Steady-state solutions for Gierer-Meinhardt type systems with Dirichlet boundary condition

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Abstract. This paper is concerned with the following Gierer-Meinhardt type systems subject to Dirichlet boundary conditions

$$\begin{cases} \Delta u - \alpha u + \dfrac{u^p}{v^q} + \rho(x) = 0, \; u > 0 & \text{in } \Omega, \\ \Delta v - \beta v + \dfrac{u^r}{v^s} = 0, \; v > 0 & \text{in } \Omega, \\ u = 0, \; v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a smooth bounded domain, $\rho(x) \geq 0$ in Ω and $\alpha, \beta \geq 0$. We are mainly interested in the case of different source terms, that is, $(p, q) \neq (r, s)$. Under appropriate conditions on the exponents p, q, r and s we establish various results of existence, regularity and boundary behavior. In the one dimensional case a uniqueness result is also presented.

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In 1972 Gierer and Meinhardt [8] proposed a mathematical model for pattern formation of spatial tissue structures in morphogenesis, a biological phenomenon discovered by Trembley [24] in 1744. The mechanism behind the Gierer-Meinhardt's model is based on the existence of two chemical substances: a slowly diffusing activator and a rapidly diffusing inhibitor. The ratio of their diffusion rates is assumed to be small.

The model introduced by Gierer and Meinhardt reads as

$$\begin{cases} u_t = d_1\Delta u - \alpha u + c\rho\dfrac{u^p}{v^q} + \rho_0\rho & \text{in } \Omega\times(0,T), \\ v_t = d_2\Delta v - \beta v + c'\rho'\dfrac{u^r}{v^s} & \text{in } \Omega\times(0,T), \end{cases} \tag{0.1}$$

subject to Neumann boundary conditions in a smooth bounded domain Ω. Here the unknowns u and v stand for the concentration of activator and inhibitor with the source distributions ρ and ρ' respectively. In system (0.1), d_1, d_2 are the diffusion coefficients and $\alpha, \beta, c, c', \rho_0$ are positive constants. The exponents $p, q, r, s > 0$ verify the relation $qr > (p-1)(s+1) > 0$.

The model introduced by Gierer and Meinhardt has been used with satisfactory quantitative results for modelling the head regeneration process of hydra, an animal of few millimeters in length, consisting of 100,000 cells of about 15 different types and having a polar structure.

The Gierer-Meinhardt system originates in the Turing's one [23] introduced in 1952 as a mathematical model for the development of complex organisms from a single cell. It has been emphasized that localized peaks in concentration of chemical substances, known as inducers or morphogens, could be responsible for a group of cells developing differently from the surrounding cells. Turing discovered through linear analysis that a large difference in relative size of diffusivities for activating and inhibiting substances carries instability of the homogeneous, constant steady state, thus leading to the presence of nontrivial, possibly stable stationary configurations.

A global existence result for a more general system than (0.1) is given in the recent paper of Jiang [10]. It has also been shown that the dynamics of the system (0.1) exhibit various interesting behaviors such as periodic solutions, unbounded oscillating global solutions, and finite time blow-up solutions. We refer the reader to Ni, Suzuki and Takagi [18] for the entire description of dynamics concerning the system (0.1).

Many works have been devoted to the study of the steady-state solutions of (0.1), that is, solutions of the stationary system

$$\begin{cases} d_1\Delta u - \alpha u + c\rho\dfrac{u^p}{v^q} + \rho_0\rho = 0 & \text{in } \Omega, \\ d_2\Delta v - \beta v + c'\rho'\dfrac{u^r}{v^s} = 0 & \text{in } \Omega, \end{cases} \tag{0.2}$$

subject to Neumann boundary conditions. The main difficulty in the treatment of (0.2) is the lack of variational structure. Another direction of research is to consider the *shadow system* associated to (0.2), an idea due to Keener [11]. This system is obtained dividing by d_2 in the second equation and then letting $d_2 \to \infty$. It has been shown that nonconstant solutions of the shadow system associated to (0.2) exhibit interior or boundary concentrating points. Among the large number of works in this direction we refer the interested reader to [19], [20], [21], [25], [26] as well as to the survey papers of Ni [16], [17].

In this paper new features of Gierer-Meinhardt type systems are emphasized. More exactly, we shall be concerned with systems of the following type

$$\begin{cases} \Delta u - \alpha u + \dfrac{u^p}{v^q} + \rho(x) = 0, \;\; u > 0 & \text{in } \Omega, \\ \Delta v - \beta v + \dfrac{u^r}{v^s} = 0, \;\; v > 0 & \text{in } \Omega, \\ u = 0, \;\; v = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.3}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 1$). Here u and v represent the concentration of the activator and inhibitor and $\rho \in C^{0,\gamma}(\overline{\Omega})$ ($0 < \gamma < 1$) represents the source distribution of the activator. We assume that $\rho \geq 0$ in Ω, $\rho \not\equiv 0$ and α, β are nonnegative real numbers. The case $\rho \equiv 0$ is more delicate and involves a more careful analysis of the Gierer-Meinhardt system. This situation has been analyzed in the recent works [1], [2], [18], [21], [25], [26].

We are mainly interested in this paper in the case where the activator and inhibitor have different source terms, that is, $(p, q) \neq (r, s)$.

Let us notice that the homogeneous Dirichlet boundary condition in (0.3) (instead of Neumann's one as in (0.2)) turns the system singular in the sense that the nonlinearities $\frac{u^p}{v^q}$ and $\frac{u^r}{v^s}$ become unbounded around the boundary.

The existent results in the literature for (0.3) concern the case of common sources of the concentrations, that is, $(p, q) = (r, s)$. If $p = q = r = s = 1$ and $\rho \equiv 0$, the system (0.3) was studied in Choi and McKenna [1]. In Kim [12], [13] it is studied the system (0.3) with $p = r$ and $q = s$. In the case of common sources, a *decouplization* of system is suitable in order to provide *a priori* estimates for the unknowns u and v. More precisely, if $p = r$ and $q = s$ then, subtracting the two equations in (0.3) and letting $w = u - v$ we get the following equivalent form

$$\begin{cases} \Delta w - \alpha w + (\beta - \alpha) w v + \rho(x) = 0, & \text{in } \Omega, \\ \Delta v - \beta v + \dfrac{(v + w)^p}{v^q} = 0 & \text{in } \Omega, \\ v = w = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.4}$$

Thus, the study of system (0.3) amounts to the study of (0.4) in which the first equation is linear. This is more suitable to derive upper and lower barriers for u and v (see [1], [12], [13]). For more applications of decouplization method in the context of elliptic systems we refer the reader to [14]. We also mention here the paper of Choi and McKenna [2] where the existence of radially symmetric solutions in the case $p = r > 1$, $q = 1$, $s = 0$ and $\Omega = B_1 \subset \mathbb{R}^2$ is discussed. In [2], *a priori* bounds for concentrations u and v are obtained through sharp estimates for the associated Green's function.

In our case, such a decouplization is not possible due to the fact that $(p, q) \neq (r, s)$. In order to overcome this lack, we shall exploit the boundary behavior of solutions of single singular equations associated to system (0.3). In turn, this approach requires uniqueness or suitable comparison principles for single singular equations that come from our system. These features are usually associated with nonlinearities having a sublinear growth and that is why we restrict our attention to the case $p < 1$. Our results extend those presented in [5], [6] and give precise answers to some questions raised in Choi and McKenna [1], [2] and Kim [12], [13]. Also the approach we give in this paper enables us to deal with various type of exponents. For instance, we shall consider the case $p < 0$ (see Theorems 1.4 and 1.5) which means that the nonlinearity in the first equation of (0.3) is singular in both its variables u and v. Furthermore, these results can be successfully applied to treat the case $-1 < s \leq 0$ (see Remark **??**).

1. Main results

We are interested in the following range of exponents

$$
\begin{aligned}
&-\infty < p < 1, \\
&q, r, s > 0 \ \text{ and } \ s \geq r - 1. \qquad (1.1)
\end{aligned}
$$

In our approach we do not require any order relation between the nonnegative numbers α and β. Also we do not impose any growth condition on the source distribution $\rho(x)$ of the activator. A major role in our analysis will be played by the number

$$
\sigma = \min\left\{1, \frac{2+r}{1+s}\right\}. \qquad (1.2)
$$

First we are concerned with the case $0 \leq p < 1$. The existence result in this case is the following.

Theorem 1.1. *Assume that $0 \leq p < 1$, $q\sigma < p + 1$ and q, r, s satisfy (1.1). Then the system (0.3) has at least one classical solution and there exist $c_1, c_2 > 0$ such that any solution (u, v) of (0.3) satisfies the following estimates in Ω:*

$$
c_1 d(x) \leq u \leq c_2 d(x) \quad \text{in } \Omega,
$$

and

$$
\begin{aligned}
c_1 d(x) \leq v \leq c_2 d(x) \quad &\text{if } s < r+1, \\
c_1 d(x)\,(1 + |\ln d(x)|)^{1/(1+s)} \leq v \leq c_2 d(x)\,(1 + |\ln d(x)|)^{1/(1+s)} \quad &\text{if } s = r+1, \\
c_1 d(x)^{(2+r)/(1+s)} \leq v \leq c_2 d(x)^{(2+r)/(1+s)} \quad &\text{if } s > r+1,
\end{aligned}
$$

where $d(x) = dist(x, \partial\Omega)$.

Further regularity of the solution to (0.3) can be obtained using the same arguments as in Gui and Lin [9]. More precisely, it is proved in [9] that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $-\Delta u = u^{-\nu}$ in a smooth bounded domain Ω and $u = 0$ on $\partial\Omega$, then $u \in C^{1,1-\nu}(\overline{\Omega})$. Using the conclusion in Theorem 1.1 we have

Corollary 1.2. *Assume that $0 \le p < 1$ and q, r, s satisfy (1.1).*

(i) If $q \le p$ and $s \le r$, then the system (0.3) has at least one classical solution. Moreover, any solution of (0.3) belongs to $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$;

(ii) If $-1 < p-q < 0$ and $-1 < r-s < 0$, then the system (0.3) has at least one classical solution. Moreover, any solution (u, v) of (0.3) satisfies $u \in C^2(\Omega) \cap C^{1,1+p-q}(\overline{\Omega})$ and $v \in C^2(\Omega) \cap C^{1,1+r-s}(\overline{\Omega})$.

The issue of uniqueness is a delicate matter even in one dimension. In this case the system (0.3) reads

$$\begin{cases} u'' - \alpha u + \dfrac{u^p}{v^q} + \rho(x) = 0 & \text{in } (0,1), \\ v'' - \beta v + \dfrac{u^r}{v^s} = 0 & \text{in } (0,1), \\ u(0) = u(1) = 0, \ v(0) = v(1) = 0. \end{cases} \tag{1.3}$$

In [1] it is proved that the system (1.3) has a unique solution provided that $p = q = r = s = 1$. The main idea is to write (1.3) as a linear system with smooth coefficients and then to use the $C^2[0,1] \times C^2[0,1]$ regularity of the solution. This approach has been used in [5] (see also [4] or [6, Theorem 2.7]) in the case $\beta \le \alpha$, $0 < q \le p \le 1$ and $r - p = s - q \ge 0$.

In this paper we are able to show that the uniqueness of the solution to (1.3) still holds provided that

$$-1 < p - q < 1, \quad -1 < r - s < 1. \tag{1.4}$$

Note that for the above range of exponents, the solutions of (1.3) do not necessarily belong to $C^2[0,1] \times C^2[0,1]$. We prove that a $C^{1+\delta}$-regularity up to the boundary of the solution suffices in order to have uniqueness. Therefore, we prove

Theorem 1.3. *Let $\Omega = (0,1)$, $0 \le p < 1$ and $q, r, s > 0$ verify (1.4). Then the system (1.3) has a unique classical solution.*

Unlike the Neumann boundary condition, in which a large multiplicities of solutions are observed, the uniqueness in the above result seems to be a particular feature of the Dirichlet boundary condition together with the sublinear character of the first equation in the system (1.3).

Next, we are concerned with the case $-\infty < p < 0$. First, we prove the following nonexistence result.

Theorem 1.4. *Suppose $-\infty < p < 0$, $q, r, s > 0$ and one of the following hold*

(i) $q \ge 2$ and $s < 1$;

(ii) $q > 2$ and $s = 1$;

(iii) $q > s + 1$ and $s > 1$.

Then the system (0.3) has no classical solutions.

The corresponding existence result in this case is the following.

Theorem 1.5. *Assume that $-\infty < p < 0$, q, r, s satisfy (1.1) and $q\sigma < 2$. Then, the system (0.3) has classical solutions. Moreover, if $q < p + 1$ and $s < r + 1$ any classical solution (u, v) of (0.3) satisfies $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$.*

In proving Theorems 1.1 and 1.5 we rely on Schauder's fixed point theorem. The main point is to provide *a priori* bounds which allows us to control the map whose fixed points are the solutions of (0.3). To this aim, boundary estimates for solutions to single singular elliptic equations associated to (0.3) will be used.

The outline of the paper is as follows. In Section 3 we collect some auxiliary results concerning boundary estimates and comparison principles for elliptic equations involving singular nonlinearities. The proofs of the above results will be separately given in Sections 4 and 5 for the case $0 \le p < 1$ and $-\infty < p < 0$ respectively. In the Appendix we provide an extension of Lemma 8 in [1] which is a useful tool in proving the uniqueness of the solution in one dimension.

2. Auxiliary results

Throughout this paper $\|\cdot\|_\infty$ denotes the $L^\infty(\Omega)$ norm. Also we denote by λ_1 and φ_1 the first eigenvalue and the first normalized eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ with $\|\varphi_1\|_\infty = 1$. As it is well known, $\varphi_1 \in C^2(\overline{\Omega})$, $\varphi_1 > 0$ in Ω, and there exists $C > 0$ such that

$$C\, d(x) \le \varphi_1 \le \frac{1}{C}\, d(x) \quad \text{in } \Omega, \tag{2.1}$$

We also recall the following useful result which is due to Lazer and McKenna.

Lemma 2.1. (Lazer and McKenna [15]) *$\int_\Omega \varphi_1^\tau dx < \infty$ if and only if $\tau > -1$.*

Basic to our approach is the following comparison result which is suitable for singular nonlinearities. We refer the reader to [7, Lemma 2.1] for a complete proof.

Lemma 2.2. *Let $\Psi : \Omega \times (0, \infty) \to \mathbb{R}$ be a Hölder continuous function such that the mapping $(0, \infty) \ni t \longmapsto \frac{\Psi(x,t)}{t}$ is strictly decreasing for each $x \in \Omega$. Assume that there exist v_1, $v_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

(a) $\Delta v_1 + \Psi(x, v_1) \le 0 \le \Delta v_2 + \Psi(x, v_2)$ in Ω;
(b) $v_1, v_2 > 0$ in Ω and $v_1 \ge v_2$ on $\partial\Omega$;
(c) $\Delta v_1 \in L^1(\Omega)$ or $\Delta v_2 \in L^1(\Omega)$.
Then $v_1 \ge v_2$ in Ω.

Another useful tool is the following result which is a direct consequence of the maximum principle.

Lemma 2.3. *Let $k \in C(0, \infty)$ be a positive decreasing function and $a_1, a_2 \in C(\Omega)$ with $0 < a_2 \le a_1$ in Ω. Assume that there exist $\beta \ge 0$, $v_1, v_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $v_1, v_2 > 0$ in Ω, $v_1 \ge v_2$ on $\partial\Omega$ and*

$$\Delta v_1 - \beta v_1 + a_1(x)k(v_1) \le 0 \le \Delta v_2 - \beta v_2 + a_2(x)k(v_2) \quad \text{in } \Omega.$$

Then $v_1 \ge v_2$ in Ω.

Proposition 2.4. *Let $0 \le p < 1$, $0 < q < p+1$ and $a \in C^{0,\gamma}(\Omega)$ $(0 < \gamma < 1)$ be such that*

$$a_1\varphi_1^{-q}(x) \le a(x) \le a_2\varphi_1^{-q}(x) \quad \text{in } \Omega, \tag{2.2}$$

for some $a_1, a_2 > 0$. Then, the problem

$$\begin{cases} \Delta u - \alpha u + a(x)u^p + \rho(x) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Moreover, there exist $m_1, m_2 > 0$ such that

$$m_1\varphi_1 \le u \le m_2\varphi_1 \quad \text{in } \Omega. \tag{2.4}$$

Proof.

Let w be the unique solution of

$$\begin{cases} \Delta w - \alpha w + \rho(x) = 0 & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

By standard elliptic arguments and maximum principle we have $w \in C^2(\overline{\Omega})$. Obviously $\underline{u} := w$ is a sub-solution of (2.3). Furthermore, by virtue of (2.1) we can find $c_1, c_2 > 0$ such that

$$c_1\varphi_1 \le w \le c_2\varphi_1 \quad \text{in } \Omega. \tag{2.6}$$

Since $q < p+1$, by a result in Wei [27], there exists $h \in C^2(0,1) \cap C^1[0,1]$ such that

$$\begin{cases} -h''(t) = t^{-q}h^p(t), & \text{for all } 0 < t < 1, \\ h > 0 & \text{in } (0,1), \\ h(0) = h(1) = 0. \end{cases}$$

Using the fact that $h'(0) > 0$ we have

$$c_3 t \le h(t) \le c_4 t, \tag{2.7}$$

for $t > 0$ small enough and for some $c_3, c_4 > 0$. Furthermore, we may find $c > 0$ such that $h'(c\varphi_1) > 0$ in $\overline{\Omega}$.

We are looking for a super-solution of (2.3) in the form $\overline{u} := Mh(c\varphi_1) + w$, for $M > 1$ large enough. For this purpose we have to check that the inequality $-\Delta\overline{u} + \alpha\overline{u} \ge a(x)\overline{u}^p + \rho(x)$ holds in Ω provided that $M > 1$ is sufficiently large.

We have

$$\begin{aligned} -\Delta\overline{u} + \alpha\overline{u} &\ge -\Delta(Mh(c\varphi_1)) + \rho(x) \\ &= Mc^{2-q}\varphi_1^{-q}h^p(c\varphi_1)|\nabla\varphi_1|^2 + M\lambda_1 c\varphi_1 h'(c\varphi_1) + \rho(x) \quad \text{in } \Omega. \end{aligned}$$

By (2.7) we may write

$$-\Delta\overline{u} + \alpha\overline{u} \ge Mc^{2+p-q}c_3^p\varphi_1^{p-q}|\nabla\varphi_1|^2 + M\lambda_1 c\varphi_1 h'(c\varphi_1) + \rho(x) \quad \text{in } \Omega. \tag{2.8}$$

On the other hand, by (2.2), (2.6) and (2.7) we have

$$a(x)\overline{u}^p \le a_2\varphi_1^{-q}(Mh(c\varphi_1)+w)^p \le a_2\varphi_1^{p-q}(Mcc_4+c_2)^p \quad \text{in } \Omega. \tag{2.9}$$

Using the Hopf's maximum principle, there exist $\omega \subset\subset \Omega$ and $\delta > 0$ such that

$$|\nabla\varphi_1| > \delta \ \text{ in } \Omega\setminus\omega \quad \text{and} \ \ \varphi_1 > \delta \text{ in } \omega. \tag{2.10}$$

Since $0 \le p < 1$, we may choose $M > 1$ such that

$$Mc^{2+p-q}c_3^p\delta^2 > a_2(Mcc_4+c_2)^p, \tag{2.11}$$

$$M\lambda_1 c \min_{\overline{\omega}} \varphi_1 h'(c\varphi_1) \ge a_2(Mcc_4+c_2)^p \max_{\overline{\omega}} \varphi_1^{p-q}. \tag{2.12}$$

Combining (2.8), (2.9) and (2.11) we obtain

$$-\Delta\overline{u}+\alpha\overline{u} \ge Mc^{2+p-q}c_3^p\varphi_1^{p-q}|\nabla\varphi_1|^2+\rho(x) \ge a(x)\overline{u}^p+\rho(x) \quad \text{in } \Omega\setminus\omega. \tag{2.13}$$

Furthermore, by (2.8), (2.9) and (2.12) we deduce

$$-\Delta\overline{u}+\alpha\overline{u} \ge M\lambda_1 c\varphi_1 h'(c\varphi_1)+\rho(x) \ge a(x)\overline{u}^p+\rho(x) \quad \text{in } \omega. \tag{2.14}$$

Now the claim follows by (2.13) and (2.14). Thus, the problem (2.3) has a solution $u \in C^2(\Omega)\cap C(\overline{\Omega})$ such that $\underline{u} \le u \le \overline{u}$ in Ω. By (2.6) and (2.7) we obtain the estimate (2.4). This also implies that

$$C_1\varphi_1^{p-q} \le a(x)u^p \le C_2\varphi_1^{p-q} \quad \text{in } \Omega,$$

for some $C_1, C_2 > 0$. Since $p-q > -1$, by Lemma 2.1 we get $a(x)u^p \in L^1(\Omega)$ which finally yields $\Delta u \in L^1(\Omega)$. Now the uniqueness follows by Lemma 2.2. This concludes the proof of Proposition 2.4.

We next consider the problem

$$\begin{cases} \Delta v - \beta v + a(x)v^{-s} + b(x) = 0 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

where $a \in C^{0,\gamma}(\Omega)$ $(0 < \gamma < 1)$ satisfies

$$a_1\varphi_1^r(x) \le a(x) \le a_2\varphi_1^r(x) \quad \text{in } \Omega, \tag{2.16}$$

for some $a_1, a_2 > 0$ and $r \in \mathbb{R}$. We also assume that $b \in C^{0,\gamma}(\overline{\Omega})$, $\beta \ge 0$ and $s > 0$.

For the convenience, let us introduce $\Gamma_{s,r} : (0,\infty) \to (0,\infty)$ defined by

$$\Gamma_{s,r}(t) = \begin{cases} t & , \quad \text{if } s < r+1, \\ t(1+|\log t|)^{1/(1+s)} & , \quad \text{if } s = r+1, \\ t^{(2+r)/(1+s)} & , \quad \text{if } s > r+1, \end{cases} \tag{2.17}$$

for all $r > -2$ and $s > 0$. It is easy to see that

$$\Gamma_{s,r}(t) \ge t^\sigma \quad \text{for all } t > 0, \tag{2.18}$$

where σ is defined in (1.2). Moreover, for all $m > 0$ there exists $m_1, m_2 > 0$ such that

$$m_1\Gamma_{s,r}(t) \le \Gamma_{s,r}(mt) \le m_2\Gamma_{s,r}(t) \quad \text{for all } t > 0. \tag{2.19}$$

Proposition 2.5. (i) *If $r \le -2$ then the problem (2.15) has no classical solutions;*

(ii) *If $r > -2$ then the problem (2.15) has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Moreover, there exist $c_1, c_2 > 0$ such that*

$$c_1 \Gamma_{r,s}(\varphi_1) \le v \le c_2 \Gamma_{r,s}(\varphi_1) \quad \text{in } \Omega. \tag{2.20}$$

A general nonexistence result for singular elliptic equations with unbounded potentials can be found in [3]. Also a nonexistence result in the case $b \equiv 0$, $\beta = 0$ and $r \le -2$ is presented in [28, Theorem 1.2]. Concerning the existence part in Proposition 2.5, a similar result can be found in [9] in the case $b \equiv 0$, $\beta = 0$ and $r \ge 0$. We shall give here a different proof which relies on a direct construction of a sub and super-solution. This will provide the estimate (2.20).

Proof. (i) Assume that there exist $r \ge -2$ and $v \in C^2(\Omega) \cap C(\overline{\Omega})$ a classical solution of (2.15). For $0 < \varepsilon < 1$ consider the problem

$$\begin{cases} \Delta z - \beta z + a_1(\varphi_1 + \varepsilon)^r (z + \varepsilon)^{-s} = 0 & \text{in } \Omega, \\ z > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.21}$$

Obviously, $\underline{z} = 0$ is a sub-solution and $\overline{z} = v$ is a super-solution of (2.21). Hence, for all $0 < \varepsilon < 1$ there exists $z_\varepsilon \in C^2(\overline{\Omega})$ a solution of (2.21) such that $0 < z_\varepsilon \le v$ in Ω. Multiplying by φ_1 in (2.21) and then integrating over Ω we get

$$(\beta + \lambda_1) \int_\Omega z_\varepsilon \varphi_1 dx = a_1 \int_\Omega \varphi_1 (\varphi_1 + \varepsilon)^r (z_\varepsilon + \varepsilon)^{-s} dx.$$

Since $z_\varepsilon \le v$ in Ω, the above equality yields

$$(\beta + \lambda_1) \int_\Omega v \varphi_1 dx \ge a_1 (1 + \|v\|_\infty)^{-s} \int_\Omega \varphi_1 (\varphi_1 + \varepsilon)^r dx.$$

This implies

$$\int_\omega \varphi_1 (\varphi_1 + \varepsilon)^r dx < M \quad \text{for all } \omega \subset\subset \Omega,$$

where $M > 0$ does not depend on ε. Passing to the limit with $\varepsilon \to 0$ in the above inequality we find $\int_\omega \varphi_1^{1+r} dx < M$, for all $\omega \subset\subset \Omega$, that is, $\int_\Omega \varphi_1^{1+r} dx < \infty$. Since $r \ge -2$, the last inequality contradicts Lemma 2.1. Therefore, the problem (2.15) has no classical solutions if $r \ge -2$.

(ii) Let $r > -2$ and $s \ge r - 1$. According to [22, Theorem 1], there exists $H \in C^2(0,1) \cap C[0,1]$ such that

$$\begin{cases} -H''(t) = t^r H^{-s}(t), & \text{for all } 0 < t < 1, \\ H > 0 & \text{in } (0,1), \\ H(0) = H(1) = 0. \end{cases} \tag{2.22}$$

Since H is concave, there exists $H'(0+) > 0$. Hence, taking $0 < \eta < 1$ sufficiently small, we can assume that $H' > 0$ in $(0, \eta)$. From [22, p. 904] (see also Theorem 3.5 in [3]), there exist $c_1, c_2 > 0$ such that

$$c_1 \Gamma_{s,r}(t) \le H(t) \le c_2 \Gamma_{s,r}(t) \quad \text{in } (0, \eta). \tag{2.23}$$

As a consequence of (2.23) and the fact that $s \geq r-1$ we derive

$$H(t)^{s+1} \leq c_3 t^r \quad \text{in } (0,\eta), \tag{2.24}$$

for some positive constant $c_3 > 0$. Let $c > 0$ be such that $c\varphi_1 < \eta$ in Ω. We claim that we can find $0 < m < 1$ small enough such that $\underline{v} := mH(c\varphi_1)$ satisfies

$$2\beta\underline{v}^{1+s} \leq a(x) \quad \text{in } \Omega \quad \text{and} \quad -\Delta\underline{v} \leq \frac{1}{2}a(x)\underline{v}^{-s} \quad \text{in } \Omega. \tag{2.25}$$

Then, from (2.25) we deduce

$$\Delta\underline{v} - \beta\underline{v} + a(x)\underline{v}^{-s} \geq 0 \quad \text{in } \Omega, \tag{2.26}$$

that is, $\underline{v}$ is a sub-solution of (2.15).

By virtue of (2.16) and (2.24) we have

$$2\beta\underline{v}^{1+s} = 2\beta m^{1+s}H^{1+s}(c\varphi_1) \leq 2\beta m^{1+s}c_3(c\varphi_1)^r \leq \frac{2\beta m^{1+s}c^r c_3}{a_1}a(x) \quad \text{in } \Omega.$$

Let us chose now $m > 0$ such that $2\beta m^{1+s}c^r c_3 < a_1$. This concludes the first inequality in (2.25).

In order to establish the second inequality in (2.25), a straightforward computation yields

$$\begin{aligned}
-\Delta\underline{v} &= -mc^2|\nabla\varphi_1|^2 H''(c\varphi_1) + m\lambda_1 c\varphi_1 H'(c\varphi_1) \\
&= mc^{2+r}\varphi_1^r|\nabla\varphi_1|^2 H^{-s}(c\varphi_1) + m\lambda_1 c\varphi_1 H'(c\varphi_1) \\
&= m^{1+s}c^{2+r}\varphi_1^r|\nabla\varphi_1|^2\underline{v}^{-s} + m\lambda_1 c\varphi_1 H'(c\varphi_1) \quad \text{in } \Omega.
\end{aligned}$$

Since H' is decreasing on $(0,\eta)$, it follows that $tH'(t) \leq H(t)$ for all $t \in (0,\eta)$. Furthermore, from (2.24) we deduce

$$c\varphi_1 H'(c\varphi_1) \leq H(c\varphi_1) \leq c_3(c\varphi_1)^r H^{-s}(c\varphi_1) \quad \text{in } \Omega. \tag{2.27}$$

Combining (2.27) and (2.27), for $0 < m < 1$ we obtain

$$\begin{aligned}
-\Delta\underline{v} &\leq m^{1+s}c^{2+r}\varphi_1^r|\nabla\varphi_1|^2\underline{v}^{-s} + m\lambda_1 c^r c_3\varphi_1^r H^{-s}(c\varphi_1) \\
&\leq mc^{2+r}\varphi_1^r|\nabla\varphi_1|^2\underline{v}^{-s} + m\lambda_1 c^r c_3\varphi_1^r\underline{v}^{-s} \\
&= mc^r\varphi_1^r\underline{v}^{-s}(c^2|\nabla\varphi_1|^2 + c_3\lambda_1) \\
&\leq \frac{mc^r}{a_1}(c^2\|\nabla\varphi_1\|_\infty^2 + c_3\lambda_1)a(x)\underline{v}^{-s} \quad \text{in } \Omega.
\end{aligned}$$

Now, it suffices to choose $0 < m < 1$ such that $\frac{mc^r}{a_1}(c^2\|\nabla\varphi_1\|_\infty^2 + c_3\lambda_1) < \frac{1}{2}$. This establishes the second inequality in (2.25) and the fact that $\underline{v}$ is a sub-solution of (2.15).

Next we provide a super-solution $\overline{v}$ of (2.15) such that $\underline{v} \leq \overline{v}$ in Ω. To this aim we first claim that there exists $M > 1$ large enough such that $z := MH(c\varphi_1)$ satisfies

$$\Delta z + a(x)z^{-s} \leq 0 \quad \text{in } \Omega. \tag{2.28}$$

As before we have

$$\Delta z = -M^{1+s}c^{2+r}\varphi_1^r|\nabla\varphi_1|^2 z^{-s} - M\lambda_1 c\varphi_1 H'(c\varphi_1) \quad \text{in } \Omega. \tag{2.29}$$

Let $\omega \subset\subset \Omega$ and $\delta > 0$ be such that (2.10) holds and let us consider $M > 1$ such that

$$M^{1+s} c^{2+r} \delta^2 > a_2, \tag{2.30}$$

$$M^{1+s} c \lambda_1 \min_{\overline{\omega}} \varphi_1 H'(c\varphi_1) \geq a_2 \max_{\overline{\omega}} \varphi_1^r H^{-s}(c\varphi_1). \tag{2.31}$$

Then, as in the proof of Proposition 2.4, by (2.10) and (2.30)-(2.31) we get

$$\begin{aligned}
\Delta z + a(x) z^{-s} &\leq -M^{1+s} c^{2+r} \varphi_1^r |\nabla \varphi_1|^2 z^{-s} + a(x) z^{-s} \\
&\leq -M^{1+s} c^{2+r} \delta^2 \varphi_1^r z^{-s} + a_2 \varphi_1^r z^{-s} \\
&= -\Big(M^{1+s} c^{2+r} \delta^2 - a_2 \Big) \varphi_1^r z^{-s} \leq 0 \quad \text{ in } \Omega \setminus \omega,
\end{aligned}$$

and

$$\begin{aligned}
\Delta z + a(x) z^{-s} &\leq -M \lambda_1 c \varphi_1 H'(c\varphi_1) + a(x) z^{-s} \\
&\leq -M \lambda_1 c \varphi_1 H'(c\varphi_1) + a_2 \varphi_1^r z^{-s} \\
&= -\frac{1}{M^s} \Big(M^{1+s} c \lambda_1 \varphi_1 H'(c\varphi_1) - a_2 \varphi_1^r H^{-s}(c\varphi_1) \Big) \\
&\leq 0 \quad \text{ in } \omega.
\end{aligned}$$

Hence, we have obtained the inequality in (2.28).

Let $\tilde{w} \in C^2(\overline{\Omega})$ be the unique solution of

$$\begin{cases}
\Delta \tilde{w} - \beta \tilde{w} + b(x) = 0 & \text{in } \Omega, \\
\tilde{w} > 0 & \text{in } \Omega, \\
\tilde{w} = 0 & \text{on } \partial\Omega.
\end{cases}$$

Then $\overline{v} := z + \tilde{w}$ satisfies $\overline{v} > 0$ in Ω, $\overline{v} = 0$ on $\partial\Omega$ and by (2.28) we have

$$\Delta \overline{v} - \beta \overline{v} + a(x) \overline{v}^{-s} + b(x) \leq \Delta z - \beta z + a(x) z^{-s} \leq 0 \quad \text{in } \Omega.$$

Hence, $\overline{v}$ is a super-solution of (2.15) and clearly we have $\underline{v} \leq \overline{v}$ in Ω. It follows that problem (2.15) has a classical solution $v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\underline{v} \leq v \leq \overline{v}$ in Ω.

On the other hand, since $\tilde{w} \in C^2(\overline{\Omega})$, we deduce that there exists $\tilde{c}_1 > 0$ such that $\tilde{w} \leq \tilde{c}_1 \varphi_1$ in Ω. This implies $\tilde{w} \leq \tilde{c}_2 \Gamma_{s,r}(\varphi_1)$ in Ω, for some $\tilde{c}_2 > 0$. Finally, using the last inequality, the definition of $\underline{v}, \overline{v}$ and (2.23), we get the estimate (2.20). The uniqueness of the solution follows by Lemma 2.3. This finishes the proof of Proposition 2.5.

3. Case $0 \leq p < 1$

3.1. Proof of Theorem 1.1

Let $0 < \varepsilon_0 < 1$ and set

$$\Omega_\varepsilon := \{x \in \Omega : d(x) > \varepsilon\}, \quad \text{ for all } 0 < \varepsilon < \varepsilon_0. \tag{3.1}$$

For ε_0 small enough, Ω_ε remains a smooth domain. The existence of a solution to (0.3) will be proved by considering the approximated system

$$\begin{cases} \Delta u - \alpha u + \dfrac{u^p}{v^q} + \rho(x) = 0, \ u > 0 & \text{in } \Omega_\varepsilon, \\ \Delta v - \beta v + \dfrac{u^r}{v^s} = 0, \ v > 0 & \text{in } \Omega_\varepsilon, \\ u = \varepsilon, \ \ v = \Gamma_{s,r}(\varepsilon) & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3.2}$$

The existence of a classical solution to (3.2) is obtained by using the Schauder's fixed point theorem. For $0 < \varepsilon < \varepsilon_0$ and $m_1, m_2 < 1 < M_1, M_2$ consider the set $\mathcal{A}_\varepsilon$ of all $(u,v) \in C(\overline{\Omega}_\varepsilon) \times C(\overline{\Omega}_\varepsilon)$ such that

$$\begin{aligned} & m_1\varphi_1 \le u \le M_1\varphi_1 && \text{in } \Omega_\varepsilon, \\ & m_2\Gamma_{s,r}(\varphi_1) \le v \le M_2\Gamma_{s,r}(\varphi_1) && \text{in } \Omega_\varepsilon, \\ & u = \varepsilon, \ v = \Gamma_{s,r}(\varepsilon) && \text{on } \partial\Omega_\varepsilon \end{aligned}$$

Next, we define the mapping $\mathcal{T} : \mathcal{A}_\varepsilon \to C(\overline{\Omega}_\varepsilon) \times C(\overline{\Omega}_\varepsilon)$ as follows. For $(u,v) \in \mathcal{A}_\varepsilon$ we set

$$\mathcal{T}(u,v) = (Tu, Tv), \tag{3.3}$$

where Tu and Tv satisfy

$$\begin{cases} \Delta(Tu) - \alpha(Tu) + \dfrac{(Tu)^p}{v^q} + \rho(x) = 0, \ Tu > 0 & \text{in } \Omega_\varepsilon, \\ \Delta(Tv) - \beta(Tv) + \dfrac{u^r}{(Tv)^s} = 0, \ Tv > 0 & \text{in } \Omega_\varepsilon, \\ Tu = \varepsilon, \ \ Tv = \Gamma_{s,r}(\varepsilon) & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3.4}$$

Using the definition of $\mathcal{A}_\varepsilon$, by sub and super-solution method combined with Lemma 2.2 and Lemma 2.3, the above system has a unique solution (Tu, Tv) with $Tu, Tv \in C^2(\overline{\Omega}_\varepsilon)$. Basic to our approach are the following two results which allows us to apply Schauder's fixed point theorem.

Lemma 3.1. *There exist $m_1 < 1 < M_1$ and $m_2 < 1 < M_2$ which are independent of ε such that $\mathcal{T}(\mathcal{A}_\varepsilon) \subseteq \mathcal{A}_\varepsilon$, for all $0 < \varepsilon < \varepsilon_0$.*

Proof. Let $w \in C^2(\overline{\Omega})$ be the unique solution of problem (2.5). In view of (2.1) and (2.6) we have

$$w(x) \le c_2\varphi_1 \le \frac{c_2}{C} d(x) = \frac{c_2}{C}\varepsilon \quad \text{on } \partial\Omega_\varepsilon.$$

Hence, if $\delta_1 = \min\{1, \frac{C}{c_2}\}$ then $\delta_1 w \le \varepsilon$ on $\partial\Omega_\varepsilon$. Furthermore,

$$\Delta(Tu) - \alpha(Tu) + \rho(x) \le 0 \le \Delta(\delta_1 w) - \alpha(\delta_1 w) + \rho(x) \quad \text{in } \Omega_\varepsilon,$$

$$Tu = \varepsilon \ge \delta_1 w \quad \text{on } \partial\Omega_\varepsilon.$$

By maximum principle, we obtain $Tu \geq \delta_1 w$ in Ω_ε. In view of (2.6), let us choose $m_1 = \delta_1 c_1$ in the definition of $\mathcal{A}_\varepsilon$ (where c_1 is the constant in (2.6)). Then, (2.6) combined with the last estimates yields

$$Tu \geq m_1\varphi_1 \quad \text{in } \Omega_\varepsilon. \tag{3.5}$$

From the second equation in (3.4) and the fact that $u \geq m_1\varphi_1$ in Ω_ε we have

$$\Delta(Tv) - \beta(Tv) + \frac{m_1^r\varphi_1^r}{(Tv)^s} \leq 0 \quad \text{in } \Omega_\varepsilon. \tag{3.6}$$

Let us consider the problem

$$\begin{cases} \Delta\xi - \beta\xi + \varphi_1^r\xi^{-s} = 0 & \text{in } \Omega, \\ \xi > 0 & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Using Proposition 2.5 (ii), there exists $\xi \in C^2(\Omega) \cap C(\overline{\Omega})$ a unique solution of (3.7) with the additional property

$$c_3\Gamma_{s,r}(\varphi_1) \leq \xi \leq c_4\Gamma_{s,r}(\varphi_1) \quad \text{in } \Omega, \tag{3.8}$$

for some $c_3, c_4 > 0$. Moreover, by (3.8), (2.1) and the property (2.19) of $\Gamma_{s,r}$ we can find $c_5, c_6 > 0$ such that

$$c_5\Gamma_{s,r}(d(x)) \leq \xi \leq c_6\Gamma_{s,r}(d(x)) \quad \text{in } \Omega. \tag{3.9}$$

Let $\delta_2 = \min\{1, m_1^{r/(1+s)}, \frac{1}{c_6}\}$. Then

$$\Delta(\delta_2\xi) - \beta(\delta_2\xi) + m_1^r\varphi_1^r(\delta_2\xi)^{-s} \geq \delta_2\Big(\Delta\xi - \beta\xi + \varphi_1^r\xi^{-s}\Big) = 0 \quad \text{in } \Omega, \tag{3.10}$$

and by (3.9) we have

$$\delta_2\xi \leq \delta_2 c_6\Gamma_{s,r}(d(x)) \leq \Gamma_{s,r}(\varepsilon) \quad \text{on } \partial\Omega_\varepsilon. \tag{3.11}$$

Therefore, from (3.6), (3.10) and (3.11) we have obtained

$$\Delta(Tv) - \beta(Tv) + m_1^r\varphi_1^r(Tv)^{-s} \leq 0 \leq \Delta(\delta_2\xi) - \beta(\delta_2\xi) + m_1^r\varphi_1^r(\delta_2\xi)^{-s} \quad \text{in } \Omega,$$

$$Tv = \Gamma_{s,r}(\varepsilon) \geq \delta_2\xi \quad \text{on } \partial\Omega_\varepsilon.$$

By Lemma 2.3 it follows that $Tv \geq \delta_2\xi$ in Ω_ε. In view of (3.8), the last inequality leads us to $Tv \geq \delta_2 c_3\Gamma_{s,r}(\varphi_1)$ in Ω_ε. Thus, we consider

$$m_2 = \min\{1, \delta_2 c_3\} > 0$$

in the definition of the set $\mathcal{A}_\varepsilon$. Note that m_2 is independent of ε and $Tv \geq m_2\Gamma_{s,r}(\varphi_1)$ in Ω_ε.

The definition of $\mathcal{A}_\varepsilon$ and (2.18) yield

$$v \geq m_2\Gamma_{s,r}(\varphi_1) \geq m_2\varphi_1^\sigma \quad \text{in } \Omega_\varepsilon.$$

Using the estimate $v \geq m_2\varphi_1^\sigma$ in the first equation of (3.4) we get

$$\Delta(Tu) - \alpha(Tu) + m_2^{-q}\varphi_1^{-q\sigma}(Tu)^p + \rho(x) \geq 0 \quad \text{in } \Omega_\varepsilon. \tag{3.12}$$

As above, we next consider the problem

$$\begin{cases} \Delta\zeta - \alpha\zeta + m_2^{-q}\varphi_1^{-q\sigma}\zeta^p + \rho(x) = 0 & \text{in } \Omega, \\ \zeta > 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

Since $q\sigma < p+1$, by Proposition 2.4 there exists $\zeta \in C^2(\Omega) \cap C(\overline{\Omega})$ a unique solution of (3.13) such that

$$c_7\varphi_1 \le \zeta \le c_8\varphi_1 \quad \text{in } \Omega, \tag{3.14}$$

for some $c_7, c_8 > 0$. Note that $q\sigma < p+1$, (3.14) and Lemma 2.1 imply $\Delta\zeta \in L^1(\Omega)$. Let $A_1 = \max\{1, \frac{1}{Cc_7}\}$. Then

$$\Delta(A_1\zeta) - \alpha(A_1\zeta) + m_2^{-q}\varphi_1^{-q\sigma}(A_1\zeta)^p + \rho(x) \le 0 \quad \text{in } \Omega_\varepsilon.$$

Also by (2.1) and (3.14) we have

$$A_1\zeta \ge A_1c_7\varphi_1 \ge A_1Cc_7d(x) \ge \varepsilon \quad \text{on } \partial\Omega_\varepsilon.$$

Define

$$\Psi(x,t) = -\alpha t + m_2^{-q}\varphi_1^{-q\sigma}(x)(A_1t)^p + \rho(x), \quad (x,t) \in \Omega_\varepsilon \times (0,\infty).$$

Then Ψ satisfies the hypotheses in Lemma 2.2 and

$$\Delta(A_1\zeta) + \Psi(x, A_1\zeta) \le 0 \le \Delta(Tu) + \Psi(x, Tu) \quad \text{in } \Omega_\varepsilon,$$

$$Tu, A_1\zeta > 0 \;\text{ in } \Omega_\varepsilon, \;\; Tu = \varepsilon \le A_1\zeta \;\text{ on } \Omega_\varepsilon,$$

$$\Delta(A_1\zeta) \in L^1(\Omega_\varepsilon).$$

By Lema 2.2 it follows that $Tu \le A_1\zeta$ in Ω_ε. In view of (3.14), let us take $M_1 := \max\{1, A_1c_8\}$ in the definition of the set $\mathcal{A}_\varepsilon$. Then M_1 does not depend on ε and by (3.14) we have

$$Tu \le M_1\varphi_1 \quad \text{in } \Omega_\varepsilon.$$

The definition of $\mathcal{A}_\varepsilon$ yields $u \le M_1\varphi_1$ in Ω_ε. Then, the second equation of system (3.4) produces

$$\Delta(Tv) - \beta(Tv) + \frac{M_1^r\varphi_1^r}{(Tv)^s} \ge 0 \quad \text{in } \Omega_\varepsilon. \tag{3.15}$$

Let $A_2 = \max\{1, M_1^r, \frac{1}{c_5}\}$. If ξ is the unique solution of (3.7), then

$$\Delta(A_2\xi) - \beta(A_2\xi) + M_1^r\varphi_1^r(A_2\xi)^{-s} \le 0 \quad \text{in } \Omega_\varepsilon,$$

and, by (3.9) we also have

$$A_2\xi \ge A_2c_5\Gamma_{s,r}(d(x)) \ge \Gamma_{s,r}(\varepsilon) \quad \text{on } \partial\Omega_\varepsilon.$$

Therefore, by Lemma 2.3 it follows that $Tv \le A_2\xi$ in Ω_ε. Now, we take $M_2 := \max\{1, A_2c_4\}$ in the definition of the set $\mathcal{A}_\varepsilon$. It follows that M_2 is independent of ε and, by virtue of (3.8), we obtain $Tv \le M_2\Gamma_{s,r}(\varphi_1)$ in Ω_ε. This finishes the proof of our Lemma 3.1.

Lemma 3.2. *The mapping $\mathcal{T} : \mathcal{A}_\varepsilon \to \mathcal{A}_\varepsilon$ defined in (3.3)-(3.4) is compact and continuous.*

Proof. Let us fix $(u, v) \in \mathcal{A}_\varepsilon$. Then u, v, Tu and Tv are bounded away from zero in $\overline{\Omega}_\varepsilon$ which yields

$$\left\| \frac{(Tu)^p}{v^q} \right\|_{L^\infty(\Omega_\varepsilon)}, \left\| \frac{u^r}{(Tv)^s} \right\|_{L^\infty(\Omega_\varepsilon)} \le c_\varepsilon = c(\varepsilon, m_1, m_2, M_1.M_2, p, q, r, s).$$

Hence, by Hölder estimates, for all $\tau > N$ we obtain

$$\|Tu\|_{W^{2,\tau}(\Omega_\varepsilon)}, \|Tv\|_{W^{2,\tau}(\Omega_\varepsilon)} \le c_{1,\varepsilon},$$

for some $c_{1,\varepsilon} > 0$ independent of u and v. Since the embedding $W^{2,\tau}(\Omega_\varepsilon) \hookrightarrow C^{1,\gamma}(\overline{\Omega}_\varepsilon)$, $0 < \gamma < 1 - N/\tau$ is compact, we derive that the mapping $\mathcal{T} : \mathcal{A}_\varepsilon \to \mathcal{A}_\varepsilon \subset C(\overline{\Omega}_\varepsilon) \times C(\overline{\Omega}_\varepsilon)$ is also compact.

It remains to prove that $\mathcal{T}$ is continuous. To this aim, let $\{(u_n, v_n)\}_{n\ge1} \subset \mathcal{A}_\varepsilon$ be such that $u_n \to u$ and $v_n \to v$ in $C(\overline{\Omega}_\varepsilon)$ as $n \to \infty$. Since $\mathcal{T}$ is compact, there exists $(U, V) \in \mathcal{A}_\varepsilon$ such that up to a subsequence we have

$$\mathcal{T}(u_n, v_n) \to (U, V) \quad \text{in } \mathcal{A}_\varepsilon \text{ as } n \to \infty.$$

Using the $L^\infty(\Omega_\varepsilon)$ bounds of $\left(\frac{(Tu_n)^p}{v_n^q}\right)_{n\ge1}$ and $\left(\frac{u_n^r}{(Tv_n)^s}\right)_{n\ge1}$, it follows that $(Tu_n)_{n\ge1}$ and $(Tv_n)_{n\ge1}$ are bounded in $W^{2,\tau}(\Omega_\varepsilon)$ for all $\tau > N$. As before, this implies that $(Tu_n)_{n\ge1}$ and $(Tv_n)_{n\ge1}$ are bounded in $C^{1,\gamma}(\overline{\Omega}_\varepsilon)$ $(0 < \gamma < 1 - N/\tau)$. Next, by Schauder estimates, it follows that $(Tu_n)_{n\ge1}$ and $(Tv_n)_{n\ge1}$ are bounded in $C^{2,\gamma}(\overline{\Omega}_\varepsilon)$. Since $C^{2,\gamma}(\overline{\Omega}_\varepsilon)$ is compactly embedded in $C^2(\overline{\Omega}_\varepsilon)$, we deduce that up to a subsequence, we have that

$$Tu_n \to U \quad \text{and} \quad Tv_n \to V \quad \text{in } C^2(\overline{\Omega}_\varepsilon) \text{ as } n \to \infty.$$

Passing to the limit in (3.4) we get that (U, V) satisfies

$$\begin{cases} \Delta U - \alpha U + \dfrac{U^p}{v^q} + \rho(x) = 0, \; U > 0 & \text{in } \Omega_\varepsilon, \\ \Delta V - \beta V + \dfrac{u^r}{V^s} = 0, \; V > 0 & \text{in } \Omega_\varepsilon, \\ U = \varepsilon, \;\; V = \Gamma_{s,r}(\varepsilon) & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Using the uniqueness of (3.4), it follows that $Tu = U$ and $Tv = V$. Thus, we have obtained that any subsequence of $\{\mathcal{T}(u_n, v_n)\}_{n\ge1}$ has a subsequence converging to $\mathcal{T}(u, v)$ in $\mathcal{A}_\varepsilon$. But this implies that the entire sequence $\{\mathcal{T}(u_n, v_n)\}_{n\ge1}$ converges to $\mathcal{T}(u, v)$ in $\mathcal{A}_\varepsilon$, whence the continuity of $\mathcal{T}$. The proof of Lemma 3.2 is now complete.

Proof of Theorem 1.1 completed

According to Lemma 3.1 and 3.2 we are now in position to apply the Schauder's fixed point theorem. Thus, for all $0 < \varepsilon < \varepsilon_0$, there exists $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_\varepsilon$ such that $\mathcal{T}(u_\varepsilon, v_\varepsilon) = (u_\varepsilon, v_\varepsilon)$. By standard elliptic regularity arguments, we deduce $u_\varepsilon, v_\varepsilon \in C^2(\overline{\Omega}_\varepsilon)$. Therefore, for all $0 < \varepsilon < \varepsilon_0$ we have proved the existence of a solution $(u_\varepsilon, v_\varepsilon) \in C^2(\overline{\Omega}_\varepsilon) \times C^2(\overline{\Omega}_\varepsilon)$ of system (3.2). Next, we extend $u_\varepsilon = \varepsilon$, $v_\varepsilon = \Gamma_{s,r}(\varepsilon)$ in $\Omega \setminus \overline{\Omega}_\varepsilon$. Furthermore, by the definition of $\mathcal{A}_\varepsilon$ we have

$$m_1\varphi_1 \le u_\varepsilon \le M_1\varphi_1 + \varepsilon \le M_1\varphi_1 + \varepsilon_0 \quad \text{in } \Omega, \tag{3.16}$$

$$m_2\Gamma_{s,r}(\varphi_1) \le v_\varepsilon \le M_2\Gamma_{s,r}(\varphi_1) + \Gamma_{s,r}(\varepsilon) \le M_1\varphi_1 + c_{\varepsilon_0} \quad \text{in } \Omega. \tag{3.17}$$

As above, L^∞ bounds together with Hölder estimates yield $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}, (v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ are bounded in $W^{2,\tau}_{loc}(\Omega)$, for all $\tau > N$. With similar arguments, there exist $u, v \in C^2(\Omega)$ such that for all $\omega \subset\subset \Omega$, $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ and $(v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ converge up to a subsequence to u and v respectively in $C^2(\overline{\omega})$ as $\varepsilon \to 0$. Passing to the limit with $\varepsilon \to 0$ in (3.2) and (3.16)-(3.17) we get

$$\begin{cases} \Delta u - \alpha u + \dfrac{u^p}{v^q} + \rho(x) = 0 & \text{in } \Omega, \\ \Delta v - \beta v + \dfrac{u^r}{v^s} = 0 & \text{in } \Omega, \end{cases}$$

and

$$m_1\varphi_1 \le u \le M_1\varphi_1 \quad \text{in } \Omega, \tag{3.18}$$

$$m_2\Gamma_{s,r}(\varphi_1) \le v \le M_2\Gamma_{s,r}(\varphi_1) \quad \text{in } \Omega. \tag{3.19}$$

Now, we extend $u = v = 0$ on $\partial\Omega$. From (3.18) and (3.19) we deduce that $u, v \in C(\overline{\Omega})$. Hence, the system (0.3) has a classical solution (u, v).

It remains to establish the boundary estimates of the solution to (0.3). This follows essentially by using the same arguments as above. Let (u, v) be an arbitrary solution of (0.3). Then $\Delta u - \alpha u + \rho(x) \le 0$ in Ω which implies that $u \ge w$ in Ω, where w is the unique solution of (2.5). By (2.6) it follows that $u \ge c_1\varphi_1$ in Ω. Using this inequality in the second equation of (0.3) we deduce $\Delta v - \beta v + c_2\varphi_1^r v^{-s} \le 0$ in Ω for some $c_2 > 0$ (we actually have $c_2 = c_1^r > 0$). Next, let ξ be the unique solution of (3.7). A similar argument to that used in Step 1 yields $v \ge c_3\xi$ in Ω. In view of estimate (2.20) in Proposition 2.5 we derive that $v \ge c_4\Gamma_{s,r}(\varphi_1)$ in Ω for some $c_4 > 0$. According to (2.18) it follows that $v \ge c_5\varphi_1^\sigma$ in Ω. This inequality combined with the first equation in system (0.3) produces $\Delta u - \alpha u + c_6\varphi_1^{-q\sigma}u^p + \rho(x) \ge 0$ in Ω.

Consider the problem

$$\begin{cases} \Delta z - \alpha z + c_6\varphi_1^{-q\sigma}z^p + \rho(x) = 0 & \text{in } \Omega, \\ z > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

Since $q\sigma < p + 1$, by Proposition 2.4 there exists a unique solution of (3.20) such that $z \le c_7\varphi_1$ in Ω. Thus, by Lemma 2.2 we get $u \le z \le c_7\varphi_1$ in Ω. Using this last inequality in the second equation of (0.3) we finally obtain $\Delta v - \beta v + c_8\varphi_1^r v^{-s} \ge 0$ in Ω for some $c_8 > 0$. By virtue of Proposition 2.5 we have $v \le c_9\Gamma_{s,r}(\varphi_1)$ in Ω. Thus, we have obtained

$$m_1\varphi_1 \le u \le m_2\varphi_1 \quad \text{in } \Omega,$$

$$m_1\Gamma_{s,r}(\varphi_1) \le u \le m_2\Gamma_{s,r}(\varphi_1) \quad \text{in } \Omega,$$

for some fixed constants $m_1, m_2 > 0$. Now, the boundary estimates in Theorem 1.1 follows from the above inequalities combined with (2.1). This concludes the proof. □

3.2. Proof of Corollary 1.2

Let (u, v) be a classical solution of (0.3). We rewrite the system (0.3) in the form

$$\Delta u = f_1(x) \text{ in } \Omega, \quad \Delta v = f_2(x) \text{ in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega,$$

where

$$f_1(x) = \alpha u(x) - \frac{u^p(x)}{v^q(x)} - \rho(x),\ f_2(x) = \beta v(x) - \frac{u^r(x)}{v^s(x)}, \quad \text{for all } x \in \Omega.$$

Note that in our settings we have $\sigma = 1$ in (1.2) and by virtue of Theorem 1.1 there exist $c_1, c_2 > 0$ such that $c_1 d(x) \le u, v \le c_2 d(x)$ in Ω. Hence,

$$|f_1(x)| \le m_1 d^{p-q}(x) + \rho(x) \quad \text{in } \Omega \quad \text{and} \quad |f_2(x)| \le m_2 d^{r-s}(x) \quad \text{in } \Omega. \tag{3.21}$$

(i) Since $0 \le p - q$ and $0 \le r - s$, by (3.21) we get $f_1, f_2 \in L^\infty(\Omega)$. Next, standard elliptic arguments leads us to $u, v \in C^2(\overline{\Omega})$.

(ii) Let us assume that $-1 < p - q < 0$ and $-1 < r - s < 0$. From (3.21) we derive

$$|f_1(x)| \le c d^{p-q}(x) \quad \text{in } \Omega \quad \text{and} \quad |f_2(x)| \le c d^{r-s}(x) \quad \text{in } \Omega,$$

for some positive constant $c > 0$.

If $N = 1$ then, for all $x_1, x_2 \in \Omega$ we have

$$|u'(x_1) - u'(x_2)| \le \left| \int_{x_1}^{x_2} |f_1(t)| dt \right| \le c \left| \int_{x_1}^{x_2} d^{p-q}(t) dt \right| \le \tilde{c}|x_1 - x_2|^{1+p-q},$$

where $\tilde{c} > 0$ does not depend on x_1, x_2 . This yields $u \in C^{1,1+p-q}(\overline{\Omega})$ and similarly $v \in C^{1,1+r-s}(\overline{\Omega})$.

If $N \ge 2$, the conclusion follows exactly in the same way as in [9]. More precisely, let $\mathcal{G}$ denote the Green's function for the Laplace operator. Then for all $x \in \Omega$ we have

$$u(x) = \int_\Omega \mathcal{G}(x, y) f_1(y) dy, \qquad v(x) = \int_\Omega \mathcal{G}(x, y) f_2(y) dy,$$

and

$$\nabla u(x) = \int_\Omega \mathcal{G}_x(x, y) f_1(y) dy, \qquad \nabla v(x) = \int_\Omega \mathcal{G}_x(x, y) f_2(y) dy.$$

Then, for all $x_1, x_2 \in \Omega$, $x_1 \ne x_2$ we have

$$\begin{aligned} |\nabla u(x_1) - \nabla u(x_2)| &\le \int_\Omega |\mathcal{G}_x(x_1, y) - \mathcal{G}_x(x_2, y)||f_1(y)| dy \\ &\le c \int_\Omega |\mathcal{G}_x(x_1, y) - \mathcal{G}_x(x_2, y)| d^{p-q}(y) dy, \end{aligned}$$

and similarly

$$|\nabla v(x_1) - \nabla v(x_2)| \le c \int_\Omega |\mathcal{G}_x(x_1, y) - \mathcal{G}_x(x_2, y)| d^{r-s}(y) dy.$$

From now on, we need only to employ the sharp estimates given in [9, Theorem 1.1] in order to obtain $u \in C^{1,1+p-q}(\overline{\Omega})$ and $v \in C^{1,1+r-s}(\overline{\Omega})$. This finishes the proof of Corollary 1.2. □

3.3. Proof of Theorem 1.3

Let (u, v) be a classical solution of (1.3). Then, by virtue of Corollary 1.2, we have

$$u, v \in C^2[0,1] \times C^2[0,1] \quad \text{if } 0 \le p - q,\ 0 \le r - s,$$

and

$$u \in C^2(0,1) \cap C^{1,1+p-q}[0,1],\ v \in C^2(0,1) \cap C^{1,1+r-s}[0,1],$$

if $-1 < p - q < 0$, $-1 < r - s < 0$. Furthermore, by Hopf's maximum principle we also have that $u'(0) > 0$, $v'(0) > 0$, $u'(1) < 0$ and $v'(1) < 0$.

Assume that there exist (u_1, v_1) and (u_2, v_2) two different solutions of (1.3).

First we claim that we can not have $u_2 \ge u_1$ or $v_2 \ge v_1$ in [0,1]. Assume by contradiction that $u_2 \ge u_1$ in [0,1]. Then

$$v_2'' - \beta v_2 + \frac{u_2^r}{v_2^s} = 0 = v_1'' - \beta v_1 + \frac{u_1^r}{v_1^s} \quad \text{in } (0,1),$$

and by Lemma 2.3 we get $v_2 \ge v_1$ in $[0, 1]$. This implies that

$$u_1'' - \alpha u_1 + \frac{u_1^p}{v_2^q} + \rho(x) \le 0 = u_2'' - \alpha u_2 + \frac{u_2^p}{v_2^q} + \rho(x) \quad \text{in } (0,1). \tag{3.22}$$

On the other hand, the mapping $\Psi(x,t) = -\alpha t + \frac{t^p}{v_2^q(x)} + \rho(x)$, $(x,t) \in (0,1) \times (0,\infty)$ satisfies the hypotheses in Lemma 2.2. Hence $u_2 \le u_1$ in $[0, 1]$, that is $u_1 \equiv u_2$. This also implies $v_1 \equiv v_2$, which is a contradiction. Replacing u_1 by u_2 and v_1 by v_2, we also get that the situation $u_1 \ge u_2$ or $v_1 \ge v_2$ in [0,1] is not possible.

Set $U = u_2 - u_1$ and $V = v_2 - v_1$. From the above arguments, both U and V change sign in $(0, 1)$. The key result in our approach is the following.

Proposition 3.3. *U and V vanish only at finitely many points in the interval* $[0, 1]$.

Proof. Subtracting the corresponding equations for (u_1, v_1) and (u_2, v_2) we obtain the following linear problem

$$\begin{cases} \mathbf{W}''(x) + A(x)\mathbf{W}(x) = \mathbf{0} & \text{in } (0,1), \\ \mathbf{W}(0) = \mathbf{W}(1) = \mathbf{0}, \end{cases} \tag{3.23}$$

where $\mathbf{W} = (U, V)^T$ and $A(x) = (A_{ij}(x))_{1\le i,j\le 2}$ is a 2×2 matrix defined as

$$A_{11}(x) = -\alpha + \begin{cases} \dfrac{1}{v_2^q(x)} \cdot \dfrac{u_2^p(x) - u_1^p(x)}{u_2(x) - u_1(x)}\,, & u_1(x) \neq u_2(x) \\ p\dfrac{u_1^{p-1}(x)}{v_1^q(x)}\,, & u_1(x) = u_2(x) \end{cases}$$

$$A_{12}(x) = \begin{cases} -\dfrac{u_1^p(x)}{v_1^q(x)v_2^q(x)} \cdot \dfrac{v_2^q(x) - v_1^q(x)}{v_2(x) - v_1(x)}\,, & v_1(x) \neq v_2(x) \\ -q\dfrac{u_1^p(x)}{v_1^{q+1}(x)}\,, & v_1(x) = v_2(x) \end{cases}$$

$$A_{21}(x) = \begin{cases} \dfrac{1}{v_2^s(x)} \cdot \dfrac{u_2^r(x) - u_1^r(x)}{u_2(x) - u_1(x)}\,, & u_1(x) \neq u_2(x) \\ r\dfrac{u_1^{r-1}(x)}{v_1^s(x)}\,, & u_1(x) = u_2(x) \end{cases}$$

$$A_{22}(x) = -\beta - \begin{cases} \dfrac{u_1^r(x)}{v_1^s(x)v_2^s(x)} \cdot \dfrac{v_2^s(x) - v_1^s(x)}{v_2(x) - v_1(x)}\,, & v_1(x) \neq v_2(x) \\ s\dfrac{u_1^r(x)}{v_1^{s+1}(x)}\,, & v_1(x) = v_2(x). \end{cases}$$

Lemma 3.4. *We have*

(i) $A_{ij} \in C(0,1)$, *for all* $1 \le i,j \le 2$;

(ii) $A_{12}(x) \neq 0$ *and* $A_{21}(x) \neq 0$ *for all* $x \in (0,1)$;

(iii) $d^{1-(p-q)}(x)A_{1j} \in L^\infty(0,1)$ *and* $d^{1-(r-s)}(x)A_{2j} \in L^\infty(0,1)$, *for* $j = 1, 2$.

Proof. The claims in (i) and (ii) are easy to verify. We prove only the statement in (iii). To this aim, let us notice first that by the regularity of solutions, there exist $c_1, c_2 > 0$ such that

$$c_1 d(x) \le u_i, v_i \le c_2 d(x) \quad \text{in } (0,1),\ 1 \le i \le 2. \tag{3.24}$$

By (3.24) and the fact that

$$|a^q - b^q| \le q|a-b| \max\{a^{q-1}, b^{q-1}\} \qquad \text{for all } a, b > 0,$$

we have

$$\begin{aligned} d(x)|A_{12}(x)| &\le q d(x) \frac{u_1^p(x)}{v_1^q(x)v_2^q(x)} \max\{v_1^{q-1}(x), v_2^{q-1}(x)\} \\ &\le q d^{p-q}(x) \left(\frac{u_1(x)}{d(x)}\right)^p \max\left\{\left(\frac{d(x)}{v_1(x)}\right)^{q+1}, \left(\frac{d(x)}{v_2(x)}\right)^{q+1}\right\} \\ &\le c d^{p-q}(x) \quad \text{for all } 0 < x < 1. \end{aligned}$$

Hence $d^{1-(p-q)}(x)A_{12} \in L^\infty(0,1)$. We obtain similar estimates for $d^{1-(p-q)}(x)A_{11}$ and $d^{1-(r-s)}(x)A_{2j}$, $j = 1, 2$. This concludes the proof.

Next, Lemma 3.4 (i)-(ii) allows us to employ the following result which is proved in [1, Lemma 7].

Lemma 3.5. (see [1]) *Let $0 < a < b < 1$ and $A = (A_{ij})_{1 \le i,j \le 2}$ be such that*

(i) $A_{ij} \in C[a,b]$, for all $1 \le i,j \le 2$;

(ii) $A_{12}(x) \ne 0$ and $A_{21}(x) \ne 0$ for all $x \in [a,b]$.

Assume that there exists $\mathbf{W} = (U,V)^T \in C^2[a,b] \times C^2[a,b]$ such that $\mathbf{W} \not\equiv 0$ and $\mathbf{W}''(x) + A(x)\mathbf{W}(x) = \mathbf{0}$ in $[a,b]$. Then, neither U nor V can have infinitely many zeros in $[a,b]$.

As a consequence, we deduce that if $\mathbf{W}$ is a solution of (3.23) and $\mathbf{W}$ vanishes for infinitely many times in an interval $[a,b] \subset (0,1)$ then, applying Lemma 3.5 in $[\varepsilon, 1-\varepsilon]$ for all $\varepsilon > 0$ sufficiently small, we get $\mathbf{W} \equiv \mathbf{0}$.

It remains to show that U and V can not vanish for infinitely many times in the neighborhood of $x = 0$ and $x = 1$. We shall consider only the case $x = 0$, the situation where U or V have infinitely many zeros near $x = 1$ can be handled in the same manner.

Without loosing the generality, we may assume that V has infinitely many zeros in a neighborhood of $x = 0$. By the continuity of V it follows that $V(0) = 0$. Furthermore, since $V \in C^2(0,1) \cap C^1[0,1]$, by Rolle's Theorem we get that both V' and V'' have infinitely many zeros near $x = 0$. Therefore, $V'(0) = 0$, that is, $v_1'(0) = v_2'(0)$.

If $U'(0) = 0$, then $\mathbf{W}(0) = \mathbf{W}'(0) = \mathbf{0}$. Let $\gamma = \min\{0, p-q, r-s\}$. Then $-1 < \gamma \le 0$ and by Lemma 3.4 (iii) it follows that $x^{1-\gamma} A_{ij} \in L^\infty(0,1/2)$. Thus, we can use Proposition 4.2 in the Appendix in order to get that $\mathbf{W} \equiv \mathbf{0}$ in $[0,1/2]$. Then, by Lemma 3.5 we obtain $\mathbf{W} \equiv 0$ in $[0,1]$, which is a contradiction. Hence $U'(0) \ne 0$. Subtracting the second equation corresponding to v_1 and v_2 in the system (1.3) we have

$$\begin{aligned} V''(x) &= \beta V(x) + \frac{u_1^r(x)}{v_1^s(x)} - \frac{u_2^r(x)}{v_2^s(x)} \\ &= x^{r-s}\left\{\beta\frac{V(x)}{x^{r-s}} + \left(\frac{u_1(x)}{x}\right)^r \left(\frac{x}{v_1(x)}\right)^s - \left(\frac{u_2(x)}{x}\right)^r \left(\frac{x}{v_2(x)}\right)^s\right\}. \end{aligned}$$

Since $r - s < 1$, $v_1'(0) = v_2'(0) > 0$ and $u_1'(0) \ne u_2'(0)$ we get

$$\begin{aligned} &\lim_{x\to 0^+} \left\{\beta\frac{V(x)}{x^{r-s}} + \left(\frac{u_1(x)}{x}\right)^r \left(\frac{x}{v_1(x)}\right)^s - \left(\frac{u_2(x)}{x}\right)^r \left(\frac{x}{v_2(x)}\right)^s\right\} \\ &\qquad = \frac{u_1'^r(0) - u_2'^r(0)}{v_1'^s(0)} \ne 0. \end{aligned}$$

From the above equalities we derive that V'' has constant sign in a small neighborhood of $x = 0$ which contradicts the fact that V vanishes for infinitely many times in the neighborhood of $x = 0$. This finishes the proof of Proposition 3.3.

Proof of Theorem 1.3 completed.

Let us define

$$\mathcal{I}^+ = \{x \in [0,1] : U(x) \ge 0\}, \quad \mathcal{I}^- = \{x \in [0,1] : U(x) \le 0\},$$

$$\mathcal{J}^+ = \{x \in [0,1] : V(x) \ge 0\}, \quad \mathcal{J}^- = \{x \in [0,1] : V(x) \le 0\}.$$

Since both U and V have a finite number of zeros, it follows that the above sets consist of finitely many disjoint closed intervals. Therefore, $\mathcal{I}^+ = \cup_{i=1}^m I_i^+$. For our convenience, let I^+ denote any interval I_i^+ and similar notations will be used for I^-, J^+ and J^-. We have

Lemma 3.6. *For all intervals I^+, I^-, J^+ and J^- defined above, the following situations can not occur:*

(i) $I^+ \subset J^+$; (ii) $I^- \subset J^-$; (iii) $J^+ \subset I^-$; (iv) $J^- \subset I^+$.

Proof. (i) Assume by contradiction that $I^+ \subset J^+$. This yields $u_2 \geq u_1$ and $v_2 \geq v_1$ in I^+. Furthermore we have

$$u_1'' - \alpha u_1 + \frac{u_1^p}{v_2^q} + \rho(x) \leq 0 = u_2'' - \alpha u_2 + \frac{u_2^p}{v_2^q} + \rho(x) \quad \text{in } I^+,$$

$$u_1, u_2 > 0 \quad \text{in } I^+, \ \ u_1 = u_2 = 0 \quad \text{on } \partial I^+, \ \ u_1'' \in L^1(0,1).$$

Thus, by Lemma 2.2 with $\Psi(x,t) = -\alpha t + \frac{t^p}{v_2^q(x)} + \rho(x)$, $(x,t) \in I^+ \times (0,\infty)$, it follows that $u_2 \leq u_1$ in I^+. Since $u_2 \geq u_1$ in I^+, we deduce $u_1 = u_2$ in I^+, that is, $U \equiv 0$ which contradicts Proposition 3.3. Replacing u_1, v_1 with u_2, v_2 in the above arguments we deduce the statement (ii).

(iii) Suppose that $J^+ \subset I^-$. Then $v_2 \geq v_1$ and $u_1 \geq u_2$ in J^+ which yield

$$\frac{u_1^r}{v_1^s} \geq \frac{u_2^r}{v_2^s} \quad \text{in } J^+.$$

Hence $V = v_2 - v_1$ satisfies

$$\begin{cases} V'' - \beta V = \dfrac{u_1^r}{v_1^s} - \dfrac{u_2^r}{v_2^s} \geq 0 & \text{in } \ J^+, \\ V = 0 & \text{on } \partial J^+. \end{cases}$$

Therefore, by maximum principle, we have $V \leq 0$ in J^+. Since $V \geq 0$ in J^+, it follows that $V \equiv 0$ in J^+ which again contradicts Proposition 3.3. The proof of (iv) follows in a similar way.

From now on, the proof of Theorem 1.3 follows in the same manner as in [1, Theorem 6]. □

4. Case $p < 0$

4.1. Proof of Theorem 1.4.

Assume that the system (0.3) has a classical solution (u,v) and set $M = \|u\|_\infty$. Then v satisfies

$$\Delta v - \beta v + c_1 v^{-s} \geq 0 \quad \text{in } \Omega,$$

where $c_1 = M^r > 0$. By Lemma 2.3 we get $v \le z$, where z is the unique solution of the problem

$$\begin{cases} \Delta z - \beta z + c_1 z^{-s} = 0 & \text{in } \Omega, \\ z > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, from the estimate (2.20) in Proposition 2.5 (with $r = 0$) there exists $c_2 > 0$ such that $v \le z \le c_2 \Gamma_{s,0}(\varphi_1)$ in Ω, which yields

$$\begin{cases} v \le c_2 \varphi_1 \text{ in } \Omega & \text{if } s < 1, \\ v \le c_2 \varphi_1 (1 + |\log \varphi_1|)^{1/(1+s)} \text{ in } \Omega & \text{if } s = 1, \\ v \le c_2 \varphi_1^{2/(1+s)} \text{ in } \Omega & \text{if } s > 1. \end{cases} \tag{4.1}$$

If $s = 1$ and $q > 2$, we fix $0 < \theta < 1$ such that $q\theta \ge 2$. Let us set

$$k = \begin{cases} 1, & \text{if } \ s < 1, \\ \theta, & \text{if } \ s = 1, \\ 2/(s+1), & \text{if } \ s > 1. \end{cases}$$

Then $qk \ge 2$ and by (4.1) we get $v \le c_3 \varphi_1^k$ in Ω, for some $c_3 > 0$. Using this inequality in the first equation of (0.3) we deduce $\Delta u - \alpha u + c\varphi_1^{-qk} u^p + \rho(x) \le 0$ in Ω, where $c = c_3^{-q}$. This means that u is a super-solution of the following problem

$$\begin{cases} \Delta z - \alpha z + c\varphi_1^{-qk} z^p + \rho(x) = 0 & \text{in } \Omega, \\ z > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Note that $w \in C^2(\overline{\Omega})$ defined as the unique solution of (2.5) is a sub-solution of (4.2). By standard maximum principle it is easy to get $u \ge w$ in Ω. Hence, the problem (4.2) has classical solutions, but this contradicts Proposition 2.5 (i), since $qk \ge 2$. Therefore, the system (0.3) has no solutions. The proof of Theorem 1.4 is now complete. $\square$

4.2. Proof of Theorem 1.5

For all $0 < \varepsilon < \varepsilon_0$ let Ω_ε be defined as in (3.1). For $m_1 < 1 < M_1$ and $m_2 < 1 < M_2$ we consider the set $\mathcal{B}_\varepsilon$ of all $(u, v) \in C(\overline{\Omega}_\varepsilon) \times C(\overline{\Omega}_\varepsilon)$ such that

$$m_1 \varphi_1 \le u \le M_1 \varphi_1^\nu \text{ in } \Omega_\varepsilon,$$

$$m_2 \Gamma_{s,r}(\varphi_1) \le v \le M_2 \varphi_1^\tau \text{ in } \Omega_\varepsilon,$$
$$u = \varepsilon, \ v = \Gamma_{s,r}(\varepsilon) \text{ on } \partial\Omega_\varepsilon,$$

where

$$\nu = \begin{cases} 1, & \text{if } q\sigma < 1 + p \\ 1/2, & \text{if } q\sigma = 1 + p \\ \dfrac{2 - q\sigma}{1 - p}, & \text{if } q\sigma > 1 + p \end{cases} \quad \text{and } \tau = \begin{cases} 1, & \text{if } \ s < 1 + r\nu \\ 1/2, & \text{if } \ s = 1 + r\nu \\ \dfrac{2 + r\nu}{1 + s}, & \text{if } \ s > 1 + r\nu \end{cases} . \tag{4.3}$$

In order to prove that $\mathcal{B}_\varepsilon$ is not empty, we first remark that $\nu \le 1$. Therefore, we only need to check that

$$\Gamma_{s,r}(t) \le c_0 t^\tau \quad \text{for all } 0 < t \le 1, \tag{4.4}$$

for some fixed $c_0 > 0$. To this aim we analyze the cases $s < 1+r$, $s = 1+r$ and $s > 1+r$.

If $s < 1 + r$, since $\tau \le 1$ we have $\Gamma_{s,r}(t) = t \le t^\tau$ for all $0 < t \le 1$.

If $s > 1 + r$, from $\nu \le 1$ we have $s > 1 + r\nu$ which implies $\tau = \frac{2+r\nu}{1+s} \le \frac{2+r}{1+s}$. Hence

$$\Gamma_{s,r}(t) = t^{(2+r)/(1+s)} \le t^{(2+r\nu)/(1+s)} = t^\tau \quad \text{for all } 0 < t \le 1.$$

Finally, if $s = 1 + r$ then $s \ge 1 + r\nu$ which implies $\tau = 1/2$ or $\tau = \frac{2+r\nu}{1+s}$. In both cases we have $\tau < 1$. Then

$$\Gamma_{s,r}(t) = t(1 + |\ln t|)^{1/(1+s)} \le c_0 t^\tau \quad \text{for all } 0 < t \le 1,$$

and for some fixed $c_0 > 0$.

Remark 4.1. *Since $t^\theta(1 + |\log t|)^{1/(s+1)} \to 0$ as $t \to 0$, for all $\theta > 0$, we could replace the value $1/2$ in the definition of ν and τ by any number $\theta \in (0, 1)$ in the case $q\sigma = p+1$ and $s = 1 + r\nu$ respectively.*

Therefore, for small $0 < m_1, m_2 < 1$ and for large values of $M_1, M_2 > 1$ the set $\mathcal{B}_\varepsilon$ is not empty.

As in the previous section, for all $(u, v) \in \mathcal{B}_\varepsilon$ let us denote by (Tu, Tv) the unique solution of

$$\begin{cases} \Delta(Tu) - \alpha(Tu) + \dfrac{(Tu)^p}{v^q} + \rho(x) = 0,\ Tu > 0 & \text{in } \Omega_\varepsilon, \\ \Delta(Tv) - \beta(Tv) + \dfrac{u^r}{(Tv)^s} = 0,\ Tv > 0 & \text{in } \Omega_\varepsilon, \\ Tu = \varepsilon,\ Tv = \Gamma_{s,r}(\varepsilon) & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{4.5}$$

In this way we have defined a mapping

$$\mathcal{T} : \mathcal{B}_\varepsilon \to C(\overline{\Omega}_\varepsilon) \times C(\overline{\Omega}_\varepsilon),\ \mathcal{T}(u, v) = (Tu, Tv).$$

Now, we proceed as in the proof of Theorem 1.1. The main point is to show that there exist $0 < m_1, m_2 < 1$ and $M_1, M_2 > 1$ which are independent of ε such that $\mathcal{T}(\mathcal{B}_\varepsilon) \subseteq \mathcal{B}_\varepsilon$. This allows us to employ the Schauder's fixed point theorem.

Following the proof of Lemma 3.1 we get the existence of $m_1, m_2 \in (0, 1)$ which are independent of ε and such that

$$Tu \ge m_1\varphi_1,\ Tv \ge m_2\Gamma_{s,r}(\varphi_1) \quad \text{in } \Omega_\varepsilon.$$

Since $\Gamma_{s,r}(t) \ge t^\sigma$ for all $0 < t \le 1$, the definition of $\mathcal{B}_\varepsilon$ yields $v \ge m_2\varphi_1^\sigma$ in Ω_ε. Furthermore, the first equation in (4.5) produces

$$\Delta(Tu) - \alpha(Tu) + m_2^{-q}\varphi_1^{-q\sigma}(Tu)^p + \rho(x) \ge 0 \quad \text{in } \Omega_\varepsilon.$$

Let $\zeta \in C^2(\Omega) \cap C(\overline{\Omega})$ be the unique solution of (3.13). Since $p < 0$ and $q\sigma < 2$, we shall make use of Proposition 2.5 (ii) instead of Proposition 2.4 as we did in the proof of Theorem 1.1. Therefore, there exist $c_1, c_2 > 0$ such that

$$c_1\Gamma_{-p,-q\sigma}(\varphi_1) \le \zeta \le c_2\Gamma_{-p,-q\sigma}(\varphi_1) \quad \text{in } \Omega. \tag{4.6}$$

Note that $\Gamma_{-p,-q\sigma}(t) \ge t$ for all $0 < t \le 1$. Hence, by (2.1) and (4.6) we get

$$\zeta \ge c_1\varphi_1 \ge Cc_1 d(x) \quad \text{in } \Omega.$$

Let us fix $A > 1$ such that $ACc_1 > 1$. Since $p < 0$ we find

$$\Delta(A\zeta) - \alpha(A\zeta) + m_2^{-q}\varphi_1^{-q\sigma}(A\zeta)^p + \rho(x) \le 0 \quad \text{in } \Omega_\varepsilon,$$

$$A\zeta \ge \varepsilon = Tu \quad \text{on } \partial\Omega_\varepsilon.$$

In view of Lemma 2.3 we derive $A\zeta \ge Tu$ in Ω_ε and by (4.6) it follows that

$$Tu \le Ac_2\Gamma_{-p,-q\sigma}(\varphi_1) \quad \text{in } \Omega_\varepsilon.$$

Note that $\Gamma_{-p,-q\sigma}(t) \le \tilde{c}t^\nu$ for all $0 < t \le 1$ and for some fixed constant $\tilde{c} > 0$. Therefore, we can find $M_1 > 1$ sufficiently large such that $Tu \le M_1\varphi_1^\nu$ in Ω_ε.

Using the estimate $u \le M_1\varphi_1^\nu$ in Ω_ε, from the second equation in (4.5) we deduce

$$\Delta(Tv) - \beta(Tv) + M_1^r\varphi_1^{r\nu}(Tv)^{-s} \ge 0 \quad \text{in } \Omega_\varepsilon.$$

Since $\nu \le 1$, we can easily prove that $\Gamma_{s,r}(t) \le c_0\Gamma_{s,r\nu}(t)$, for all $0 < t \le 1$ and for some positive constant c_0. This implies that

$$Tv = \Gamma_{s,r}(\varepsilon) \le c_0\Gamma_{s,r\nu}(\varepsilon) \quad \text{on } \partial\Omega_\varepsilon.$$

Next, similar arguments to those in the proof of Lemma 3.1 yield $Tv \le c\Gamma_{s,r\nu}(\varphi_1)$ in Ω_ε. It remains to notice that $\Gamma_{s,r\nu}(t) \le \bar{c}t^\tau$ for all $0 < t \le 1$ and for some $\bar{c} > 0$. Hence, $Tv \le M_2\varphi_1^\tau$ in Ω_ε for some $M_2 > 1$ independent of ε. Therefore $\mathcal{T}(\mathcal{B}_\varepsilon) \subseteq \mathcal{B}_\varepsilon$. From now on, we proceed exactly in the same way as in the proof of Theorem 1.1.

Assume next that $q < p + 1$ and $s < r + 1$. Then, by (1.2) and (4.3) we get $\sigma = \nu = \tau = 1$. With the same arguments as in the proof of Theorem 1.1 we get $m_1 d(x) \le u, v \le m_2 d(x)$ in Ω, for some $m_1, m_2 > 0$ and for all solutions (u, v) of (0.3). Then we use the same approach as in Corollary 1.2 in order to get that $u, v \in C^2(\Omega) \cap C^{1,\gamma}(\overline{\Omega})$, for some $0 < \gamma < 1$. This finishes the proof of Theorem 1.5. □

Remark. The approach used in this paper can be employed to extend the study of system (0.3) to the following class of exponents

$$0 \le p < 1, \; 0 < q < p + 1, \; r > 0, \; -1 < s \le 0.$$

In this sense, we need the *smooth* variant of Proposition 2.5 concerning the sublinear case $-1 < s \le 0$. Taking into account the fact that $r > 0$, if $-1 < s \le 0$ then the problem (2.15) has a unique solution $v \in C^2(\overline{\Omega})$. One can show that system (0.3) has classical solutions and any solution (u, v) of (0.3) satisfies

$$c_1 d(x) \le u, v \le c_2 d(x) \quad \text{in } \Omega,$$

for some $c_1, c_2 > 0$. Furthermore, with the same idea as in the proof of Corollary 1.2 we get

(i) if $p \ge q$ then $u, v \in C^2(\overline{\Omega})$;

(ii) if $-1 < p - q < 0$ then $u \in C^2(\Omega) \cap C^{1,1+p-q}(\overline{\Omega})$ and $v \in C^2(\overline{\Omega})$.

Appendix

In this part we prove a result which generalizes Lemma 8 in [1]. More precisely we have

Proposition 4.2. *Let $0 < a < 1$, $-1 < \gamma \le 0$ and $A = (A_{ij})_{1\le i,j\le 2}$ be a 2×2 matrix such that for all $1 \le i, j \le 2$ we have*

$$A_{ij} \in C(0, a] \quad \textit{and} \quad x^{1-\gamma} A_{ij} \in L^\infty(0, a).$$

Assume that there exists $\mathbf{W} = (W_1, W_2)^T \in (C^2(0, a] \cap C^1[0, a])^2$ a solution of

$$\begin{cases} \mathbf{W}''(x) + A(x)\mathbf{W}(x) = \mathbf{0} & \text{in } (0, a], \\ \mathbf{W}(0) = \mathbf{W}'(0) = \mathbf{0}. \end{cases}$$

Then $\mathbf{W} \equiv \mathbf{0}$ in $[0, a]$.

Proof. First we need the following result whose proof is a simple exercise of calculus.

Lemma 4.3. *Let $f \in C(0, a] \cap L^{1+\delta}(0, a)$ for some $a, \delta > 0$ and $u \in C^2(0, a] \cap C^1[0, a]$ be such that $u(0) = u'(0) = 0$ and $u'' = f$ in $(0, a)$. Then*

$$u(x) = \int_0^x (x - t) f(t) dt, \qquad \textit{for all } 0 \le x \le a.$$

Since $\mathbf{W} \in C^1[0, 1] \times C^1[0, 1]$ we have $A\mathbf{W} \in C(0, a] \cap L^{1+\delta}(0, a)$ provided that $0 < \delta < -1 - \gamma^{-1}$. Therefore, by Lemma 4.3 we get

$$\mathbf{W}(x) = -\int_0^x (x - t) A(t) \mathbf{W}(t) dt \quad \text{for all } 0 \le x \le a. \tag{4.7}$$

Define $B = (B_{ij})_{1\le i,j\le 2}$ by $B_{ij}(x) = x^{1-\gamma} A_{ij}(x)$, $0 < x \le a$, $1 \le i, j \le 2$. Then $B_{ij} \in C(0, a] \cap L^\infty(0, a)$. Set

$$M = \max_{1\le i,j\le 2} \|B_{ij}\|_\infty, \quad k = \max\left\{ \frac{|\mathbf{W}(x)|}{x}\, ; 0 < x \le a \right\},$$

where $|\mathbf{W}(x)| = \max\{\, |W_1(x)|, |W_2(x)|\, \}$. Notice that both M and k are finite, since $\mathbf{W} \in C^1[0, a]$. From (4.7) we have

$$\mathbf{W}(x) = -\int_0^x (x-t)B(t)\frac{\mathbf{W}(t)}{t}\,t^\gamma dt \quad \text{ for all } 0 \le x \le a,$$

which yields

$$|\mathbf{W}(x)| \le M\int_0^x (x-t)\frac{|\mathbf{W}(t)|}{t}\,t^\gamma dt \quad \text{ for all } 0 \le x \le a. \tag{4.8}$$

It follows that

$$|\mathbf{W}(x)| \le Mk\int_0^x (x-t)\,t^\gamma dt = \frac{Mk}{(1+\gamma)(2+\gamma)}\,x^{2+\gamma} \quad \text{ for all } 0 \le x \le a. \tag{4.9}$$

Using (4.9) in (4.8) we obtain

$$\begin{aligned} |\mathbf{W}(x)| &\le \frac{M^2k}{(1+\gamma)(2+\gamma)}\int_0^x (x-t)t^{1+2\gamma}dt \\ &= \frac{M^2k}{(1+\gamma)(2+\gamma)(2+2\gamma)(3+2\gamma)}\,x^{3+2\gamma} \\ &\le \frac{M^2k}{2(1+\gamma)^2}\,x^{3+2\gamma} \quad \text{ for all } 0 \le x \le a. \end{aligned}$$

By induction, we deduce that for all $n \ge 2$ we have

$$|\mathbf{W}(x)| \le \frac{M^n k}{n!(1+\gamma)^n}\,x^{n+1+n\gamma} \quad \text{ for all } 0 \le x \le a.$$

Since $-1 < \gamma \le 0$, we can pass to the limit in the last inequality in order to get $\mathbf{W} \equiv \mathbf{0}$. This completes the proof.

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