Green’s function for the Boundary Value Problems (BVP)\(^1\)

1. Dirac Delta Function and Heaviside Step Function

Definition:

\[
\delta(x) = \begin{cases} 
  A > 0, & x = 0 \\
  0, & \text{otherwise}
\end{cases}
\]

Two main properties of \(\delta\)-function:

\[
\int_{-\infty}^{\infty} \delta(x) dx = 1
\]

and

\[
\int_{-\infty}^{\infty} f(x) \delta(x - a) dx = \int_{-\infty}^{\infty} f(a) \delta(x - a) dx = f(a) \int_{-\infty}^{\infty} \delta(x - a) dx = f(a)
\]

for any analytic \(f(x)\).

From a purely mathematical viewpoint, the Dirac delta is not strictly a function, because any extended-real function that is equal to zero everywhere but a single point must have total integral zero. The \(\delta\)-function only makes sense as a mathematical object when it appears inside an integral. While from this perspective the Dirac delta can usually be manipulated as though it were a function, formally it must be defined as a distribution.

There are the number of known analytic approximations of \(\delta\)-function that satisfy the same properties:

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\(^1\)Based on Section 1.5 of textbook “Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory” by C. M. Bender and S. A. Orszag. Springer, 1999.
1. The sequence of functions
\[ \delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}, \quad n = 1, 2, \ldots. \]

Example:
\[
\int_{-\infty}^{\infty} \delta_n(x) \, dx = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-n^2x^2} \, dx,
\]
\[
= \frac{n}{\sqrt{\pi}} \frac{1}{n} \int_{-\infty}^{\infty} e^{-z^2} \, dz,
\]
\[
= \frac{n}{\sqrt{\pi}} \sqrt{\pi} = 1.
\]

2. The family of functions
\[ \delta_L(x) = \int_{-L}^{L} e^{ikx} \, dk, \quad L > 0, \]
which could be re-written by doing the integral:
\[
\delta_L(x) = \frac{1}{2\pi i x} \left( e^{ixL} - e^{-ixL} \right),
\]
\[
= \frac{1}{2\pi} \cdot 2L \cdot \left( \frac{e^{ixL} - e^{-ixL}}{2iLx} \right),
\]
\[
= \frac{L \sin(Lx)}{\pi Lx}.
\]

More approximations exist and their use is usually motivated by the particular problem.

Alternatively \( \delta(x) \) could be defined as a derivative of the Heaviside step function \( H(x) \):
\[
H(x) = \begin{cases} 
0, & x < 0 \\
1/2, & x = 0 \\
1, & x > 0.
\end{cases}
\]

Then \( \delta(x) = dH(x)/dx \).

Note that Heaviside step function is “smoother” than the Dirac delta function, as integration is a smoothing operation. Furthermore, the integral of the Heaviside function is a ramp function:
\[
r(x) = \int_{-\infty}^{\infty} H(x) \, dx \begin{cases} 
0, & x \leq 0 \\
x, & x \geq 0.
\end{cases}
\]
2. Application to differential equations

Consider the 2nd order linear differential equation:

\[ L[y] = y'' + p(x)y' + q(x)u = f(x) \]

where \( p(x) \) and \( q(x) \) are continuous functions.

The Green’s function approach could be applied to the solution of linear ODEs of any order, however, we showcase it on the 2nd order equations, due to the vast areas of their applications in physics and engineering.

The Green’s function \( G(x, \xi) \) associated with the inhomogeneous equation \( L[y] = f(x) \) satisfies the differential equation:

\[ L[G(x, \xi)] = \delta(x - \xi) \]

Once \( G(x, \xi) \) is known then the solution of \( L[y] = f(x) \) is

\[ y(x) = \int_{-\infty}^{\infty} G(x, \xi) f(\xi) d\xi \]  \hspace{1cm} (1)

because

\[ L[y] = L \left[ \int_{-\infty}^{\infty} G(x, \xi) f(\xi) d\xi \right] = \int_{-\infty}^{\infty} L[G(x, \xi)] f(\xi) d\xi = \]

\[ \int_{-\infty}^{\infty} \delta(x - \xi) f(\xi) d\xi = f(x) \]

Also note that this is true for

\[ y(x) = \int_{I} G(x, \xi) f(\xi) d\xi \]

over any interval \( I \) that contains \( x \) on which \( f(x) \) is continuous.

The idea behind Green’s function approach is to replace the direct solution of the inhomogeneous equation \( L[y(x)] = f(x) \), which could be cumbersome by computing Green’s function that satisfies equation \( L[G(x, \xi)] = \delta(x - \xi) \) and eventually deriving \( y(x) \) from the integral (1).

First we consider the homogeneous equation \( L[y] = 0 \). That has a general solution \( y = C_1 y_1 + C_2 y_2 \). When \( x \neq \xi, L[G(x, \xi)] = 0 \). So we could write:

\[ G(x, \xi) = \begin{cases} ay_1 + by_2, & x < \xi \\ cy_1 + dy_2, & x > \xi \end{cases} \]

To find four unknown parameters we need four equations. Two of them are based on the following conditions:
1. Continuity at $x = \xi$:

$$ay_1(\xi) + by_2(\xi) = cy_1(\xi) + dy_2(\xi) \quad (2)$$

2. $\partial G(x, \xi)/\partial x$ has a finite jump (of size 1) at $x = \xi$

To verify this property we integrate $L[G(x, \xi)]$ in the close vicinity of $\xi$ (on the interval $[\xi - \varepsilon, \xi + \varepsilon]$) and look through the integral term by term:

$$I_1 + I_2 + I_3 = \int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-\xi)dx = \int_{-\infty}^{\infty} \delta(x-\xi)dx = \delta(x = \xi) = 1$$

We use the definition of $\delta(x - \xi)$, which is equal to 0 everywhere except $x = \xi$.

From the condition of continuity follows that when $\varepsilon \to 0$

$$I_3 = \int_{\xi-\varepsilon}^{\xi+\varepsilon} q(x)G(x, \xi)dx = 0$$

$$I_2 = \int_{\xi-\varepsilon}^{\xi+\varepsilon} p(x)G_x(x, \xi)dx = 0.$$ 

Both $I_3$ and $I_2$ represent continuous functions at $x = \xi$. Thus the only candidate “to be discontinuous” is

$$I_1 = G_x(x, \xi)|_{\xi-\varepsilon}^{\xi+\varepsilon} = \int_{\xi-\varepsilon}^{\xi+\varepsilon} G_{xx}(x, \xi)dx$$

$I_1$ corresponds to the Heaviside step function, which has a jump discontinuity of size 1 at $x = \xi$. This yields one more equation to for computing the coefficients of the Green’s function:

$$cy_1'(\xi) + dy_2'(\xi) - ay_1'(\xi) - by_2'(\xi) = 1 \quad (3)$$

The remaining two equations for unknown parameters of Greens function we formulate from the initial or boundary conditions.

Combining Eqs. (2) and (3) we get:

$$b - a = -\frac{y_2(\xi)}{W(y_1, y_2)(\xi)} \quad \quad \quad \quad \quad d - c = \frac{y_1(\xi)}{W(y_1, y_2)(\xi)}$$

where $W(y_1, y_2)$ is a Wronskian of the homogeneous problem.

The Green’s function for IVP was explained in the previous set of notes (and derived using the method of variation of parameter). Here we consider the BVP. The Green’s function approach is particularly better to solve boundary-value problems, especially when the operator $L$ and the
boundary conditions are fixed but the RHS may vary. It is easy for solving boundary value problem with homogeneous boundary conditions. For inhomogeneous boundary conditions, for which the BVP has solutions (an open question!!!), some transformations of the variable are needed to homogenize the boundary conditions.

The algorithm of the BVP solution includes the following steps:

1. First we solve homogeneous problem $L[y] = 0$. The fundamental solutions $y_1$ and $y_2$ should satisfy the following properties:

$$L[y_1] = 0, \quad B_0[y_1] = 0$$
$$L[y_2] = 0, \quad B_1[y_2] = 0.$$  

One can also check, for instance, that $B_0[y_2] \neq 0$ (equivalent to the condition of linear independence). Based on $y_1, y_2$ and their Wronskian we formulate first two equations to find the Green’s function coefficients.

2. Then we apply continuity and jump conditions: (2) and (3) to form two remaining equations.

3. By solving all four equations we find the coefficients and construct the Green’s function of the BVP $G(x, \xi)$.

4. The solution of BVP is computed using integral (1) and known $f(x)$.

3. Examples

**Question 1**: Compute the Green’s function of the BVP: $y'' = f(x)$, with $y(0) = y'(1) = 0$.

The Green’s function for this problem is satisfying

$$G_{xx}(x, \xi) = \delta(x - \xi)$$
$$G(0, \xi) = G_x(1, \xi) = 0$$

And the solution of BVP is

$$y(x) = \int_0^1 G(x, \xi)f(\xi)d\xi$$

Since

$$y(0) = \int_0^1 G(0, \xi)f(\xi)d\xi = 0$$
$$y'(1) = \int_0^1 G_x(1, \xi)f(\xi)d\xi = 0$$
If \( x \neq \xi \) \( G_{xx}(x, \xi) = 0 \). Therefore

\[
G(x, \xi) = \begin{cases} 
ax + b, & x < \xi \\
 cx + d, & x > \xi 
\end{cases}
\]

where \( x \in (0, 1) \). We start with BC:

\[
G(0, \xi) = 0 \Rightarrow b = 0 \Rightarrow G(x, \xi) = ax, \ x < \xi \\
G'(1, \xi) = 0 \Rightarrow c = 0 \Rightarrow G(x, \xi) = d, \ x > \xi
\]

From the continuity follows: \( a\xi = d \) and from the jump condition: \( 0 - a = 1 \). Therefore \( a = -1 \) and \( d = -\xi \). The result is

\[
G(x, \xi) = \begin{cases} 
-x, & x < \xi \\
-\xi, & x > \xi 
\end{cases}
\]

**Question 1a:** Suppose that \( f(x) = x^2 \). Solve the BVP from the previous question.

\[
y(x) = \int_0^1 G(x, \xi)\xi^2 d\xi = \int_0^x (-\xi)\xi^2 + \int_x^1 (-x)\xi^2 d\xi = -\frac{\xi^4}{4} \bigg|_0^x - \frac{\xi^3}{3} \bigg|_x^1 = \frac{x^4}{12} - \frac{x}{3}
\]

**Question 2:** Compute the Green’s function of the BVP: \( y'' - y = f(x) \), with \( y(\pm\infty) = 0 \).

The Green’s function for this problem is satisfying

\[
G_{xx}(x, \xi) - G(x, \xi) = \delta(x - \xi) \\
G(-\infty, \xi) = G(\infty, \xi) = 0
\]

And the solution of BVP is

\[
y(x) = \int_{-\infty}^{\infty} G(x, \xi)f(\xi)d\xi
\]

Since

\[
y(-\infty) = \int_{-\infty}^{\infty} G(-\infty, \xi)f(\xi)d\xi = 0 \\
y(\infty) = \int_{-\infty}^{\infty} G(\infty, \xi)f(\xi)d\xi = 0
\]

The homogeneous equation has a solution \( y = C_1e^x + C_2e^{-x} \). Therefore

\[
G(x, \xi) = \begin{cases} 
ae^x + be^{-x}, & x < \xi \\
ce^x + de^{-x}, & x > \xi 
\end{cases}
\]
From BC follows that $b = 0$ and $c = 0$ (we substitute $x = -\infty$ into the top equation and $x = \infty$ into the bottom).

The continuity and the jump in derivative conditions yield

$$ae^\xi = de^{-\xi}$$

$$-de^{-\xi} - ae^\xi = 1$$

Solving the linear system, we obtain

$$a = -\frac{1}{2}e^{-\xi}$$

$$c = -\frac{1}{2}e^{\xi}$$

and

$$G(x, \xi) = \begin{cases} 
-\frac{1}{2}e^{x-\xi}, & x < \xi \\
-\frac{1}{2}e^{\xi-x}, & x > \xi 
\end{cases} = -\frac{1}{2}e^{-|x-\xi|}$$