The nonnegative inverse eigenvalue problem

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(Dated: May 29, 2012)

Introduction

An \( n \times n \) matrix \( A = (a_{ij}) \) is (entrywise) nonnegative, denoted \( A \geq 0 \), if its entries \( a_{ij} \geq 0 \). Such matrices arise naturally in probability theory, economics, ergodic theory, graph theory and the theory of internet search engines. By inverse eigenvalue problems for matrices we shall mean the following problem:

given a property \( X \) applicable to the entries of a matrix, and a list of complex numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) (including multiplicities), find necessary and sufficient conditions for the existence of an \( n \times n \) complex matrix \( A \) having property \( X \) and spectrum (list of eigenvalues) \( \lambda_1, \lambda_2, \ldots, \lambda_n \).

For example if \( X \) is the property of being complex, the problem is quite easy since in this case we can choose \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \). If \( X \) is the property of being real, then \( A \) is required to be a real matrix and the condition required is that the list \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be closed under complex conjugation. If \( X \) is the property that the entries be either 0 or 1, then the problem is the spectral theory of graphs. Choosing \( X \) to be the property of being an integer requires that \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be a set of algebraic integers closed under the action of the Galois group of the algebraic closure \( \overline{\mathbb{Q}} \) of the rational number field \( \mathbb{Q} \) (see Masha Vlasenko’s talk from yesterday for more on algebraic integers). For the purposes of this discussion we require that property \( X \) be that the entries are nonnegative.

\[ \text{NIEP} \]

Thus the nonnegative inverse eigenvalue problem is the following:

\( \text{NIEP} \): Find a complete set of necessary and sufficient conditions on a list of \( n \)
complex numbers so that this list is the set of eigenvalues of an $n \times n$ nonnegative matrix $A$. Such an $A$ is called a realizing matrix.

The problem is completely solved only in the case $n \leq 4$ and in the case $n = 5$ when $\text{trace}(A) = 0$, [6], where $\text{trace}(A)$ means the sum of the diagonal entries of $A$. For the purposes of the NIEP we consider irreducible matrices.

**Definition 1.** An $n \times n$ matrix $A$ is said to be (permutationally) reducible if $A$ is the zero matrix for $n = 1$, or for $n \geq 2$, there exists a permutation matrix $P$ with

$$P^{-1}AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where $B$, $D$ are square matrices of order $r$ and $s$ respectively where $0 < r, s < n$. Otherwise, $A$ is said to be (permutationally) irreducible.

Since the spectrum of a reducible matrix is just the union of the spectra of the irreducible blocks, we may restrict ourselves to considering only irreducible matrices. Let $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then we define

$$s_k := \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k$$

and note that if $A \geq 0$ and $A$ has spectrum $\sigma$ then

$$s_k = \text{trace}(A^k) \geq 0$$

is a necessary condition for NIEP.

Another necessary condition is that

$$(\lambda_1, \ldots, \lambda_n) = (\overline{\lambda_1}, \ldots, \overline{\lambda_n}),$$

where $\overline{\cdot}$ denotes complex conjugation, since $A$ is required to be real and so the characteristic polynomial of $A$ has real coefficients.

The Perron-Frobenius theorem yields a further necessary condition, namely that if

$$\rho = \max\{|\lambda_i| : i = 1, 2, \ldots, n\}$$
then $\rho \in \{\lambda_1, \ldots, \lambda_n\}$. In fact for irreducible nonnegative $A$, $\rho$ occurs just once. A very useful set of necessary conditions (known as the JLL conditions) was discovered independently by Johnson [3], and Loewy and London [5], that is

$$n^{m-1}s_{km} \geq s_k^m \text{ for } k, m = 1, 2, \ldots$$

For a new necessary condition in the NIEP see [2].

**Suleimanova’s theorem**

The first major result towards answering the NIEP was discovered in 1949 by Suleimanova [9]. It says

**Theorem 2.** If $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a list of real numbers with

$$\lambda_1 \geq 0 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

then $\sigma$ is realizable if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n \geq 0.$$

A proof of this result given by Friedland uses the concept of companion matrices. If $f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n$, then the companion matrix of $f$ is

$$A(f) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
-p_n & -p_{n-1} & \cdots & -p_2 & -p_1
\end{bmatrix}$$

Note that the companion matrix has the property that its characteristic polynomial and minimal polynomial are equal to $f(x)$. Clearly, $A(f)$ is nonnegative if and only if $p_i \leq 0$ for $i = 1, \ldots, n$.

**Sufficient conditions**
Perhaps the most celebrated result in the area of the NIEP is due to Boyle and Handelman from 1993, [1]. They used ergodic theory and symbolic dynamics to show that:

**Theorem 3.** If \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) is a list of complex numbers satisfying \(\lambda_1 > |\lambda_m|\) (all \(m > 1\)) with \(s_k \geq 0\) for all positive integers \(k\) (and \(s_m = 0\) for some \(m \Rightarrow s_d = 0\) for all \(d \mid m\)), then there exists a positive integer \(N\) and a nonnegative \(N \times N\) matrix \(A\) with spectrum \((\lambda_1, \lambda_2, \ldots, \lambda_n, 0, 0, \ldots, 0)\).

The proof of this result does not yield a useable algorithm to construct \(A\). Laffey, in a recent result [4], gave a constructive approach to the Boyle-Handelman theorem. It states that if \(\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)\) satisfies \(\lambda_1 > |\lambda_m|\) (all \(m > 1\)), \(s_1 \geq 0\) and \(s_k > 0\) for \(k \geq 2\), then \(\sigma\) with \(N - n\) zeros appended is the spectrum of a nonnegative matrix \(A\). Starting with a polynomial \(q(x) = \prod_{j=1}^{n} (x - \mu_j) = x^n + q_1x^{n-1} + \cdots + q_{n-1}x + q_n\), Laffey examines the matrix

\[
X_n = \begin{bmatrix} x_1 & 1 & 0 & \ldots & 0 \\ x_2 & x_1 & 2 & 0 & \ldots & 0 \\ x_3 & x_2 & x_1 & 3 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \ddots \\ x_{n-1} & x_{n-2} & \cdots & x_2 & x_1 & n - 1 \\ x_n & x_{n-1} & \cdots & x_3 & x_2 & x_1 \end{bmatrix}
\]

where \(x_k := \mu_1^k + \mu_2^k + \cdots + \mu_n^k, k = 1, 2, \ldots\)

He shows that the characteristic polynomial of \(X_n\) is

\[
Q(x) = x^n + nq_1x^{n-1} + n(n - 1)q_2x^{n-2} + \cdots + n!q_n
\]

where

\[
q(x) = \prod_{i=1}^{n} (x - \mu_i) = x^n + q_1x^{n-1} + \cdots + q_n.
\]

He also shows that \(\sigma\) with \(N - n\) zeros added may be realized by an appropriate \(X_N\), and obtains a bound on the minimum \(N\) for which this is possible.
NIEP solutions for \( n \leq 5 \)

For \( n = 2 \), let \( \sigma = (\lambda_1, \lambda_2) \). The characteristic polynomial of the nonnegative matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

is

\[
f(x) = x^2 - (a + d)x + ad - bc,
\]

which has roots

\[
a + d \pm \frac{\sqrt{(a + d)^2 - 4(ad - bc)}}{2} = \frac{a + d \pm \sqrt{(a - d)^2 + 4bc}}{2}.
\]

So, \( A \) has real eigenvalues \( \lambda_1 \geq \lambda_2 \) with \( \lambda_1 + \lambda_2 = a + d \geq 0 \).

Conversely, if \( \lambda_1 \geq \lambda_2 \) are real numbers with \( \lambda_1 + \lambda_2 \geq 0 \), then

\[
\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}
\]

is a nonnegative matrix with spectrum \( \sigma \) in the case where \( \lambda_1 \geq 0, \lambda_2 \geq 0 \), and

\[
\begin{pmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1 + \lambda_2 \end{pmatrix}
\]

is a nonnegative matrix with spectrum \( \sigma \) in the case where \( \lambda_1 > 0 > \lambda_2 \) with respect to the above matrices.

If \( \lambda_1 + \lambda_2 \geq 0 \) where \( \lambda_1 \geq |\lambda_2| \), then the following symmetric nonnegative matrix realizes \( \sigma \)

\[
A = \frac{1}{2} \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{pmatrix}.
\]

For \( n = 3 \), we must consider a number of cases. Firstly, suppose \( \sigma = (\lambda_1, \lambda_2, \lambda_3) \) is a decreasing list of real numbers with \( \lambda_1 + \lambda_2 + \lambda_3 \geq 0 \) and \( \lambda_1 \geq |\lambda_j| \) for \( j = 2, 3 \).
Then

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\]

realizes \( \sigma \) if \( \lambda_2, \lambda_3 \) are nonnegative,

\[
\begin{pmatrix}
\lambda_2 & 0 & 0 \\
0 & 0 & \sqrt{|\lambda_1 \lambda_3|} \\
0 & \sqrt{|\lambda_1 \lambda_3|} & \lambda_1 + \lambda_3
\end{pmatrix}
\]

realizes \( \sigma \) if \( \lambda_2 \geq 0 \geq \lambda_3 \), and

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\lambda_1 \lambda_2 \lambda_3 & -(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) & \lambda_1 + \lambda_2 + \lambda_3
\end{pmatrix}
\]

realizes \( \sigma \) if \( \lambda_2 \leq 0 \). For \( n = 3 \), Loewy and London [5] completely solved the NIEP for lists of length three.

**Theorem 4.** Let \( \sigma = (\lambda_1, \lambda_2, \lambda_3) \) be three complex numbers. Then \( \sigma \) is realizable if and only if:

1. \( \max \{ |\lambda_i| : \lambda_i \in \sigma \} \in \sigma \),
2. \( \sigma = \overline{\sigma} \),
3. \( s_1 \geq 0 \), and
4. \( 3s_2 \geq s_1^2 \).

Note that condition 4 is the JLL condition for \( m = 2 \) and \( k = 1 \). As all four of these conditions are homogeneous, if \( \sigma \) is non-real, we can multiply \( \sigma \) by \( \frac{1}{|\lambda_2|} \) to get \( \hat{\sigma} = (\rho, e^{i\theta}, e^{-i\theta}) \), where \( \rho \geq 1 \) and \( 0 < \theta < \pi \). Then the following nonnegative circulant matrix \( A \) realizes \( \hat{\sigma} \):

6
\[ A = \frac{1}{3} \begin{pmatrix} 
\rho + 2\cos\theta & \rho - 2\cos\left(\frac{\pi}{3} + \theta\right) & \rho - 2\cos\left(\frac{\pi}{3} - \theta\right) \\
\rho - 2\cos\left(\frac{\pi}{3} - \theta\right) & \rho + 2\cos\theta & \rho - 2\cos\left(\frac{\pi}{3} + \theta\right) \\
\rho - 2\cos\left(\frac{\pi}{3} + \theta\right) & \rho - 2\cos\left(\frac{\pi}{3} - \theta\right) & \rho + 2\cos\theta 
\end{pmatrix}. \]

Note that \( A \) is in fact a doubly stochastic matrix, where a matrix is stochastic (doubly stochastic) if it is nonnegative and has row sums (and column sums) equal to one. The Newton identities [7] state that

\[ s_k + s_{k-1}p_1 + s_{k-2}p_2 + \cdots + s_1p_{k-1} + kp_k = 0 \]

for \( k = 1, 2, \ldots, n \), where \( p(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n \) is a polynomial with roots \( \lambda_1, \lambda_2, \ldots, \lambda_n \), and \( s_k = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k \).

It is worth noting that if \( \sigma \) is not real and \( \sigma' = (\lambda_1 - \frac{a_1}{3}, \lambda_2 - \frac{a_1}{3}, \lambda_2 - \frac{a_1}{3}) \), the characteristic polynomial of a matrix with spectrum \( \sigma' \) has the form

\[ c(x) = x^3 + p_2x + p_3, \]

where \( p_3 = -(\lambda_1 - \frac{a_1}{3})(\lambda_2 - \frac{a_1}{3})(\lambda_2 - \frac{a_1}{3}) \leq 0 \), and so, (using the Newton identities), \( p_2 = -\frac{1}{6}(3s_2 - s_1^2) \leq 0 \), whereby \( \sigma \) is realizable by \( \frac{a_1}{3}I + C \), where \( C \) is the companion matrix of \( c(x) \). Hence, we have shown

**Theorem 5.** If \( \sigma = (\lambda_1, \lambda_2, \lambda_2) \) with \( \lambda_1 \geq |\lambda_2| \). Then \( \sigma \) is realizable if and only if \( \sigma - \frac{a_1}{3} := (\lambda_1 - \frac{a_1}{3}, \lambda_2 - \frac{a_1}{3}, \lambda_2 - \frac{a_1}{3}) \) is realizable.

and its corollary

**Corollary 6.** Let \( \sigma = (\lambda_1, \lambda_2, \lambda_2) \). Then \( \sigma \) is realizable if and only if \( \sigma \) is realizable by \( A = \frac{a_1}{3}I + C \) where \( C \) is the companion matrix of

\[ c(x) = (x - (\lambda_1 - \frac{a_1}{3}))(x - (\lambda_2 - \frac{a_1}{3}))(x - (\lambda_2 - \frac{a_1}{3})) = x^3 + p_2x + p_3. \]

Note that this result does not hold for real spectra of length three, since the list \( \sigma = (1, 1, -1) \) is the spectrum of

\[ \begin{pmatrix} 
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 
\end{pmatrix}. \]
but \( \sigma - \frac{1}{3} = (\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}) \) is not realizable, as it does not have the Perron property. For \( n = 4 \) and \( s_1 = 0 \), Reams [8] showed that \( s_2 \geq 0, s_3 \geq 0 \) and \( 4s_4 - s_2^2 \geq 0 \) are necessary and sufficient conditions for \( \sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) to be realizable.

Observe that if \( A \) is a \( 4 \times 4 \) matrix with \( \text{trace}(A) = 0 \), then the characteristic polynomial \( c(x) \) of \( A \) can be written (using Newton’s identities) as

\[
c(x) = x^4 - \frac{s_2x^2}{2} - \frac{s_3x}{3} - \left( \frac{s_4}{4} - \frac{s_2^2}{8} \right)\]

\[
= \left( x^2 - \frac{s_2}{4} \right)^2 - \frac{s_3x}{3} - \left( \frac{4s_4 - s_2^2}{16} \right),
\]

and so \( c(x) \) is the characteristic polynomial of the matrix

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{s_2}{4} & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{4s_4 - s_2^2}{16} & \frac{s_3}{3} & \frac{s_2}{4} & 0
\end{pmatrix},
\]

with \( A \) being nonnegative under the stated necessary conditions.

Again, note that here we require only one JLL inequality and three nonnegative power sums for realizability.

Note that Corollary 6 does not hold for \( \sigma = (\rho, a + ib, a - ib, \mu) \) where \( \mu \neq 0 \). For example the list \( (4, 2 + i, 2 - i, 3) \) is realizable by

\[
\begin{pmatrix}
\frac{8}{3} & 1 & 0 & 0 \\
0 & \frac{8}{3} & 1 & 0 \\
\frac{52}{27} & \frac{1}{3} & \frac{8}{3} & 0 \\
0 & 0 & 0 & 3
\end{pmatrix}
\]

but the list \( (\frac{5}{4}, \frac{-3}{4} + i, \frac{-3}{4} - i, \frac{1}{4}) \) is not realizable by \( \frac{11}{4} I_4 + C \) where \( C \) is a nonnegative trace zero companion matrix since

\[
f(x) = \left( x - \frac{5}{4} \right) \left( x - \frac{1}{4} \right) \left( \left( x + \frac{3}{4} \right)^2 + 1 \right)
\]

\[
= x^4 - \frac{3x^2}{8} - \frac{15x}{8} + \frac{125}{256}
\]
has its constant term positive and so the companion matrix of \( f(x) \) is not non-negative. However we do have the following:

**Theorem 7.** Spectra of the form \( \sigma = (\rho, a+ib, a-ib, 0) \) are realizable by \( \frac{1}{4}I_4 + C \) for \( C \) a nonnegative companion matrix if and only if \( \rho \geq 2a \).

In general a nonnegative companion matrix cannot have more than one positive real eigenvalue. For a general list of four numbers, Laffey and Meehan [6] and Torre-Mayo et al. in [10] have independently found full solutions to the problem but they are complicated and beyond the requirements of this discussion.

**References**


1. For the following lists, determine whether or not they form the set of eigenvalues of an $n \times n$ nonnegative matrix. If they do, provide a realizing matrix and if they do not, justify your reason.

   i. $\sigma = \{1 \pm i\}$

   ii. $\sigma = \{1, -1\}$

   iii. $\sigma = \{1, 0, i\}$

   iv. $\sigma = \{3, i, -i\}$

   v. $\sigma = \{5, 2 + 3i, 2 - 3i, -6\}$

   vi. $\sigma = \{2, 1 + i, 1 - i, 0\}$

   vii. $\sigma = \{3, 3, -2, -2, -2\}$

   viii. $\sigma = \{8, -2, -2, -2, -2\}$

   ix. $\sigma = \{\sqrt{2}, \sqrt{2}, i, -i\}$

2. Find the minimum $t > 0$ for which $\sigma = \{3 + t, 3, -2, -2, -2\}$ is realizable.

   This problem is unsolved, but we do know that

   
   $0.396711738 \cdots \leq t \leq 0.519310982048 \cdots$
1. No, since for $n = 2$, $\sigma \subset \mathbb{R}$

2. Yes, for example,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

works.

3. No, since $\sigma \neq \overline{\sigma}$

4. Yes, for example

$$A = I_3 + C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}$$

works.

5. No, since $\rho = \max\{|\lambda| : \lambda \in \sigma\} = 6 \notin \sigma$

6. Yes, for example

$$A = I_4 + C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

works.

7. No, since $\rho = 3$ occurs twice, a realizing matrix must be reducible and no such (nonnegative) matrix can exist since $s_1 \geq 0$ must hold.

8. Yes, since this list satisfies Suleimanova's condition ($\lambda_1 > 0 \geq \lambda_2 \geq \cdots \geq \lambda_5$) we have that the companion matrix $A$ of

$$\prod_{n=1}^{5} (x - \lambda_i)$$

works, i.e.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 128 & 240 & 160 & 40 & 0 \end{pmatrix}$$

works.

9. No, again a realizing matrix must be reducible and since JLL requires $n^{m-1} s_{km} \geq s_k^m$ for this $3 \times 3$ matrix we have that (for $m = 2, k = 1$) $s_k(0) \geq 2$ which is false.