values of p. This will be carried out through a bootstrap argument which is reminescent of the classical Moser iteration method, up to one major difference: we will take advantage of both the standard Sobolev inequality and the stability inequality (1.3). To be more precise, rather than (1.3), the following interpolated version of it will be used.

Lemma 6.1. *Let Ω be an open set of $\mathbb{R}^N$, $N \ge 1$. Assume that the stability inequality* (1.3) *holds. Then, for every $s \in [0,1]$,*

$$\int_\Omega e^{su} \varphi^2 \, dx \le \int_\Omega |(-\Delta)^s \varphi|^2 \, dx \quad \textit{for all } \varphi \in H^2(\Omega) \cap H^1_0(\Omega).$$

In particular, for $s = \frac{1}{2}$,

$$\int_\Omega e^{\frac{u}{2}} \varphi^2 \, dx \le \int_\Omega |\nabla \varphi|^2 \, dx \quad \textit{for all } \varphi \in H^1_0(\Omega). \tag{6.1}$$

Proof Consider first the case where $\Omega = \mathbb{R}^N$. We apply complex interpolation between the family of spaces X_s, Y_s given for $0 \le s \le 1$ by

$$X_s = L^2((2\pi)^{-N} |\xi|^{4s} d\xi)\,, \quad Y_s = L^2(e^{su} dx).$$

Recall that the inverse Fourier transform $\mathcal{F}^{-1} : X_0 \to Y_0$ satisfies $||\mathcal{F}||_{\mathcal{L}(X_0,Y_0)} = 1$. Furthermore, by the stability inequality (1.3) and Plancherel's theorem, we have

$$\int_{\mathbb{R}^N} e^u \varphi^2 \, dx \le \int_{\mathbb{R}^N} |\Delta \varphi|^2 \, dx = (2\pi)^{-N} \int_{\mathbb{R}^N} |\xi|^4 |\mathcal{F}(\phi)|^2 \, d\xi.$$

Thus, $\mathcal{F}^{-1} : X_1 \to Y_1$ satisfies $||\mathcal{F}||_{\mathcal{L}(X_1,Y_1)} \le 1$. By the complex interpolation theorem (see e.g. [29, Theorem 2]), we deduce that $||\mathcal{F}||_{\mathcal{L}(X_s,Y_s)} \le 1$ for all $0 \le s \le 1$.

In the case where Ω is a bounded open set, simply repeat the above proof, using the spectral decomposition of the Laplace operator in place of the Fourier transform. More precisely, let (λ_k) denote the eigenvalues of the Laplace operator (with domain $D = H^2(\Omega) \cap H^1_0(\Omega)$), let $\hat{\varphi}_k$ be the k-th component of φ in the Hilbert basis of eigenfunctions associated to (λ_k) and interpolate between the family of weighted L^2-spaces X_s, Y_s corresponding to the norms

$$||\varphi||^2_{X_s} = \sum_k \lambda_k^{2s} |\hat{\varphi}_k|^2, \quad ||\varphi||^2_{Y_s} = \int_\Omega e^{su}\varphi^2 \, dx.$$

In the case where Ω is an unbounded proper open set, take $k > 0$, let $\Omega_k = \Omega \cap B_k$ and

$$-\mu_k = \inf\left\{\int_{\Omega_k} \left(|\Delta\varphi|^2 - e^u\varphi^2\right) \, dx \; : \; \varphi \in H^2(\Omega_k) \cap H^1_0(\Omega_k), ||\varphi||_{L^2(\Omega_k)} = 1\right\}.$$

Then, the previous analysis leads to

$$\int_{\Omega_k} \left[(e^u - \mu_k)_+\right]^s \varphi^2 \, dx \leq \int_{\Omega_k} |(-\Delta)^s\varphi|^2 \, dx \qquad \text{for all } \varphi \in H^2(\Omega_k) \cap H^1_0(\Omega_k).$$

By (1.3), $\lim_{k\to+\infty} -\mu_k \geq 0$ and the result follows. $\square$

Our next lemma is simply the first step in the Moser iteration method: we multiply the equation by a power of its right-hand side, localize, and integrate.

Lemma 6.2. *Let Ω be an open set of $\mathbb{R}^N$, $N \geq 1$. Assume $(u, v) \in C^2(\Omega)^2$ solves*

$$\begin{cases} -\Delta u = v & \text{in } \Omega, \\ -\Delta v = e^u & \text{in } \Omega. \end{cases}$$

Take $\alpha > \frac{1}{2}$, $\varphi \in C^1_c(\Omega)$. Then, there exists a constant C depending on α only, such that

$$\frac{\sqrt{2\alpha - 1}}{\alpha} ||\nabla(v^\alpha\varphi)||_{L^2(\Omega)} \leq ||e^{\frac{u}{2}} v^{\alpha - \frac{1}{2}}\varphi||_{L^2(\Omega)} + C||v^\alpha\nabla\varphi||_{L^2(\Omega)}. \tag{6.2}$$

and

$$\frac{2}{\sqrt{\alpha}} ||\nabla(e^{\frac{\alpha}{2}u}\varphi)||_{L^2(\Omega)} \leq ||e^{\frac{\alpha}{2}u} v^{\frac{1}{2}}\varphi||_{L^2(\Omega)} + C||e^{\frac{\alpha}{2}u}\nabla\varphi||_{L^2(\Omega)}. \tag{6.3}$$

Proof. Since the computations are very similar, we prove only (6.2). We multiply $-\Delta v = e^u$ by $v^{2\alpha-1}\varphi^2$ and we integrate. We obtain

$$\begin{aligned}
\int_\Omega e^u v^{2\alpha-1}\varphi^2 \, dx &= \int_\Omega \nabla v \cdot \nabla(v^{2\alpha-1}\varphi^2) \, dx \\
&= \frac{2\alpha-1}{\alpha^2} \int_\Omega |\nabla v^\alpha|^2\varphi^2 \, dx + 2\int_\Omega v^{2\alpha-1}\varphi\nabla v \cdot \nabla\varphi \, dx \\
&= \frac{2\alpha-1}{\alpha^2} \left(\int_\Omega |\nabla(v^\alpha\varphi)|^2 \, dx - \int_\Omega v^{2\alpha}|\nabla\varphi|^2 \, dx\right) \\
&\quad - \frac{2(\alpha-1)}{\alpha^2} \int_\Omega v^\alpha\varphi\nabla\varphi \cdot \nabla v^\alpha \, dx.
\end{aligned}$$

In the last term of the right hand side, replace $\varphi\nabla v^\alpha$ by $\nabla(v^\alpha\varphi) - v^\alpha\nabla\varphi$. Then,

$$\int_\Omega e^u v^{2\alpha-1}\varphi^2\,dx = \frac{2\alpha-1}{\alpha^2}\int_\Omega |\nabla(v^\alpha\varphi)|^2\,dx - \frac{1}{\alpha^2}\int_\Omega v^{2\alpha}|\nabla\varphi|^2\,dx$$
$$- \frac{2(\alpha-1)}{\alpha^2}\int_\Omega v^\alpha\nabla\varphi\nabla(v^\alpha\varphi)\,dx,$$

which we rewrite as

$$\begin{aligned}(2\alpha-1)\int_\Omega |\nabla(v^\alpha\varphi)|^2\,dx &= 2(\alpha-1)\int_\Omega v^\alpha\nabla\varphi\nabla(v^\alpha\varphi)\,dx + \\ &\int_\Omega v^{2\alpha}|\nabla\varphi|^2\,dx + \alpha^2\int_\Omega e^u v^{2\alpha-1}\varphi^2\,dx.\end{aligned}\tag{6.4}$$

By the Cauchy-Schwarz inequality,

$$\left|\int_\Omega v^\alpha\nabla\varphi\nabla(v^\alpha\varphi)\,dx\right| \le \left(\int_{\mathbb{R}^n}|\nabla(v^\alpha\varphi)|^2\,dx\right)^{\frac12}\left(\int_\Omega v^{2\alpha}|\nabla\varphi|^2\,dx\right)^{\frac12},$$

Plugging this in (6.4), we obtain a quadratic inequality of the form

$$(2\alpha-1)X^2 \le 2|\alpha-1|AX + A^2 + B^2$$

where

$X = ||\nabla(v^\alpha\varphi)||_{L^2(\Omega)}$, $\quad A = ||v^\alpha\nabla\varphi||_{L^2(\Omega)}$ and $B = \alpha||e^{\frac12 u}v^{\alpha-\frac12}\varphi||_{L^2(\Omega)}$.

Solving the quadratic inequality, we deduce that

$$X \le \frac{|\alpha-1|A + \sqrt{|\alpha-1|^2A^2 + (2\alpha-1)(A^2+B^2)}}{2\alpha-1} \le \frac{B}{\sqrt{2\alpha-1}} + C_\alpha A.$$

(6.2) follows by replacing A, B and X with their values. ☐

We next refine the estimates obtained in Lemma 6.2 for stable solutions to (1.2).

Lemma 6.3. *Make the same assumptions as in Lemma 6.2. Assume in addition that* (6.1) *holds for every* $\varphi \in C^1_c(\Omega)$. *Let* $\alpha^\sharp, \alpha^*$ *denote the largest two roots of the polynomial* $X^3 - 8X + 4$. *Then, for every* $\alpha \in (\alpha^\sharp, \alpha^*)$, *there exists a constant* C *depending on* α *only such that*

$$\int_\Omega |\nabla(v^\alpha\varphi)|^2\,dx \le C\int_\Omega v^{2\alpha}|\nabla\varphi|^2\,dx,\tag{6.5}$$

or

$$\int_\Omega |\nabla(e^{\frac{\alpha}{2}u}\varphi)|^2\,dx \le C\int_\Omega e^{\alpha u}|\nabla\varphi|^2\,dx,\tag{6.6}$$

for any $\varphi \in C^1_c(\Omega)$.

Proof. By Hölder's inequality,

$$\int_\Omega e^u v^{2\alpha-1}\varphi^2\,dx \le \left(\int_\Omega e^{\frac{2\alpha+1}{2}u}\varphi^2\,dx\right)^{\frac{1}{2\alpha}}\left(\int_\Omega e^{\frac{u}{2}}v^{2\alpha}\varphi^2\,dx\right)^{\frac{2\alpha-1}{2\alpha}}.$$

Using the stability inequality (6.1), we deduce that

$$\int_\Omega e^u v^{2\alpha-1}\varphi^2\,dx \le H^{\frac{1}{\alpha}}K^{2-\frac{1}{\alpha}},$$

where we set

$$H = ||\nabla(e^{\frac{\alpha}{2}u}\varphi)||_{L^2(\Omega)} \quad\text{and}\quad K = ||\nabla(v^\alpha\varphi)||_{L^2(\Omega)}.$$

Similarly,

$$\int_\Omega e^{\alpha u}v\varphi^2\,dx \le K^{\frac{1}{\alpha}}H^{2-\frac{1}{\alpha}}.$$

Combining with (6.2)-(6.3), this gives

$$\frac{\sqrt{2\alpha-1}}{\alpha}H \le K^{\frac{1}{2\alpha}}H^{1-\frac{1}{2\alpha}} + C||e^{\frac{\alpha}{2}u}\nabla\varphi||_{L^2(\Omega)}, \tag{6.7}$$

$$\frac{2}{\sqrt{\alpha}}K \le H^{\frac{1}{2\alpha}}K^{1-\frac{1}{2\alpha}} + C||v^\alpha\nabla\varphi||_{L^2(\Omega)}. \tag{6.8}$$

Multiply (6.7) by (6.8). Then,

$$\left(\frac{2\sqrt{2\alpha-1}}{\alpha\sqrt{\alpha}}-1\right)HK \le aH^{\frac{1}{2\alpha}}K^{1-\frac{1}{2\alpha}} + bK^{\frac{1}{2\alpha}}H^{1-\frac{1}{2\alpha}} + ab, \tag{6.9}$$

where

$$a = C||e^{\frac{\alpha}{2}u}\nabla\varphi||_{L^2(\Omega)} \quad\text{and}\quad b = C||v^\alpha\nabla\varphi||_{L^2(\Omega)}.$$

Note that for $\alpha \in (\alpha^\sharp, \alpha^*)$,

$$\delta := \frac{2\sqrt{2\alpha-1}}{\alpha\sqrt{\alpha}} - 1 > 0.$$

Introduce

$$X = K^{\frac{1}{2\alpha}}H^{1-\frac{1}{2\alpha}} \quad\text{and}\quad Y = H^{\frac{1}{2\alpha}}K^{1-\frac{1}{2\alpha}}.$$

Then, (6.9) can be rewritten as

$$\delta XY \le aY + bX + ab,$$

and so, either X is bounded by a multiple of a or Y by a multiple of b. In the former case, recalling (6.7), we obtain (6.6). In the latter case, (6.8) implies (6.5). □

In the two previous lemmata, we have used successively the equation and the stability assumption. Now, we apply the Sobolev inequality to set up a bootstrap procedure.

Lemma 6.4. *Make the same assumptions as in Lemma 6.3. Take* $\alpha \in (\alpha^\sharp, \alpha^*)$ *and for* $R > 0$*, let* B_R *denote a ball of radius* R *(resp. an annulus of radii* R *and* $R/2$*) contained in* Ω*. Assume that there exists a constant* C *depending on* N *and* α *only, such that for all* $R > 0$ *(resp. for all* R *large enough)*

$$\int_{B_R} (e^{\alpha u} + v^{2\alpha})\, dx \le C R^{N-4\alpha}. \tag{H_α}$$

Then, $\left(H_{\frac{N}{N-2}\alpha}\right)$ *also holds.*

Remark 6.5. The inequality (H_α) can be further simplified if the boundary values/limiting behavior at infinity of the solution is known. See in particular Proposition A.1.

Proof. Assume (H_α) is valid. By Lemma 6.3, either (6.5) or (6.6) holds. Assume that (6.5) is valid (the other case is similar). Using the Sobolev embedding, we obtain

$$\left(\int_{\mathbb{R}^N} v^{2^*\alpha}\varphi^{2^*}\, dx\right)^{\frac{2}{2^*}} \le \int_{\mathbb{R}^N} |\nabla(v^\alpha\varphi)|^2 dx \lesssim \int_{\mathbb{R}^N} v^{2\alpha}|\nabla\varphi|^2 dx.$$

Take a standard cut-off function $\psi \in C^1_c(B_2)$ such that $0 \le \psi \le 1$, $\psi = 1$ in B_1, and $\psi = 0$ outside B_2. Apply the above inequality with $\varphi(x) = \psi(x/R)$ and use (H_α). Then,

$$\int_{B_R} v^{2^*\alpha} dx \lesssim R^{N-2.2^*\alpha}. \tag{6.10}$$

Going back to (6.7), we deduce similarly that

$$\int_{B_R} e^{\frac{2^*\alpha}{2}u} dx \lesssim R^{N-2^*.2\alpha}. \tag{6.11}$$

This concludes the proof of our Lemma. □

Bootstrapping Lemma 6.4, we find:

Corollary 6.6. *Make the same assumptions as in Lemma 6.3. Assume that* (H_α) *holds for some* $\alpha \in (\alpha^\sharp, \alpha^*)$*. Then,*

$$\begin{aligned} &\int_{B_R} e^{pu}\, dx \le C R^{N-4p}, \qquad \text{for all } p < p^* := \alpha^* + \frac{1}{2}, \\ &\int_{B_R} v^q\, dx \le C R^{N-2q}, \qquad \text{for all } q < q^* := \frac{2N}{N-2}\alpha^*. \end{aligned} \tag{6.12}$$

Proof. By Hölder's inequality, if (H_α) holds for some α, then (H_β) holds for all $\beta \le \alpha$. So, bootstrapping (H_α), we easily deduce that it holds for all $\alpha < \frac{N}{N-2}\alpha^*$. Fix at last $\alpha < \alpha^*$ and apply now stability (6.1), with test function $e^{\frac{\alpha}{2}u}\psi(x/R)$, to deduce that (6.12) holds for all $p = \alpha + \frac{1}{2} < p^* = \alpha^* + \frac{1}{2}$. □

7. The Liouville theorem

We prove here Theorem 1.7. Assume by contradiction that there exists a solution u of (1.2) which is stable outside a compact set $K \subset B_{R_0}$ and such that $\overline{v}(\infty) = 0$.

Step 1. $e^u \in L^p(\mathbb{R}^N)$, for every $p < p^*$.

By Lemma 5.1, we have

$$\int_{A_R} v\, dx \le CR^{N-2} \quad \text{for every } R > 2R_0. \tag{7.1}$$

In addition, stability (1.3) implies that

$$\int_{A_R} e^u\, dx \le CR^{N-4} \quad \text{for every } R > 2R_0. \tag{7.2}$$

Recall now the following standard elliptic estimate : for $p \in [1, \frac{N}{N-2})$,

$$||v||_{L^p(B_2\setminus B_1)} \lesssim ||\Delta v||_{L^1(B_2\setminus B_1)} + ||v||_{L^1(B_4\setminus B_{1/2})},$$

and its rescaled version

$$R^{-\frac{N}{p}}||v||_{L^p(A_R)} \lesssim R^{-N}\left(R^2||\Delta v||_{L^1(A_R)} + ||v||_{L^1(B_{4R}\setminus B_{R/2})}\right).$$

Applying this estimate respectively to u and v, we deduce from (7.1), (7.2) that (H_α) holds for any $\alpha \in [1, \frac{N}{N-2})$ and all large R. Hence, (6.12) holds. By a straightforward covering argument, Step 1 follows.

The rest of the proof is very similar to the one given in [9].

Step 2.

$$\lim_{|x|\to+\infty} |x|^4 e^u = 0.$$

By Step 1, given any $\delta > 0$, we can choose $\tilde{R}$ large enough such that

$$\int_{|y|\ge\tilde{R}} e^{pu} dx \le \delta^p. \tag{7.3}$$

Let $x \in \mathbb{R}^N\setminus\overline{B_{R_0}}$, $|x| \ge 4\tilde{R}$. Set $R = \frac{2}{3}|x|$. This yields

$$B_{R/4}(x) \subset A_R = B_{2R}\setminus\overline{B_R} \subset \{y \in \mathbb{R}^N \ : \ |y| \ge \tilde{R}\}.$$

Thus, we have

$$\begin{aligned} &\int_{B_{R/4}(x)} e^{pu}\, dx \le CR^{N-4p}, &&\text{for all } p < p^* := \alpha^* + \frac{1}{2},\\ &\int_{B_{R/4}(x)} v^q\, dx \le CR^{N-2q}, &&\text{for all } q < q^* := \frac{2N}{N-2}\alpha^*. \end{aligned} \tag{7.4}$$

Next fix $\delta > 0$ and consider $w = e^u$. By Kato's inequality, w satisfies

$$-\Delta w - vw \le 0 \quad \text{in } \mathbb{R}^N. \tag{7.5}$$

Take ε small enough such that $\frac{N}{2-\varepsilon} < \frac{2N}{N-2}\alpha^*$. Here, we have used the assumption $5 \le N \le 12$. The Serrin-Trudinger inequality [27], [30] for subsolutions to (7.5) ensures that for any $p < p^*$ we have

$$||w||_{L^\infty(B_{R/8}(x))} \le CR^{-\frac{N}{p}}||w||_{L^p(B_{R/4}(x))},$$

where $C > 0$ depends only on N, p and

$$R^\varepsilon ||v||_{L^{\frac{N}{2-\varepsilon}}(B_{R/4}(x))}.$$

In particular, for $p = \frac{N}{4}$ and using (7.4) we find

$$e^{u(x)} \le CR^{-4}||e^u||_{L^{N/4}(B_{R/4}(x))}, \tag{7.6}$$

where $C > 0$ depends only on N. Combining (7.3) and (7.6) we infer that

$$e^{u(x)} \le C\delta R^{-4} \lesssim \delta |x|^{-4},$$

which proves Step 2.

Step 3. By Step 2, there exists $R_1 > 2R_0$ such that

$$-\Delta \overline{v} \le \frac{1}{2r^4} \quad \text{for all } r > R_1.$$

Hence,

$$\overline{v}'(r) \ge \frac{C(N)}{r^{N-1}} - \frac{1}{2(N-4)r^3} \quad \text{for all } r > R_1.$$

Integrating between r and $+\infty$, this yields

$$-\Delta \overline{u}(r) = \overline{v}(r) = \overline{v}(r) - \overline{v}(\infty) \le \frac{1}{2r^2}\left(\frac{1}{2(N-4)} - \frac{C'(N)}{r^{N-4}}\right) \quad \text{for all } r > R_1.$$

Hence

$$-\Delta \overline{u}(r) \le \frac{1}{2r^2} \quad \text{for all } r > R_1.$$

In the same way as above, we find $R_2 > R_1$ such that

$$\overline{u}'(r) \ge -\frac{1}{r} \quad \text{for all } r > R_2.$$

Integrating this latter estimate and taking the exponential, we obtain

$$r^4 e^{\overline{u}(r)} \ge cr^3 \quad \text{for all } r > R_2,$$

where the constant $c > 0$ does not depend on r. By Jensen's inequality, we have for any $r > R_2$,

$$cr^3 \le r^4 e^{\overline{u}(r)} \le r^4 \overline{e^u}(r) \le \max_{|x|=r}\left[|x|^4 e^{u(x)}\right].$$

This contradicts Step 2. $\square$

8. Partial regularity of the extremal solution in dimension $N \geq 13$.

In this section, we prove the second part of Theorem 1.9. As in the previous sections, we interpret our equation as the system (4.2). Here, our task consists in showing that if a rescaled L^p norm of e^u is small on some ball B, then it remains small on any ball of smaller radius, which is included in B. This provides an estimate in Morrey spaces, for which an ϵ-regularity theorem is available, thanks to the Moser-Trudinger inequality.

Let u be the extremal solution of (1.8). By scaling, we may assume that $B_2 \subset\subset \Omega$. For any $x \in B_{1/2}$, and $0 < r \leq 1 - |x|$ we define

$$E(x,r) = r^{4-N} \int_{B_{2r}(x)} e^u dy \quad \text{and} \quad F(x,r) = r^{4-N} \int_{B_{2r}(x)} v^2 dy. \tag{8.1}$$

Lemma 8.1. *There exists $\varepsilon_0 \in (0,1)$ and $\eta \in (0, \frac{1}{4})$ such that if for some $\varepsilon \in (0, \varepsilon_0)$ we have*

$$E(0,1) + F(0,1) < \varepsilon, \tag{8.2}$$

then

$$E(x,r) + F(x,r) \leq C(N)\varepsilon \quad \textit{for all } x \in B_{1/2},\ 0 < r \leq \eta, \tag{8.3}$$

where $C(N) > 0$ is a positive constant depending on N only.

Proof. We shall perform the proof of Lemma 8.1 along six steps.

Step 1. Decoupling.

Let $u = u_1 + u_2$ and $v = v_1 + v_2$, where

$$\begin{cases} -\Delta u_1 = 0 & \text{in } B_1(0), \\ u_1 = u & \text{on } \partial B_1(0), \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 = v & \text{in } B_1(0), \\ u_2 = 0 & \text{on } \partial B_1(0), \end{cases} \tag{8.4}$$

$$\begin{cases} -\Delta v_1 = 0 & \text{in } B_1(0), \\ v_1 = v & \text{on } \partial B_1(0), \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v_2 = e^u & \text{in } B_1(0), \\ v_2 = 0 & \text{on } \partial B_1(0). \end{cases} \tag{8.5}$$

Since $v > 0$, by the maximum principle, $v_1, v_2 > 0$ in B_1.

Step 2. Estimates on u_2, v_2.

Using (8.2) and (8.4) we find

$$||\Delta u_2||_{L^2(B_1)} = ||v||_{L^2(B_1)} \leq C\varepsilon^{\frac{1}{2}}. \tag{8.6}$$

To estimate v_2, we first use (8.2), (8.5) and standard elliptic estimates to obtain

$$||v_2||_{L^1(B_1)} \leq C||e^u||_{L^1(B_1)} \leq C\varepsilon. \tag{8.7}$$

We need further an estimate on the L^2 norm of v_2. To this aim, let $\varphi \in C^1_c(B_2)$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ in B_1. Then, by (8.2) and a boostrap argument as in Lemma 6.3 we have

$$\int_{B_2} \varphi^2 (e^{\frac{3}{2}u} + v^2 e^{\frac{u}{2}}) \le C \int_{B_2} (e^u + v^2)|\nabla\varphi|^2 \le C\varepsilon.$$

By Cauchy-Schwarz inequality we next obtain

$$||ve^u||_{L^1(B_1)} \le C\varepsilon. \tag{8.8}$$

Observe now that $-\Delta v_2^2 = -2|\nabla v_2|^2 + 2v_2 e^u$ in B_1 and $v_2^2 = 0$ on ∂B_1. Thus, by (8.8) we have

$$||\Delta v_2^2||_{L^1(B_1)} \le 2\int_{B_1} |\nabla v_2|^2 dy + 2\int_{B_1} v_2 e^u dy \le 4\int_{B_1} ve^u dy \le C\varepsilon.$$

By standard elliptic estimates, for $p \in (1, \frac{N}{N-2})$,

$$||v_2^2||_{L^p(B_1)} \le C\varepsilon.$$

Interpolating this last estimate with the L^1 estimate in (8.7) we derive,

$$||v_2||_{L^2(B_1)} \le ||v_2||^{\frac{p-1}{2p-1}}_{L^1(B_1)} ||v_2||^{\frac{p}{2p-1}}_{L^{2p}(B_1)} \le C\varepsilon^{\frac{1}{2}+\gamma}, \tag{8.9}$$

where $\gamma = \frac{p-1}{2(2p-1)} > 0$.

Step 3. Estimates on $E(x,r)$.
Let $\theta > 0$ that will be chosen later. We decompose $E(x,r) = E_1(x,r) + E_2(x,r)$, where

$$E_1(x,r) = r^{4-N}\int_{B_{2r}(x)\cap[u_2\le\theta]} e^u dy\,, \quad E_2(x,r) = r^{4-N}\int_{B_{2r}(x)\cap[u_2>\theta]} e^u dy.$$

Using the fact that e^{u_1} is subharmonic, for all $0 < r \le 1/4$ we have

$$\begin{aligned} E_1(x,r) &\le 2^N e^\theta r^4 \Big[(2r)^{-N}\int_{B_{2r}(x)} e^{u_1} dy\Big] \le 2^{2N} e^\theta r^4 \int_{B_{1/2}(x)} e^{u_1} dy \\ &\le 2^{2N} e^\theta r^4 \int_{B_1} e^u dy \le 2^{2N} e^\theta r^4 \varepsilon. \end{aligned}$$

Also by (8.6) and (8.9) we derive

$$\begin{aligned} E_2(x,r) = r^{4-N}\int_{B_{2r}(x)\cap[u_2>\theta]} e^u dy &\le \frac{r^{4-N}}{\theta}\int_{B_{2r}(x)} e^u u_2 dy \\ &\le \frac{r^{4-N}}{\theta}\int_{B_1} u_2(-\Delta v_2) dy = \frac{r^{4-N}}{\theta}\int_{B_1} (-\Delta u_2) v_2 dy \\ &\le \frac{r^{4-N}}{\theta} ||\Delta u_2||_{L^2(B_1)} ||v_2||_{L^2(B_1)} \le \frac{Cr^{4-N}}{\theta}\varepsilon^{1+\gamma}. \end{aligned}$$

Hence

$$E(x,r) \le 2^{2N} e^\theta r^4 \varepsilon + \frac{Cr^{4-N}}{\theta}\varepsilon^{1+\gamma}. \tag{8.10}$$

Step 4. Estimates on $F(x,r)$.
Remark first that

$$F(x,r) \le 2\Big[r^{4-N}\int_{B_{2r}(x)} v_1^2 dy + r^{4-N}\int_{B_{2r}(x)} v_2^2 dy\Big].$$

Let

$$F_i(x,r) = 2r^{4-N}\int_{B_{2r}(x)} v_i^2 dy\,, \quad i=1,2.$$

Since v_1^2 is subharmonic in B_1 we have

$$\begin{aligned} F_1(x,r) &= 2^{N+1}r^4\Big[(2r)^{-N}\int_{B_{2r}(x)} v_1^2 dy\Big] \le 2^{2N+1}r^4\int_{B_{1/2}(x)} v_1^2 dy \\ &\le 2^{2N+1}r^4\int_{B_1} v^2 dy \le 2^{2N+1}r^4\varepsilon. \end{aligned}$$

Using next the estimates (8.6) and (8.9) we have

$$\begin{aligned} F_2(x,r) &= 2r^{4-N}\int_{B_{2r}(x)} v_2^2 dy \le 2r^{4-N}\int_{B_1} v_2 v dy = 2r^{4-N}\int_{B_1} v_2(-\Delta u_2)dy \\ &\le 2r^{4-N}||\Delta u_2||_{L^2(B_1)}||v_2||_{L^2(B_1)} \le Cr^{4-N}\varepsilon^{1+\gamma}. \end{aligned}$$

Hence

$$F(x,r) \le 2^{2N+1}r^4\varepsilon + Cr^{4-N}\varepsilon^{1+\gamma}. \tag{8.11}$$

Step 5. Estimates on $E(x,\eta)+F(x,\eta)$.
We combine (8.10) and (8.11) for $r=\eta\le 1/4$. We find

$$E(x,\eta)+F(x,\eta) \le \Big[2^{2N}e^{\theta}\eta^4 + \frac{C\eta^{4-N}}{\theta}\varepsilon^{\gamma} + 2^{2N+1}\eta^4 + C\eta^{4-N}\varepsilon^{\gamma}\Big]\varepsilon. \tag{8.12}$$

Fix $\eta\in(0,1/4)$ such that $2^{2N+1}\eta^4 \le 1/8$. Fix $\varepsilon_0>0$ small enough such that

$$e^{\varepsilon_0^{\gamma/2}} \le 2 \quad \text{and} \quad C\eta^{4-N}\varepsilon_0^{\gamma/2} \le \frac{1}{8},$$

and let $\theta = \varepsilon_0^{\gamma/2} > 0$. Then, from (8.12) we find

$$E(x,\eta)+F(x,\eta) \le \varepsilon \quad \text{for all } 0<\varepsilon\le\varepsilon_0. \tag{8.13}$$

Step 6. Conclusion of the proof.
Let $U(x)=u(\eta x)+4\ln(\eta)$, $x\in\eta^{-1}\Omega$ and $V(x)=-\Delta U$. Then $\Delta^2 U = e^U$ in $\eta^{-1}\Omega$ and by (8.13) we have

$$\int_{B_2} e^U dy + \int_{B_2} V^2 dy = E(0,\eta)+F(0,\eta) \le \varepsilon.$$

Thus, by Step 5 we have

$$\eta^{4-N}\int_{B_{2\eta}(x)} e^U dy + \eta^{4-N}\int_{B_{2\eta}(x)} V^2 dy < \varepsilon, \quad \text{for all } x\in B_{1/2},$$

that is,

$$E(x,\eta^2)+F(x,\eta^2) < \varepsilon, \quad \text{for all } x\in B_{1/2}.$$

Working inductively we obtain

$$E(x,\eta^k)+F(x,\eta^k)<\varepsilon, \quad \text{for all } x\in B_{1/2} \text{ and } k\geq 1.$$

Let now $x\in B_{1/2}$ and $r\leq\eta$. Then $\eta^{k+1}<r\leq\eta^k$ for some $k\geq 1$. By the above estimates we find

$$\begin{aligned} E(x,r)+F(x,r) &= r^{4-N}\int_{B_{2r}(x)} e^{u(y)}dy + r^{4-N}\int_{B_{2r}(x)} v^2(y)dy \\ &\leq (\eta^{k+1})^{4-N}\int_{B_{2\eta^k}(x)} e^{u(y)}dy + (\eta^{k+1})^{4-N}\int_{B_{2\eta^k}(x)} v^2(y)dy \\ &= \eta^{4-N}(E(x,\eta^k)+F(x,\eta^k)) \\ &< \eta^{4-N}\varepsilon, \end{aligned}$$

which concludes the proof. □

Proof of Theorem 1.9 completed. Let Σ be the singular set of u and let $1<p<p^*$ be fixed. We claim that

$$\Sigma\subset A\cup B, \tag{8.14}$$

where

$$A=\left\{x\in\Omega : \limsup_{r\to 0} r^{4p-N}\int_{B_r(x)} e^{pu}>0\right\},$$
$$B=\left\{x\in\Omega : \limsup_{r\to 0} r^{4p-N}\int_{B_r(x)} v^{2p}>0\right\}.$$

Assume to the contrary that there exists $x_0\in\Sigma\setminus(A\cup B)$ such that

$$\lim_{r\to 0} r^{4p-N}\int_{B_r(x_0)} e^{pu} = \lim_{r\to 0} r^{4p-N}\int_{B_r(x_0)} v^{2p}=0.$$

By translation, we may assume $x_0=0$. By Hölder's inequality it follows that

$$\lim_{r\to 0} E(0,r)=\lim_{r\to 0} F(0,r)=0,$$

where E and F are the quantities defined in (8.1). Using a scaling argument, we may next assume that

$$E(0,1)+F(0,1)<\varepsilon_0,$$

where ε_0 is the positive constant introduced in Lemma 8.1. Thus, by Lemma 8.1 it follows that

$$\int_{B_r(x)} e^u(y)dy\leq C(N)\varepsilon_0 r^{N-4} \quad \text{for all } x\in B_{1/2} \text{ and } 0<r\leq 2\eta,$$

where $C(N)$ is a positive constant depending on N only. In particular, this implies that e^u belongs to the Morrey space $M^{\frac{N}{4}}(B_\eta)$ and

$$||e^u||_{M^{\frac{N}{4}}(B_\eta)}\leq\varepsilon_0. \tag{8.15}$$

Let

$$w(x) = \int_{\mathbb{R}^N} |x-y|^{4-N} e^{u(y)} dy\,, \quad x \in B_{1/2},$$

and $\tilde{w} = u - w$. By Lemma 7.20 in [20] it follows that $e^{\beta w} \in L^1(B_\eta)$ for all

$$0 < \beta < \frac{C(N)}{||e^u||_{M^{\frac{N}{4}}(B_\eta)}},$$

where $C(N) > 0$ depends on N only.

Using next the estimate (8.15) and the fact that $\tilde{w}$ is biharmonic in $B_{1/2}$, it follows that $e^u \in L^\beta(B_\eta)$ for all

$$0 < \beta < C(N)\varepsilon_0^{-1}.$$

Letting $\varepsilon_0 << 1$ small, we have $e^u \in L^\beta(B_\eta)$ for some $\beta > N/4$, which, by standard regularity theory, yields $u \in L^\infty(B_\eta)$ and contradicts $0 \in \Sigma$.

Hence, (8.14) holds for all $1 < p < p^*$. By [12, Lemma 5.3.4], it follows that $\mathcal{H}^{N-4p}(A) = \mathcal{H}^{N-4p}(B) = 0$, so that $\mathcal{H}^{N-4p}(\Sigma) = 0$ for all $1 < p < p^*$. Thus $\mathcal{H}_{dim}(\Sigma) \leq N - 4p^*$. $\square$

9. A scaling argument

This last section is devoted to the proof of Theorem 1.11. By rescaling, we may always assume that $\lambda = 1$. By standard elliptic regularity, it suffices to show that $u \leq C$ in Ω. We assume to the contrary that there exists a sequence of solutions u_n of fixed index k, such that $M_n := \max_{\overline{\Omega}} u_n \to +\infty$, as $n \to +\infty$. Let x_n denote a corresponding point of maximum of u_n. Passing to a subsequence if necessary, we may assume that there exists $x_0 \in \overline{\Omega}$, such that $x_n \to x_0$, as $n \to +\infty$. Since Ω is convex, we can also assert that u_n is uniformly bounded in a fixed neighborhood of the boundary i.e. $x_n \in \omega \subset\subset \Omega$ for large n. See e.g. [34] for this standard boundary estimate.

We use a scaling argument. Let $r_n = e^{-M_n/4}$ and $U_n(x) = u_n(x_n + r_n x) - M_n$, for $x \in \Omega_n := \frac{1}{r_n}(\Omega - x_n)$. Then, U_n solves

$$\begin{cases} \Delta^2 U_n = e^{U_n} & \text{in } \Omega_n, \\ U_n + M_n = \Delta U_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{9.1}$$

Since $x_n \in \omega \subset\subset \Omega$, we have $\Omega_n \to \mathbb{R}^N$ as $n \to +\infty$. We claim that $\{U_n\}$ is uniformly bounded on compact sets of $\mathbb{R}^N$. To see this, fix a ball B_R and n so large that $B_R \subset \Omega_n$. Write $-\Delta U_n = V_n = V_n^1 - V_n^2$, where V_n^1 solves

$$\begin{cases} -\Delta V_n^1 = e^{U_n} & \text{in } B_R, \\ V_n^1 = 0 & \text{on } \partial B_R, \end{cases}$$

and V_n^2 is harmonic in B_R. Since $U_n \le 0$, V_n^1 is uniformly bounded in B_R. Using the assumption (1.9), we also have

$$\int_{B(x_n,r_nR)} v_n\,dx \le C(r_nR)^{N-2}. \tag{9.2}$$

In other words, $V_n = -\Delta U_n$ is bounded in $L^1(B_R)$. Since V_n^1 is uniformly bounded in B_R, it follows that V_n^2 is bounded in $L^1(B_R)$. Since V_n^2 is harmonic, V_n^2 is uniformly bounded in $B_{R/2}$. Similarly, write $U_n = U_n^1 - U_n^2$, where

$$\begin{cases} -\Delta U_n^1 = V_n & \text{in } B_{R/2}, \\ U_n^1 = 0 & \text{on } \partial B_{R/2}, \end{cases}$$

and $U_n^2 \ge 0$ is harmonic in $B_{R/2}$. Since V_n is bounded in $B_{R/2}$, so is U_n^1. Since $U_n^2(0) = U_n^1(0)$, we may now apply Harnack's inequality to conclude that U_n^2 is uniformly bounded in $B_{R/4}$. Hence, U_n is uniformly bounded on compact subsets of $\mathbb{R}^N$. So, we may pass to the limit in the first line of (9.1) and find a solution U of finite Morse index to (1.2). Thanks to (9.2), if $V = -\Delta U$ and $R > 0$, then

$$\int_{B_R} V\,dx \le CR^{N-2},$$

which is possible only if $\overline{V}(\infty) = 0$. This contradicts Theorem 1.7. □

Appendix A. Appendix

The following result is a consequence of Kato's inequality. Our proof[5] builds upon a similar result of P. Souplet [28], as well as similar inequalities on bounded domains subsequently found by C. Cowan, P. Esposito and N. Ghoussoub [7].

Proposition A.1. *Let u be a stable solution of (1.2) and $v = -\Delta u$. Then,*

$$v \ge \sqrt{2}e^{\frac{u}{2}} \quad \text{in } \mathbb{R}^N.$$

Proof. Let $w := \sqrt{2}e^{\frac{u}{2}} - v$. A straightforward calculation yields

$$\Delta w \ge \frac{1}{\sqrt{2}} e^{\frac{u}{2}} w \quad \text{in } \mathbb{R}^N. \tag{A.1}$$

Since $v > 0$ (see the proof of Lemma 5.1), we have $w^+ \le \sqrt{2}e^{\frac{u}{2}}$ in $\mathbb{R}^N$. We multiply (A.1) by w^+ and integrate over $B_R(0)$, for $R > 0$. We obtain

$$\int_{B_R} |\nabla w^+|^2 + \frac{1}{\sqrt{2}} \int_{B_R} e^{\frac{u}{2}} |w^+|^2 \le \int_{\partial B_R} w^+ \frac{\partial w^+}{\partial R}. \tag{A.2}$$

[5]N. Ghoussoub informed us that his collaborators and him obtained the same result prior to us in unpublished work.

For all $R > 0$ define

$$f(R) := \frac{1}{2} \int_{\partial B_1} |w^+(Rx)|^2 d\sigma(x).$$

Then (A.2) reads

$$\int_{B_R} |\nabla w^+|^2 + \frac{1}{\sqrt{2}} \int_{B_R} e^{\frac{u}{2}} |w^+|^2 \leq R^{N-1} f'(R). \tag{A.3}$$

Since u is stable and $e^u \geq \frac{1}{2}|w^+|^2$ in $\mathbb{R}^N$, we have

$$R^{N-4} \gtrsim \int_{B_R} e^u ds \geq \int_0^R r^{N-1} f(r) dr.$$

In particular, f cannot be strictly increasing in a given interval $[S, +\infty)$. Hence, there exists an increasing sequence $\{R_j\}$ such that $R_j \to \infty$ and $f'(R_j) \leq 0$. Now, letting $R = R_j$ in (A.3), we find $w^+ \equiv 0$, that is, $v \geq \sqrt{2} e^{\frac{u}{2}}$ in $\mathbb{R}^N$. □

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