Partition identities and representation theory

Jehanne Dousse

CNRS and Université Lyon 1

Workshop on Enumerative Combinatorics 2021
University College Dublin
March 10, 2021

Integer partitions

Definition

A *partition* π of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \ldots, \lambda_m$ such that $\lambda_1 + \cdots + \lambda_m = n$. The integers $\lambda_1, \ldots, \lambda_m$ are called the *parts* of the partition.

Example

There are 5 partitions of 4:

$$4, 3+1, 2+2, 2+1+1 \text{ and } 1+1+1+1.$$

Generating functions

Notation : $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n \in \mathbb{N} \cup \{\infty\}$.

Let $Q(n,k;m,N)$ be the number of partitions of n into k distinct parts congruent to $m \mod N$. Then

$$1 + \sum_{n\geq 1}\sum_{k\geq 1} Q(n,k;m,N) z^k q^n = (1+zq^m)(1+zq^{N+m})(1+zq^{2N+m})\cdots$$
$$= (-zq^m;q^N)_\infty.$$

Generating functions

Notation : $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n \in \mathbb{N} \cup \{\infty\}$.

Let $Q(n,k;m,N)$ be the number of partitions of n into k distinct parts congruent to $m \mod N$. Then

$$1 + \sum_{n\geq 1}\sum_{k\geq 1} Q(n,k;m,N) z^k q^n = (1+zq^m)(1+zq^{N+m})(1+zq^{2N+m})\cdots$$
$$= (-zq^m;q^N)_\infty.$$

Let $p(n,k;m,N)$ be the number of partitions of n into k parts congruent to $m \mod N$. Then

$$1 + \sum_{n\geq 1}\sum_{k\geq 1} p(n,k;m,N) z^k q^n = \prod_{\ell\geq 0}\left(1 + zq^{\ell N+m} + z^2 q^{2(\ell N+m)} + \cdots\right)$$
$$= \frac{1}{(zq^m;q^N)_\infty}.$$

The first Rogers–Ramanujan identity

Theorem (Rogers 1894, Rogers–Ramanujan 1919)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

The first Rogers–Ramanujan identity

Theorem (Rogers 1894, Rogers–Ramanujan 1919)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

Theorem (Partition version)

For every positive integer n, the number of partitions of n such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.

Some quick definitions on Lie algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with Cartan subalgebra $\mathfrak{h}$.
The corresponding (derived) affine Lie algebra $\hat{\mathfrak{g}}$ is constructed as

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $\mathbb{C}[t, t^{-1}]$ is the complex vector space of Laurent polynomials in the indeterminate t, and $\mathbb{C}c$ is $\hat{\mathfrak{g}}$'s center (one-dimensional).

Some quick definitions on Lie algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with Cartan subalgebra $\mathfrak{h}$.

The corresponding (derived) affine Lie algebra $\hat{\mathfrak{g}}$ is constructed as

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $\mathbb{C}[t, t^{-1}]$ is the complex vector space of Laurent polynomials in the indeterminate t, and $\mathbb{C}c$ is $\hat{\mathfrak{g}}$'s center (one-dimensional).

If V is an irreducible highest weight module of $\hat{\mathfrak{g}}$, the central element c acts on V by multiplication by a scalar k, which is called the **level** of V.

Some quick definitions on Lie algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with Cartan subalgebra $\mathfrak{h}$.
The corresponding (derived) affine Lie algebra $\hat{\mathfrak{g}}$ is constructed as

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $\mathbb{C}[t, t^{-1}]$ is the complex vector space of Laurent polynomials in the indeterminate t, and $\mathbb{C}c$ is $\hat{\mathfrak{g}}$'s center (one-dimensional).

If V is an irreducible highest weight module of $\hat{\mathfrak{g}}$, the central element c acts on V by multiplication by a scalar k, which is called the **level** of V.
The **character** $\mathrm{ch}(V)$ of V is defined as

$$\mathrm{ch}(V) = \sum_{\mu} \dim(V_\mu) e^{\mu},$$

where the sum is over the weights μ of V,
$V_\mu := \{v \in V : \forall H \in \mathfrak{h}, \quad H \cdot v = \mu(H)v\}$ is a weight space,
and e^μ is a formal exponential satisfying $e^{\mu} e^{\mu'} = e^{\mu+\mu'}$.

Representation theoretic interpretation

Lepowsky and Wilson 1984: representation theoretic interpretation

$$\frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q^2)_\infty}\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$$

Obtained by giving two different formulations for the principal specialisation

$$e^{-\alpha_0}\mapsto q,\quad e^{-\alpha_1}\mapsto q$$

of $e^{-\Lambda}\mathrm{ch}(L(\Lambda))$ where $L(\Lambda)$ is an irreducible highest weight $A_1^{(1)}$-module of level 3.

RHS: principal specialisation of the Weyl-Kac character formula

LHS: comes from the construction of a basis of $L(\Lambda)$ using vertex operators

Representation theoretic interpretation

Lepowsky and Wilson 1984: representation theoretic interpretation

$$\frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^2)_\infty}\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$$

LHS: comes from the construction of a basis of $L(\Lambda)$ using vertex operators.

Idea:

- Start with a spanning set of $L(\Lambda)$: here, monomials of the form $Z_1^{f_1}\dots Z_s^{f_s}$ for $s, f_1, \dots, f_s \in \mathbb{N}_{\geq 0}$.
- Using Lie theory, reduce this spanning set: here, it allows one to remove all monomials containing Z_j^2 or $Z_j Z_{j+1}$.
- Show that the obtained set is a basis of the representation (very difficult).

Some other identities from representation theory

Studying other representations or other Lie algebras leads to new, **often conjectural**, partition identities:

- Capparelli (conj. 1992, proof 1996): level 3 standard modules of $A_2^{(2)}$
- Nandi 2014: level 4 standard modules of $A_2^{(2)}$
- Meurman and Primc 1987-1999: higher levels of $A_1^{(1)}$
- Siladić 2002: twisted level 1 modules of $A_2^{(2)}$
- Primc and Šikić 2016: level k standard modules of $C_n^{(1)}$

Some other identities from representation theory

Studying other representations or other Lie algebras leads to new, **often conjectural**, partition identities:

- Capparelli (conj. 1992, proof 1996): level 3 standard modules of $A_2^{(2)}$
- Nandi 2014: level 4 standard modules of $A_2^{(2)}$
- Meurman and Primc 1987-1999: higher levels of $A_1^{(1)}$
- Siladić 2002: twisted level 1 modules of $A_2^{(2)}$
- Primc and Šikić 2016: level k standard modules of $C_n^{(1)}$

Sometimes, **combinatorial proofs** and refinements are found (often simpler than Lie algebraic proofs):

- Andrews (1992), Alladi–Andrews–Gordon (1995): combinatorial proofs and refinement of Capparelli's identity
- D. (2017): combinatorial proof and refinement of Siladić's identity, Konan (2019): bijection

Crystals: "combinatorial representations" of Lie algebras

Crystal for the vector representation of the affine Lie algebra $A_{n-1}^{(1)}$:

$$\mathcal{B}: \quad \begin{tikzcd} \boxed{0} \arrow[r, "1"] & \boxed{1} \arrow[r, "2"] & \cdots \arrow[r, "n-2"] & \boxed{n-2} \arrow[r, "n-1"] & \boxed{n-1} \arrow[llll, bend left, "0"] \end{tikzcd}$$

Given a crystal $\mathcal{B}$, one can be define an **energy function** $H : \mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}$. The value of $H(b_1 \otimes b_2)$ determines the values $H(b_1' \otimes b_2')$ of all the vertices $b_1' \otimes b_2'$ which are in the same connected component as $b_1 \otimes b_2$.

The $(\mathrm{KMN})^2$ crystal base character formula

To each dominant weight λ, one can associate a **ground state path**

$$\mathfrak{p}_\lambda = (g_k)_{k=0}^{\infty} = \cdots \otimes g_{k+1} \otimes g_k \otimes \cdots \otimes g_1 \otimes g_0,$$

where $g_i \in \mathcal{B}$ for all i.
A tensor product $\mathfrak{p} = (p_k)_{k=0}^{\infty} = \cdots \otimes p_{k+1} \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0$ of elements $p_k \in \mathcal{B}$ is said to be a *λ-path* if $p_k = g_k$ for k large enough. Let $\mathcal{P}(\lambda)$ denote the set of λ-paths .

Theorem ($(\mathrm{KMN})^2$ 1992)

Let $L(\lambda)$ be an irreducible highest weight module of weight λ.

$$\mathrm{ch}(L(\lambda)) = \sum_{\mathfrak{p} \in \mathcal{P}(\lambda)} e^{\mathrm{wt}\,\mathfrak{p}},$$

where $\mathrm{wt}\,\mathfrak{p}$ is defined in terms of the energy function and the simple roots.

Let P be the energy function in $(\mathcal{B} \otimes \mathcal{B}^\vee) \otimes (\mathcal{B} \otimes \mathcal{B}^\vee)$ for $A_1^{(1)}$.
Partitions in four colours a, b, c, d, with the order

$$1_a < 1_b < 1_c < 1_d < 2_a < 2_b < 2_c < 2_d < \cdots ,$$

and difference conditions

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}.$$

Conjecture (Primc 1999)

After performing the specialisations

$$k_a \to 2k-1, k_b \to 2k, k_c \to 2k, k_d \to 2k+1,$$

the generating function for these partitions (not keeping track of the colours) becomes $\frac{1}{(q;q)_\infty}$.

Proof (combinatorial) and refinement: D.–Lovejoy 2017.

A combinatorial non-specialised character formula

Definition

A *multi-grounded partition* with ground $c_{g_0}, \dots, c_{g_{t-1}}$ and relation $\gg$ is a coloured partition $\pi = (\pi_0, \cdots, \pi_{s-1}, u_{c_{g_0}}^{(0)}, \dots, u_{c_{g_{t-1}}}^{(t-1)})$ such that for all i, $\pi_i \gg \pi_{i+1}$, and $(\pi_{s-t}, \cdots, \pi_{s-1}) \neq (u_{c_{g_0}}^{(0)}, \dots, u_{c_{g_{t-1}}}^{(t-1)})$.

Let $\mathcal{P}^{\gg}_{c_{g_0},\dots,c_{g_{t-1}}}$ be the set of grounded partitions with ground c_g and relation $\gg$ defined by
$k_{c_b} \gg k'_{c_{b'}}$ if and only if $k - k' \geq H(b' \otimes b)$.

Theorem (D.–Konan 2021)

Let $L(\lambda)$ be an irreducible highest weight module of weight λ. Setting $q = e^{-\delta/d_0}$ and $c_b = e^{\mathrm{wt}\, b}$ for all $b \in \mathcal{B}$,

$$\sum_{\pi \in \mathcal{P}^{\gg}_{c_{g_0},\dots,c_{g_{t-1}}}} C(\pi) q^{|\pi|} = \frac{e^{-\lambda}\mathrm{ch}(L(\lambda))}{(q;q)_\infty}.$$

Thank you for your attention!