University College Dublin
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School of Mathematics and Statistics

Mathematics of Sustainability & Environment (ACM41010)

Dr. Graham Benham

Lecture notes on Maths of Sust. & Enviroment 2024

Mathematics of Sustainability & Environment (ACM41010)

- School: Mathematics and Statistics
- Module Coordinator: Dr. Graham Benham
- Credits: 5
- Level: 4
- Semester: Autmun

What will I learn?

On successful completion of this module the student should be able to:

1. Build mathematical models based on conservation laws
2. Non-dimensionalise these models
3. Analyse these models using dimensional scaling arguments
4. Convert PDE's into ODE's via similarity transformations
5. Solve simple PDE systems using the method of characteristics
6. Analyse discontinuities and shocks
7. Apply all of the above techniques to problems in sustainability and the environment

Assessment

1. Class test: There will be one 45 minutes examination during the lecture slot in week 6 of the trimester. The test will be worth 20% of your module grade.

2. Assignment: There will be one assignment worth 20% of your module grade. This will take the form of a research project, conducted in teams of 2-3 students. The assessment will take place during the lecture slot in week 12 of the trimester in the form of a 10 minute presentation plus 3 minutes of questions.

3. Final Exam: 2 hours examination in December 2024. This accounts for 60% of your module grade

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Acknowledgements: These notes were typed up by Maria Semerkina, a former student of ACM41010.

Contents

Continuum Mechanics

1.1 Introduction

We will be studying how material properties vary in space and time e.g: density, temperature, pressure, etc.

We are making a continuum approximation and we will treat a material as having a continuous distribution of mass. This will apply on scales much larger than a molecule.

Each point in space is ascribed different properties and is governed by physical principles. This is the conservation of mass, momentum and energy.

1.2 Conservation of Mass

Beginning with the conservation of mass we will look at an example.

For a fluid, the conservation of mass is given as shown below. This is also known as the continuity equation.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1.1}$$

The fluid velocity and density are functions of position and time as shown: $\mathbf{u}(\mathbf{x}, t)$ and $\rho(\mathbf{x}, t)$.

If ρ is constant, it is an incompressible fluid i.e. $\nabla \cdot \mathbf{u} = 0$

If we consider a concentration of a substance like salt in water we can use:

$$\frac{\partial \mathbf{C}}{\partial t} + \nabla \cdot \mathbf{J} = \mathbf{G} - \mathbf{U} \tag{1.2}$$

- Concentration is $\mathbf{C}(\mathbf{x}, t)$ [ ] (no dimension)
- Flux of Concentration $\mathbf{J}(\mathbf{x}, t)$ [ms^{-1}]
- Source of Concentration $\mathbf{G}(\mathbf{x}, t)$ [s^{-1}]
- Sink of Concentration $\mathbf{U}(\mathbf{x}, t)$ [s^{-1}]

Concentration is conserved unless there are some sources/sinks.

If we focus on the on the flux: $\mathbf{J}$ is often decomposed into advective and diffusive components:

$$\mathbf{J} = \mathbf{Q_A} + \mathbf{Q_D} \tag{1.3}$$

Diffusion $\rightarrow$ Molecular Diffusion or Brownian Motion

Advection $\rightarrow$ Another word for transported

Plotting in 1D

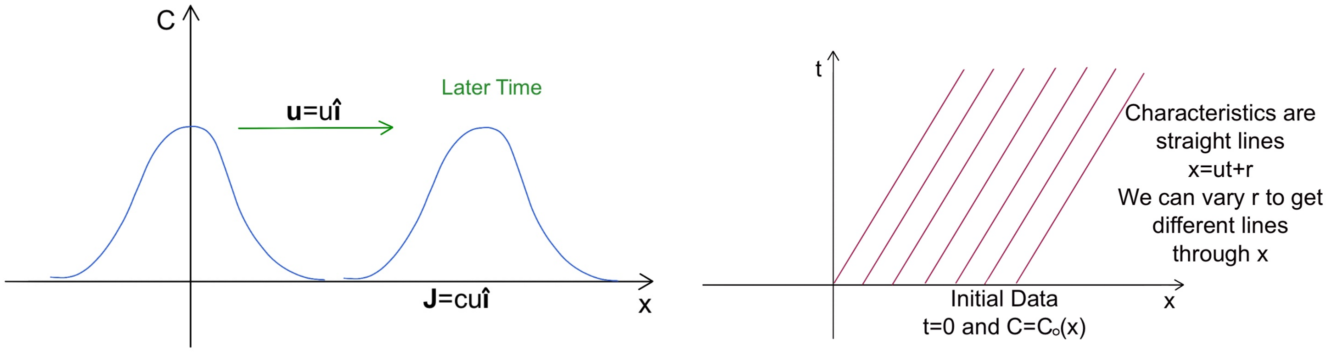
Consider 1D advection at velocity $u\hat{\boldsymbol{\imath}}$:

$$\mathbf{J} = \mathbf{Q_A} = Cu\hat{\boldsymbol{\imath}} \tag{1.4}$$

In ACM30220 we saw the method of characteristics and we can draw the characteristic projections. We have the PDE

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = 0 \tag{1.5}$$

This is known as the transport equation for constant speed u.

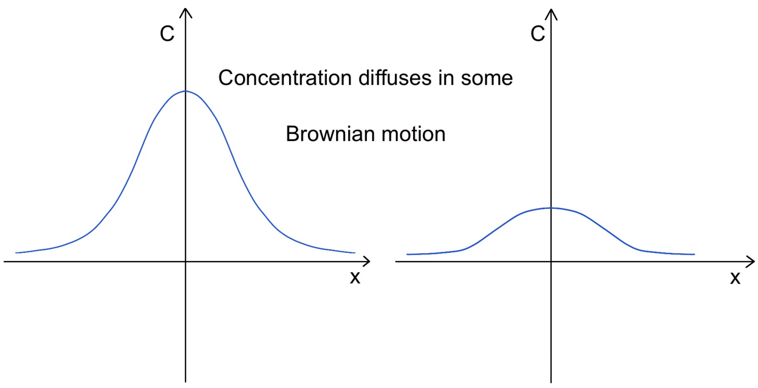


Now consider a diffusive flux with a constant diffusion coefficient D_o:

$$\mathbf{J} = \mathbf{Q_D} = -D_o\frac{\partial C}{\partial x}\hat{\boldsymbol{\imath}} \tag{1.6}$$

The conservation of mass becomes the heat equation as shown below:

$$\frac{\partial C}{\partial t} - D_o\frac{\partial^2 C}{\partial x^2} = 0 \tag{1.7}$$



We can solve it in two cases:

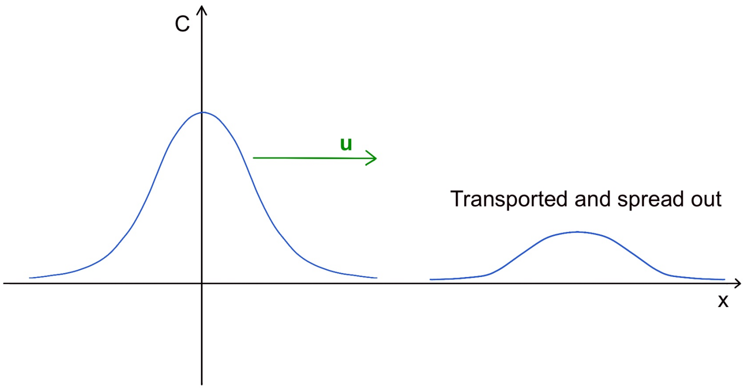
- Infinite domain - Fourier Transform
- Finite domain - Separation of Variables

Example

Next consider the case $\mathbf{J} = uC\hat{\boldsymbol{\imath}} - D_o\frac{\partial C}{\partial x}\hat{\boldsymbol{\imath}}$, so we have advection and diffusion

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D_0\frac{\partial^2 C}{\partial x^2} \tag{1.8}$$

This is the advection-diffusion equation which is hard to solve analytically. General behaviour is shown below:



1.3 Conservation of Momentum

Next looking at the conservation of momentum, we start with the Navier-Stokes equations. These are the momentum equations for an incompressible fluid.

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\,\mathbf{u}\right)=\nabla\cdot\underline{\underline{\sigma}}+\mathbf{F} \tag{1.9}$$

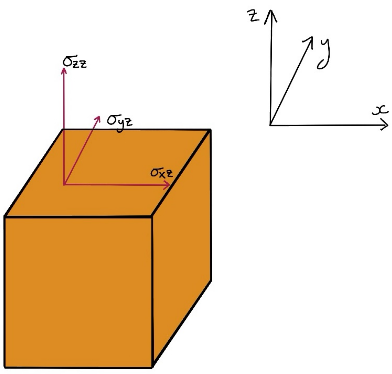
Rate of change of momentum = Forces applied

$\mathbf{F}$ = external force (per unit volume) [N/m^3], e.g: gravity $\mathbf{F}=-\rho g\hat{\mathbf{k}}$
$\underline{\underline{\sigma}}$ = stress tensor [N/m^2]

In index notation this is σ_{ij} which is an $n\times n$ matrix in n dimensions.

Aside about the Stress Tensor

Consider a fluid element. E.g. σ_{yz}= stress on the z-surface in the y direction



Let $S_i=\sigma_{ij}n_j$, this is the stress acting on the surface with outward pointing normal $\mathbf{n}$.

So for the cubic element the upper surface has $\mathbf{n}=(0,0,1)$.

$\therefore\mathbf{S}=(\sigma_{xz},\sigma_{yz},\sigma_{zz})$

We require a few more steps to get to the Navier Stokes equations.

For a Newtonian fluid, stress $\underline{\underline{\sigma}}$ is proportional to strain $\underline{\underline{e}}$:

$$\sigma_{ij}=-p\delta_{ij}+2\mu\dot{e}_{ij} \tag{1.10}$$

$$\dot{e}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.11}$$

This tensor $\underline{\underline{\dot{e}}}$ is a symmetric rate-of-strain tensor that tells is how the fluid is deforming.

Exercise:

If we take the divergence:

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla p + \mu \nabla^2 \mathbf{u} \tag{1.12}$$

Then we have the Navier Stokes equations:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} - \rho g\hat{\mathbf{k}} \tag{1.13}$$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{u} - \rho g\hat{\mathbf{k}} \tag{1.14}$$

The $\frac{D}{Dt}$ is an advective derivative or a material derivative. This comes about because we are in an Eulerian reference frame. An Eulerian reference frame is a stationary frame through which a fluid is moving.

1.4 Conservation of Energy

If we look at the temperature in a solid $T(\mathbf{x}, t)$, this satisfies the heat equation (conservation of energy):

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) \tag{1.15}$$

- T= temperature [K]
- ρ = density [kg/m^3]
- c_p = specific heat capacity [J/ kg K]
- k = conductivity [J/m K s]

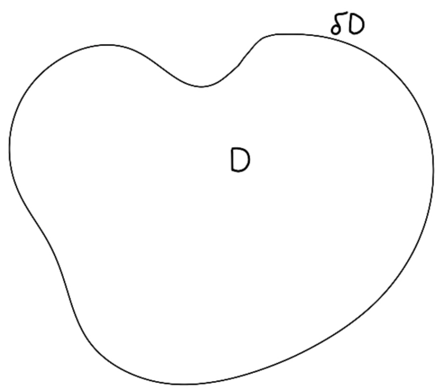
Each conservation law is a PDE. To complete the system we would require boundary and initial conditions.

1.5 Dimensional Analysis

It is often useful to reduce/remove the physical parameters in a problem. It makes it simpler and more general. This is done by re-scaling dependant and independent variables.

Example: The heat equation

We have a domain D,



and here ρ, c_p, k are all constants. We have:

$$\rho c_p T_t = k\nabla^2 T \qquad \text{on } D \tag{1.16}$$
$$T = T_1 \qquad \text{on } \delta D \tag{1.17}$$
$$T = T_0 \qquad \text{at } t = 0 \tag{1.18}$$

This is a well defined problem but first we will remove dimensions. Taking L as the characteristic length scale. An obvious choice would be to re-scale.

$$\mathbf{x} = L\hat{\mathbf{x}} \qquad t = \tau\hat{t} \qquad T = T_0 + (T_1 - T_0)\hat{T} = T_0 + \Delta T\ \hat{T} \tag{1.19}$$

Now inserting this into the heat equation,

$$\frac{\rho c_p \Delta T}{\tau}\hat{T}_{\hat{t}} = \frac{\Delta T k}{L^2}\hat{\nabla}^2\hat{T} \tag{1.20}$$

Rearranging this into,

$$\hat{T}_{\hat{t}} = \frac{\tau k}{\rho c_p L^2}\hat{\nabla}^2\hat{T} \tag{1.21}$$

This indicates that the characteristic timescale must be,

$$\tau = \frac{\rho c_p L^2}{k} \tag{1.22}$$

So now our PDE is

$$\hat{T}_{\hat{t}} = \hat{\nabla}^2\hat{T} \tag{1.23}$$

Now we must look at the initial and boundary conditions and non dimensionalise these.

$$\hat{T} = 0 \qquad \text{at } \hat{t} = 0 \tag{1.24}$$
$$\hat{T} = 1 \qquad \text{at } \delta\hat{D} \tag{1.25}$$

In summary:

$$\hat{T}_{\hat{t}} = \hat{\nabla}^2\hat{T} \tag{1.26}$$
$$\hat{T} = 0 \qquad \text{at } \hat{t} = 0 \tag{1.27}$$
$$\hat{T} = 1 \qquad \text{at } \delta\hat{D} \tag{1.28}$$

Now our problem has no parameters so we only need to solve once for all the parameter values. i.e. We can use the same solution for a variety of materials.

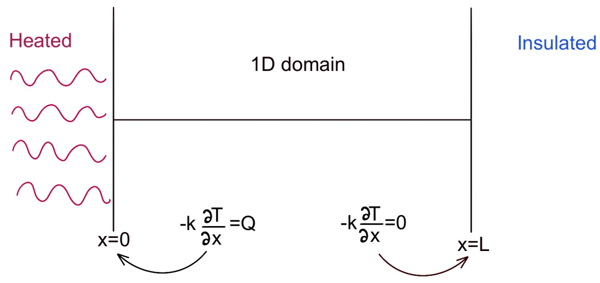
Example: Heat Diffusion in a 1D Ice Sheet

Mathematically we have,

$$-k\frac{\partial T}{\partial x} = Q \qquad \text{at } x = 0 \tag{1.29}$$

Our heat problem is,

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} = \alpha\frac{\partial^2 T}{\partial x^2} \tag{1.30}$$



where $\alpha = \frac{k}{\rho c_p}$ is known as the thermal diffusivity [m^2/s]. The boundary conditions:

$$-kT_x = Q \qquad \text{at } x = 0 \tag{1.31}$$
$$-kT_x = 0 \qquad \text{at } x = L \tag{1.32}$$
$$T = T_0 \qquad \text{at } t = 0 \tag{1.33}$$

Now we pose the question, how long does it take to move the mean ice temperature from T_0 to T_1?

First we non-dimensionalise,

$$x = L\hat{x} \qquad t = \tau\hat{t} \qquad T = T_0 + (T_1 - T_0)\hat{T} = T_0 + \theta\hat{T} \tag{1.34}$$

leaving the constants θ, τ undetermined for now. So our heat problem becomes,

$$\frac{\theta}{\tau}\hat{T}_{\hat{t}} = \frac{\alpha\theta}{L^2}\hat{T}_{\hat{x}\hat{x}} \tag{1.35}$$
$$\frac{-k\theta}{L}\hat{T}_{\hat{x}} = Q \qquad \text{at } \hat{x} = 0 \tag{1.36}$$
$$\frac{-k\theta}{L}\hat{T}_{\hat{x}} = 0 \qquad \text{at } \hat{x} = 1 \tag{1.37}$$
$$\hat{T} = 0 \qquad \text{at } \hat{t} = 0 \tag{1.38}$$

How do we choose our unknown constants?

Let $\theta = \frac{LQ}{k}$ and $\tau = \frac{L^2}{\alpha}$

$$\hat{T}_{\hat{t}} = \hat{T}_{\hat{x}\hat{x}} \tag{1.39}$$
$$-\hat{T}_{\hat{x}} = 1 \qquad \text{at } \hat{x} = 0 \tag{1.40}$$
$$-\hat{T}_{\hat{x}} = 0 \qquad \text{at } \hat{x} = 1 \tag{1.41}$$
$$\hat{T} = 0 \qquad \text{at } \hat{t} = 0 \tag{1.42}$$

We could solve this using separation of variables for example and it would be general. Note sometimes it is not possible to remove all the parameters of the problem. E.g. Consider the problem:

$$T_t = \alpha T_{xx} \tag{1.43}$$
$$T = T_1 \qquad \text{at } x = 0 \tag{1.44}$$
$$T = T_2 \qquad \text{at } x = L \tag{1.45}$$
$$T = T_0 \qquad \text{at } t = 0 \tag{1.46}$$

Here we can't remove all the parameters because there are too many. If we choose,

$$T = T_0 + (T_1 - T_0)\hat{T} \qquad \text{then:} \quad \hat{T} = \frac{T_2 - T_0}{T_1 - T_0} \quad \text{at} \quad \hat{x} = 1 \tag{1.47}$$

We will end up with one dimensionless parameter $\lambda = \frac{T_2 - T_0}{T_1 - T_0}$ (or something equivalent). It is still solvable but no longer generalised. If the parameter λ changes, a new solution needs to be calculated.

1.6 Scaling Arguments

Dimensional scaling arguments can be a powerful tool to help gain insight and physical intuition without doing a lot of maths. Revisiting our previous example:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1.48}$$

$$-\frac{\partial T}{\partial x} = \frac{Q}{k} \qquad \text{at } x = 0 \tag{1.49}$$

$$-\frac{\partial T}{\partial x} = 0 \qquad \text{at } x = L \tag{1.50}$$

$$T = T_0 \qquad \text{at } t = 0 \tag{1.51}$$

Now we need to consider what are the physical parameters. We can make a list.

- α [m^2/s]
- $\eta = \frac{Q}{k}$ [K/m] (Note: Q and k only appear as a fraction, so we write this as a new parameter η)
- L [m]
- T_0 [K]

We asked the question how long does it take to increase the mean temperature from $T_0 \to T_1$.

Consider $\Delta T = T_1 - T_0$ [K]

We want a timescale τ [s] that depends on the other parameters of the problem, such that $\tau = \tau(\alpha, \eta, L, \Delta T)$.

To form a scaling argument, we are going to look at all the dimensional parameters, and try to rearrange in such a way to just get dimensions of seconds.

We have two routes to take:

1. $\tau = \frac{L^2}{\alpha}$ [s]

The problem here is, we want our timescale to predict something about ΔT. Here we have mo dependence on temperature change, so this won't work. Then we can try,

2. $\tau = \frac{L \Delta T}{\alpha \eta}$ [s]

Now we have a characteristic timescale that links time with ΔT.

Claim: Approximate time to heat the ice by ΔT is

$$\tau \approx \frac{L \Delta T}{\alpha \eta} \tag{1.52}$$

To check this we can solve the problem.

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{1.53}$$

$$-\frac{\partial T}{\partial x} = \frac{Q}{k} \qquad \text{at } x = 0 \tag{1.54}$$

$$-\frac{\partial T}{\partial x} = 0 \qquad \text{at } x = L \tag{1.55}$$

$$T = T_0 \qquad \text{at } t = 0 \tag{1.56}$$

Integrating equation (1.55) from $x = 0$ to $x = L$ gives,

$$\int_0^L \frac{\partial T}{\partial t} dx = \alpha \left[\frac{\partial T}{\partial x}\right]_{x=0}^{x=L} \tag{1.57}$$

$$\frac{d}{dt}\int_0^L T dx = \alpha\frac{Q}{k} = \frac{kQ}{\rho c_p k} = \frac{Q}{\rho c_p} \tag{1.58}$$

We want the mean temperature so we seek,

$$\frac{1}{L}\int_0^L T dx = \overline{T} \tag{1.59}$$

So our equation (1.58) is:

$$L\frac{d\overline{T}}{dt} = \frac{Q}{\rho c_p} \tag{1.60}$$

$$\overline{T} = \frac{Q}{\rho c_p L} t + \text{constant} \tag{1.61}$$

Using the initial condition $T = T_0$ at $t = 0$, we find

$$\overline{T} = T_0 + \frac{Q}{\rho c_p L} t \tag{1.62}$$

Now if we have $\overline{T} = T_1$

$$t = \frac{\rho c_p L(T_1 - T_0)}{Q} \tag{1.63}$$

Thus our timescale is $\tau = \frac{\rho c_p L \Delta T}{Q}$

We predicted,

$$\tau \approx \frac{L\Delta T}{\alpha\eta} = \frac{L\Delta T}{\frac{k}{\rho c_p}\frac{Q}{k}} = \frac{\rho c_p L \Delta T}{Q} \tag{1.64}$$

So it matches!

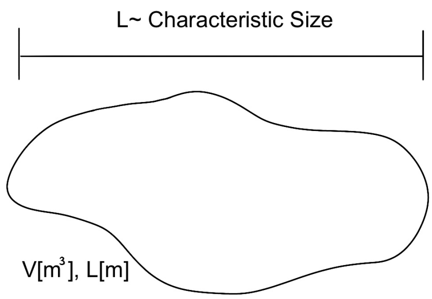
In this case our scaling argument matches exactly but normally this wouldn't happen. In general scaling arguments give an approximate answer based on the size of the parameters. In this example we got lucky and we managed to solve without any maths. Generally, one should always include a dimensionless pre-factor in the scaling argument.

$$\text{e.g.} \qquad t = \beta\left(\frac{\Delta T \rho c_p L}{Q}\right) \tag{1.65}$$

Now β here is just a number, such as 1 or 7 or $\pi/2$.

Another example

What if we have some arbitrary 3D shape and ask, what is its volume?



One can guess with the scaling argument:

$$V \sim L^3 \qquad \text{or} \qquad V = \beta L^3 \tag{1.66}$$

For example with a sphere $V = \frac{4\pi}{3}L^3$, where L is the radius and β is $\frac{4\pi}{3}$.

Water Waves

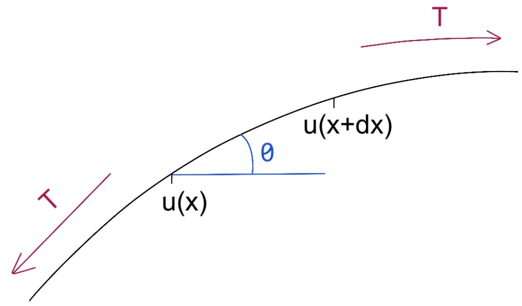
2.1 Introduction

We will first begin with the fundamentals and then we will move on to describe water. Consider the wave equation in one dimension.

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.1}$$

So for a displacement, $u(x,t)$, of a wire under tension T, and with density ρ we have a wave speed:

$$c = \left(\frac{T}{\rho}\right)^{1/2} \tag{2.2}$$

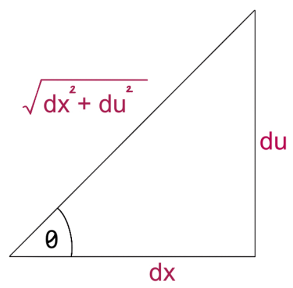


Then with Newton's second law in the vertical direction ($F = ma$)

$$m = \rho\,\mathrm{d}x \qquad a = \frac{\partial^2 u}{\partial t^2} \tag{2.3}$$

The resultant force is:

$$F = T\sin\theta|_{x+\mathrm{d}x} - T\sin\theta|_{x} \tag{2.4}$$



$$\sin\theta = \frac{\mathrm{d}u}{\sqrt{\mathrm{d}x^2+\mathrm{d}u^2}} = \frac{\frac{\partial u}{\partial x}}{\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}} \tag{2.5}$$

For small displacements we can approximate since $\frac{\partial u}{\partial x} \ll 1$ thus,

$$\sin\theta \approx \frac{\partial u}{\partial x} \tag{2.6}$$

From Newton's second law, we can derive our wave equation.

$$\rho \mathrm{d}x \frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial u}{\partial x}|_{x+\mathrm{d}x} - \frac{\partial u}{\partial x}|_x\right) \tag{2.7}$$

Dividing by $\mathrm{d}x$ and taking $\lim_{\mathrm{d}x\to 0}$:

$$\rho \frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} \tag{2.8}$$

In previous modules, we showed the general solution to the wave equation is:

$$u(x,t) = f(x-ct) + g(x+ct) \tag{2.9}$$

To get f and g we need initial conditions:

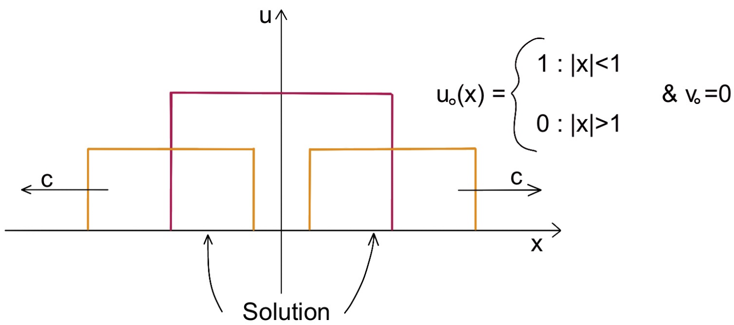
$$u(x,0) = u_0(x) \tag{2.10}$$

$$\frac{\partial u}{\partial t}(x,0) = v_0(x) \tag{2.11}$$

These leads us to d'Alembert's solution.

$$u(x,t) = \frac{1}{2}\left(u_0(x-ct) + u_0(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(x')dx' \tag{2.12}$$

Example



Red is the initial conditions. D'Alembert's solution tells us that $u_0(x+ct)$ is left travelling and $u_0(x-ct)$ is right travelling. The solution is made up of two waves with half the initial amplitude. The splitting hints towards conservation of energy (i.e. K.E. + P.E. = 0)

If we now consider $E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\rho(u_t)^2 + \frac{1}{2}T(u_x)^2 dx$. The first term here is the kinetic energy and the second term is the potential energy which adds up to make the total energy.

We can show energy is conserved as:

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} (\rho u_t u_{tt} + T u_x u_{xt}) dt \tag{2.13}$$

We can sub in the wave equation for u_{tt}.

$$= \int_{-\infty}^{\infty} \left(\rho u_t \left(\frac{T}{\rho} u_{xx} \right) + T u_x u_{xt} \right) dx \tag{2.14}$$

$$= T \int_{-\infty}^{\infty} \left(u_t u_{xx} + u_x u_{xt} \right) dx \tag{2.15}$$

$$= T \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left(u_t u_x \right) dx \tag{2.16}$$

$$= T [u_t u_x]_{-\infty}^{\infty} = 0 \tag{2.17}$$

This has to be zero since the wave moves at a finite speed and the limits at $\pm\infty$ are always ahead of the wave.

$$u_x, u_t \to 0 \qquad \text{as} \qquad x \to \pm\infty \tag{2.18}$$

Therefore energy is conserved: If $E(0) = E_0$, then $E(t) = E_0 \, \forall t$.

Example: Find $E(t)$ for a plane wave

If we have a plane wave with solution,

$$u(x,t) = A \sin\left(kx - \omega t\right) \tag{2.19}$$

We can check it satisfies the wave equation (2.1) if

$$-\frac{\omega^2}{c^2} = -k^2 \to \omega = \pm kc \tag{2.20}$$

We can calculate the kinetic energy.

$$\text{K.E.} = \int \frac{\rho}{2} (u_t)^2 dx \tag{2.21}$$

$$= \frac{1}{2} \rho \int \left(A\omega \cos\left(kx - \omega t\right) \right)^2 dx \tag{2.22}$$

$$= \frac{1}{2} \frac{\rho A^2 \omega^2}{k} \int_{-\pi}^{\pi} \cos^2(x') dx' \tag{2.23}$$

Here we have chosen $\pm\pi$ since we are looking at one travelling wavefront (between $x = \pi - ct$ and $x = \pi + ct$). Hence, using (2.20)

$$\text{K.E.} = \frac{1}{4} \rho A^2 c^2 k \tag{2.24}$$

Similarly for potential energy,

$$\text{P.E.} = \int \frac{1}{2} T \left(Ak \cos(kx - \omega t) \right)^2 dx = \frac{1}{2} \frac{T A^2 k^2}{k} \int_{-\pi}^{\pi} \cos^2(x') dx' \tag{2.25}$$

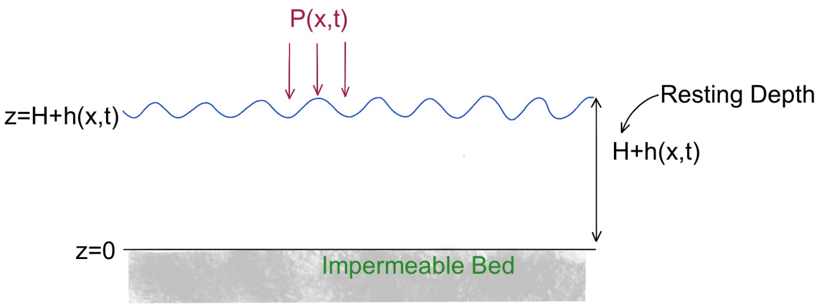
$$= \frac{1}{4} T A^2 k \tag{2.26}$$

$$\therefore \text{Total Energy} = E = \frac{1}{4} A^2 k \left(T + \rho c^2 \right) = \frac{1}{2} A^2 k T \tag{2.27}$$

where in the final step we have used (2.2).

2.2 Waves in Shallow Water

Now let's consider wave energy in shallow water. Consider a thin layer of water with a pressure force applied to the surface.



Later we will use the pressure $p = P(x,t)$ at the surface to represent a wave energy generator. We want to show that the water surface satisfies the wave equation with a source term,

$$\frac{1}{c^2}\frac{\partial^2 h}{\partial t^2} - \frac{\partial^2 h}{\partial x^2} = \frac{1}{\rho g}\frac{\partial^2 P}{\partial x^2} \tag{2.28}$$

where the shallow water wave speed satisfies $c^2 = gH$.

To show this we will begin with the **Euler Equations** (Navier-Stokes equations with $\mu = 0$)

$$\nabla \cdot \mathbf{u} = 0 \tag{2.29}$$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p - \rho g\hat{\mathbf{k}} \tag{2.30}$$

We can note our boundary conditions as,

$$\frac{D}{Dt}\left(z - (h(x,t) + H)\right) = 0 \qquad \text{on} \qquad z = H + h(x,t) \tag{2.31}$$

which is our kinematic boundary condition;

$$p = P(x,t) \qquad \text{on} \qquad z = H + h(x,t) \tag{2.32}$$

which is our dynamic boundary condition, and

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \text{on} \qquad z = 0 \tag{2.33}$$

which means the bottom surface (with normal $\mathbf{n}$) is impermeable. Now we have our equations and boundary conditions, we can non-dimensionalise according to the scalings:

$$x = L\hat{x}, \quad (z,h) = H(\hat{z},\hat{h}), \quad (p,P) = p_0(\hat{p},\hat{P}), \quad u = U\hat{u}, \quad w = W\hat{w}, \quad t = \tau\hat{t} \tag{2.34}$$

Shallow water waves exist in the limit $\epsilon \ll 1$, where $\epsilon = \frac{H}{L}$ is the aspect ratio.

We insert these values into our governing equations, beginning with the continuity equation (2.29),

$$\nabla\cdot\mathbf{u} = 0 \rightarrow \frac{U}{L}\hat{u}_{\hat{x}} + \frac{W}{H}\hat{w}_{\hat{z}} = 0 \tag{2.35}$$

We can set $W = \epsilon U$ to reduce things to:

$$\hat{u}_{\hat{x}} + \hat{w}_{\hat{z}} = \hat{\nabla}\cdot\hat{\mathbf{u}} = 0 \tag{2.36}$$

Now we consider the momentum equation (2.30) in the $\hat{x}$ direction.

$$\rho\left(u_t + uu_x + wu_z\right) = -\, p_x \tag{2.37}$$

$$\rho\left(\frac{U}{\tau}\hat{u}_{\hat{t}} + \frac{U^2}{L}\hat{u}\hat{u}_{\hat{x}} + \frac{WU}{H}\hat{w}\hat{u}_{\hat{z}}\right) = -\, \frac{p_0}{L}\hat{p}_{\hat{x}} \tag{2.38}$$

$$\rho\left(\frac{U}{\tau}\hat{u}_{\hat{t}} + \frac{U^2}{L}\hat{u}\hat{u}_{\hat{x}} + \frac{U^2}{H}\frac{H}{L}\hat{w}\hat{u}_{\hat{z}}\right) = -\, \frac{p_0}{L}\hat{p}_{\hat{x}} \tag{2.39}$$

$$\rho\left(\frac{UL}{\tau}\hat{u}_{\hat{t}} + U^2\left(\hat{u}\hat{u}_{\hat{x}} + \hat{w}\hat{u}_{\hat{z}}\right)\right) = -\, p_0\hat{p}_{\hat{x}} \tag{2.40}$$

$$\frac{L}{U\tau}\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} + \hat{w}\hat{u}_{\hat{z}} = -\, \frac{p_0}{\rho U^2}\hat{p}_{\hat{x}} \tag{2.41}$$

This equation suggests we take two natural scales,

$$\tau = \frac{L}{U}, \quad p_0 = \rho U^2 \tag{2.42}$$

such that

$$\hat{u}_{\hat{t}} + \left(\hat{\mathbf{u}} \cdot \hat{\nabla}\right)\hat{u} = -\hat{p}_{\hat{x}} \tag{2.43}$$

Now we focus on (2.30) in the z-direction,

$$\rho\left(w_t + uw_x + ww_z\right) = -p_z - \rho g \tag{2.44}$$

$$\rho\epsilon\frac{U^2}{L}\left(\hat{w}_{\hat{t}} + \hat{u}\hat{w}_{\hat{x}} + \hat{w}\hat{w}_{\hat{z}}\right) = \rho\frac{U^2}{H}\hat{p}_{\hat{z}} - \rho g \tag{2.45}$$

Dividing by $\frac{\rho U^2}{H}$:

$$\epsilon^2\left(\hat{w}_{\hat{t}} + \hat{u}\hat{w}_{\hat{x}} + \hat{w}\hat{w}_{\hat{z}}\right) = -\, \hat{p}_{\hat{z}} - \frac{gH}{U^2} \tag{2.46}$$

Which suggests the velocity scale $U = \sqrt{gH}$.

Hence, the z-momentum equation becomes:

$$\epsilon^2\left(\hat{w}_{\hat{t}} + \left(\hat{\mathbf{u}} \cdot \hat{\nabla}\right)\hat{w}\right) = -\hat{p}_{\hat{z}} - 1 \tag{2.47}$$

Taking the limit $\epsilon \to 0$ we have the system of equations which are the dimensionless shallow water equations:

$$\hat{\nabla} \cdot \hat{\mathbf{u}} = 0 \tag{2.48}$$

$$\hat{u}_{\hat{t}} + \left(\hat{\mathbf{u}} \cdot \hat{\nabla}\right)\hat{u} = -\, \hat{p}_{\hat{x}} \tag{2.49}$$

$$0 = -\, \hat{p}_{\hat{z}} - 1 \tag{2.50}$$

Now we non-dimensionalise the boundary conditions: The kinematic condition (2.31),

$$w - h_t - uh_x = 0 \qquad : z = H + h \tag{2.51}$$

$$\epsilon U\left(\hat{w} - \hat{h}_{\hat{t}} - \hat{u}\hat{h}_{\hat{x}}\right) = 0 \qquad : \hat{z} = 1 + \hat{h} \tag{2.52}$$

$$\hat{w} = \hat{h}_{\hat{t}} + \hat{u}\hat{h}_{\hat{x}} = 0 \qquad : \hat{z} = 1 + \hat{h} \tag{2.53}$$

the dynamic condition (2.32),

$$p = P \qquad : z = H + h \tag{2.54}$$
$$\hat{p} = \hat{P} \qquad : \hat{z} = 1 + \hat{h} \tag{2.55}$$

and the impermeability condition (2.33),

$$w = 0 \qquad : z = 0 \tag{2.56}$$
$$\hat{w} = 0 \qquad : \hat{z} = 0 \tag{2.57}$$

Now that we have everything non-dimensionalised we can look at the simplest case.

Case 1: Resting water solution

Let $\hat{P}(\hat{x}, \hat{t}) = 0$. Our equations become:

$$\hat{\nabla} \cdot \hat{\mathbf{u}} = 0 \tag{2.58}$$
$$\hat{u}_{\hat{t}} + \left(\hat{\mathbf{u}} \cdot \hat{\nabla}\right) \hat{u} = -\hat{p}_x \tag{2.59}$$
$$0 = -\hat{p}_{\hat{z}} - 1 \tag{2.60}$$

Our boundary conditions are:

$$\hat{w} = \hat{h}_{\hat{t}} + \hat{u}\hat{h}_{\hat{x}} \qquad : \hat{z} = 1 + \hat{h} \tag{2.61}$$
$$\hat{p} = 0 \qquad : \hat{z} = 1 + \hat{h} \tag{2.62}$$
$$\hat{w} = 0 \qquad : \hat{z} = 0 \tag{2.63}$$

Before jumping in and solving, we can first make an educated guess based on what we know.

Since there is no applied pressure $P(x, t) = 0$, we expect a resting water solution.

So we will try $\hat{u} = \hat{w} = 0, \hat{h} = 0$

The continuity equation (2.58) is trivially satisfied. Meanwhile the conservation of momentum gives,

$$\hat{p}_{\hat{x}} = 0 \rightarrow \hat{p} = \hat{p}(\hat{z}) \tag{2.64}$$
$$0 = -\hat{p}_{\hat{z}} - 1 \tag{2.65}$$

This tells us that:

$$\hat{p}(z) = -\hat{z} + C \tag{2.66}$$

We double check our boundary conditions:

$$\hat{p} = 0 \qquad \text{on} \qquad \hat{z} = 1 + 0 \tag{2.67}$$

Hence,

$$C = 1 \tag{2.68}$$
$$\rightarrow \hat{p}(\hat{z}) = 1 - \hat{z} \tag{2.69}$$

Case 2: Perturbation to resting water

Now let's take our resting water solution and add a small perturbation.

First we will need to recall asymptotic expansions. Let's consider a small perturbation $\hat{P} = \epsilon\tilde{P}(\hat{x}, \hat{t})$, where $\epsilon = \frac{H}{L} \ll 1$.

Let's expand in powers of ϵ, such that

$$\hat{u} = 0 + \epsilon\tilde{u} + ... \tag{2.70}$$
$$\hat{h} = 0 + \epsilon\tilde{h} + ... \tag{2.71}$$
$$\hat{p} = (1 - \hat{z}) + \epsilon\tilde{p} + ... \tag{2.72}$$

where the first terms in the expansion are the resting water solution (zeroth order) and the next terms are the "first order perturbation".

Since the resting water solution already satisfies our equations, we focus on the first order perturbation terms.

Ignoring second order terms that are proportional to ϵ^2 and higher powers, our governing equations are

$$\epsilon\hat{\nabla} \cdot \tilde{\mathbf{u}} = 0 \tag{2.73}$$
$$\epsilon\tilde{u}_{\hat{t}} + \cancel{\epsilon^2 (\tilde{\mathbf{u}} \cdot \nabla)}\, \tilde{u} = -\epsilon\tilde{p}_{\hat{z}} \tag{2.74}$$
$$0 = \cancel{1} - \epsilon\tilde{p}_{\hat{z}} - \cancel{1} = -\epsilon\tilde{p}_{\hat{z}} \tag{2.75}$$

This gives us a perturbed system which we call the linearised equations.

$$\hat{\nabla} \cdot \tilde{\mathbf{u}} = 0 \tag{2.76}$$
$$\tilde{u}_{\hat{t}} = -\tilde{p}_{\hat{x}} \tag{2.77}$$
$$0 = -\tilde{p}_{\hat{z}} \tag{2.78}$$

Now we need to do similar steps for the boundary conditions, starting with the kinematic condition,

$$\epsilon\tilde{w} = \epsilon\tilde{h}_{\hat{t}} + \cancel{\epsilon^2\tilde{u}\tilde{h}_{\hat{x}}} \qquad : \hat{z} = 1 + \epsilon\tilde{h} \tag{2.79}$$

We take $\tilde{w}(\hat{x}, 1 + \epsilon\tilde{h}, \hat{t})$ and Taylor expand:

$$\tilde{w}\left(\hat{x}, 1 + \epsilon\tilde{h}, \hat{t}\right) \approx \tilde{w}(\hat{x}, 1, \hat{t}) + \epsilon\tilde{h}\frac{\partial\tilde{w}}{\partial\hat{z}}(\hat{x}, 1, \hat{t}) \tag{2.80}$$

which gives us (at leading order):

$$\tilde{w} = \tilde{h}_{\hat{t}} \qquad : \hat{z} = 1 \tag{2.81}$$

Meanwhile, the dynamic boundary condition becomes

$$\left(1 - (1 + \epsilon\tilde{h})\right) + \epsilon\tilde{p} = \epsilon\tilde{P} \qquad : \hat{z} = 1 \tag{2.82}$$
$$\rightarrow \tilde{p} = \tilde{P} + \tilde{h} \qquad : \hat{z} = 1 \tag{2.83}$$

We have again Taylor expanded and imposed at $\hat{z} = 1$.

Finally we have the impermeability condition:

$$\tilde{w} = 0 \qquad : \hat{z} = 0 \tag{2.84}$$

From here, we can solve the system of equations and get the wave equation.

Thus integrating (2.78) and applying (2.83) :

$$\tilde{p}(\hat{x},\hat{z},\hat{t}) = \tilde{P}(\hat{x},\hat{t}) + \tilde{h}(\hat{x},\hat{t}) \tag{2.85}$$

This can now be inserted into (2.77)

$$\tilde{u}_{\hat{t}} = -\tilde{P}_{\hat{x}} - \tilde{h}_{\hat{x}} \tag{2.86}$$

Next we can integrate equation (2.76) vertically (from $\hat{z} = 0$ to $\hat{z} = 1$)

$$\int_0^1 \tilde{u}_{\hat{x}} + \tilde{w}_{\hat{z}} d\hat{z} = 0 \tag{2.87}$$

$$\int_0^1 \tilde{u}_{\hat{x}} d\hat{z} + [\tilde{w}]_0^1 = 0 \tag{2.88}$$

$$\int_0^1 \tilde{u}_{\hat{x}} d\hat{z} + \tilde{h}_{\hat{t}} - 0 = 0 \tag{2.89}$$

We can take the time derivative,

$$\int_0^1 \tilde{u}_{\hat{x}\hat{t}} d\hat{z} + \tilde{h}_{\hat{t}\hat{t}} = 0 \tag{2.90}$$

Also we can use the x-derivative of equation (2.86) to substitute since

$$\tilde{u}_{\hat{x}\hat{t}} = -\tilde{P}_{\hat{x}\hat{x}} - \tilde{h}_{\hat{x}\hat{x}} \tag{2.91}$$

We can insert this into the integral, noting that these are all functions of $\hat{x}, \hat{t}$ only:

$$\left(-\tilde{P}_{\hat{x}\hat{x}} - \tilde{h}_{\hat{x}\hat{x}}\right) \int_0^1 d\hat{z} + \tilde{h}_{\hat{t}\hat{t}} = 0 \tag{2.92}$$

$$-\tilde{P}_{\hat{x}\hat{x}} - \tilde{h}_{\hat{x}\hat{x}} + \tilde{h}_{\hat{t}\hat{t}} = 0 \tag{2.93}$$

Now we have a dimensionless wave equation with a source term.

Now we can re-dimensionalise,

$$h = H\hat{h} \qquad t = \frac{L}{U}\hat{t} \qquad x = L\hat{x} \qquad P = \rho U^2 \hat{P} \tag{2.94}$$

where $U = \sqrt{gH}$. So we get

$$-\frac{L^2}{\rho U^2} P_{xx} - \frac{L^2}{H} h_{xx} + \frac{L^2}{U^2 H} h_{tt} = 0 \tag{2.95}$$

Then we divide by $\frac{gL^2}{U^2} = \frac{L^2}{H}$ to get,

$$-\frac{P_{xx}}{\rho g} - h_{xx} + \frac{1}{gH} h_{tt} = 0 \tag{2.96}$$

$$\frac{1}{c^2} h_{tt} - h_{xx} = \frac{P_{xx}}{\rho g} \tag{2.97}$$

which is the linear wave equation with a source term and with $c = \sqrt{gH}$.

2.3 Wave Energy

We can now ask, what is the energy of this wave? The wave has both kinetic and potential energy.

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty}\left(\rho\left(\frac{\partial h}{\partial t}\right)^2 + \rho g H\left(\frac{\partial h}{\partial x}\right)^2\right)dx \tag{2.98}$$

We can check the dimension of that:

$$\left[\rho(h_t)^2 dx\right] = \left[\frac{kg}{m^3}\frac{m^2}{s^2}m\right] = \left[kg\, s^{-2}\right] \tag{2.99}$$

This is energy per unit area.

In the practice problems, we show that

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} H P_{xx} h_t dx \tag{2.100}$$

So if $P = 0$, we are not applying any force and energy is conserved $\rightarrow E = E(0)$.

2.4 Periodic Waves

Starting now with the shallow water wave equation we had earlier:

$$\frac{1}{c^2}h_{tt} - h_{xx} = \frac{P_{xx}}{\rho g} \tag{2.101}$$

Periodic waves (with frequency ω) would be of the form

$$P(x,t) = Re\left(\hat{P}(x)e^{i\omega t}\right) \qquad h(x,t) = Re\left(\hat{h}(x)e^{i\omega t}\right) \tag{2.102}$$

where $\hat{P}$ and $\hat{h}$ are complex functions. We can insert it into the wave equation:

$$-\frac{\omega^2}{c^2}\hat{h}(x) - \hat{h}''(x) = \frac{\hat{P}''(x)}{\rho g} \tag{2.103}$$

As per usual we non-dimensionalise:

$$x = \frac{c}{\omega}\tilde{x} \qquad \hat{h} = \frac{c}{\omega}\tilde{h} \qquad \hat{P} = \rho g\frac{c}{\omega}\tilde{P} \tag{2.104}$$

$$-\frac{\omega^2}{c^2}\frac{c}{\omega}\tilde{h}(\tilde{x}) - \frac{c}{\omega}\frac{\omega^2}{c^2}\tilde{h}''(\tilde{x}) = \frac{\rho g \frac{c}{\omega}\frac{\omega^2}{c^2}}{\rho g}\tilde{P}''(\tilde{x}) \tag{2.105}$$

$$\tilde{h}''(\tilde{x}) + \tilde{h}(\tilde{x}) = -\tilde{P}''(\tilde{x}) \tag{2.106}$$

This equation can be solved using Green's functions.

As a quick recap consider ODE's of the form:

$$p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x) = S(x) \tag{2.107}$$

If we let the two independent solutions to the homogeneous equations be $u(x)$, $v(x)$, then we can write general solutions of the form:

$$y(x) = a_0 u(x) + b_0 v(x) + \int_0^x G(x,x')S(x')dx' \tag{2.108}$$

G is our Green's function,

$$G(x,x') = \frac{u(x')v(x) - u(x)v(x')}{p_2(x)W(x)} \tag{2.109}$$

$W(x)$ is the Wronskian and is given by:

$$W(x) = u(x)v'(x) - u'(x)v(x) \tag{2.110}$$

If we consider the shallow water wave equation (2.106),

$$p_2 = 1 \qquad p_1 = 0 \qquad p_0 = 1 \qquad S = -P''(\tilde{x}) \tag{2.111}$$

So all we must do is solve the homogenous case:

$$\tilde{h}''(\tilde{x}) + \tilde{h}(\tilde{x}) = 0 \qquad u = \cos\tilde{x} \qquad v = \sin\tilde{x} \tag{2.112}$$

And hence our general solution is:

$$\tilde{h}(\tilde{x}) = a_0\cos\tilde{x} + b_0\sin\tilde{x} + \int_0^{\tilde{x}} (\cos x\sin\tilde{x} - \sin x\cos\tilde{x})\,(-P''(x))\,dx \tag{2.113}$$

Note our Wronskian is $W = \cos^2(x) + \sin^2(x) = 1$.

Remember $\tilde{h}(\tilde{x})$ is a complex function so $a_0, b_0 \in \mathbb{C}$, so it is easier to rewrite in terms of complex exponentials.

$$u = e^{i\tilde{x}} \qquad v = e^{-i\tilde{x}} \tag{2.114}$$

$$\tilde{h}(\tilde{x}) = Ae^{i\tilde{x}} + Be^{-i\tilde{x}} + \frac{i}{2}\int_0^{\tilde{x}} \tilde{P}''(x)\left(e^{i(x-\tilde{x})} - e^{i(\tilde{x}-x)}\right)dx \tag{2.115}$$

Note: Boundary conditions will determine A, $B \in \mathbb{C}$

Example

We will construct a wave energy generator mathematically of the form

$$P = \begin{cases} P(x,t): & 0 \le x \le L, \\ 0: & \text{Otherwise} \end{cases} \tag{2.116}$$

where $P(x,t)$ will be chosen to harness all the energy of an incoming right travelling wave (such that there is no wave for $x > L$).

We begin by noting that anything proportional to $e^{i\tilde{x}}$ has to be left travelling and anything proportional to $e^{-i\tilde{x}}$ is right travelling.

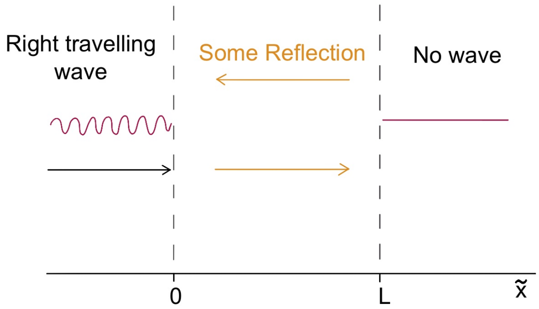
We have different regions:

For $\tilde{x} < 0$ we know $\tilde{h} = Ae^{i\tilde{x}} + Be^{-i\tilde{x}}$ from equation (2.115).

But since the wave is right travelling, this indicates $A = 0$.

For $\tilde{x} > \tilde{L} = \frac{Lw}{c}$ (the dimensionless length of the generator) we have:

$$\tilde{h} = Be^{-i\tilde{x}} + \frac{i}{2}\int_0^L \tilde{P}''(x)\left(e^{i(\tilde{x}-x)} - e^{i(x-\tilde{x})}\right)dx = 0 \tag{2.117}$$



which must be zero because the wave vanishes to the right of the generator.

Let's consider a pressure $\tilde{P}''(x)$ composed of Dirac δ distributions.

Let $\tilde{P}''(x) = \alpha\delta(x) + \beta\delta(x-L)$ for $\alpha,\ \beta \in \mathbb{C}$.

For $\tilde{x} > \tilde{L}$:

$$\tilde{h}(\tilde{x}) = Be^{-i\tilde{x}} + \frac{i}{2}\left(\alpha\left(e^{i(\tilde{x}-0)} - e^{i(0-\tilde{x})}\right) + \beta\left(e^{i(\tilde{x}-\tilde{L})} - e^{i(\tilde{L}-\tilde{x})}\right)\right) = 0 \tag{2.118}$$

Now our aim is to collect right and left travelling whilst setting $\tilde{h} = 0$:

$$e^{-i\tilde{x}}\left(B - \frac{i}{2}\alpha - \frac{i\beta e^{i\tilde{L}}}{2}\right) + e^{i\tilde{x}}\left(\frac{i}{2}\alpha + \frac{i}{2}\beta e^{-i\tilde{L}}\right) = 0 \tag{2.119}$$

The two waves must be zero independently:

Right $\rightarrow B - \frac{i\alpha}{2} - \frac{i\beta e^{i\tilde{L}}}{2} = 0$

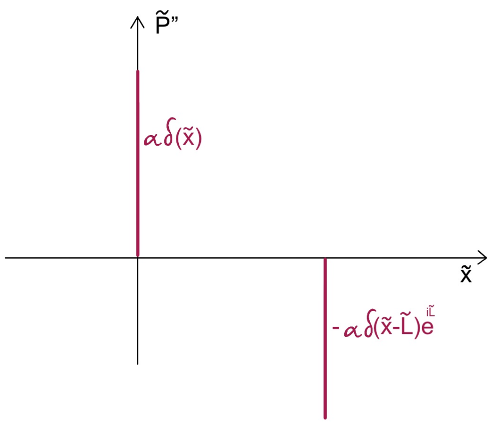
Left $\rightarrow \frac{i\alpha}{2} + \frac{i\beta e^{-i\tilde{L}}}{2} = 0$

Thus : $B = \frac{i}{2}\alpha\left(1 - e^{2i\tilde{L}}\right)$ and $\beta = -\alpha e^{i\tilde{L}}$

Now we can get α,β in terms of B to tell us how the δ-distributions behave.

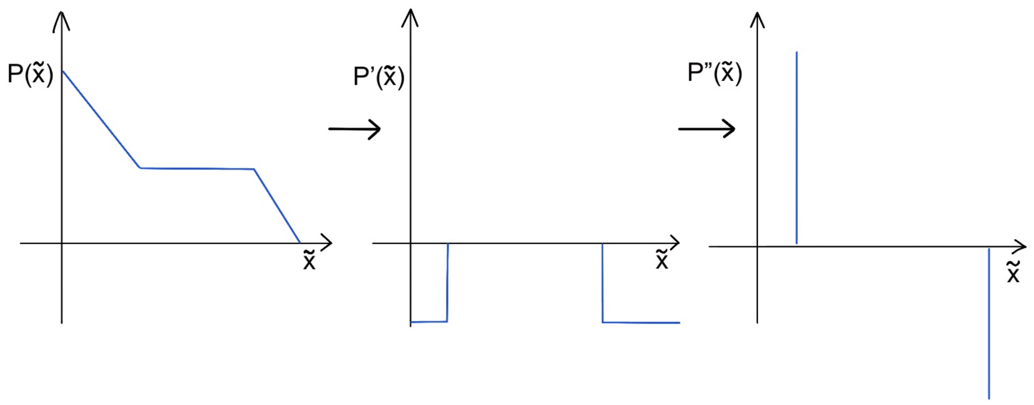
$$\alpha = \frac{2B}{i(1-e^{2i\tilde{L}})} \qquad\qquad \beta = \frac{-2Be^{i\tilde{L}}}{i(1-e^{2i\tilde{L}})} \tag{2.120}$$

Plotting this we have:



The δ are opposite in direction but have the same amplitude of $|\alpha|$. They are out of phase by an angle of $\tilde{L}$.

Here is an illusion of the pressure and its derivatives:



2.5 2D Waves

Consider the 2D Wave equation.

$$\nabla^2 h - \frac{1}{c^2} h_{tt} = -\frac{1}{\rho g} \nabla^2 P \tag{2.121}$$

We restrict our attention to monochromatic (single frequency ω) waves:

$$P = Re\left(\hat{P}(x,y)e^{iwt}\right) \tag{2.122}$$

$$h = Re\left(h(x,\hat{y})e^{iwt}\right) \tag{2.123}$$

$$\rightarrow \nabla^2 \hat{h} + \frac{w^2}{c^2}\hat{h} = -\frac{\nabla^2 \hat{P}}{\rho g} \tag{2.124}$$

We non dimensionalise as before,

$$\tilde{\nabla}^2 \tilde{h} + \tilde{h} = -\tilde{\nabla}^2 \tilde{P} = \tilde{S} \tag{2.125}$$

This is the inhomogeneous Helmholtz equation which we will solve using Greens functions.

Consider the operator $\mathcal{L} = \left(\nabla^2 + 1\right)$ i.e. $\mathcal{L}u(x,y) = S(x,y)$

We require $G(x,y,X,Y)$ such that $\mathcal{L}G(x,y,X,Y) = \delta(x-X)\delta(y-Y)$

Then we write the solution as :

$$u(x,y) = \int\int G(x,y,X,Y)S(X,Y)dXdY \tag{2.126}$$

$$\mathcal{L}u(x,y) = \int\int \mathcal{L}G(x,y,X,Y)S(X,Y)dXdY \tag{2.127}$$

$$= \int\int \delta(x-X)\delta(y-Y)S(X,Y)dXdY \tag{2.128}$$

$$= S(x,y) \tag{2.129}$$

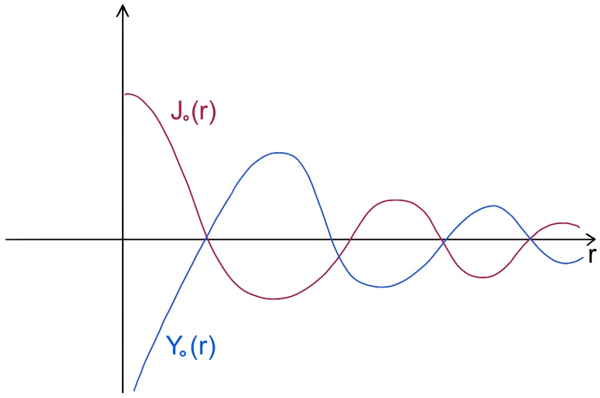
The Green's function for the 2D Helmholtz equation is the Hankel function (0th order 1st kind).

$$G(x,y,X,Y) = H_0^{(1)}(|\mathbf{x}-\mathbf{X}|) \tag{2.130}$$

$$= J_0(|\mathbf{x}-\mathbf{X}|) + iY_0(|\mathbf{x}-\mathbf{X}|) \tag{2.131}$$

$$= \text{Bessel J} + \text{Bessel Y} \tag{2.132}$$

Example:



Let $-\tilde{\nabla}^2 \tilde{P} = \tilde{S} = \delta(\tilde{x})\delta(\tilde{y})$ which is a point at the origin.

$$\tilde{h} = \int\int G(\tilde{x}, \tilde{y}, \tilde{X}, \tilde{Y})\delta(\tilde{X})\delta(\tilde{Y})d\tilde{X}d\tilde{Y} \tag{2.133}$$

$$= G(\tilde{x}, \tilde{y}, 0, 0) \tag{2.134}$$

$$\tilde{h} = J_0(|\tilde{\mathbf{x}}|) + iY_0(|\tilde{\mathbf{x}}|) \tag{2.135}$$

$$\tilde{h} = J_0(\tilde{r}) + iY_0(\tilde{r}) \tag{2.136}$$

(2.137)

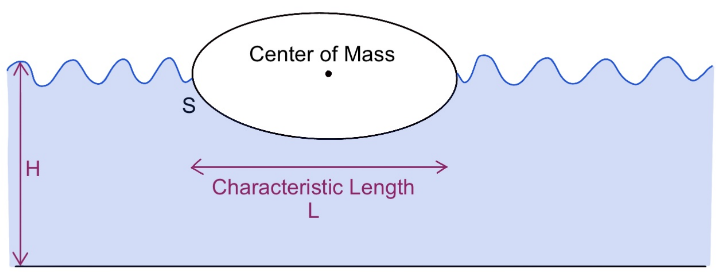
The time-dependant solution is:

$$h = Re\left(\frac{c}{w}\tilde{h}\left(\frac{rw}{c}\right)e^{iwt}\right) h(r,t) = \frac{c}{w}Re\left(\left(J_0\left(\frac{rw}{c}\right) + iY_0\left(\frac{rw}{c}\right)\right)e^{iwt}\right) \tag{2.138}$$

Exercise: Plot this as an animation to see the radially symmetric solution.

2.6 Wave-Body Interactions

Motivated by wave energy generators, let's consider a floating body (e.g an ellipse shape) with a water wave causing it to move up and down.



It has a centre of mass $\mathbf{x} = \mathbf{X}(t)$ and surface S.

The equations of motion for the centre of mass are:

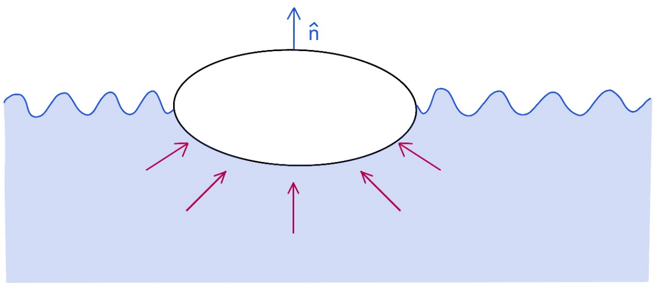
$$m\ddot{\mathbf{X}} = \sum \text{Forces on body} = -\int\int_S (p - p_a)\,\hat{\mathbf{n}}dA - mg\hat{\mathbf{k}} \tag{2.139}$$

We have $(p - p_a)$ because on the upper surface $p = p_a$ since we only have atmospheric pressure there.

Why is this a force from the fluid?

Pressure acts on the body in the normal direction.

If we draw an outward unit normal, the pressure force is $-(p-p_a)\hat{\mathbf{n}}$.



Total force is integrated over the surface.

In the steady state (no waves), the position of the centre of mass is just at $z=H$.

In 2D coordinates (x,z), S is given by $z=H-f(x)$ for some function of f.

In the fluid we have the Euler equations with $\mathbf{u}=0$.

$$0=-\nabla p-\rho g\hat{\mathbf{k}} \tag{2.140}$$

$$\frac{\partial p}{\partial x}=0 \tag{2.141}$$

$$\frac{\partial p}{\partial z}=-\rho g \tag{2.142}$$

Using the boundary condition $p=p_a$ at $z=H$ we find,

$$p=p_a-\rho g(z-H) \tag{2.143}$$

So on S: $p=p_a+\rho g f(x)$. The equations of motion come from (2.139).

We can split the vector equation into:

$$m\ddot{X}=-\int\rho g f(x)(\hat{\mathbf{n}}\cdot\hat{\mathbf{\imath}})dl \tag{2.144}$$

$$m\ddot{Z}=-\int\rho g f(x)(\hat{\mathbf{n}}\cdot\hat{\mathbf{k}})dl-mg \tag{2.145}$$

where we are integrating over the arclength l, for which $dl^2=dx^2+df^2$. Thus $dl=(1+f'(x)^2)^{1/2}dx$.

The normal vector $\hat{\mathbf{n}}=\frac{1}{(1+f'(x)^2)^{1/2}}\begin{pmatrix}f'(x)\\-1\end{pmatrix}$

Thus equation (2.144) becomes

$$0=\int_{-L/2}^{L/2}\frac{f(x)f'(x)}{(1+f'(x)^2)^{1/2}}\left(1+f'(x)^2\right)^{1/2}dx \tag{2.146}$$

$$0=[\frac{1}{2}f(x)^2]_{-L/2}^{L/2} \tag{2.147}$$

which is true by the definition of $f(x)$, since the edges of the body must be at zero depth.

Also, using equation (2.145)

$$0 = \int_{-L/2}^{L/2} \rho g f(x) dx - mg \tag{2.148}$$

$$m = \rho \int_{-L/2}^{L/2} f(x) dx \tag{2.149}$$

$$m = \rho(\text{Volume}) \tag{2.150}$$

i.e. In the steady state the level of the body $f(x)$ is set by the mass.

Now let's consider a small vertical perturbation to this:

$$\text{Let} \quad Z = Z_0 + \eta(t) \quad : \quad \frac{\eta}{Z_0} \ll 1 \tag{2.151}$$

We insert into (2.145):

$$m\ddot{\eta} = -\int_S (p - p_a)\, \hat{\mathbf{n}} \cdot \hat{\mathbf{k}} dl - mg \tag{2.152}$$

On the surface $Z = H - f(x) + \eta(t)$:

$$p = p_a - \rho g \left(-f(x) + \eta(t)\right) + \tilde{p}(x, z, t) \tag{2.153}$$

where $\tilde{p}$ is the perturbation to the pressure. Inserting this into (2.152) we get

$$m\ddot{\eta} = -\int_{-L/2}^{L/2} \left(-\rho g \left(-f + \eta\right) + \hat{p}\right)(-dx) - mg \tag{2.154}$$

$$= \rho g \int_{-L/2}^{L/2} \left(f(x) - \eta(t)\right) dx + \int_{-L/2}^{L/2} \tilde{p} dx - mg \tag{2.155}$$

$$m\ddot{\eta} = -\rho g L \eta + \int_{-L/2}^{L/2} \tilde{p} dx \tag{2.156}$$

The term $\int_{-L/2}^{L/2} \tilde{p} dx$ is often parameterised in terms of "added mass" m_a and acceleration $\ddot{\eta}$

$$\int_{-L/2}^{L/2} \tilde{p} dx = -m_a \ddot{\eta} \tag{2.157}$$

The equation becomes,

$$(m + m_a)\, \ddot{\eta} = \rho g L \eta \tag{2.158}$$

$$\ddot{\eta} + \frac{\rho g L}{m + m_a} \eta = 0 \tag{2.159}$$

$$\ddot{\eta} + \omega^2 \eta = 0 \tag{2.160}$$

We rename $\omega = \left(\frac{\rho g L}{m + m_a}\right)^{1/2}$ as our natural frequency of the body. These equations show the use of simple harmonic motion.

Consider the equations of motion for η:

$$m\ddot{\eta} = -\rho g L \eta + \int_{-L/2}^{L/2} \tilde{p} dx \tag{2.161}$$

which can be compared to Newton's 2nd law ($ma = F$). The first term on the right hand side is the restorative force (i.e a spring $F = -kx$) and the last term is the pressure force due to the fluid acceleration.

In this, generally we need a model for the perturbed pressure $\tilde{p}$. Here we will use a simple scaling argument.

The dimensions of the integral are:

$$\left[\int_{-L/2}^{L/2} \tilde{p}dx\right] = \left[\frac{\text{kg m}}{\text{m s}^2}\right] = \left[\frac{\text{kg}}{\text{s}^2}\right] \tag{2.162}$$

Whereas the other dimensions are:

$$[L] = [\text{m}] \qquad [\rho] = [\text{kg/m}^3] \qquad [\ddot{\eta}] = [\text{m/s}^2] \qquad [\rho\ddot{\eta}L^2] = \left[\frac{\text{kg}}{\text{s}^2}\right] \tag{2.163}$$

Hence we arrive at the following scaling argument: $\int_{-L/2}^{L/2} \tilde{p}dx \sim -\rho L^2\ddot{\eta}$. Note m has dimensions mass per unit length since we are in 2D.

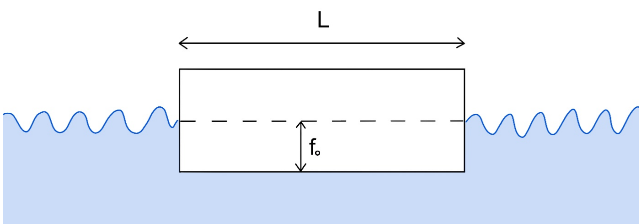
We wrote: $\int_{-L/2}^{L/2} \tilde{p}dx = -m_a\ddot{\eta} \sim -\rho L^2\ddot{\eta}$

Hence we write $m_a = \alpha\rho L^2$ for dimensionless constant α.

The ODE for η then is: $(m + m_a)\ddot{\eta} + \rho g L\eta = 0$

Next let's apply this problem to a rectangular floating body.

Rectangular Floating Body



The ODE in this case becomes:

$$\ddot{\eta} + \omega^2\eta = 0 \tag{2.164}$$

$$\omega^2 = \frac{\rho g L}{m + m_a} = \left(\frac{\rho g L}{\rho L f_0 + \rho\alpha L^2}\right) = \left(\frac{g}{f_0 + \alpha L}\right) \tag{2.165}$$

For slender bodies $f_0 \ll \alpha L$ and so $\omega = \left(\frac{g}{\alpha L}\right)^{1/2}$ Then ω is the natural frequency of the body and clearly larger bodies oscillate more slowly.

Forcing from a Wave: Consider forcing from a wave of Amplitude F_0 and frequency Ω.

We will consider the ODE:

$$(m + m_a)\ddot{\eta} = -\rho g L\eta + F_0 \sin\Omega t \tag{2.166}$$

$$\ddot{\eta} = \omega^2\eta + G\sin\Omega t \tag{2.167}$$

$$G = \frac{F_0}{m + m_a} \tag{2.168}$$

Case 1: $\Omega \neq \omega$ (Forcing frequency is not equal to the natural frequency of the body)

Homogeneous solution to the equation: $\eta = A\cos\omega t + B\sin\omega t$.

We use the particular solution: $\eta = C\sin\Omega t$ as the ansatz

Then,

$$\dot{\eta} = \Omega C \cos \Omega t \tag{2.169}$$
$$\ddot{\eta} = -\Omega^2 C \sin \Omega t \tag{2.170}$$

Putting these coefficients together:

$$-\Omega^2 C + \omega^2 C = G \tag{2.171}$$
$$C\left(\omega^2 - \Omega^2\right) = G \tag{2.172}$$
$$C = \frac{G}{\omega^2 - \Omega^2} \tag{2.173}$$

The general solution can be written as:

$$\eta = A \cos \omega t + B \sin \omega t + \frac{G}{\omega^2 - \Omega^2} \sin \Omega t \tag{2.174}$$

Then for our initial condition, $\eta(0) = \dot{\eta}(0) = 0$ we can show:

$$\eta(t) = \frac{G}{\omega\left(\omega \sin \Omega t - \Omega \sin \omega t\right)} \tag{2.175}$$

Case 2: $\omega = \Omega$ (Forcing frequency = natural body frequency)

It would be expected that in this case the amplitude would grow.

$$\ddot{\eta} + \omega^2 \eta = G \sin \omega t \tag{2.176}$$

The homogeneous solution is $\eta = A \cos \omega t + B \sin \omega t$

The particular solution: if we try $C \sin \omega t$ or $C \cos \omega t$ then the equation will vanish and so instead we will try:

$$\eta(t) = Ct \cos \omega t + Dt \sin \omega t \tag{2.177}$$
$$\dot{\eta}(t) = C \cos \omega t + D \sin \omega t - Ct\omega \sin \omega t + Dt\omega \cos \omega t \tag{2.178}$$
$$\ddot{\eta}(t) = -2C\omega \sin \omega t + 2D\omega \cos \omega t + C\omega^2 t \cos \omega t - D\omega^2 t \sin \omega t \tag{2.179}$$

Hence, our ODE becomes

$$\ddot{\eta} + \omega^2 \eta = -2C\omega \sin \omega t + 2D\omega \cos \omega t = G \sin \omega t \tag{2.180}$$

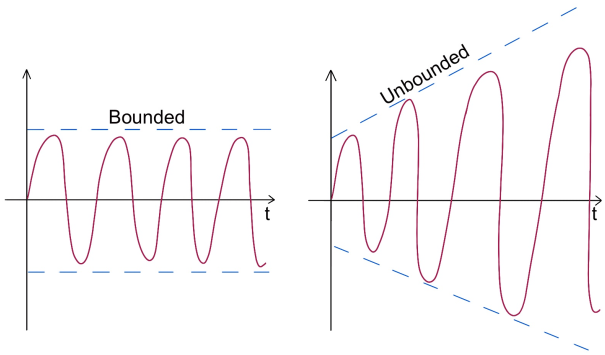
Hence, comparing coefficients we find

$$D = 0 \qquad\qquad C = -\frac{G}{2\omega} \tag{2.181}$$

The general solution is then:

$$\eta(t) = A \cos \omega t + B \sin \omega t - \frac{G}{2\omega} t \cos \omega t \tag{2.182}$$

Since the last term in this equation is multiplied by t it is unbounded and will grow in amplitude with time.



Solving for the coefficients with initial conditions $\eta(0) = \dot{\eta}(0) = 0$.

$$\eta(t) = A\cos\omega t + B\sin\omega t - \frac{G}{2\omega}t\cos\omega t \tag{2.183}$$

$$\eta(0) = A = 0 \tag{2.184}$$

$$\dot{\eta}(t) = \omega B\cos\omega t - \frac{G}{2\omega}\cos\omega t + \frac{Gt\omega}{2\omega}\sin\omega t \tag{2.185}$$

$$\dot{\eta}(0) = \omega B\frac{G}{2\omega} + 0 = 0 \rightarrow B = \frac{G}{2\omega^2} \tag{2.186}$$

$$\eta(t) = \frac{G}{2\omega^2}(\sin\omega t - \omega t\cos\omega t) \tag{2.187}$$

In practice we want to damp the oscillations to extract the energy of the wave:

$$\ddot{\eta} + \nu\dot{\eta} + \omega^2\eta = G\sin\omega t \tag{2.188}$$

In this equation ν is the damping coefficient [s^{-1}]

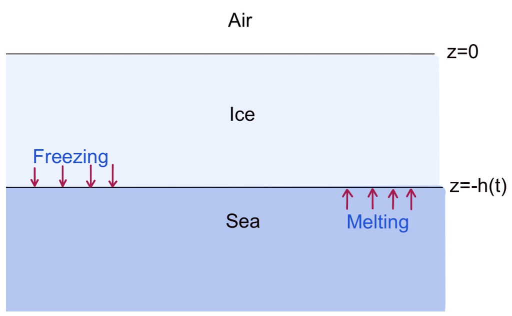
Then the power generated $(m\nu\dot{\eta}) \cdot \dot{\eta} =$ Force x Velocity

The energy generated over one period $=\int_0^T m\nu\dot{\eta}^2 dt$

Ice and Glaciers

3.1 Melting/Solidification of Sea Ice

Consider the set up:



We consider the melting/freezing of sea ice that is floating in the form of a layer of uniform thickness $h(t)$.

Mathematical Model

We begin with conservation of energy (the heat equation).

Given the temperature of sea/ice is $T(z,t)$, we have

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2} \tag{3.1}$$

with the following constant physical parameters:

- ρ (density) [kg/m^3]
- c_p (specific heat capacity) [J/kg K]
- k (thermal conductivity) [J/m K s]

Since ρ, c_p, κ are different for ice and water, we use subscript notation, where subscripts w and i refer to water and ice, respectively. We also define the thermal diffusivities for each phase as:

$$\alpha_w = \frac{k_w}{\rho_w c_{pw}} \quad [\mathrm{m}^2/s] \qquad\qquad \alpha_i = \frac{k_i}{\rho_i c_{pi}} \quad [\mathrm{m}^2/s] \tag{3.2}$$

which is analogous to a diffusion coefficient for temperature (as discussed in the Introduction). Hence we will write the heat equation in the form:

$$\frac{\partial T}{\partial t} = \begin{cases} \alpha_i \frac{\partial^2 T}{\partial z^2} & : -h < z < 0 \\ \alpha_w \frac{\partial^2 T}{\partial z^2} & : -\infty < z < -h \end{cases} \tag{3.3}$$

There are different types of boundary conditions we will consider:

1. $-k\frac{\partial T}{\partial z} = q$: This describes the heat flux q [J/m^{2}s]
   - $q = q_0$: This gives a prescribed heating/cooling.
   - $q = h_c\,(T_e - T)$: This is Newton's Law of heating/cooling, given an external temperature T_e and a heat transfer coefficient h_c [J/K m^{2}s].

So in essence

$$\begin{cases} \text{if} \quad T > T_e \to q < 0 & \text{(Cooling)} \\ \text{if} \quad T < T_e \to q > 0 & \text{(Heating)} \end{cases} \tag{3.4}$$

2. We may also prescribe a temperature boundary condition of the form:

$$T = T_0 \tag{3.5}$$

3. Another possible boundary condition is the 'Stefan condition', which encapsulates phase change.

$$-\rho_i L \frac{dh}{dt} = -k_w \frac{\partial T}{\partial z}|_{z=-h_-} + k_i \frac{\partial T}{\partial z}|_{z=-h_+} \tag{3.6}$$

where L is the latent heat [J/kg] and we use the notation

$$z = -h_+ \iff z \to -h \text{ from } z > -h \tag{3.7}$$

$$z = -h_- \iff z \to -h \text{ from } z < -h \tag{3.8}$$

And in a more compact form:

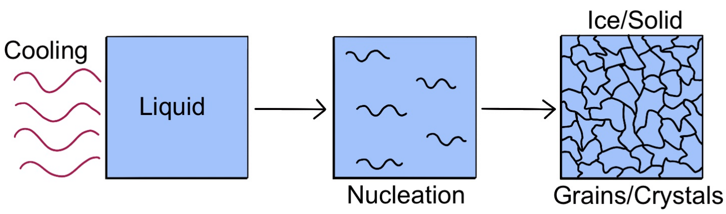
$$-\rho_i L \dot{h} = q_- - q_+ \tag{3.9}$$

Hence, if $q_- > q_+ \implies -\dot{h} > 0 \implies \dot{h} < 0$ (ice melting)

This means the ice melts due to more heating from below than above.

So physically speaking, the Stefan condition imposes conservation of energy across the moving boundary.

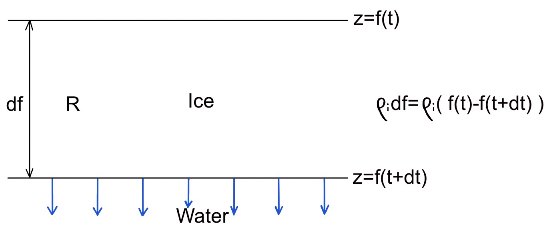
A change in heat flow across the moving boundary induces a release or sink of latent heat which is the energy (per unit mass), associated with a change of phase of matter.



The liquid loses energy as the crystals form and this energy is called latent heat.

Derivation of the Stefan condition

Consider the case of freezing (without loss of generality). Let R denote the region between two instances of the moving front, $z = f(t)$ and $z = f(t + dt)$, as shown below:

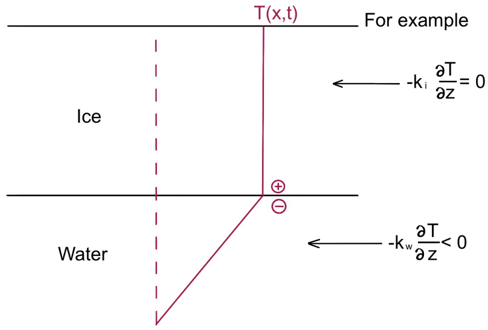


The negative heat energy required to convert a region R from a liquid to a solid is given by:

$$-\rho_i L(f(t) - f(t + dt)) \quad [\text{J/m}^2] \tag{3.10}$$

This must be supplied by a difference in heat flux across R.

Consider the example illustrated below in which the ice is at constant (freezing) temperature $T = T_m$, whilst it is being cooled by a temperature gradient in the water below.



Hence, we have $\frac{\partial T}{\partial z} = 0$ in the ice region and $\frac{\partial T}{\partial z} > 0$ in the water region. Taking limits from either side of the interface, we have

$$-k_i \frac{\partial T}{\partial z}|_+ > -k_w \frac{\partial T}{\partial z}|_- \tag{3.11}$$

$$q_+ > q_- \tag{3.12}$$

We can create an energy balance:

$$-\rho_i L(f(t) - f(t + dt)) = (q_- - q_+)\, dt < 0, \tag{3.13}$$

which indicates that $f(t + dt) < f(t)$, and hence we have ice growth.

To derive the Stefan limit we divide by dt and take the limit $dt \to 0$:

$$\rho_i L \frac{df}{dt} = q_- - q_+ \tag{3.14}$$

This gives (3.9) if we set $f = -h$. i.e. We have

$$\rho_i L \frac{dh}{dt} = [q]_-^+ \tag{3.15}$$

Initial Conditions

We also need initial conditions for the temperature $T(x, t)$ and the ice thickness $h(t)$.

$$\text{e.g. } T = T_i \text{ at } t = 0, \quad h(0) = 0 \tag{3.16}$$

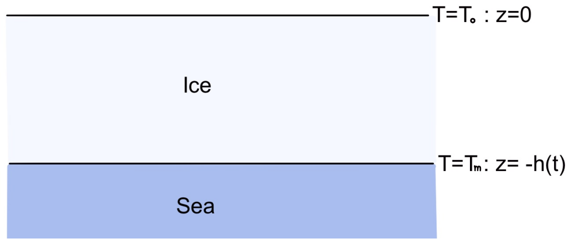
which corresponds with zero ice initially.

Example Let's consider the freezing of ice during winter, where the upper surface is held at a constant temperature below freezing, such that $T = T_0 < T_m$ at $z = 0$. The sea is assumed to be at the freezing temperature T_m everywhere initially. Hence, the ice will begin to grow downwards (from zero thickness) due to cooling from the top.

The boundary conditions at the interface are

$$\rho_i L\dot{h}(t) = [q]_-^+ : \quad z = -h(t) \tag{3.17}$$

$$T = T_m : \quad z = -h(t) \tag{3.18}$$



We also impose the far field condition

$$T \to T_m \quad : \quad z \to -\infty \tag{3.19}$$

We assume that the sea is initially at the freezing temperature T_m and grows from zero thickness, such that our initial conditions are

$$T = T_m \quad : \quad t = 0, \;\; \forall z < 0, \;\; \text{and} \;\; h(0) = 0 \tag{3.20}$$

We will start by non-dimensionalising,

$$T = T_m + (T_m - T_0)\,\hat{T} \qquad\qquad (z, h) = H(\hat{z}, \hat{h}) \qquad\qquad t = \tau\hat{t} \tag{3.21}$$

Currently H and τ are unknown since we have no obvious choices. First we will work with the governing equations:

$$\frac{(T_m - T_0)}{\tau}\hat{T}_{\hat{t}} = \begin{cases} \frac{\alpha_i (T_m - T_0)}{H^2}\hat{T}_{\hat{z}\hat{z}} & -\hat{h} < \hat{z} < 0 \\ \frac{\alpha_w (T_m - T_0)}{H^2}\hat{T}_{\hat{z}\hat{z}} & -\infty < \hat{z} < -\hat{h} \end{cases} \tag{3.22}$$

Our boundary conditions become:

$$\hat{T} = -1 : \quad \hat{z} = 0 \tag{3.23}$$

$$\hat{T} = 0 : \quad \hat{z} = -\hat{h} \tag{3.24}$$

$$\hat{T} \to 0 : \quad \hat{z} \to -\infty \tag{3.25}$$

whilst the Stefan condition is:

$$\rho_i \frac{LH}{\tau}\hat{h}'(\hat{t}) = -\frac{k_i\,(T_m - T_0)}{H}\frac{\partial \hat{T}}{\partial \hat{z}}|_+ + \frac{k_w\,(T_m - T_0)}{H}\frac{\partial T}{\partial z}|_- \tag{3.26}$$

This suggests we should take

$$\tau = \frac{\rho_i L H^2}{k_i\,(T_m - T_0)} \tag{3.27}$$

so we can write our Stefan condition as:

$$-\hat{h}'(\hat{t}) = -\hat{T}_{\hat{z}}|_+ + \frac{k_w}{k_i}\hat{T}_{\hat{z}}|_- \tag{3.28}$$

Finally, our initial condition is

$$\hat{T} = \hat{h} = 0 \text{ at } \hat{t} = 0 \tag{3.29}$$

Writing our governing equations with the new τ:

$$\frac{k_i\,(T_m - T_0)}{\rho_i L \alpha_i}\hat{T}_{\hat{t}} = \begin{cases} \hat{T}_{\hat{z}\hat{z}} : & \text{Ice} \\ \frac{\alpha_w}{\alpha_i}\hat{T}_{\hat{z}\hat{z}} : & \text{Water} \end{cases} \tag{3.30}$$

Since $\alpha_i = \frac{k_i}{\rho_i c_{pi}}$, we find

$$\mathrm{St}\,\hat{T}_{\hat{t}} = \begin{cases} \hat{T}_{\hat{z}\hat{z}} : & \text{Ice} \\ \frac{\alpha_w}{\alpha_i}\hat{T}_{\hat{z}\hat{z}} : & \text{Water} \end{cases} \tag{3.31}$$

where the Stefan number St is defined as follows

$$\mathrm{St} = \frac{(T_m - T_0)c_{pi}}{L} \tag{3.32}$$

which measures the ratio between the *applied heat* and the *latent heat.*

We notice that the vertical length scale H is in fact not needed (notice how it has cancelled out everywhere). Hence, we conclude that there is no natural length scale in this problem. An alternative version of this problem has a non-zero initial ice thickness $h(0) = H$, in which case this would set the vertical length scale.

To begin solving, let's start with the water region ($-\infty < \hat{z} < -\hat{h}$):

$$\mathrm{St}\,\hat{T}_{\hat{t}} = \frac{\alpha_w}{\alpha_i}\hat{T}_{\hat{z}\hat{z}} \tag{3.33}$$

$$\hat{T} = 0 \quad : \quad \hat{z} = -\hat{h} \tag{3.34}$$

$$\hat{T} \to 0 \quad : \quad \hat{z} \to -\infty \tag{3.35}$$

$$\hat{T} = 0 \quad : \quad \hat{t} = 0 \tag{3.36}$$

Clearly the solution in the water region is simply $\hat{T} = 0$ everywhere.

We can now move onto the ice region ($-\hat{h} < \hat{z} < 0$):

$$\mathrm{St}\,\hat{T}_{\hat{t}} = \hat{T}_{\hat{z}\hat{z}} \tag{3.37}$$

$$\hat{T} = -1 \quad : \quad \hat{z} = 0 \tag{3.38}$$

$$\hat{T} = 0 \quad : \quad \hat{z} = -\hat{h}(\hat{t}) \tag{3.39}$$

$$\hat{h}'(\hat{t}) = -\hat{T}_{\hat{z}}|_+ \quad : \quad \hat{z} = -\hat{h}(\hat{t}) \tag{3.40}$$

$$\hat{T} = 0 \quad : \quad \hat{t} = 0 \tag{3.41}$$

$$\hat{h}(0) = 0 \tag{3.42}$$

noting that $-\hat{T}_{\hat{z}}|_- = 0$ because the temperature in the water region is constant (as shown above).

Since we have no natural length scale, this suggests a similarity solution of the form

$$\hat{T}(\hat{z}, \hat{t}) = \hat{t}^a f(\eta) \quad \text{where} \quad \eta = \frac{\hat{z}}{t^b} \quad \text{for some } a \text{ and } b \tag{3.43}$$

If we can find a solution of this form, we can reduce the problem to an ODE for the function $f(\eta)$ rather than a PDE for $\hat{T}$. In order to do so, we must remove all explicit dependence on $\hat{x}$ and $\hat{t}$ (by choosing a and b appropriately) such that the problem only depends on the similarity variable η.

First, let's look at the expressions for $\hat{T}_{\hat{t}}$ and $\hat{T}_{\hat{z}\hat{z}}$:

$$\hat{T}_{\hat{z}} = \frac{\partial}{\partial \hat{z}}\left(\hat{t}^a f\left(\frac{\hat{z}}{\hat{t}^b}\right)\right) = \hat{t}^{a-b} f'(\eta) \tag{3.44}$$

$$\hat{T}_{\hat{z}\hat{z}} = \hat{t}^{a-2b} f''(\eta) \tag{3.45}$$

$$\hat{T}_{\hat{t}} = a\hat{t}^{a-1} f(\eta) - b\hat{t}^a f'(\eta)\left(\frac{\hat{z}}{\hat{t}^{b+1}}\right) \tag{3.46}$$

$$= a\hat{t}^{a-1} f(\eta) - b\hat{t}^{a-1} f'(\eta)\eta \tag{3.47}$$

Hence, the heat equation becomes:

$$\mathrm{St}\,(af(\eta) - b\eta f'(\eta))\,\hat{t}^{a-1} = \hat{t}^{a-2b} f''(\eta) \tag{3.48}$$

To remove the dependence on $\hat{t}$ (i.e. under the assumption of self-similarity (3.43)) we require:

$$a - 1 = a - 2b \tag{3.49}$$

$$\rightarrow b = \frac{1}{2} \tag{3.50}$$

The boundary condition (3.38) becomes:

$$\hat{t}^a f(0) = -1 \tag{3.51}$$

Again, there cannot be any $\hat{t}$ dependence, so we must have: $a = 0$.

The boundary condition (3.39) at $\hat{z} = -\hat{h}(\hat{t})$ has $\eta = \frac{-\hat{h}(\hat{t})}{\hat{t}^{1/2}}$ which must be a constant to avoid explicit time-dependence in the solution. Hence, the moving boundary is located at $\eta = -\lambda$, where λ is an unknown constant that we must find. In terms of space and time, this boundary is at $\hat{h}(\hat{t}) = \lambda\hat{t}^{1/2}$, indicating that the ice grows like the square-root of time - something that be observed in reality. Hence, the boundary condition (3.39) becomes

$$f(-\lambda) = 0 \tag{3.52}$$

The Stefan condition becomes:

$$-\frac{1}{t^{1/2}} f'(-\lambda) = \frac{1}{2}\lambda t^{-1/2} \tag{3.53}$$

$$\rightarrow f'(-\lambda) = -\frac{\lambda}{2} \tag{3.54}$$

The governing heat equation now takes the form

$$-\frac{1}{2}\mathrm{St}\,\eta f'(\eta) = f''(\eta) \tag{3.55}$$

which can be integrated once to give

$$f'(\eta) = Ae^{-\mathrm{St}\,\eta^2/4} \tag{3.56}$$

for some constant A. If we integrate this again we get the error function:

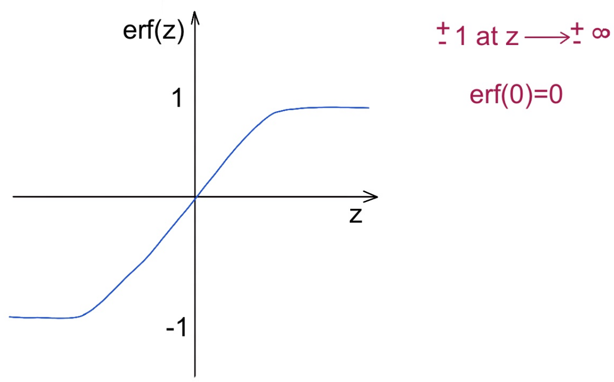
$$f(\eta) = C\,\mathrm{erf}\left(\eta\frac{\sqrt{\mathrm{St}}}{2}\right) + D \tag{3.57}$$

for some constants C, D.

The error function $\mathrm{erf}(z)$ is defined as

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,\mathrm{d}t \tag{3.58}$$

and is associated with the cumulative probability distribution for a normal distribution. It asymptotes to $\mathrm{erf}(z) \to \pm 1$ at $z \to \pm\infty$ and satisfies $\mathrm{erf}(0) = 0$:



The boundary condition (3.51) implies

$$D = -1 \tag{3.59}$$

Meanwhile, (3.52) implies

$$C = \frac{1}{\mathrm{erf}(-\lambda\frac{\sqrt{\mathrm{St}}}{2})} \tag{3.60}$$

This means our solution is :

$$f(\eta) = \frac{\mathrm{erf}\left(\eta\frac{\sqrt{\mathrm{St}}}{2}\right)}{\mathrm{erf}\left(-\lambda\frac{\sqrt{\mathrm{St}}}{2}\right)} - 1 \tag{3.61}$$

We still have to determine λ using the Stefan condition (3.54), which gives

$$\frac{\sqrt{\mathrm{St}}}{2}\frac{\mathrm{erf}'\left(-\lambda\frac{\sqrt{\mathrm{St}}}{2}\right)}{\mathrm{erf}\left(-\lambda\frac{\sqrt{\mathrm{St}}}{2}\right)} = \frac{-\lambda}{2} \tag{3.62}$$

We can rearrange this to give the following transcendental equation for λ:

$$\frac{\lambda\sqrt{\pi}}{2\,\mathrm{St}}\mathrm{erf}\left(-\lambda\frac{\sqrt{\mathrm{St}}}{2}\right)e^{\lambda^2\mathrm{St}/4} = -1 \tag{3.63}$$

which must be solved numerically.

Next we convert back to dimensional coordinates,

$$T = T_m + (T_m - T_0)\,\hat{T} \tag{3.64}$$

$$= T_m + (T_m - T_0)\left(-1 + \frac{\mathrm{erf}\left(\eta\frac{\sqrt{\mathrm{St}}}{2}\right)}{\mathrm{erf}\left(-\lambda\frac{\sqrt{\mathrm{St}}}{2}\right)}\right) \tag{3.65}$$

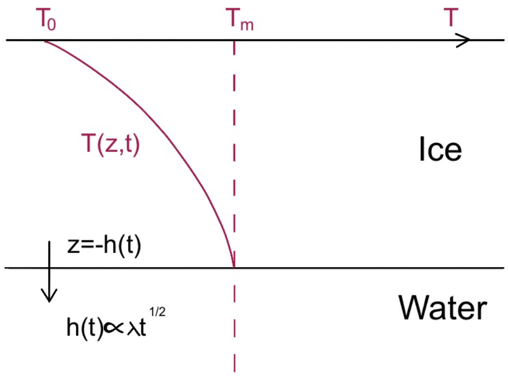
where

$$\eta = \frac{\hat{z}}{\hat{t}^{1/2}} = \frac{z}{H}\left(\frac{\tau}{t}\right)^{1/2} = \frac{z}{(\alpha_i \mathrm{St}\, t)^{1/2}} \tag{3.66}$$

Hence, the solution for the temperature in the ice $T(z,t)$ is given by

$$T(z,t) = T_0 + \frac{\operatorname{erf}\left(\frac{z}{2\sqrt{\alpha_i t}}\right)}{\operatorname{erf}\left(-\lambda\frac{\sqrt{\mathrm{St}}}{2}\right)} \tag{3.67}$$

We can sketch our solution here:



Quasi Steady Solution Let us now consider the Stefan number again.

$$\mathrm{St} = \frac{c_p \Delta T}{L} = \frac{\text{Applied Heat}}{\text{Latent Heat}} \tag{3.68}$$

For sea ice we may take typical parameter values as:

$$\Delta T = 10^\circ \quad [\mathrm{K}] \tag{3.69}$$
$$c_{pi} = 2090 \quad [\mathrm{J/kg\ K}] \tag{3.70}$$
$$c_{pw} = 4184 \quad [\mathrm{J/kg\ K}] \tag{3.71}$$
$$L = 3.36 \times 10^5 \quad [\mathrm{J/kg}] \tag{3.72}$$

Hence, we find

$$\mathrm{St} \approx \frac{10^3 10}{10^5} \approx 0.1 \tag{3.73}$$

Therefore, for small temperature differences $\Delta T < 10^\circ$ we may approximate the heat equation as

$$\mathrm{St}\,\hat{T}_{\hat{t}} = \hat{T}_{\hat{z}\hat{z}} \approx 0 \tag{3.74}$$

i.e. The "curvature" of $\hat{T}$ (2nd derivative in $\hat{z}$) becomes very small.

In this case, known as the quasi-steady limit, the problem becomes independent of time except through the motion of the boundary $\hat{h}(\hat{t})$. Hence, integrating (3.74) with respect to $\hat{z}$ twice we get

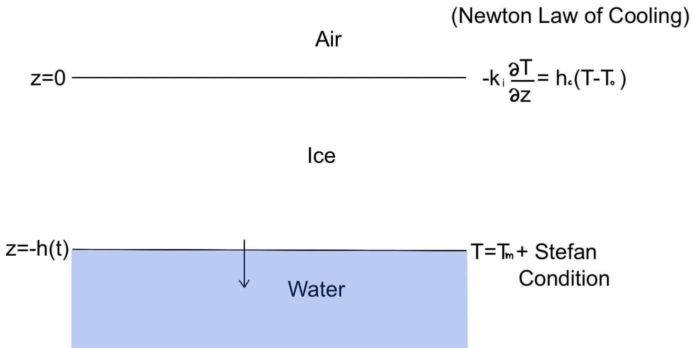
$$\hat{T} = A(\hat{t}) + B(\hat{t})\hat{z} \tag{3.75}$$

for some functions A, B.

We follow the same example as before, except with a different boundary condition at the upper surface of the ice. Specifically, we consider a Newton cooling boundary condition, which in dimensional form is given by

$$-k_i \frac{\partial T}{\partial z} = h_c (T - T_0) : \quad z = 0 \tag{3.76}$$

for some constant heat transfer coefficient h_c [J/m^2 K s]. This indicates that the surface ice temperature is



cooled towards an external temperature of T_0.

Non-dimensionalising as before, we find

$$-k_i \frac{\Delta T}{H} \hat{T}_{\hat{z}} = h_c \Delta T \left(\hat{T} + 1 \right) \qquad : \hat{z} = 0 \tag{3.77}$$

This provides a natural length scale choice, $H = \frac{k_i}{h_c}$, such that

$$-\hat{T}_{\hat{z}} = \hat{T} + 1 \qquad : z = 0 \tag{3.78}$$

Meanwhile, the boundary condition at the moving interface is the same as before:

$$\hat{T} = 0 \qquad : \hat{z} = -\hat{h}(\hat{t}) \tag{3.79}$$

Hence, inserting (3.75) into the boundary conditions (3.78), (3.79), we find

$$B(\hat{t}) = \frac{-1}{1 + \hat{h}(\hat{t})} \qquad A(\hat{t}) = \frac{-\hat{h}(\hat{t})}{1 + \hat{h}(\hat{t})} \tag{3.80}$$

The solution therefore is:

$$\hat{T}(\hat{z}, \hat{t}) = \frac{-1}{1 + \hat{h}(\hat{t})} \left(\hat{h}(\hat{t}) + \hat{z} \right) \tag{3.81}$$

To find $\hat{h}(\hat{t})$ we use the Stefan condition:

$$\hat{h}'(\hat{t}) = -\hat{T}_{\hat{z}}|_+ \qquad : \hat{z} = -\hat{h}(\hat{t}) \tag{3.82}$$

$$\hat{h}'(\hat{t}) = \frac{1}{1 + \hat{h}(\hat{t})} \tag{3.83}$$

$$\hat{h}' + \hat{h}\hat{h}' = 1 \tag{3.84}$$

$$\hat{h} + \frac{1}{2}\hat{h}^2 = \hat{t} + C \tag{3.85}$$

for some constant C. Using our initial conditions $\hat{h}(0) = 0$ we find that $C = 0$

Our solution for the ice thickness is then:

$$\hat{h} = -1 + \left(1 + 2\hat{t}\right)^{1/2} \tag{3.86}$$

We note that a positve square root is chosen (as opposed to a negative one) because this must be consistent with the initial conditions $\hat{h}(0) = 0$.

Finally converting back to dimensional coordinates, our dimensional solution for the temperature in the ice and the ice thickness is:

$$T(z,t) = T_m - \frac{(T_m - T_0)}{H + h(t)}(z + h(t)) \tag{3.87}$$

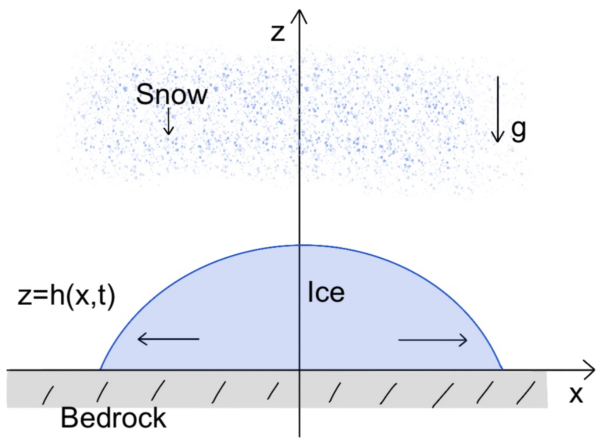
$$h(t) = \left(-1 + \left(1 + \frac{2t}{\tau}\right)^{1/2}\right) H \tag{3.88}$$

where the constants H and τ are given by

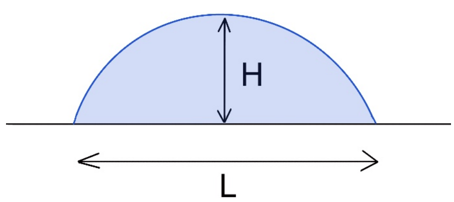
$$H = \frac{k_i}{h_c} \qquad \tau = \frac{\rho_i L H^2}{k_w (T_m - T_0)} \tag{3.89}$$

3.2 Glacier Dynamics

Over long timescales glaciers can be observed to flow like a viscous fluid (as illustrated in the sketch below), whose horizontal length scale is around ∼100 - 1000 km, and whose vertical length scale is around ∼1 km.



We consider ice with constant density ρ and viscosity μ. To model the flow of such a glacier we will use the "lubrication approximation" of the Navier-Stokes equations (see Appendix). The flow within this type of approximation is long and thin assuming $\epsilon = \frac{H}{L} \ll 1$. The Reynolds number is $\mathrm{Re} = \frac{\rho U L}{\mu}$ and within a lubrication approximation it is assumed $\mathrm{Re}\,\epsilon^2 \ll 1$.



Under the lubrication approximation, the governing equations for the ice (conservation of mass and momentum) are:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad \text{Continuity Equation} \tag{3.90}$$

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} \qquad x\text{-momentum} \tag{3.91}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g \qquad z\text{-momentum} \tag{3.92}$$

which are to be solved in a region $0 \leq z \leq h(x,t)$, where $h(x,t)$ is the ice thickness.

The corresponding boundary conditions are:

$$p = p_a \qquad : z = h(x,t) \quad \text{(Dynamic boundary condition)} \tag{3.93}$$

$$\mu\frac{\partial u}{\partial z} = 0 \qquad : z = h(x,t) \quad \text{(Zero stress)} \tag{3.94}$$

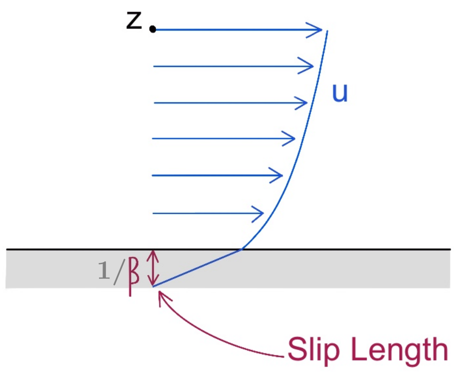
$$w = h_t + uh_x - a \qquad : z = h(x,t) \quad \text{(Kinematic boundary condition)} \tag{3.95}$$

$$w = 0 \qquad : z = 0 \quad \text{(Impermeability of bedrock)} \tag{3.96}$$

$$\mu\frac{\partial u}{\partial z} = \mu\beta u \qquad : z = 0 \quad \text{(Slip condition)} \tag{3.97}$$

Most of these we have seen before. A new constant term a [m/s] has been added to the kinematic condition to represent the accumulation of snowfall (that turns to ice) on the upper surface of the glacier. A new boundary condition called the Slip condition has been introduced at the base of the glacier with a parameter β [1/m] known as the inverse slip length.

According to the 'slip length model', the velocity of the glacier at the base ($z = 0$) is not equal to zero as it would be with a regular viscous fluid. In fact, the base of the glacier slips according to some friction relation. We approximate this with (3.97), which assumes that the velocity can be linearly extrapolated 'below' the impermeable boundary at $z = 0$. We imagine that $u = (\beta z + 1)\, u_0$ for $z < 0$, where u_0 is the velocity at $z = 0$. In this way, if extrapolated below $z < 0$, the velocity would reach zero at a negative length scale $z = -\frac{1}{\beta}$. Hence, we call $1/\beta$ [m] as the slip length, as illustrated below:



Now we can begin solving our problem. We integrate (3.92) and use (3.93):

$$p = -\rho g\,(z - h) + p_a \tag{3.98}$$

We then insert this into (3.91) to get

$$\mu\frac{\partial^2 u}{\partial z^2} = \rho g\frac{\partial h}{\partial x} \tag{3.99}$$

By integrating this twice we find

$$u = \frac{1}{2}\frac{\rho g}{\mu}\frac{\partial h}{\partial x}z^2 + az + b \tag{3.100}$$

where the constants a and b are found using (3.94) and (3.97). Hence, the velocity profile within the ice is given by

$$u = \frac{\rho g}{2\mu}\frac{\partial h}{\partial x}\left(z^2 - 2zh - \frac{2h}{\beta}\right) \tag{3.101}$$

Integrating the continuity equation (3.90) from $z = 0$ to $z = h(x,t)$, we get

$$\int_0^h u_x + w_z dx = 0 \tag{3.102}$$

$$\int_0^h u_x dx + w|_0^h = 0 \tag{3.103}$$

Using the boundary conditions (3.95) and (3.96) and rearranging the integrals, we find

$$\frac{\partial}{\partial x}\int_0^h udz - u\frac{\partial h}{\partial x}|_{x=h} + h_t + u\frac{\partial h}{\partial x}|_{z=h} - a = 0 \tag{3.104}$$

$$\rightarrow \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h udz = a \tag{3.105}$$

If we let $Q = \int_0^h u\,dz$ represent the horizontal flux of ice [m^2/s], the governing PDE can be written as

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = a \tag{3.106}$$

Note that the step between (3.103) and (3.104) uses the Leibniz rule for the differentiation of integrals with varying limits.
Leibniz Rule Recap:

$$\frac{d}{dx}\left(\int_{b(x)}^{a(x)} f(x,t)dt\right) = f(x,a(x))a'(x) - f(x,b(x))b'(x) + \int_{b(x)}^{a(x)} \frac{\partial}{\partial x} f(x,t)dt \tag{3.107}$$

Next, we need to evaluate the flux $Q = \int_0^h u\,dz$. This is given by inserting the expression for the velocity (3.101) into the integral, giving

$$Q = -\frac{\rho g}{3\mu}\frac{\partial h}{\partial x}h^2\left(h + \frac{3}{\beta}\right) \tag{3.108}$$

Hence, the governing equation for shape of the glacier (3.106) becomes:

$$\frac{\partial h}{\partial t} - \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^2\left(h + \frac{3}{\beta}\right)\frac{\partial h}{\partial x}\right) = a \tag{3.109}$$

We now require boundary conditions and initial conditions for $h(x,t)$. It is assumed that the glacier is symmetric about the z axis and extends between $x = -x_n(t)$ and $x = x_n(t)$, where $x_n(t)$ represents the 'nose' of the glacier (i.e. it's propogating front).

Even though (3.109) is only a second order PDE in x, we actually require 3 boundary conditions in x because the nose position $x_n(t)$ is unknown. Also, we will only consider half of the total domain, $0 < x < x_n(t)$, for simplicity since the glacier is symmetric. The boundary conditions we impose are

$$Q = -\frac{\rho g}{3\mu}\frac{\partial h}{\partial x}h^2\left(h + \frac{3}{\beta}\right) = 0 : \quad x = 0 \tag{3.110}$$

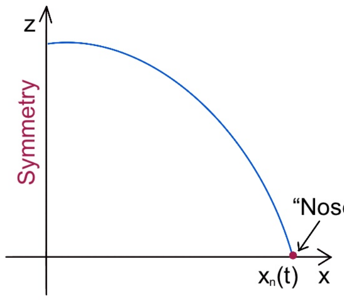
$$Q = -\frac{\rho g}{3\mu}\frac{\partial h}{\partial x}h^2\left(h + \frac{3}{\beta}\right) = 0 : \quad x = x_n(t) \tag{3.111}$$

$$h = 0 : \quad x = x_n(t) \tag{3.112}$$

The first condition (3.110) is a consequence of symmetry, the second (3.111) imposes that no ice is added or removed from the edges of the glacier, and the third (3.112) imposes that the ice thickness is zero at the nose.

As initial conditions, we take some initial shape $h_0(x)$, such that

$$h(x,0) = h_0(x) \quad \text{at} \quad t = 0 \tag{3.113}$$



For the rest of this analysis, we will restrict our attention to the case where $a = 0$ (no snowfall), in which the PDE (3.109) can be integrated across the domain to give a statement of conservation of mass:

$$\int_0^{x_n(t)} h(x,t)dx = V_0 \tag{3.114}$$

where V_0 is the total volume of ice (constant). The above condition can be used instead of one of the boundary conditions, e.g. (3.110) or (3.111).

In this problem we have two distinct regimes, depending on the ratio between the ice sheet thickness h and the characteristic length scale $3/\beta$, which represents the relative order of magnitude of the different flux terms in the governing PDE (3.109).

Case 1: $h \gg \frac{3}{\beta}$ "No slip driven"

In the case where the ice thickness h is much larger than the slip length, the flux (3.108) is approximated by

$$Q \approx -\frac{\rho g}{3\mu} h^3 h_x \tag{3.115}$$

In this case, the governing PDE for the glacier shape is

$$h_t - \frac{\rho g}{3\mu}\left(h^3 h_x\right)_x = 0 \tag{3.116}$$

We can show that if we replace the slip boundary condition (3.97) with a no slip boundary condition, $u = 0$ at $z = 0$, and follow the derivation from the beginning we end up with the same result (3.116). Hence, we refer to this case as the "no slip driven" case: *the ice sheet is thick and dominated by friction as if it were not able to move at the base.*

We can non-dimensionalise as follows:

$$(x,h) = V_0^{1/2}(\hat{x},\hat{h}), \quad t = \frac{\mu}{\rho g V_0^{1/2}}\hat{t} \tag{3.117}$$

In this case, the governing equation and boundary conditions become:

$$\hat{h}_{\hat{t}} - \frac{1}{3}\left(\hat{h}^3\hat{h}_{\hat{x}}\right)_{\hat{x}} = 0 \tag{3.118}$$

$$-\frac{1}{3}\hat{h}^3\hat{h}_{\hat{x}} = 0 : \quad \hat{x} = 0 \tag{3.119}$$

$$\hat{h} = 0 : \quad \hat{x} = \hat{x}_n(\hat{t}) \tag{3.120}$$

$$\int_0^{\hat{x}_n} \hat{h}(\hat{x},\hat{t})\,\mathrm{d}\hat{x} = 1 \tag{3.121}$$

By seeking a similarity solution of the form

$$\hat{h}(\hat{x},\hat{t}) = \hat{t}^a f(\eta), \quad \eta = \frac{\hat{x}}{\hat{t}^b} \tag{3.122}$$

we can show that self-similarity is only possible if we choose

$$a = -1/5, \quad b = 1/5 \tag{3.123}$$

Hence, the position of the nose is given by

$$\hat{x}_n(\hat{t}) = \eta_n \hat{t}^{1/5} \tag{3.124}$$

where η_n is a constant that must be found.

We may derive the governing ODE and boundary conditions

$$-\frac{1}{5}(f(\eta) + f'(\eta)\eta) = \frac{1}{3}(f(\eta)^3 f'(\eta))' \tag{3.125}$$

$$-\frac{1}{3}f(0)^3 f'(0) = 0 \tag{3.126}$$

$$f(\eta_n) = 0 \tag{3.127}$$

$$\int_0^{\eta_n} f(\eta)\,\mathrm{d}\eta = 1 \tag{3.128}$$

This has a solution

$$f(\eta) = \left(\frac{9}{10}\left(\eta_n^2 - \eta^2\right)\right)^{1/3} \tag{3.129}$$

The value of the constant η_n can be calculated numerically by inserting the above into (3.121), giving $\eta_n \approx 1.33$.

Case 2: $h \ll \frac{3}{\beta}$ "Slip Driven"

In the case where the ice thickness h is much smaller than the slip length, the flux (3.108) is approximated by

$$Q \approx -\frac{\rho g}{\mu\beta} h^2 h_x \tag{3.130}$$

In this case, the governing PDE for the glacier shape is

$$h_t - \frac{\rho g}{\mu\beta}\left(h^2 h_x\right)_x = 0 \tag{3.131}$$

The PDE is clearly dominated by the term multiplied by the slip length. Hence, we refer to this case as the "slip driven" case: *the ice sheet is thin and acts as if it is slipping at the base*

We non-dimensionalise exactly as before (3.117). Hence, the governing equation and boundary conditions become:

$$\hat{h}_{\hat{t}} - \frac{1}{B}\left(\hat{h}^2\hat{h}_{\hat{x}}\right)_{\hat{x}} = 0 \tag{3.132}$$

$$-\frac{1}{B}\hat{h}^2\hat{h}_{\hat{x}} = 0: \quad \hat{x} = 0 \tag{3.133}$$

$$\hat{h} = 0: \quad \hat{x} = \hat{x}_n(\hat{t}) \tag{3.134}$$

$$\int_0^{\hat{x}_n} \hat{h}(\hat{x},\hat{t})\,\mathrm{d}\hat{x} = 1 \tag{3.135}$$

where $B = \beta V_0^{1/2}$ is a dimensionless parameter.

By seeking a similarity solution of the form

$$\hat{h}(\hat{x},\hat{t}) = \hat{t}^a f(\eta), \quad \eta = \frac{\hat{x}}{\hat{t}^b} \tag{3.136}$$

we can show that self-similarity is only possible if we choose

$$a = -1/4, \quad b = 1/4 \tag{3.137}$$

Hence, the position of the nose is given by

$$\hat{x}_n(\hat{t}) = \eta_n \hat{t}^{1/4} \tag{3.138}$$

where η_n is a constant that must be found.

We may derive the governing ODE and boundary conditions

$$-\frac{1}{4}(f(\eta) + f'(\eta)\eta) = \frac{1}{B}(f(\eta)^2 f'(\eta))' \tag{3.139}$$

$$-\frac{1}{B}f(0)^2 f'(0) = 0 \tag{3.140}$$

$$f(\eta_n) = 0 \tag{3.141}$$

$$\int_0^{\eta_n} f(\eta)\,\mathrm{d}\eta = 1 \tag{3.142}$$

This has a solution

$$f(\eta) = \left(\frac{B}{4}\left(\eta_n^2 - \eta^2\right)\right)^{1/2} \tag{3.143}$$

The value of the constant η_n can be calculated numerically by inserting the above into (3.135), giving $\eta_n = (8/\pi)^{1/2}B^{-1/4}$.

Transition between cases Next we consider the time it takes to transition from Case 1 to Case 2, as described above. We can use the similarity solution in Case 1 (for example) to predict the maximum thickness of the glacier at $x = 0$:

$$\hat{h}(0,\hat{t}) = \hat{t}^{-1/5} f(0) = \left(\frac{\rho g V_0^{1/2}\, t}{\mu}\right)^{-1/5} f(0) \tag{3.144}$$

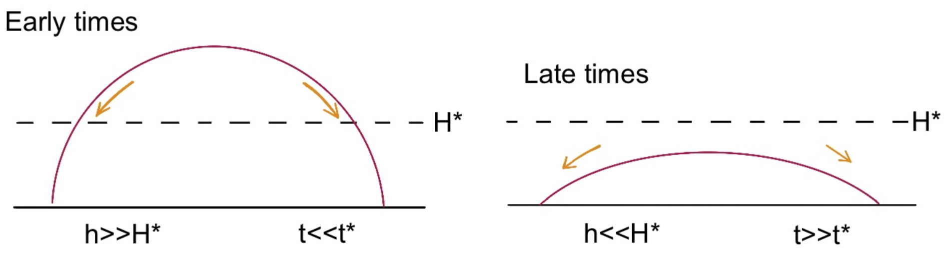
which must equal the dimensionless transition thickness

$$\hat{h}(0,\hat{t}) = \frac{3}{\beta V_0^{1/2}} \tag{3.145}$$

Hence, by equating (3.144) to (3.145) we find the time to transition

$$t^* = \frac{\mu V_0^2}{\rho g}\left(\frac{\beta f(0)}{3}\right)^5 \tag{3.146}$$

where $f(0) = \left(\frac{9\eta_n^2}{10}\right)^{1/3}$. Below is an illustration of the transition between Case 1 at early times and Case 2 at late times. Early and late times are defined as $t \ll t^*$ (or $h \gg H^*$) and $t \gg t^*$ (or $h \ll H^*$), where $H^* = 3/\beta$ is the transition thickness.



3.3 Plastic Model for an Ice Sheet

For a Newtonian fluid, stress is proportional to the rate-of-strain (1.10). In reality ice is non-Newtonian, which means that stress and rate-of-strain may have a more complicated relationship. In the following section we will follow a common model for ice sheets in which the ice is treated as a 'yield stress' fluid undergoing 'plastic' deformations. The assumption is that the ice will not deform unless a critical stress τ_b is reached, and will deform freely for any stress larger than this.

The 'deviatoric' part of the stress tensor τ_{ij} is the part related to deformations of the fluid and is given by

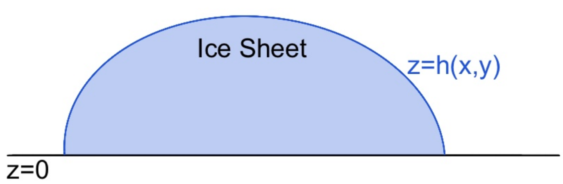
$$\tau_{ij} = 2\mu\dot{e}_{ij} \tag{3.147}$$

where $\dot{e}_{ij}$ is given by (1.11). Our model assumes that the modulus of the deviatoric stress vector at the base of the glacier ($z = 0$) is always equal to the yield stress, such that

$$|\underline{\tau}| = \tau_b, \quad \text{at} \quad z = 0 \tag{3.148}$$

Also, since the fluid is assumed to have yielded according to this stress, we anticipate a steady state with no time-dependence.

We will consider a 3D model for shallow ice flow under the lubrication approximation. In this 3D model the ice sheet occupies a region $0 \leq z \leq h(x, y)$, as illustrated below.



Under the steady lubrication approximation, the governing equations for the flow of ice can be written as

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}(\tau_{xz}) \qquad (x\text{-momentum}) \tag{3.149}$$

$$0 = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}(\tau_{yz}) \qquad (y\text{-momentum}) \tag{3.150}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g \qquad (z\text{-momentum}) \tag{3.151}$$

The boundary conditions are:

$$p = p_a \qquad : z = h(x,y) \qquad \text{(Dynamic)} \tag{3.152}$$

$$\tau_{xz} = \tau_{yz} = 0 \qquad : z = h(x,y) \qquad \text{(Zero stress)} \tag{3.153}$$

Integrating (3.151) and using (3.152),

$$p = p_a + \rho g \left(h(x,y) - z\right) \tag{3.154}$$

Inserting this into (3.149),

$$\frac{\partial}{\partial z}\left(\tau_{xz}\right) = \rho g \frac{\partial h}{\partial x} \tag{3.155}$$

We now integrate this and use (3.153),

$$\tau_{xz} = \rho g \frac{\partial h}{\partial x}\left(z - h\right) \tag{3.156}$$

In a similar procedure,

$$\tau_{yz} = \rho g \frac{\partial h}{\partial y}\left(z - h\right) \tag{3.157}$$

Consider the stress vector at the base of the glacier,

$$\underline{\tau} = \left(\tau_{xz}, \tau_{yz}, \tau_{zz}\right) = \rho g \left(z - h\right)\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, 0\right) \tag{3.158}$$

Next we calculate the norm of this,

$$\tau = |\underline{\tau}| = \rho g |z - h| \left(h_x^2 + h_y^2\right)^{1/2} \tag{3.159}$$

If we have constant basal friction (friction at the base) τ_b, we impose:

$$\tau = \tau_b : \qquad z = 0 \tag{3.160}$$

This gives us,

$$h|\nabla h| = H_0 \tag{3.161}$$

where

$$H_0 = \frac{\tau_b}{\rho g} \tag{3.162}$$

is the vertical length scale of the glacier which is set by the balance between gravity and basal friction. We write this more conveniently as

$$\left|\nabla\left(\frac{h^2}{2}\right)\right| = H_0 \tag{3.163}$$

or, letting $u = \frac{h^2}{2H_0}$, we get

$$u_x^2 + u_y^2 = 1 \tag{3.164}$$

which is known as the "Eikonal equation".

We wish to solve this using the method of characteristics where we have a non-linear first order PDE. Hence, we will use Charpit's equations (see ACM30220).

We write the PDE as a function F of the form

$$F(x, y, u, p, q) = 0 \tag{3.165}$$

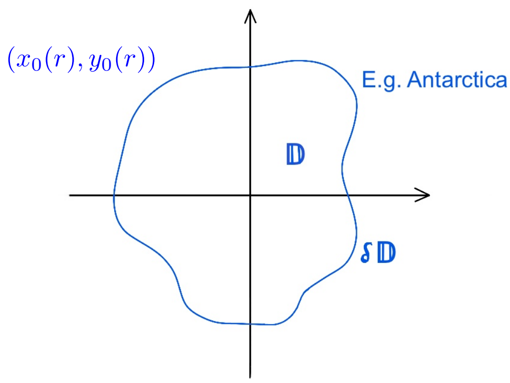
where $p = u_x$ and $q = u_y$. Hence, in the case of the Eikonal equation we have

$$F = p^2 + q^2 - 1 = 0 \tag{3.166}$$

Charpits equations are given by

$$\frac{dx}{ds} = \frac{\partial F}{\partial p} = 2p \tag{3.167}$$
$$\frac{dy}{ds} = \frac{\partial F}{\partial q} = 2q \tag{3.168}$$
$$\frac{du}{ds} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} = 2(p^2 + q^2) = 2 \tag{3.169}$$
$$\frac{dp}{ds} = -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial u} = 0 \tag{3.170}$$
$$\frac{dq}{ds} = -\frac{\partial F}{\partial y} - q\frac{\partial F}{\partial u} = 0 \tag{3.171}$$

The following ODE's need some initial data. In our case, the initial data is given on the boundary $\partial\mathbb{D}$ of our domain $\mathbb{D}$. On the boundary, the ice is assumed to have zero thickness $h = 0$ since this is where it meets the ocean and transforms into icebergs. Since $u \propto h^2$, this mean that $u = 0$ on the boundary too. We will parameterise the boundary shape $\partial\mathbb{D}$ as some 2D curve $(x_0(r), y_0(r))$.



Along this curve, the initial data (when $s = 0$) is given by

$$x(s = 0) = x_0(r) \tag{3.172}$$
$$y(s = 0) = y_0(r) \tag{3.173}$$
$$u(s = 0) = u_0(r) = 0 \tag{3.174}$$
$$p(s = 0) = p_0(r) \tag{3.175}$$
$$q(s = 0) = q_0(r) \tag{3.176}$$

In addition to $u_0(r) = 0$, it is assumed that $x_0(r)$ and $y_0(r)$ are known (i.e. given by the prescribed boundary), whereas $p_0(r)$ and $q_0(r)$ are unknown and therefore need to be found. These are given by Charpit's equations for the initial data, which are

$$F(x_0(r), y_0(r), u_0(r), p_0(r), q_0(r)) = 0 \tag{3.177}$$
$$u_0'(r) = x_0'(r)p_0(r) + y_0'(r)q_0(r) \tag{3.178}$$

which provides 2 equations for 2 unknowns ($p_0(r)$ and $q_0(r)$).

Let's look again at Charpit's equations (3.167)-(3.171). The equations for p and q can be integrated directly:

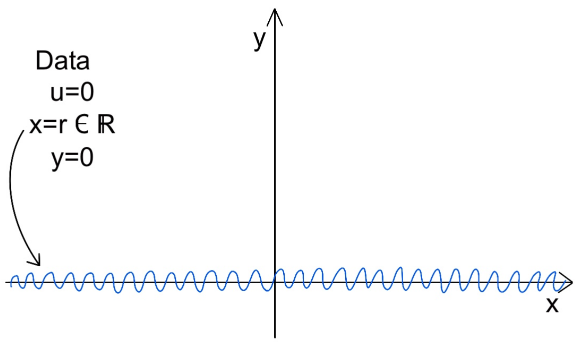
$$\frac{dp}{ds} = 0 \implies p = p_0(r) \tag{3.179}$$
$$\frac{dq}{ds} = 0 \implies q = q_0(r) \tag{3.180}$$

The rest of Charpit's equations can then be integrated to give

$$\frac{dx}{ds} = 2p_0(r) \to x = 2p_0(r)s + x_0(r) \tag{3.181}$$
$$\frac{dy}{ds} = 2q_0(r) \to y = 2q_0(r)s + y_0(r) \tag{3.182}$$
$$\frac{du}{ds} = 2 \to u = 2s \tag{3.183}$$

We now have a parametric solution for $x(r,s)$,$y(r,s)$,$u(r,s)$. However, to get our final solution in terms of x and y we need to invert by finding $r(x,y)$ and $s(x,y)$.

Example: Ice build-up near a crevasse: Let us now consider a specific example where we determine the thickness of ice near a crevasse. The crevasse is located along the x axis, along which the ice has zero thickness, such that $h = u = 0$ for $y = 0$. We can parameterise the initial data curve using a real-valued parameter



$r \in \mathbb{R}$, such that

$$x_0(r) = r \tag{3.184}$$
$$y_0(r) = 0 \tag{3.185}$$
$$u_0(r) = 0 \tag{3.186}$$

Meanwhile, the equations for the initial data for p and q (3.177)-(3.178) give us,

$$p_0(r)^2 + q_0(r)^2 - 1 = 0 \tag{3.187}$$
$$p_0(r) = 0 \tag{3.188}$$

From these, we find

$$q_0(r) = \pm 1 \quad \text{and} \quad p_0(r) = 0 \tag{3.189}$$

The two possible values of $q_0(r)$ indicate that there are multiple possible solutions to our PDE (which is a common feature of the Eikonal equation).

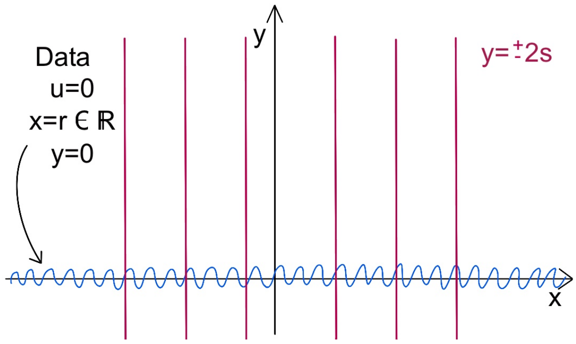
Our parametric solution is

$$x = r \tag{3.190}$$

$$y = \pm 2s \tag{3.191}$$

$$u = 2s \tag{3.192}$$

Hence, the characteristics are straight vertical lines of the form $y = \pm 2s$ for $s \in \mathbb{R}$. In other words, information flows away from the crevasse, towards the interior of the ice.



The solution is written simply as

$$u(x, y) = \pm y \tag{3.193}$$

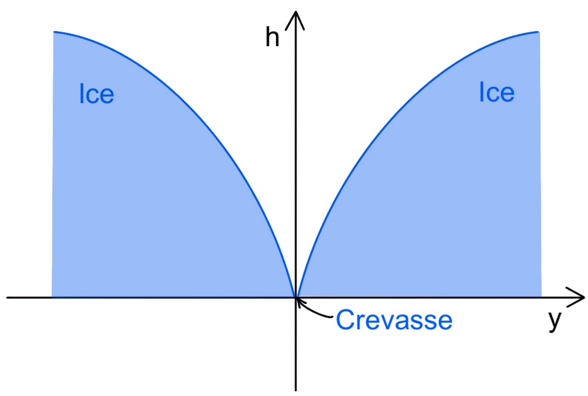
Since $u = h^2/2H_0 > 0$ must be positive, we write our solution as

$$u(x, y) = |y| \tag{3.194}$$

which is no longer multivalued. Converting this to an ice thickness, we get

$$h = (2H_0|y|)^{1/2} \tag{3.195}$$

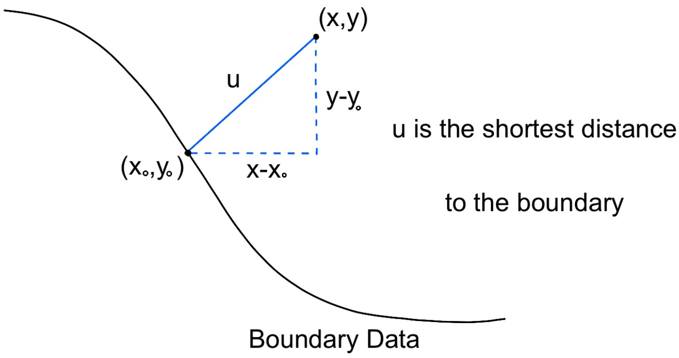
Sketching this we have:



Note: For any boundary curve, the solution for the Eikonal equation $u(x, y)$ has a geometric interpretation which is the Euclidean distance to the boundary $\partial\mathbb{D}$, since

$$u(x, y)^2 = (x - x_0(r))^2 + (y - y_0(r))^2 \tag{3.196}$$

as illustrated below:



This can be seen by inserting the expressions for $x(r,s), y(r,s), u(r,s)$, (3.181)-(3.183), into (3.196) directly and seeing that it is indeed satisfied:

$$u^2 = 4s^2 = (x - x_0(r))^2 + (y - y_0(r))^2 = 4s^2(p_0(r)^2 + q_0(r)^2) = 4s^2 \tag{3.197}$$

Porous Media

4.1 What is a porous medium?

A porous medium is one in which a fluid is distributed through the pore space between the grains of solid. For example, porous materials include soil, sandstone, and sponge.

Fluids move through a porous medium according to Darcy's Law:

$$\mathbf{u} = -\frac{k}{\mu}\left(\nabla p + \rho g \hat{\mathbf{k}}\right) \tag{4.1}$$

where we use the following definitions

- Permeability k [m^2]
- Viscosity μ [kg m^{-1} s^{-1}]
- Density ρ [kg m^{-3}]
- Darcy Velocity $\mathbf{u}$ [m s^{-1}]
- Porosity ϕ [$\sim$]
- Pressure p [kg m^{-1} s^{-2}]

Note: Permeability is often related to the "pore size" d [m] according to some relation $k \propto d^2$.

We know that for an incompressible fluid $\nabla \cdot \mathbf{u} = 0$. Hence, Darcy's Law in this case tells us

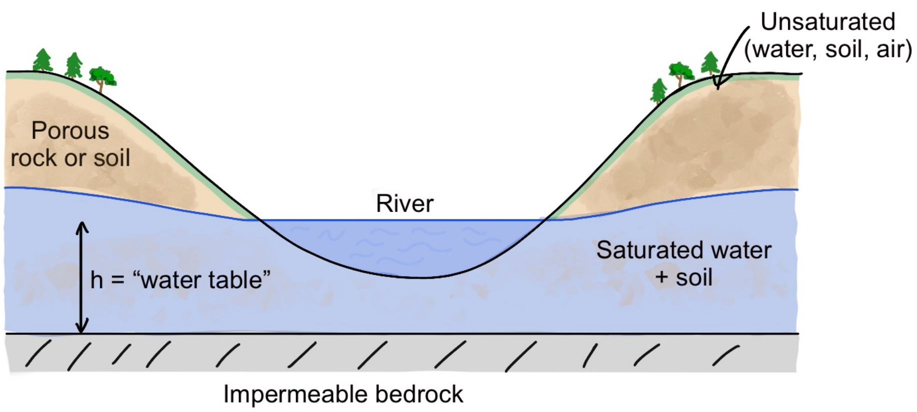
$$\nabla \cdot \mathbf{u} = \nabla \cdot \left(-\frac{k}{\mu}\left(\nabla p + \rho g \hat{\mathbf{k}}\right)\right) = 0 \tag{4.2}$$

Hence, if ρ, k, and μ are constants then we get Laplace's equation for the pressure

$$\nabla^2 p = 0 \tag{4.3}$$

4.2 Groundwater

In this section we will be studying how water resides in the ground. Let's consider a landscape as depicted below.



The water table is a region composed of soil saturated with water. Above the water table is a region of unsaturated soil, which may contain a combination of water, soil and air. Water may reside above the ground in the form of a river. The water table is typically bounded below by a layer of impermeable "bedrock".

We shall consider 2D scenarios using coordinates (x, z). Darcy's law is accompanied by boundary conditions of the form:

$$p = p_a \quad : z = h(x,t) \quad \text{(Dynamic boundary condition)} \tag{4.4}$$

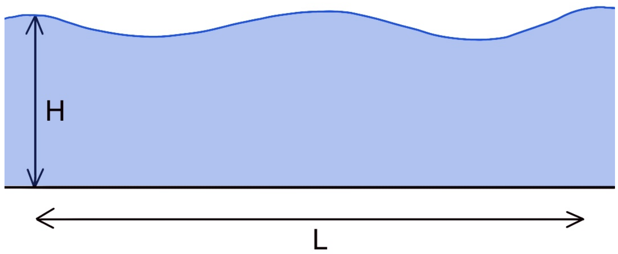
$$w = uh_x + \phi h_t \quad : z = h(x,t) \quad \text{(Kinematic boundary condition)} \tag{4.5}$$

$$w = 0 \quad : z = 0 \quad \text{(Impermeability condition)} \tag{4.6}$$

Next we non-dimensionalise according to some typical horizontal and vertical length scales, L and H, that have a long-thin aspect ratio

$$\frac{H}{L} = \epsilon \ll 1 \tag{4.7}$$

which is relevant for many groundwater problems.



We take the following scalings,

$$z = H\hat{z}, \quad x = L\hat{x}, \quad u = U\hat{u}, \quad w = W\hat{w}, \quad t = \tau\hat{t}, \quad h = H\hat{h}, \quad p = p_a + P\hat{p}, \tag{4.8}$$

where U, W, τ and P are not yet known. Under these scalings, the continuity equation becomes

$$\frac{U}{L}\hat{u}_{\hat{x}} + \frac{W}{H}\hat{w}_{\hat{z}} = 0 \tag{4.9}$$

Choosing $W = \epsilon U$ we then have,

$$\hat{u}_{\hat{x}} + \hat{w}_{\hat{z}} = 0 \tag{4.10}$$

Looking at the x-component of Darcy's law, we have

$$u = -\frac{k}{\mu}\frac{\partial p}{\partial x} \tag{4.11}$$

$$U\hat{u} = -\frac{k}{\mu}\frac{P}{L}\frac{\partial \hat{p}}{\partial \hat{x}} \tag{4.12}$$

Therefore, choosing $U = P/\mu L$ we get

$$\hat{u} = -\hat{p}_{\hat{x}} \tag{4.13}$$

Next, the z-component of Darcy's law gives us

$$w = -\frac{k}{\mu}\left(\frac{\partial p}{\partial z} + \rho g\right) \tag{4.14}$$

$$\epsilon U\hat{w} = -\frac{k}{\mu}\frac{P}{H}\left(\frac{\partial \hat{p}}{\partial \hat{z}} + \frac{\rho H g}{P}\right) \tag{4.15}$$

Let's choose $P = \rho g H$, such that

$$\epsilon^2\hat{w} = -\hat{p}_{\hat{z}} - 1 \tag{4.16}$$

Hence, in the limit $\epsilon \to 0$, the pressure is hydrostatic, such that

$$\hat{p}_{\hat{z}} = -1 \tag{4.17}$$

From the above choices of scalings, we can see that the natural velocity scale within the problem is

$$U = \frac{k\rho g H}{\mu L} = K\epsilon \tag{4.18}$$

where $K = \frac{k\rho g}{\mu}$ is known as the hydraulic conductivity [m/s].

Next let's look at the boundary conditions. The dynamic boundary condition becomes

$$\hat{p} = 0 : \qquad \hat{z} = \hat{h}(\hat{x}, \hat{t}) \tag{4.19}$$

whilst the kinematic condition is

$$\epsilon U \hat{w} = \frac{H\phi}{\tau}\hat{h}_{\hat{t}} + \epsilon U \hat{u}\hat{h}_{\hat{x}} : \qquad : \hat{z} = \hat{h}(\hat{x}, \hat{t}). \tag{4.20}$$

Choosing $\tau = \frac{\phi L}{U}$, the kinematic condition simplifies to

$$\hat{w} = \hat{h}_{\hat{t}} + \hat{u}\hat{h}_{\hat{x}} \qquad : \hat{z} = \hat{h}(\hat{x}, \hat{t}) \tag{4.21}$$

Finally, the impermeability condition becomes

$$\hat{w} = 0 : \qquad \hat{z} = 0 \tag{4.22}$$

Now let's integrate equation (4.17) and use the boundary condition (4.19), giving us

$$\hat{p} = -\hat{z} + \hat{h}(\hat{x}, \hat{t}) \tag{4.23}$$

Now we will insert this into equation (4.13) to give us the dimensionless velocity

$$\hat{u} = -\hat{h}_{\hat{x}} \tag{4.24}$$

Next, we integrate (4.10) from $\hat{z} = 0$ to $\hat{z} = \hat{h}(\hat{x}, \hat{t})$ to give

$$\int_0^{\hat{h}} u_{\hat{x}} d\hat{z} + [\hat{w}]_0^{\hat{h}} = 0 \tag{4.25}$$

Then, using the Leibniz Rule and applying the boundary conditions (4.21) and (4.22), we get

$$\frac{\partial}{\partial \hat{x}} \int_0^{\hat{h}} \hat{u} d\hat{z} - \hat{h}_{\hat{x}}\hat{u}|_{\hat{z}=\hat{h}} + \left(\hat{u}\hat{h}_{\hat{x}} + \hat{h}_{\hat{t}}\right)|_{\hat{z}=\hat{h}} - 0 = 0 \tag{4.26}$$

After simplification, this gives us the following governing equation for the water table

$$\hat{h}_{\hat{t}} = \left(\hat{h}\hat{h}_{\hat{x}}\right)_{\hat{x}} \tag{4.27}$$

which is known as the "Dupuit Approximation" for long-thin groundwater problems. This is a nonlinear diffusion (heat) equation which is second order in $\hat{x}$ and first order in $\hat{t}$ so we will need two boundary conditions and one initial condition.

Boundary Conditions A typical boundary condition we may consider is presecribing the water table at a given location

$$\hat{h} = \hat{h}_0 : \quad \hat{x} = \hat{x}_0 \tag{4.28}$$

From (4.23) we see that imposing h is equivalent to imposing the pressure, such as from a reservoir. In fact, the dimensionless water table can be related to a constant pressure boundary condition p_0 according to

$$\hat{h}_0 = \frac{p_0 - p_a}{\rho g H} \tag{4.29}$$

Another boundary condition we may consider is setting the flux Q at a given location. We can calculate the dimensional flux (depth integrated horizontal velocity) as

$$\int_0^h u dz = UH \int_0^{\hat{h}} \hat{u} d\hat{z} = -UH\hat{h}\hat{h}_{\hat{x}} = Q \tag{4.30}$$

Hence, an appropriate flux boundary condition is of the form

$$-\hat{h}\hat{h}_{\hat{x}} = \hat{Q} : \qquad\qquad \hat{x} = \hat{x}_0 \tag{4.31}$$

where the dimensionless flux is $\hat{Q} = \frac{Q}{UH}$

Finally we need some initial conditions, which are of the form

$$\hat{h}(\hat{x}, 0) = \hat{h}_i(\hat{x}) \tag{4.32}$$

for some initial water table shape $\hat{h}_i(\hat{x})$.

Steady State

After a long time we expect the water table to reach an equilibrium, such that $\frac{\partial \hat{h}}{\partial \hat{t}} \to 0$. In this case, we can integrate the governing equation (4.27) and apply the boundary condition (4.31) to get

$$\hat{h}(\hat{x})\hat{h}'(\hat{x}) = -\hat{Q} \tag{4.33}$$

Hence, the steady state has constant flux for all values of $\hat{x}$. Integrating once more yields

$$\frac{1}{2}\hat{h}^2 = -\hat{Q}\hat{x} + C \tag{4.34}$$

for some constant C. Hence, applying the boundary condition (4.28) we get

$$\hat{h} = \left(\hat{h}_0^2 - 2\hat{Q}\hat{x}\right)^{1/2} \tag{4.35}$$

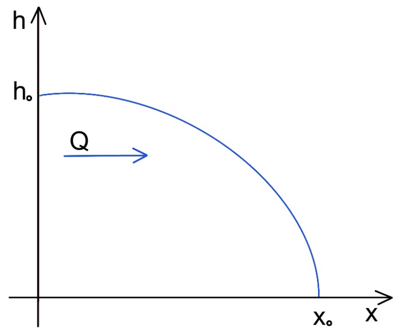
taking the positive square root since we require $\hat{h} > 0$. The above water table has $\hat{h} = h_0$ at $\hat{x}_0 = 0$ and $\hat{h} = 0$ at $\hat{x} = \frac{\hat{h}_0^2}{2\hat{Q}}$. This indicates that the water table can only exist for a finite interval before vanishing at a seepage face, for example.

The dimensional version of the water table is then:

$$h(x) = H\left(\frac{h_0^2}{H^2} - \frac{2Q}{\epsilon\kappa H}\frac{x}{L}\right)^{1/2} \tag{4.36}$$

$$= \left(h_0^2 - \frac{2Q}{\kappa}x\right)^{1/2} \tag{4.37}$$

We can see that the scalings H and L were not needed in the end since they are not present anywhere in the solution. In dimensional terms, the water table vanishes ($h = 0$) at $x = \frac{Kh_0^2}{2Q}$, which is illustrated below.

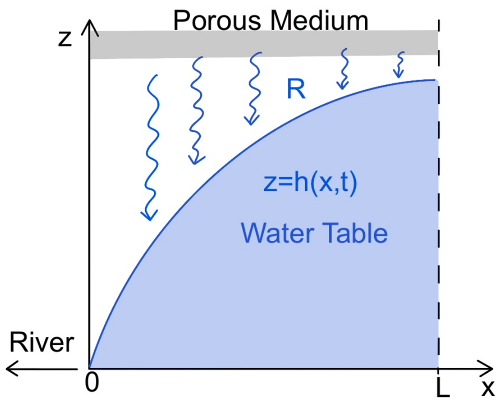


We also note that a negative slope in the water table corresponds with a positive flow direction, since

$$-\hat{h}\hat{h}_{\hat{x}} = \hat{Q} > 0 \tag{4.38}$$

4.3 Aquifer Response to Rainfall

An aquifer is a porous underground reservoir that contains groundwater. Let's consider how an aquifer responds to rainfall at a rate R [m/s], as illustrated below.



Here we will use the Dupuit approximation as before, assuming that the water table has a long-thin aspect ratio $h/L \ll 1$. However, we will now use a slightly modified kinematic boundary condition at $z = h(x,t)$ to account for rainfall reaching that water table according to

$$w = \phi h_t + u h_x - R : \quad z = h(x,t) \tag{4.39}$$

We non-dimensionalise using the same scalings as before, such that

$$\hat{w} = \hat{h}_{\hat{t}} + \hat{u}\hat{h}_{\hat{x}} - \frac{RL^2}{H^2 K} : \quad \hat{z} = \hat{h}(\hat{x},\hat{t}) \tag{4.40}$$

Unlike before, we now have an obvious natural length scale H for the water table height, set by a balance between rainfall and conductivity. As such we choose $H = \left(\frac{R}{K}\right)^{1/2} L$, giving the simplified kinematic condition

$$\hat{w} = \hat{h}_{\hat{t}} + \hat{u}\hat{h}_{\hat{x}} - 1 : \quad \hat{z} = \hat{h}(\hat{x},\hat{t}) \tag{4.41}$$

We integrate the continuity equation (4.10), using the Leibniz rule as before, but now this gives

$$\frac{\partial}{\partial x}\int_0^{\hat{h}} \hat{u}d\hat{z} - \hat{u}\hat{h}_{\hat{x}}|_{\hat{z}=\hat{h}} + [\hat{w}]_0^{\hat{h}} = 0 \tag{4.42}$$

$$\frac{\partial}{\partial x}\left(-\hat{h}\hat{h}_{\hat{x}}\right) - \hat{u}\hat{h}_{\hat{x}}|_{\hat{z}=\hat{h}} + \left(\hat{h}_{\hat{t}} + \hat{u}\hat{h}_{\hat{x}}\right)|_{\hat{z}=\hat{h}} - 1 = 0 \tag{4.43}$$

$$\hat{h}_{\hat{t}} = \left(\hat{h}\hat{h}_{\hat{x}}\right)_{\hat{x}} + 1 \tag{4.44}$$

Hence, (4.44) is the new governing equation for the water table and is accompanied by boundary conditions

$$\hat{h} = 0: \qquad \hat{x} = 0 \tag{4.45}$$

$$-\hat{h}\hat{h}_{\hat{x}} = 0: \qquad \hat{x} = 1 \tag{4.46}$$

The first of these corresponds with imposing atmospheric pressure at $\hat{x} = 0$, which is equivalent to an outflow into a river, for example. The second boundary condition imposes that there is no horizontal flux into the aquifer at $\hat{x} = 1$, which is equivalent with a flow divide at the edge of a drainage system, for example.

Next let's look for a similarity solution of the form

$$\hat{h}(\hat{x},\hat{t}) = \hat{t}^a f(\eta) \qquad \eta = \frac{\hat{x}}{\hat{t}^b} \tag{4.47}$$

Using the chain rule, we find each of the following derivative expressions

$$\hat{h}_{\hat{x}} = \hat{t}^{a-b} f'(\eta) \tag{4.48}$$

$$\left(\hat{h}\hat{h}_{\hat{x}}\right)_{\hat{x}} = \hat{t}^{2a-2b}\left(f(\eta)f'(\eta)\right)' \tag{4.49}$$

$$\hat{h}_{\hat{t}} = a\hat{t}^{a-1}f(\eta) - b\hat{t}^a f'(\eta)\frac{\hat{x}}{\hat{t}^{b+1}} \tag{4.50}$$

$$\hat{h}_{\hat{t}} = \hat{t}^{a-1}\left(af(\eta) - b\eta f'(\eta)\right) \tag{4.51}$$

Our governing PDE for $\hat{h}$ then becomes

$$\hat{t}^{a-1}\left(af - b\eta f'\right) = \hat{t}^{2a-2b}\left(ff'\right)' + 1 \tag{4.52}$$

We want this to be independent of time for a similarity solution to exist. Thus we impose

$$a - 1 = 2a - 2b = 0 \tag{4.53}$$

which has a unique solution $a = b = 1$. Hence our self-similar water table is of the form

$$\hat{h}(\hat{x},\hat{t}) = \hat{t} f\left(\frac{\hat{x}}{\hat{t}}\right) \tag{4.54}$$

and our ODE for f is

$$f - \eta f' = (ff') + 1 \tag{4.55}$$

The boundary conditions (4.45)-(4.46) then become

$$f(0) = 0 \tag{4.56}$$

$$-f(\hat{t})f'(\hat{t}) = 0 \tag{4.57}$$

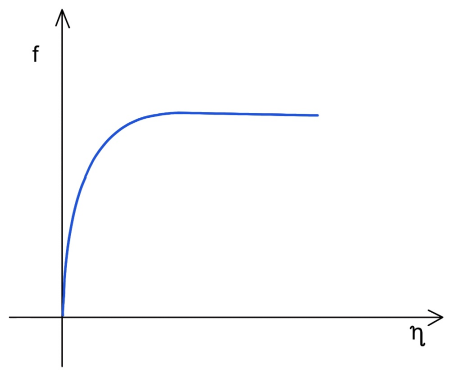
The final boundary condition is not compatible with self-similarity since it clearly depends on time explicitly. Hence, we restrict our attention to the late-time solution $\hat{t} \to \infty$, and replace this with the approximate boundary condition

$$-ff' \to 0: \quad \eta \to \infty \tag{4.58}$$

Alternatively, according to the governing equation for f (4.55), we see that (4.58) is equivalent to imposing that

$$f \to 1: \quad \eta \to \infty \tag{4.59}$$

The governing equation (4.55) with boundary conditions (4.56) and (4.59) must be solved numerically, using finite difference for example. A sketch is illustrated below.



Calculating the drainage flux into the river:

A useful output from this model is the drainage flux into the river, which is important for applications to flooding and water management. The flux into the river at $\hat{x} = 0$ is given in dimensionless terms as

$$\hat{Q}_0 = -\hat{h}\hat{h}_{\hat{x}} \qquad \text{at} \qquad \hat{x} = 0 \tag{4.60}$$

To evaluate this flux, we will consider the approximate behaviour of our similarity solution near $\eta \approx 0$. The terms on the left hand side of (4.55) correspond with the time derivative of the water table, and therefore must tend to zero near $\hat{x} \approx 0$ since the water table is $\hat{h} \approx 0$ there. Hence, the dominant behaviour of (4.55) near $\eta \approx 0$ is

$$(ff')' \approx 0 \tag{4.61}$$

After integrating this twice and imposing the boundary condition (4.56) we get

$$f^2 = C^2\eta \tag{4.62}$$

for some constant C, indicating that there is a square-root singularity near the origin. This is because we are trying to force a finite flow through an infinitely small water table near the outlet. Hence, the flux behaviour near the outlet must be of the form

$$f(0)f'(0) = \frac{1}{2}C^2 \tag{4.63}$$

Thus the flux into the river can be calculated as,

$$\hat{Q}_0 = -\hat{t}f(0)f'(0) = -\frac{C^2\hat{t}}{2} \tag{4.64}$$

The constant C can be found from the numerical solution, for example.

Drought period

Now let's suppose that at some time the rainfall stops, $R = 0$. The governing equations are then,

$$\hat{h}_{\hat{t}} = \left(\hat{h}\hat{h}_{\hat{x}}\right)_{\hat{x}} \tag{4.65}$$

$$\hat{h} = 0 \qquad : \hat{x} = 0 \tag{4.66}$$

$$-\hat{h}\hat{h}_{\hat{x}} = 0 \qquad : \hat{x} = 1 \tag{4.67}$$

$$\hat{h} = \hat{h}_0(\hat{x}) \qquad : \hat{t} = 0 \tag{4.68}$$

Let's look for a separable solution of the form

$$\hat{h} = X(\hat{x})T(\hat{t}) \tag{4.69}$$

Inserting this ansatz into the governing PDE (4.65), we get

$$XT' = \left(X'^2\right)T^2 + XX''T^2 \tag{4.70}$$

Then, dividing both sides by T^2X, we find an equation

$$\frac{T'}{T^2} = \frac{X'^2}{X} + X'' = B \tag{4.71}$$

whose left hand side depends only on time and whose right hand side depends only on space, and therefore must equal a constant B.

The time-dependent part of the solution must therefore satisfy the ODE

$$T' = BT^2 \tag{4.72}$$

Since we expect the water table to shrink over time (i.e. because the rainfall has stopped), we expect $T' < 0$ and hence this suggests, $B = -\lambda^2 < 0$ for some real constant λ. The above ODE can then be integrated to give

$$T(\hat{t}) = \frac{1}{\lambda^2\left(\hat{t} + A\right)} \tag{4.73}$$

where the constant A is a constant time scale that is determined by the initial conditions.

Now let's look at the X part of (4.71), which (after rearranging) is

$$X'^2 + XX'' + \lambda^2 X = 0 \tag{4.74}$$

The boundary conditions (4.45) and (4.46) indicate that

$$X(0) = 0 \tag{4.75}$$

$$X(1)X'(1) = 0 \tag{4.76}$$

We can eliminate λ by rescaling according to

$$\hat{X}(\hat{x}) = \lambda^2\tilde{X}(\hat{x}) \tag{4.77}$$

In this way, the governing equation and boundary conditions for $\tilde{X}$ are

$$\tilde{X}'^2 + \tilde{X}\tilde{X}'' + \tilde{X} = 0 \tag{4.78}$$
$$\tilde{X}(0) = 0 \tag{4.79}$$
$$\tilde{X}'(1) = 0 \tag{4.80}$$

where in the case of the ultimate boundary condition we have divided by a factor $\tilde{X}(1)$ since this is assumed non-zero. We can solve these equations numerically, using finite differences for example.

The solution is then given by:

$$\hat{h}(\hat{x},\hat{t}) = X(\hat{x})T(\hat{t}) \tag{4.81}$$
$$= \lambda^2 \tilde{X}(\tilde{x}) \frac{1}{\lambda^2 \left(\hat{t} + A\right)} \tag{4.82}$$
$$= \frac{\tilde{X}(\tilde{x})}{\hat{t} + A} \tag{4.83}$$

Hence, we see that the constant λ was not needed in the end since it does not appear in the solution. However, there is still one undetermined constant A in the above expression. This is determined by the initial conditions (4.32), which are assumed to be in the form:

$$\hat{h}(\hat{x}, 0) = \hat{h}_i(\hat{x}) = \frac{\tilde{X}(\tilde{x})}{A} \tag{4.84}$$

Hence, the constant A is set by the height of the water table initially.

Calculating the drainage flux into the river: Next, we shall calculate the drainage flux into the river for the drought scenario, as we did before. In this case, the form of the drainage flux is

$$\hat{Q}_0 = -\hat{h}\hat{h}_{\hat{x}} : \qquad \hat{x} = 0 \tag{4.85}$$
$$= -\frac{\tilde{X}(0)\tilde{X}'(0)}{\left(\hat{t} + A\right)^2} \tag{4.86}$$

Consider the ODE (4.78) near $\hat{x} \approx 0$. For similar reasons as before, the dominant behaviour of the ODE near the origin must be

$$\left(\tilde{X}\tilde{X}'\right)' \approx 0 \tag{4.87}$$

By integrating this twice and imposing the boundary condition (4.79) we get

$$\tilde{X}(0)\tilde{X}'(0) = \frac{1}{2} D^2 \tag{4.88}$$

for some constant D. Hence, the flux behaviour near the origin must be

$$\hat{Q}_0 = -\frac{D^2}{2\left(\hat{t} + A\right)^2} \tag{4.89}$$

The long-time behaviour of this is clearly

$$\hat{Q}_0 \approx -\frac{D^2}{2\hat{t}^2} : \quad \hat{t} \gg 1 \tag{4.90}$$

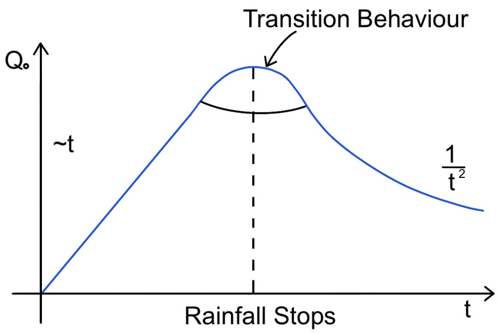
and the constant D can be calculated from the numerical solution, for example.

Now in summary, we have the following behaviours for the magnitude of the outflux into the river

$$\text{During rainfall:} \qquad |Q_0| \sim t \tag{4.91}$$

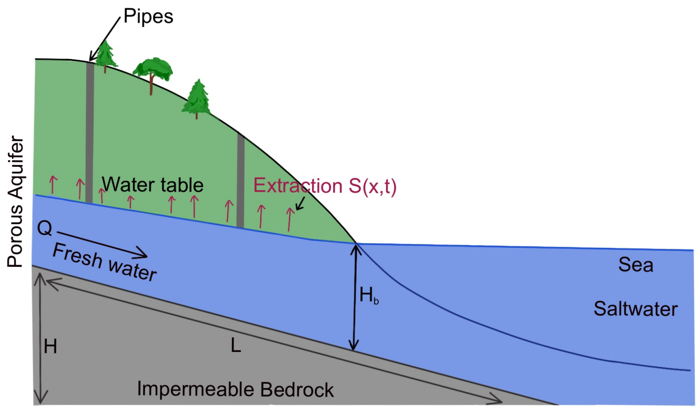
$$\text{During drought:} \qquad |Q_0| \sim \frac{1}{t^2} \tag{4.92}$$

which is illustrated in the sketch below. Around the time when the rainfall switches off, there will be some transition behaviour that we have ignored because it is difficult to model.



4.4 Coastal Groundwater Management

Many dry coastal regions get their drinking water from porous aquifers. If too much water is extracted, the seawater may intrude which corrupts the water supply.



How much can we extract without contaminating the aquifer?

The governing equations for the flow are the Darcy equations rotated by an angle α corresponding with the slope of the bedrock. Hence, these are

$$u_x + w_z = 0 \qquad \text{(Continuity)} \tag{4.93}$$

$$u = -\frac{k}{\mu}\left(p_x - \rho g \sin\alpha\right) \qquad \text{(Darcy } x\text{-component)} \tag{4.94}$$

$$w = -\frac{k}{\mu}\left(p_z + \rho g \cos\alpha\right) \qquad \text{(Darcy } z\text{-component)} \tag{4.95}$$

where $\tan\alpha = \frac{H}{L} = \epsilon$, and H and L are illustrated in the figure above.

If we consider a very small slope then

$$\tan\alpha \approx \alpha \approx \epsilon \tag{4.96}$$

$$\sin\alpha \approx \alpha \approx \epsilon \tag{4.97}$$

$$\cos\alpha \approx 1 \tag{4.98}$$

Next we non-dimensionalise the problem, using similar scalings as before

$$x = L\hat{x} \qquad z = H\hat{z} \qquad u = \epsilon K\hat{u} \qquad w = \epsilon^2 K\hat{w} \qquad p = p_a + \rho g H\hat{p} \tag{4.99}$$

If we take the limit $\epsilon \to 0$ the dimensionless governing equations become

$$\hat{u}_{\hat{x}} + \hat{w}_{\hat{z}} = 0 \tag{4.100}$$

$$\hat{u} = -\hat{p}_{\hat{x}} + 1 \tag{4.101}$$

$$0 = -\hat{p}_{\hat{z}} - 1 \tag{4.102}$$

We follow the same steps as in previous sections in order to find a governing equation for the water table at $\hat{z} = \hat{h}(\hat{x}, \hat{t})$, which is

$$\hat{h}_{\hat{t}} + \left(\hat{h}\left(1 - \hat{h}_{\hat{x}}\right)_{\hat{x}}\right) = -\hat{S} \tag{4.103}$$

where the term $-\hat{S}(\hat{x}, \hat{t})$ is much like the rainfall term in the previous problem, but here represents the extraction of drinking water from the water table. As before, this term originates from the kinematic boundary condition at the water table, such that

$$\hat{w} = \hat{h}_{\hat{t}} + \hat{u}\hat{h}_{\hat{x}} + \hat{S} \qquad : \hat{z} = \hat{h}(\hat{x}, \hat{t}) \tag{4.104}$$

where $\hat{S} = S(x, t)/\epsilon^2 K$ is assumed to vary with space and time.

The governing equation for the water table (4.103) is accompanied by boundary conditions of the form

$$\hat{h}\left(1 - \hat{h}_{\hat{x}}\right) = \hat{Q} : \quad \hat{x} = 0 \tag{4.105}$$

$$\hat{h} = \hat{H}_b : \quad \hat{x} = 1 \tag{4.106}$$

where the first condition imposes a seepage flux $\hat{Q} = \frac{Q}{\epsilon K H}$ from an upstream reservoir, and the second condition represents imposing atmospheric pressure at sea level, where $\hat{H}_b = \frac{H_b}{H}$ is the dimensionless height of the sea above the bedrock at the outlet.

After a long time, we reach a steady state $\hat{h}_{\hat{t}} \to 0$, such that the governing equation becomes

$$\left(\hat{h}\left(1 - \hat{h}_{\hat{x}}\right)\right)_{\hat{x}} = -\hat{S}(\hat{x}) \tag{4.107}$$

This can be integrated to give

$$\hat{h}\left(1 - \hat{h}_{\hat{x}}\right) = -\int_0^{\hat{x}} \hat{S}(X)dX + \hat{Q} \tag{4.108}$$

If there is no extraction we can integrate the above equations to derive an implicit solution for the water table of the form

$$\hat{h} - \hat{H}_b + \hat{Q}\log\left(\frac{\hat{h} - \hat{Q}}{\hat{H}_b - \hat{Q}}\right) = \hat{x} - 1. \tag{4.109}$$

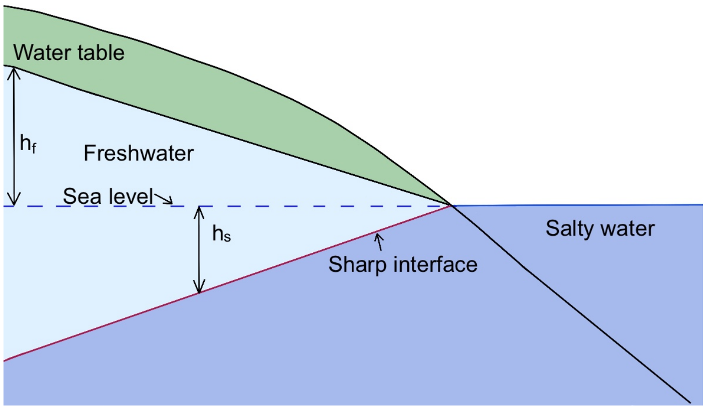
Since we want to avoid sea water intrusion, we will impose the constraint that the dimensionless flux at the outlet must be positive, such that

$$\hat{h}\left(1-\hat{h}_{\hat{x}}\right)|_{\hat{x}=1} > 0 \tag{4.110}$$

$$\implies \hat{Q} > \int_0^1 \hat{S}(X)\,dX \tag{4.111}$$

This shows we need to extract a total amount of water that is less than the influx from the reservoir $\hat{Q}$ (if we want to avoid seawater intrusion in the steady state).

So far we have only modelled the position of the freshwater table in the reservoir. In addition we may also need to model the interface between the freshwater and the seawater, as illustrated below. A simple model for seawater intrusion assumes that the interface between freshwater and saltwater is 'sharp', i.e. there is no mixing between the two bodies of water.



In this case we can use an Archimedes balance in the steady state to determine the position of the interface between freshwater and saltwater. To do so, we let $z = h_f(x)$ represent the height of the fresh water table above sea level, and $z = -h_s(x)$ represent the position of the saltwater interface below sea level. By equating the hydrostatic pressure within each body of water at the interface, we find

$$\rho_f\left(h_f + h_s\right) = \rho_s h_s \tag{4.112}$$

$$\implies h_s = h_f\left(\frac{\rho_f}{\rho_s - \rho_f}\right) \tag{4.113}$$

This Archimedes balance is known as the "Ghyben-Herzberg Relationship", and is only valid for steady state water tables. Inserting real parameter values, we find $\frac{\rho_f}{\rho_s-\rho_f} \approx 40$, which indicates that for every 1 m of freshwater above sea level, there is approximately 40 m of saltwater beneath sea level. This allows groundwater engineers to manage coastal aquifers and prevent seawater intrusion.

In reality, saltwater tends to mix with freshwater, so we don't have a sharp interface between these two regions. Instead, we could model the concentration of salt in the aquifer with an advection-diffusion equation of the form

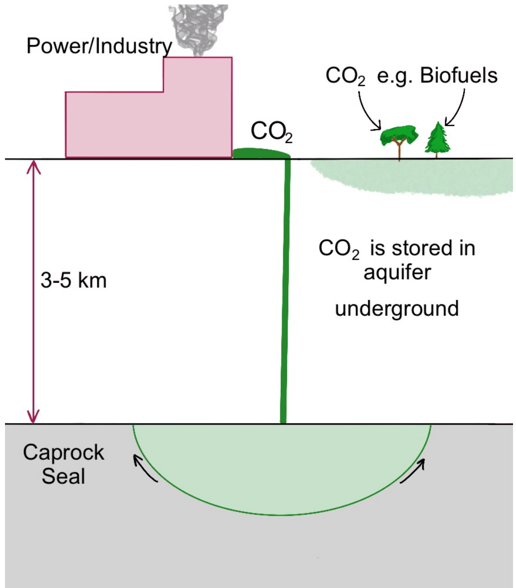
$$C_t + (\mathbf{u}\cdot\nabla)C = D_0\nabla^2 C \tag{4.114}$$

where D_0 [m^2/s] is the molecular diffusivity.

4.5 CO_2 Sequestration

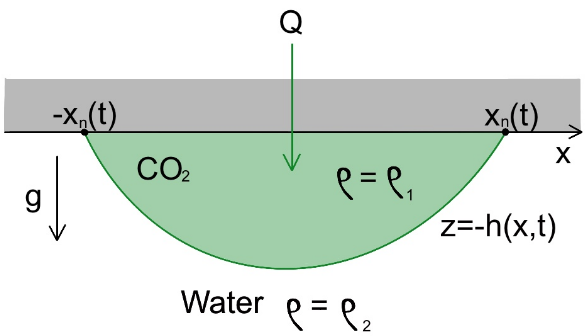
The overproduction of carbon dioxide emissions is one of biggest challenges facing humankind over the next century. As outlined in the Paris Agreement (2015), it is necessary to limit global warming to less than 2 degrees C by the year 2100 to avoid the most dangerous consequences of climate change. To meet these temperature targets it is imperative to reduce our CO_2 emissions quickly, and by as much as possible.

One of the few proposed technological solutions to this problem is carbon capture and storage (CCS) - that is, capturing CO_2 at source (e.g. power plants and factories) and injecting it into porous geological aquifers to be sequestered (stored), either by dissolution or trapping in the rock pores and boundaries. One of the most famous case studies of CO_2 storage is the ongoing Sleipner Project in the Norwegian North Sea, storing 1MT of CO_2 each year since 1996.



In many cases, CO_2 is injected into porous geological aquifers that are saturated with salty water. The CO_2 is typically injected several km beneath the ground beneath a 'caprock seal' - a layer of impermeable rock which acts as a ceiling, preventing the buoyant CO_2 from rising to the surface again.

In this section we will model the migration of CO_2 beneath the caprock as a buoyant fluid (in fact it is in a supercritical state i.e. indistinguishable between a gas or a liquid, due to high pressures). We will assume the CO_2 current is two-dimensional (for simplicity) and supplied by a constant injection rate Q [m^2/s]. As such, we will derive a governing PDE for the CO_2 thickness $h(x,t)$ beneath the caprock, as illustrated below.



The density of CO_2, which we denote ρ_1, is less than that of water, which we denote ρ_2. Hence, the density difference is

$$\Delta\rho = \rho_2 - \rho_1 > 0 \tag{4.115}$$

We model the flow using Darcy equations in a long-thin limit, as before. The only difference is that we now have two different densities. The z-component of Darcy's equations in the shallow limit within the CO_2 is

$$0 = -p_z - \rho_1 g \tag{4.116}$$

whilst the dynamic boundary conditon at the CO_2-water interface is

$$p = p_0 + \rho_2 g h : \quad z = -h(x,t) \tag{4.117}$$

where p_0 is the reference pressure of the brine at the caprock, i.e. at $z = 0$. By integrating the z-momentum equation (4.116) and applying (4.117) we find an expression for the pressure in the CO2:

$$p = -\rho_1 g\,(z + h) + \rho_2 g h + p_0 \tag{4.118}$$

Next we can insert this into the x-component of Darcy's equations, which is

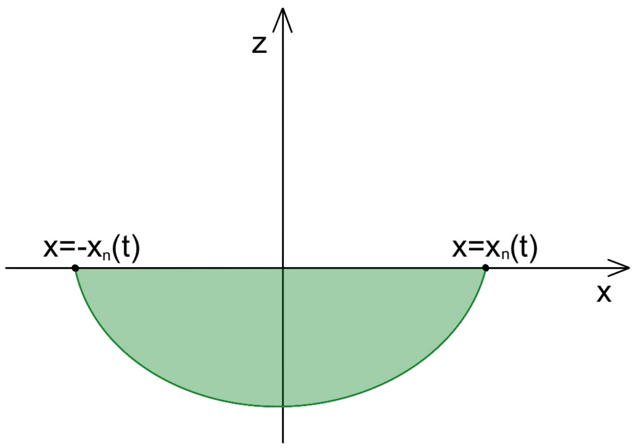
$$u = -\frac{k}{\mu} p_x = -K h_x \tag{4.119}$$

where $K = \frac{k\Delta\rho g}{\mu}$ is the hydraulic conductivity. As per usual, we acquire the governing equation for $h(x,t)$ by integrating the continuity equation to obtain

$$\phi \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \int_0^h u\,dz = 0 \tag{4.120}$$

Inserting (4.119) into the above, we obtain the governing PDE for the CO_2 thickness:

$$\phi h_t = K\,(h h_x)_x \tag{4.121}$$



Next we consider the boundary conditions for $h(x,t)$, which extends between $x = -x_n(t)$ and $x = x_n(t)$ and is assumed to be symmetric about the middle $x = 0$. Hence, for simplicity, we restrict our attention to half of the current $0 \le x \le x_n(t)$. At the origin we impose a constant input flux of CO_2, such that

$$-K h h_x = Q : \qquad x = 0 \tag{4.122}$$

Meanwhile at the edge of the current, we impose that the current thickness is zero and that there is no added flux of CO_2, such that

$$h = 0 : \qquad x = x_n(t) \tag{4.123}$$

$$-K h h_x = 0 : \qquad x = x_n(t) \tag{4.124}$$

Having described our governing equation and boundary conditions, next we will non-dimensionalise. If we look at our dimensional parameters,

$$Q \quad [\mathrm{m}^2/\mathrm{s}] \qquad\qquad K \quad [\mathrm{m/s}] \tag{4.125}$$

we can see that a length scale is formed by $L = \frac{Q}{K}$ [m]. So we choose the following scalings for our variables

$$x = L\hat{x} \qquad h = L\hat{h} \qquad t = \frac{\phi L}{K}\hat{t} \tag{4.126}$$

Thus our governing equation becomes

$$\hat{h}_{\hat{t}} = \left(\hat{h}\hat{h}_{\hat{x}}\right)_{\hat{x}} \tag{4.127}$$

whilst the boundary conditions become

$$-\hat{h}\hat{h}_{\hat{x}} = 1 : \quad \hat{x} = 0 \tag{4.128}$$

$$\hat{h} = 0 : \quad \hat{x} = \hat{x}_n(\hat{t}) \tag{4.129}$$

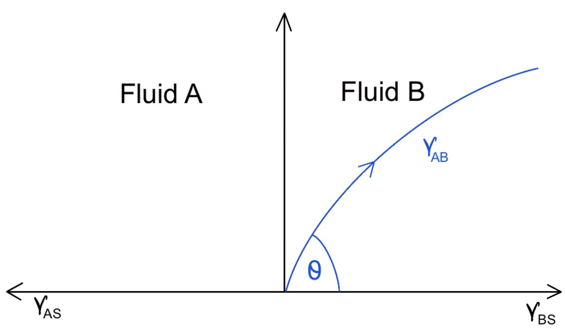
$$-\hat{h}\hat{h}_{\hat{x}} = 0 : \quad \hat{x} = \hat{x}_n(\hat{t}) \tag{4.130}$$

In practice problems 4, we will show that it is possible to find similarity solutions of the form $\hat{h} = \hat{t}^a f\left(\frac{\hat{x}}{\hat{t}^b}\right)$ for the above equation and boundary conditions.

4.6 Two-Phase Flow in Porous Media

In the problems we have looked at so far, we have assumed that the porous medium is fully saturated. For example, in the groundwater problems we assumed that below the water table the soil contained water only, and was devoid of air. Likewise, in the above model of a CO_2 plume, we assumed the plume region was composed purely of CO_2 and the aquifer was otherwise purely saturated with brine.

A more realistic model accounts for the fact that two phases may exist in a given region at a given time. This is true even if the two fluid phases are immiscible - that is to say that they do not mix together. Examples of immiscible fluid pairs are CO_2 and water, or water and air. When two immiscible phases occupy a porous medium, we must account for the junctions that occur between fluid and solid boundaries. Consider the diagram below, in which two immiscible fluid phases (A and B) are in contact with a solid phase beneath, e.g. CO_2, water, and sandstone.



The interface between fluid A and B makes an angle θ with the solid surface. This is known as the contact angle, which is defined for a given combination of phases. There are some forces per unit length γ [N/m], known as surface tension, acting on each of the 3 interfaces at the junction. We denote these γ_{AS} (between fluid A and the solid), γ_{BS} (between fluid B and the solid), and γ_{AB} (between fluid A and fluid B). We assume that each of these surface tension parameters is a constant and obeys the following trigonometric force balance

$$\gamma_{AB} \cos\theta = \gamma_{AS} - \gamma_{BS} \tag{4.131}$$

which is known as Young's law. If each of the surface tensions γ is known, then Young's law tells us the contact angle θ. For two-phase flow (e.g. CO_2 and water), is conventional to define a 'wetting' phase and a 'non-wetting' phase in terms of the contact angle. Specifically, if

$$0 < \theta < \frac{\pi}{2} \tag{4.132}$$

$$B \text{ is wetting} \, (w) \tag{4.133}$$

$$A \text{ is non wetting} \, (n) \tag{4.134}$$

whilst if

$$\frac{\pi}{2} < \theta < \pi \tag{4.135}$$

$$B \text{ is non wetting} \, (n) \tag{4.136}$$

$$A \text{ is wetting} \, (w) \tag{4.137}$$

It is assumed that the pore space is occupied entirely by a combination of (w) and (n) phases. Hence, we use notation S_i (dimensionless) to define the fraction of pore space occupied by phase i, also known as the 'saturation'. Hence, the sum of saturations satisfies

$$S_n + S_w = 1 \tag{4.138}$$

A key feature of two-phase flow in porous media is that *the flow of either phase depends on the saturation of that phase.* A good example of this is trying to pour water onto dry soil - the water usually runs off the surface of the soil if it is too dry since it cannot penetrate through. By contrast, water poured onto wet soil enters easily. Hence, we need to introduce the concept of 'relative permeability' for each phase, k_{rw} and k_{rn}. Then, we model the permeability as $k = k_0 k_{ri}(S_i)$ for phase i, where k_{ri} is a dimensionless function of the saturation. If the porous medium is fully saturated with phase i (i.e. $S_i = 1$) then $k_{ri} = 1$, and hence we retrieve the regular expression for permeability in single-phase flow in porous media, $k = k_0$.

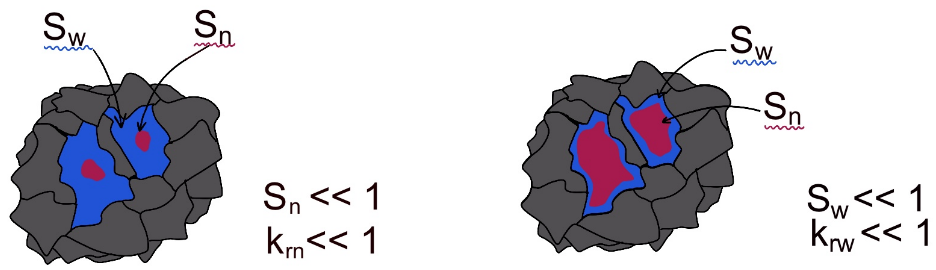
The diagram below illustrates two extreme cases of high and low saturations. The plot shows typical behaviour of the relative permeabilities k_{rw} and k_{rn} as a function of non-wetting saturation S_n. These are typically non-linear, monotone functions in the range [0,1].

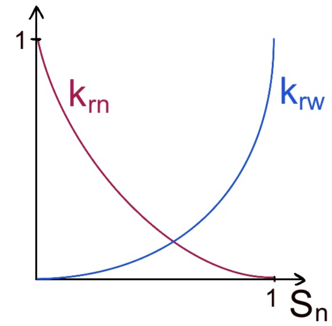
Another consequence of surface tension is that *the pressures of each phase are not equal.* This is because some energy is stored in the curvature of the interfaces between fluid phases. A simple example of this is shown below, with two fluids occupying a 'capillary tube', e.g. as seen when taking a blood sample. The interface between the two fluid phases is curved, due to the contact angle at the solid walls. This results in a pressure difference

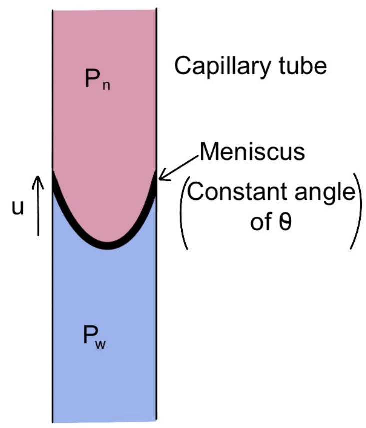
$$p_n - p_w = \gamma\kappa \tag{4.139}$$

where p_n and p_w are the pressures of each phase and κ [1/m] is the curvature of the interface. Because there is a difference in pressure, this usually drives a flow of liquid up the tube at speed u. In medicine, this technique is used to sample blood, by placing a thin tube on a drop of blood and using surface tension to drive it upwards.

In two-phase flow in porous medium, there are millions or billions of such junctions located in all the pores of



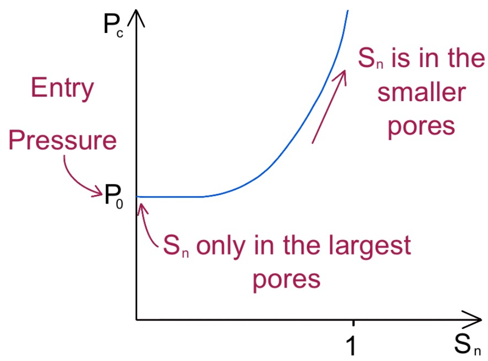




the solid. Hence, we represent the macroscopic effect of surface tension using the following relationship

$$p_n - p_w = p_c\left(S_n\right) \tag{4.140}$$

where p_c is known as the capillary pressure. Like relative permeability, the capillary pressure is also a function of the phase saturation. This is typically monotone increasing from a value p_0 towards infinity, as illustrated in the figure below.

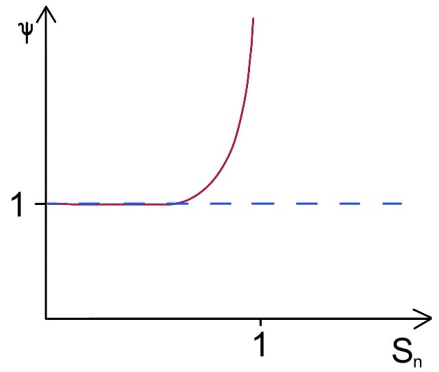


The pressure when there is no non-wetting phase is known as the 'pore entry pressure' and represents the minimum pressure required to get any non-wetting phase into the largest pores. As the pressure difference increases towards infinity, the non-wetting phase can invade smaller and smaller pores, such that S_n tends towards 1.

We represent the capillary pressure by the following relationship

$$p_c\left(S_n\right) = p_0 \psi\left(S_n\right) \tag{4.141}$$

where $\psi(S_n)$ is a dimensionless function.



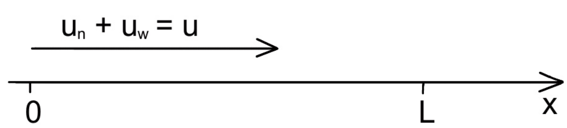
The governing equations for two-phase flow in porous media are known as the two-phase Darcy equations, which are given by

$$\phi \frac{\partial S_n}{\partial t} + \nabla \cdot \mathbf{u}_n = 0 \qquad \text{(Conservation of Mass } (n)) \tag{4.142}$$
$$\phi \frac{\partial S_w}{\partial t} + \nabla \cdot \mathbf{u}_w = 0 \qquad \text{(Conservation of Mass } (w)) \tag{4.143}$$
$$\mathbf{u}_n = -\frac{k_0 k_{rn}(S_n)}{\mu_n}\left(\nabla p_n + \rho_n g \mathbf{k}\right) \qquad \text{(Darcy Momentum } (n)) \tag{4.144}$$
$$\mathbf{u}_w = -\frac{k_0 k_{rw}(S_n)}{\mu_w}\left(\nabla p_w + \rho_w g \mathbf{k}\right) \qquad \text{(Darcy Momentum } (w)) \tag{4.145}$$
$$S_n + S_w = 1 \qquad \text{(Pore Space Occupied)} \tag{4.146}$$
$$p_n - p_w = p_0 \psi(S_n) \qquad \text{(Capillary Pressure)} \tag{4.147}$$

We note that:

- The last two equations allow us to eliminate either S_n and p_n (and work with wetting phase only), or S_w and p_w (and work with non-wetting phase only) as required.
- Each phase has different properties, such as density (ρ_w, ρ_n) and viscosity (μ_w, μ_n).
- The permeability and porosity of the medium are k_0 and ϕ, which are assumed to be constants.
- It is convention to write the relative permeabilities and the capillary pressure as functions of S_n without loss of generality since $S_n + S_w = 1$.

4.7 Buckley-Leverett Flow



The most famous example of two-phase flow in porous media is known as Buckley-Leverett flow. This problem concerns the displacement of one fluid by another in a long thin aquifer, for example CO_2 (non-wetting) injected into a porous aquifer saturated with water (wetting). Since the flow is long and thin, we will treat it as one-dimensional. Hence, we take our domain to be $x \in [0, L]$, where L is the length of the aquifer (see diagram above). The velocity vectors reduce to

$$\mathbf{u}_n = u_n(x,t)\hat{\boldsymbol{\imath}} \qquad\qquad \mathbf{u}_w = u_w(x,t)\hat{\boldsymbol{\imath}} \tag{4.148}$$

The conservation of mass equations (4.142)-(4.143) in one dimension become

$$\phi\frac{\partial S_n}{\partial t} + \frac{\partial u_n}{\partial x} = 0 \tag{4.149}$$

$$\phi\frac{\partial S_w}{\partial t} + \frac{\partial u_w}{\partial x} = 0 \tag{4.150}$$

Now if we add equations (4.149) and (4.150), using (4.146), we find

$$\phi\frac{\partial}{\partial t}\left(S_n + 1 - S_n\right) + \frac{\partial}{\partial x}\left(u_n + u_w\right) = 0 \tag{4.151}$$

Hence, we conclude that

$$u_n + u_w = f(t) \tag{4.152}$$

for some function of time $f(t)$. We impose the boundary condition,

$$u_n + u_w = u \qquad\qquad : x = 0 \tag{4.153}$$

which corresponds with constant input flux of CO_2 at the origin. Hence, due to (4.152)-(4.153), we find that

$$u_n + u_w = u \tag{4.154}$$

which applies for all time. Meanwhile Darcy's equations in one dimension give

$$u_n = -\frac{k_0 k_{rn}(S_n)}{\mu_n}\frac{\partial p_n}{\partial x} \tag{4.155}$$

$$u_w = -\frac{k_0 k_{rw}(S_n)}{\mu_w}\frac{\partial p_w}{\partial x} \tag{4.156}$$

However, we can eliminate the wetting pressure according to

$$p_w = p_n - p_0\psi(S_n) \tag{4.157}$$

Thus, we find

$$u_n + u_w = -\frac{k_0 k_{rn}(S_n)}{\mu_n}\frac{\partial p_n}{\partial x} - \frac{k_0 k_{rw}(S_n)}{\mu_w}\left(\frac{\partial p_n}{\partial x} - p_0\frac{\partial \psi}{\partial x}\right) \tag{4.158}$$

$$= -\frac{k_0 k_{rn}(S_n)}{\mu_n}\frac{\partial p_n}{\partial x} - \frac{k_0 k_{rw}(S_n)}{\mu_w}\left(\frac{\partial p_n}{\partial x} - p_0\psi'\left(S_n\right)\frac{\partial S_n}{\partial x}\right) = u \tag{4.159}$$

where we have used the chain rule in the last step. Next, we collect the $\frac{\partial p_n}{\partial x}$ terms together and rearrange, finding

$$\frac{\partial p_n}{\partial x} = -\frac{\frac{\mu_w u}{k_0}}{M k_{rn} + k_{rw}} + \frac{k_{rw} p_0}{M k_{rn} + k_{rw}}\psi'\frac{\partial S_n}{\partial x} \tag{4.160}$$

where $M = \frac{\mu_w}{\mu_n}$ is the viscosity ratio. Next going back to equation (4.149) and subbing in (4.155), we find

$$\phi \frac{\partial S_n}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{k_0 k_{rn}}{\mu_n}\frac{\partial p_n}{\partial x}\right) = 0 \tag{4.161}$$

Hence, inserting our expression for the pressure gradient (4.160), we find

$$\phi \frac{\partial S_n}{\partial t} + \frac{\partial}{\partial x}\left(u \frac{M k_{rn}}{M k_{rn} + k_{rw}}\right) - \frac{\partial}{\partial x}\left(\frac{k_0 p_0}{\mu_w}\frac{M k_{rn} k_{rw}}{M k_{rn} + k_{rw}}\psi' \frac{\partial S_n}{\partial x}\right) = 0 \tag{4.162}$$

We will now write this in the form

$$\phi \frac{\partial S_n}{\partial t} + \frac{\partial}{\partial x}\left(J(S_n)\right) = \frac{\partial}{\partial x}\left(D(S_n)\frac{\partial S_n}{\partial x}\right) = 0 \tag{4.163}$$

where we define the following functions

$$J(S_n) = u \frac{M k_n}{M k_n + k_{rw}} \tag{4.164}$$

$$D(S_n) = \frac{k_0 p_0}{\mu_w}\frac{M k_n k_{rw}}{M k_n + k_w}\psi'(S_n) \tag{4.165}$$

The first of these is called the 'flow rate fraction' and represents the fraction of non-wetting flow to the total flow, such that $J \in [0, 1]$. The second of these is a nonlinear diffusion coefficient which represents how capillary pressure causes the saturation to get spread out. It is also useful to define the advection velocity

$$V(S_n) = J'(S_n) \tag{4.166}$$

such that the governing equation for S_n can be written as

$$\phi \frac{\partial S_n}{\partial t} + V(S_n)\frac{\partial S_n}{\partial x} = \frac{\partial}{\partial x}\left(D(S_n)\frac{\partial S_n}{\partial x}\right) \tag{4.167}$$

This is an advection-diffusion equation, known as the Buckley-Leverett equation. We have seen similar such equations before in this course, but now we have nonlinear advection and diffusion coefficients V and D that depend on S_n.

To solve this, we first non dimensionalise according to the scalings

$$x = L\hat{x} \qquad t = \tau \hat{t} \qquad V = u\hat{V} \qquad D = \frac{k_0 p_0}{\mu_w}\hat{D} \tag{4.168}$$

The governing equations then are:

$$\frac{\phi}{\tau}\frac{\partial S_n}{\partial \hat{t}} + \frac{u}{L}\hat{V}(S_n)\frac{\partial S_n}{\partial \hat{x}} = \frac{k_0 p_0}{\mu_w L^2}\frac{\partial}{\partial \hat{x}}\left(\hat{D}(S_n)\frac{\partial S_n}{\partial \hat{x}}\right) \tag{4.169}$$

If we choose an advective timescale $\tau = \frac{\phi L}{u}$, then we get

$$\frac{\partial S_n}{\partial \hat{t}} + \hat{V}(S_n)\frac{\partial S_n}{\partial \hat{x}} = \frac{1}{\mathrm{Pe}}\frac{\partial}{\partial \hat{x}}\left(\hat{D}(S_n)\frac{\partial S_n}{\partial \hat{x}}\right) \tag{4.170}$$

Here we define the Peclet number Pe as

$$\mathrm{Pe} = \frac{uL}{\frac{k_0 p_0}{\mu_w}} = \frac{uL}{\alpha}, \qquad \alpha = \frac{k_0 p_0}{\mu_w} \tag{4.171}$$

where we have introduced a diffusivity α [m^2/s] for comparison with previous advection-diffusion problems.

For typical parameter values, for CO_2 storage we find that $\mathrm{Pe} \gg 1$. Hence, advection typically dominates over diffusion. We can therefore approximate equation (4.170) as

$$\frac{\partial S_n}{\partial \hat{t}} + \hat{V}(S_n)\frac{\partial S_n}{\partial \hat{x}} = 0 \tag{4.172}$$

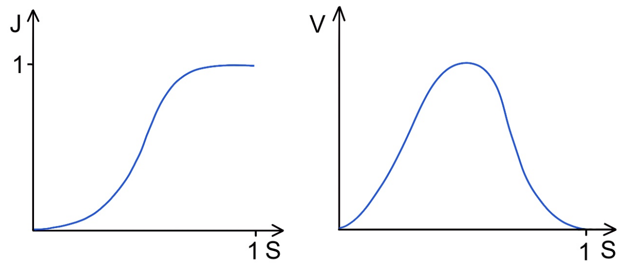
which is a nonlinear transport equation for S_n which we can solve using the method of characteristics. In order to progress, we first need definitions for k_{rn}, k_{rw} in our expressions for J and V in (4.164) and (4.166). A common model for k_w and k_n is the Corey model, which is given by

$$k_n = S_n^2 \qquad\qquad k_w = (1 - S_n)^2 \tag{4.173}$$

Under the Corey model, the dimensionless flow rate fraction and advection velocity are given by

$$\hat{J}(S_n) = \frac{Mk_{rn}}{Mk_{rn} + k_{rw}}, \quad \hat{V} = \frac{d}{dS_n}\left(\frac{Mk_{rn}}{Mk_{rn} + k_{rw}}\right) \tag{4.174}$$

These functions are illustrated below for a moderate value of M.



Henceforth, we simplify our notation to use S for the non-wetting saturation S_n. Our governing dimensionless PDE is then

$$\hat{S}_{\hat{t}} + \hat{V}(S)\hat{S}_{\hat{x}} = 0 \tag{4.175}$$

To solve this we will use the characteristic equations, with characteristic parameter τ. The characteristic equations are

$$\frac{d\hat{t}}{d\tau} = 1 \qquad\qquad \frac{d\hat{x}}{d\tau} = \hat{V}(S) \qquad\qquad \frac{dS}{d\tau} = 0 \tag{4.176}$$

which can be integrated to give

$$\hat{t} = \tau + C_1 \quad \hat{x} = \hat{V}(S)\tau + C_2, \quad S = C_3 \tag{4.177}$$

where C_1-C_3 are constants of integration. Hence, the characteristic projections are straight lines of the form

$$\hat{x} = \hat{V}(S)\hat{t} + \text{const} \tag{4.178}$$

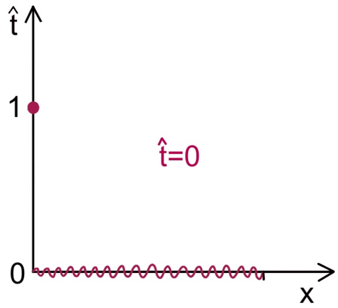
where the slope of these lines is $\hat{V}(S)$.

We shall assume initial and boundary conditions of the form

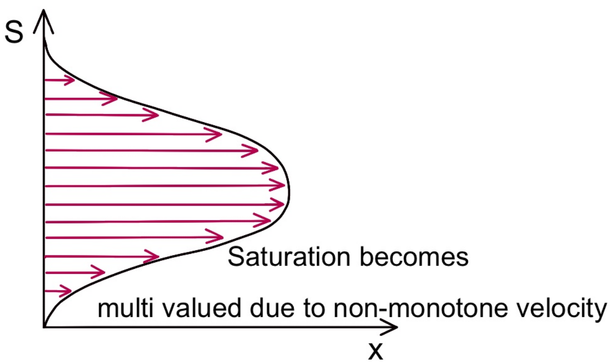
$$S(0, \hat{t}) = 1 \qquad\qquad : \hat{t} \geq 0 \tag{4.179}$$

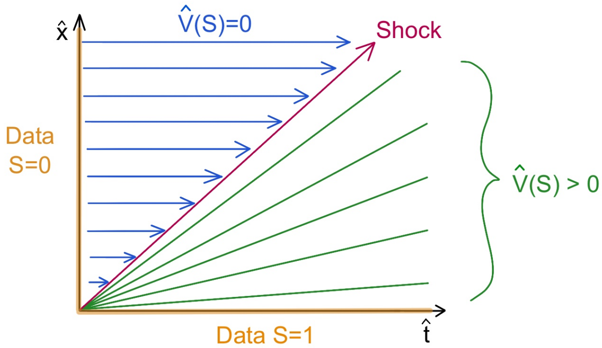
$$S(\hat{x}, 0) = 0 \qquad\qquad : \hat{x} > 0 \tag{4.180}$$

The first imposes that we are inputting CO_2 at the origin, whereas the second imposes that the aquifer is initially saturated with water.



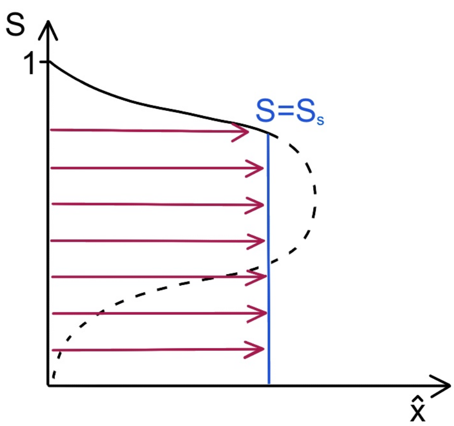
Equation (4.177) tells us that saturation is constant along characteristics. This means that the saturation profile at $\hat{x} = 0$ remains constant and is advected down the aquifer at speed $\hat{V}(S)$. However, as shown in the plot of $\hat{V}(S)$ above, we see that the advection velocity is a non-monotone function of saturation with a unique maximum in between $S = 0$ and $S = 1$. This means that intermediate values of S travel the fastest. Hence, assuming that S is continuous for $\hat{t} > 0$, the initial conditions (4.179)-(4.180) indicate that S must vary between 1 and 0 near the origin shortly after the initial time. Thereafter, saturation gets advected down the aquifer with the non-monotone velocity $\hat{V}(S)$, as depicted below.





However, there is a problem with this because the non-monotone velocity results in a multi-valued solution for the saturation which does not make sense. Hence, we must introduce a discontinuity called a 'shock' at some intermediate saturation value S_s. This shock corresponds with a crossing of the characteristics in the $\hat{x} - \hat{t}$ plane, as shown above. The shock develops as the characteristics originating at $\hat{t} = 0$, $\hat{x} > 0$ (with velocity $\hat{V}(0) = 0$) collide with the characteristics originating at $\hat{x} = 0$, $\hat{t} > 0$ (with velocity in the range $\hat{V}(S) \geq 0$ for $S \in [0, 1]$).

The corrected saturation profile with a discontinuous shock at intermediate saturation value S_s is shown below.



We can ask, at what value S_s does the shock occur? The characteristic at the shock value travels along a line $\hat{x} = \hat{V}(S_s)\hat{t}$, which is therefore the dividing characteristic shown in the plot above.

To find the shock behaviour, we need to use the "Rankine-Hugoniot conditions" from the theory of discontinuous PDE's (see appendix). For a PDE of the form

$$P_t + Q_x = 0 \tag{4.181}$$

the following condition must be satisfied across a shock:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{[Q]_-^+}{[P]_-^+} \tag{4.182}$$

which is known as the Rankine-Hugoniot condition for this PDE. This condition corresponds with imposing mass conservation across a discontinuity in the solution.

In our case our PDE for S has $P = S$ and $Q = J$, whilst the shock speed satisfies $\frac{\mathrm{d}\hat{x}}{\mathrm{d}\hat{t}} = \hat{V}(S_s)$. Hence, the Rankine-Hugoniot condition is

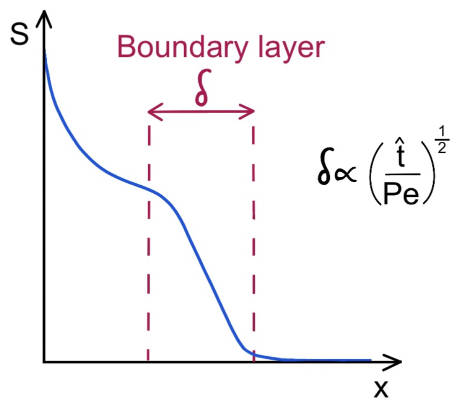
$$\hat{V}(S_s) = \frac{J(S_s) - J(0)}{S_s - 0} \tag{4.183}$$

which after rearranging gives

$$S_s \hat{V}(S_s) - J(S_s) = 0 \tag{4.184}$$

which is a nonlinear equation for S_s which must be solved with a numerical root-finder in general. As an exercise you can show that $S_s = (1+M)^{-1/2}$ in the case of the Corey Model.

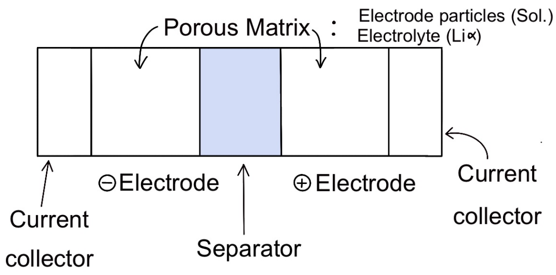
Note that near the shock there are potentially very large gradients in the saturation. Hence, our assumption of neglecting the diffusive term in (4.167) must be revisited (since it contains second derivatives of S_n). Hence, the role of the diffusion becomes important again near the shock. The shock gets smoothed over a diffusive boundary layer of width δ, which is a small length scales that grows in time. Whilst this is outside the scope of this course, an illustration is shown below.



Batteries

5.1 Basic Idea

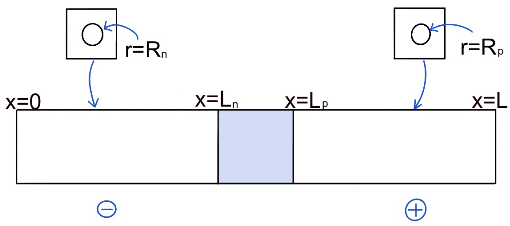
Sketch of a lithium ion battery.



- We consider the fundamental operation of a lithium ion battery (other types e.g. sodium are similar)
- Each electrode consists of a porous matrix of electrode particles (solid) and an electrolyte (liquid) containing dissolved lithium ions.
- As the battery charges/discharges lithium ions leave particles in one electrode and transport to particles in the other electrode.
- This transport of charge between the electrodes creates a potential difference (voltage) across the battery.
- There are many models of batteries, but we choose the simplest one called SPM (Single Particle Model).
- This chapter largely follows the paper by Xie, Y. and Cheng, X., 2021. *A new solution to the spherical particle surface concentration of lithium-ion battery electrodes*. Electrochimica Acta, 399, p.139391.

5.2 Single Particle Model of a Battery

We will solve for a representative particle in each electrode which de-lithiates uniformly.



The concentration C of lithium (Li) ions in the spherical electrode particle only changes due to molecular diffusion. Hence, the concentration satisfies the following (spherically symmetric) diffusion equation and boundary conditions

$$\frac{\partial C_k}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_k(C_k)\frac{\partial C_k}{\partial r}\right) \tag{5.1}$$

$$\frac{\partial C_k}{\partial r} = 0 \qquad : r = 0 \tag{5.2}$$

$$-D_k(C_k)\frac{\partial C_k}{\partial r} = j_k(t) \qquad : r = R_k \tag{5.3}$$

$$C_k = C_{k0} \qquad : t = 0 \tag{5.4}$$

Here the subscript k is either n (negative electrode) or p (positive electrode). The boundary conditons (5.2) and (5.3) indicate that the flux of lithium ions is zero at the centre and set by a surface flux density $j_k(t)$ at the particle surface.

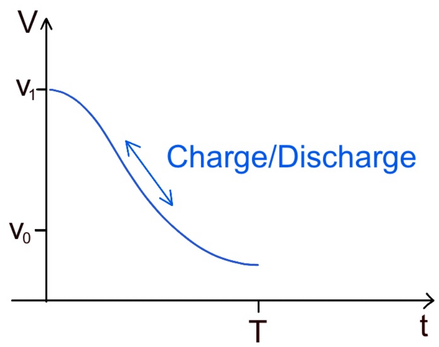
We have that,

- C_k is dimensionless
- $D_k(C_k)$ is the non-constant diffusivity of lithium ions [m^2/s]
- $j_k(t)$ is the surface flux density [m/s]

Key Idea

The voltage of a battery (as a function of time) depends on how the electrode particles de-lithiate.

$$V(t) = f\left(C_n(R_n, t), C_p(R_p, t)\right) \tag{5.5}$$



Hence, we require a model for $C_k(r,t): \quad k = n, p$, which is given by the above PDE.

Example

Consider the case where $D_n = D_p = D$ (constant), $j_n = j_p = j_0$ (constant), $R_n = R_p = R$, and $C_{n0} = C_{p0} = 0$. In this case our equations become

$$C_t = \frac{D}{r^2}\left(r^2 C_r\right)_r \tag{5.6}$$

$$C_r = 0 \qquad : r = 0 \tag{5.7}$$

$$-DC_r = j_0 \qquad : r = R \tag{5.8}$$

$$C = 0 \qquad : t = 0 \tag{5.9}$$

We will solve this using a Laplace transform by defining

$$L\{C\} = \mathcal{C}(r,s) = \int_0^\infty C(r,t)e^{-st}dt \tag{5.10}$$

Given the definition of the Laplace transform, we note some immediately useful identities (obtained via integration by parts):

$$L\{C_t\} = -C(r,0) + s\,\mathcal{C}(r,s) \tag{5.11}$$

$$L\{1\} = \frac{1}{s} \tag{5.12}$$

Hence our governing equation and boundary conditions become

$$s\mathcal{C} = D\mathcal{C}_{rr} + \frac{2D}{r}\mathcal{C}_r \tag{5.13}$$

$$\mathcal{C}_r = 0 \qquad\qquad : r = 0 \tag{5.14}$$

$$-D\mathcal{C}_r = \frac{j_0}{s} \qquad\qquad : r = R \tag{5.15}$$

Now we define a new variable $\tilde{C} = r\mathcal{C}$, such that our governing equation becomes

$$\tilde{C}_{rr} - \frac{s}{D}\tilde{C} = 0 \tag{5.16}$$

This can be integrated directly, giving the general solution

$$\mathcal{C}(r,s) = \frac{A}{r}e^{\sqrt{\frac{s}{D}}r} + \frac{B}{r}e^{-\sqrt{\frac{s}{D}}r} \tag{5.17}$$

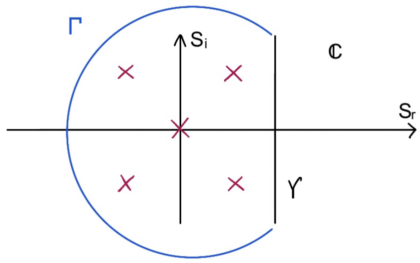
As an exercise you can show that using boundary conditions, the solution becomes:

$$\mathcal{C}(r,s) = \frac{-j_0R^2}{Drs}\frac{\sinh(r\sqrt{\frac{s}{D}})}{R\sqrt{\frac{s}{D}}\cosh(R\sqrt{\frac{s}{D}}) - \sinh(R\sqrt{\frac{s}{D}})} \tag{5.18}$$

Now we need to invert the Laplace transformation. Recall that the inverse Laplace transform is given by

$$C(r,t) = L^{-1}\{\mathcal{C}\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\mathcal{C}(r,s)e^{st}ds \tag{5.19}$$

where $\gamma \in \mathbb{R}$ is to the right of all poles of $\mathcal{C}$.



Additionally we have that:

$$\oint = \lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} + \int_{\Gamma} = 2\pi i\sum_{\mathbb{C}}\text{Res}\{\mathcal{C}(r,s)e^{st}\} \tag{5.20}$$

where the integral over the contour Γ vanishes in the limit $T \to \infty$ (Exercise: show this to be true).

Now considering the singular behaviour of $f(s) = \mathcal{C}(r,s)e^{st}$, we have poles at

$$s = 0, \quad \text{and} \quad R\sqrt{\frac{s}{D}} = \tanh(R\sqrt{\frac{s}{D}}) \tag{5.21}$$

Let's first consider the behaviour of $f(s)$ near $s \approx 0$.

Recall (from Cauchy's theorem) that for a pole of order n at $s = s_0$, the 'Residual' is given by

$$\operatorname{Res}\{f, s_0\} = \lim_{s \to s_0} \frac{d^{n-1}}{ds^{n-1}} \left((s - s_0)^n f(s) \right) \tag{5.22}$$

Here $s_0 = 0$ and $n = 2$. As an exercise in the Practice Problems, we will show that

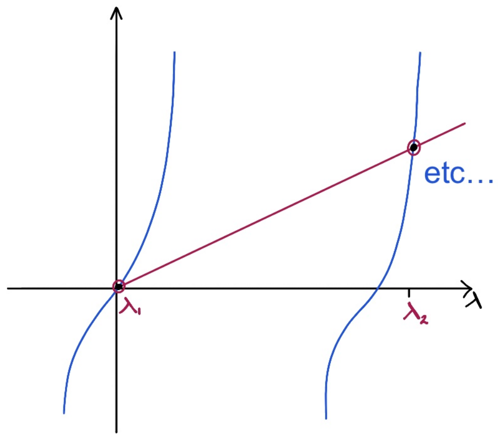
$$\lim_{s \to 0} \frac{d}{ds} \left(s^2 f(s) \right) = \frac{-3 j_0}{R} \left(t + \frac{r^2}{6D} - \frac{R^2}{10D} \right) \tag{5.23}$$

Next we can take a look the pole where $R\sqrt{\frac{s}{D}} = \tanh(R\sqrt{\frac{s}{D}})$. We let $s_0 = -\frac{\lambda^2 D}{R^2}$ for some $\lambda \in \mathbb{R}$, and we note that

$$\sinh(ix) = i \sin x \qquad \cosh(ix) = \cos x \tag{5.24}$$

Hence, we find that

$$\mathcal{C}(r, s_0) = \frac{-j_0 R^2}{D r s_0} \frac{i \sin(\lambda \frac{r}{R})}{i \lambda \cos \lambda - i \sin(\lambda)} \tag{5.25}$$



(N.B. $R\sqrt{\frac{s}{D}} = \lambda \geqslant 0$)

The above expression has poles when $\lambda = \tan \lambda$. This has infinitely many roots, which we denote

$$\lambda_m \in \mathbb{R} \geq 0, \quad m = 0, 1, 2, \ldots \tag{5.26}$$

The corresponding values of s are

$$s_m = -\frac{\lambda_m^2 D}{R^2}, \quad m = 0, 1, 2, \ldots \tag{5.27}$$

Suppose $\mathcal{C}$ has poles of order 1 at $\lambda = \lambda_m$. We will consider,

$$\operatorname{Res}\{f, s_n\} = \lim_{\substack{s \to s_m \\ \lambda \to \lambda_m}} \left(-\frac{j_0 R^2}{D r s} \frac{\sin(\lambda \frac{r}{R})}{\lambda \cos \lambda - \sin \lambda} (s - s_m) e^{st} \right) \tag{5.28}$$

Consider the Taylor expansion of the denominator near $\lambda = \lambda_m$, which is

$$\lambda \cos \lambda - \sin \lambda \approx -\lambda_m (\lambda - \lambda_m) \sin \lambda_m \tag{5.29}$$

Also note that,

$$(s - s_m) = -\frac{D}{R^2}\left(\lambda^2 - \lambda_m^2\right) \tag{5.30}$$

$$= -\frac{D}{R^2}\left(\lambda + \lambda_m\right)\left(\lambda - \lambda_m\right) \tag{5.31}$$

$$\approx -\frac{2D}{R^2}\lambda_m\left(\lambda - \lambda_m\right) \tag{5.32}$$

Putting this all together we find

$$\text{Res}\{f, s_m\} = \frac{2j_0R^2}{Dr}\frac{\sin\left(\lambda_m \frac{r}{R}\right)}{-\lambda_m\left(\lambda - \lambda_m\right)\sin\lambda_m}\left(-\frac{2D}{R^2}\lambda_m\left(\lambda - \lambda_m\right)\right)e^{-\frac{\lambda_m^2 Dt}{R^2}} \tag{5.33}$$

Now we will construct the whole solution,

$$C(r,t) = \frac{-3j_0}{R}\left(t + \frac{r^2}{6D} - \frac{R^2}{10D}\right) + \frac{2j_0R^2}{Dr}\sum_{m=1}^{\infty}\frac{\sin\left(\lambda_m \frac{r}{R}\right)}{\lambda_m^2\sin\lambda_m}e^{\frac{-\lambda^2 Dt}{R^2}} \tag{5.34}$$

The concentration at the surface $r = R$ is:

$$C_s(t) = C(R,t) = -\frac{3j_0t}{R} - \frac{1}{5}j_0\frac{R}{D} + \frac{2j_0R}{D}\sum_{m=1}^{\infty}\frac{e^{\frac{-\lambda_m^2 Dt}{R^2}}}{\lambda_m^2} \tag{5.35}$$

To gain some intuition we write

$$C_s(t) = C_{s1}(t) + C_{s2}(t) \tag{5.36}$$

where

$$C_{s1}(t) = -\frac{3j_0t}{R} \tag{5.37}$$

$$C_{s2}(t) = -\frac{1}{5}j_0\frac{R}{D} + \frac{2j_0R}{D}\sum_{m=1}^{\infty}\frac{e^{\frac{-\lambda_m^2 Dt}{R^2}}}{\lambda_m^2} \tag{5.38}$$

Here, C_{s1} defines the average concentration inside a spherical particle. The voltage across the battery is a function of $C_s(t)$ in each of the positive and negative electrodes (5.5), which is how the charge and discharge of a battery is modelled.

As a check, let's confirm that our solution satisfies the appropriate initial conditions. We find that

$$C_s(0) = C(R,0) = -\frac{1}{5}j_0\frac{R}{D} + \frac{2j_0R}{D}\sum_{m=1}^{\infty}\frac{1}{\lambda_m^2} \tag{5.39}$$

We can use the following result to simplify things:

$$\sum_{m=1}^{\infty}\frac{1}{\lambda_m^2} = \frac{1}{10} \quad \text{for} \quad \lambda_m = \tan\lambda_m \tag{5.40}$$

which we will not prove in this course, but can be easily tested numerically.

Hence our initial conditions are satisfied:

$$C(R,0) = 0 \tag{5.41}$$

Time Dependent Case

Now suppose $j_0 = j(t)$ and consider:

$$J(s) := L\{j(t)\} = \int_0^\infty j(t)e^{-st}dt \tag{5.42}$$

Under this definition we can write the Laplace transform of the concentration as

$$\mathcal{C}(r,s) = F(s)J(s) \tag{5.43}$$

where

$$F(s) = -\frac{R^2}{Dr}\frac{\sinh(r\sqrt{\frac{s}{D}})}{R\sqrt{\frac{s}{D}}\cosh(R\sqrt{\frac{r}{R}}) - \sinh(R\sqrt{\frac{r}{R}})} \tag{5.44}$$

Through the Convolution Theorem we have that the inverse Laplace transform satisfies

$$C(r,t) = L^{-1}\{F(s)J(s)\} = (f * j)(t) \tag{5.45}$$

where $*$ represents a convolution. Hence, we write our concentration as

$$C(r,t) = \int_0^t f(r,\tau)j(t-\tau)d\tau \tag{5.46}$$

where f is the inverse Laplace transform of F. It is important to note that F has simple poles at $s = 0$ (order 1) and at $s = s_m$, $m = 0, 1, 2, \ldots$.

As an exercise you can show that near $s \approx 0$ we have

$$F(s) \approx -\frac{3}{Rs}\left(1 + \left(\frac{r^2}{6D} - \frac{R^2}{10D}\right)s\right) \tag{5.47}$$

such that

$$\lim_{s\to 0}\left(sF(s)e^{st}\right) = -\frac{3}{R} \tag{5.48}$$

The poles at $s = s_m$ remain unchanged except for a factor $s_m = -\lambda_m^2\frac{D}{R^2}$. Hence, we find that

$$C(r,t) = \int_0^t \left(-\frac{3}{R} - \frac{2}{r}\sum_{m=1}^{\infty}\frac{\sin\lambda_m\frac{r}{R}}{\sin\lambda_m}e^{-\frac{\lambda_m^2 Dt}{R^2}}\right)j(t-\tau)d\tau \tag{5.49}$$

By letting $\tilde{\tau} = t - \tau$ such that $d\tilde{\tau} = -d\tau$, we can write

$$\int_0^t j(t-\tau)d\tau = \int_t^0 -j(\tilde{\tau})d\tilde{\tau} \tag{5.50}$$

In this way, we can write our solution for the concentration as

$$C(r,t) = -\frac{3}{R}\int_0^t j(\tilde{\tau})d\tilde{\tau} - \frac{2}{r}\int_0^t\left(\sum_{m=1}^{\infty}\frac{\sin(\lambda_m\frac{r}{R})}{\sin\lambda_m}e^{-\frac{\lambda_m^2 D\tau}{R^2}}\right)j(t-\tau)d\tau \tag{5.51}$$

Hence, the concentration at the surface is

$$C_s(t) = C(R,t) = C_{s1}(t) + C_{s2}(t) \tag{5.52}$$

where

$$C_{s1}(t) = -\frac{3}{R}\int_0^t j(\tilde{\tau})d\tilde{\tau} \tag{5.53}$$

$$C_{s2}(t) = -\frac{2}{R}\int_0^t \left(\sum_{m=1}^{\infty} e^{-\frac{\lambda_m^2 D\tau}{R^2}}\right) j(t-\tau)d\tau \tag{5.54}$$

This gives us the corresponding version of the surface concentration of lithium ions when the imposed flux is time-dependent.

Quasi Steady Limit: (A.K.A Fast diffusion)

Let's now consider the case where the applied flux is varying at some frequency ω, such that $j = j(wt)$. Our governing equation and boundary conditions are written as

$$C_t = \frac{D}{r^2}\left(r^2 C_r\right)_r \tag{5.55}$$

$$C_r = 0 \qquad : r = 0 \tag{5.56}$$

$$-DC_r = j(wt) \qquad : r = R \tag{5.57}$$

$$C = C_0 \qquad : t = 0 \tag{5.58}$$

Next we will non-dimensionalise this problem according to the scalings

$$t = \frac{1}{w}\hat{t} \qquad r = R\hat{r} \qquad C = C_0\hat{c} \qquad j = j_0\hat{j} \tag{5.59}$$

As such, our equations become

$$\frac{\omega R^2}{D}\hat{c}_{\hat{t}} = \frac{1}{\hat{r}^2}\left(\hat{r}^2\hat{c}_{\hat{r}}\right)_{\hat{r}} \tag{5.60}$$

$$\hat{c}_{\hat{r}} = 0 \qquad : \hat{r} = 0 \tag{5.61}$$

$$-\hat{c}_{\hat{r}} = \frac{j_0 R}{DC_0}\hat{j}(\hat{t}) \qquad : \hat{r} = 1 \tag{5.62}$$

$$\hat{c} = 1 \qquad : \hat{t} = 0 \tag{5.63}$$

Next let's consider the limit of large diffusion such that $\frac{\omega R^2}{D} \ll 1$. In other words, we are considering the case where the forcing timescale is much larger than the diffusion timescale, such that

$$\frac{1}{\omega} \gg \frac{R^2}{D} \tag{5.64}$$

We will also assume that

$$\frac{j_0}{C_0 R\omega} = O(1) \tag{5.65}$$

is an order one quantity, such that the term on the right hand side of (5.62) is small, i.e.

$$\frac{Rj_0}{DC_0} = \frac{\omega R^2}{D}\frac{j_0}{C_0 R\omega} \ll 1 \tag{5.66}$$

Hence, according to this regime, our governing system reduces to

$$\frac{1}{\hat{r}^2}\left(\hat{r}^2\hat{c}_{\hat{r}}\right)_{\hat{r}} \approx 0 \tag{5.67}$$

$$\hat{c}_{\hat{r}} \approx 0 \qquad : \hat{r} = 0, 1 \tag{5.68}$$

Our 'leading order' solution thus does not depend on $\hat{r}$, and so we write

$$\hat{c} = \hat{c}_0(\hat{t}) \tag{5.69}$$

for some unknown function $\hat{c}_0$ which is yet to be found.

Next we will write our dimensionless concentration as an asymptotic expansion in terms of the small parameter

$$\epsilon = \frac{\omega R^2}{D} \ll 1 \tag{5.70}$$

As such, we write

$$\hat{c} = \hat{c}_0 + \epsilon\hat{c}_1 + \epsilon^2\hat{c}_2 + ... \tag{5.71}$$

We insert this expansion into governing PDE, such that

$$\epsilon\left(\hat{c}_0'(\hat{t}) + \epsilon\hat{c}_1(\hat{r},\hat{t}) + ...\right) = \frac{1}{r^2}\left(r^2\left(\hat{c}_0(\hat{t}) + \epsilon\hat{c}_1(\hat{r},\hat{t}) + ...\right)\right)_{\hat{r}} \tag{5.72}$$

We will now ignore terms of order $O(\epsilon^2)$ in the above equation (i.e. we consider a 'first order' approximation). This leaves us with

$$\hat{c}_0'(\hat{t}) = \frac{1}{r^2}\left(r^2\hat{c}_{1\hat{r}}\right)_{\hat{r}} \tag{5.73}$$

Next we insert our asymptotic expansion into the boundary conditions, giving

$$\epsilon\hat{c}_{1\hat{r}} + ... = 0 \qquad : \hat{r} = 0 \tag{5.74}$$

$$\epsilon\hat{c}_{1\hat{r}} + \epsilon^2\hat{c}_{2\hat{r}} + ... = -\epsilon\frac{j_0}{C_0R\omega}\hat{j}(\hat{t}) \qquad : \hat{r} = 1 \tag{5.75}$$

Hence, a first order approximation gives us the simplified boundary conditions

$$\hat{c}_{1\hat{r}} = 0 \qquad : \hat{r} = 0 \tag{5.76}$$

$$\hat{c}_{1\hat{r}} = -\frac{j_0}{C_0R\omega}\hat{j}(\hat{t}) \qquad : \hat{r} = 1 \tag{5.77}$$

Now we will integrate equation (5.73) multiplied by $\hat{r}^2$ to get

$$\int_0^1 \hat{r}^2\hat{c}_0'(\hat{t})d\hat{r} = [\hat{r}^2\hat{c}_{1\hat{r}}]_0^1 \tag{5.78}$$

$$\implies \frac{1}{3}\hat{c}_0'(\hat{t}) = -\left(\frac{j_0}{C_0R\omega}\right)\hat{j}(\hat{t}) \tag{5.79}$$

Converting this back to dimensional coordinates, we get

$$\frac{dc_0}{dt} = -\frac{3}{R}j(t) \tag{5.80}$$

We can also write this as,

$$c_0 = -\frac{3}{R}\int_0^t j(\tau)d\tau \tag{5.81}$$

which now finally determines our leading order solution for the concentration.

We can compare this with the time dependent solution (5.51) in the limit $D \to \infty$, and we see that we get the same result.