Bijective proofs of alternating sign matrix theorems

Ilse Fischer

Universität Wien

Joint work with Matjaž Konvalinka

Some remarks

• We translate non-bijective proofs into combinatorics !

• The combinatorial point of view led to many modifications and it is hard to recognize the original proofs from our bijective proofs.

• In the original proofs, signs are unavoidable and this makes it necessary to work with signed sets. This causes the use of a generalization of the involution principle by Garsia and Milne.

• We have written a computer code that performs the bijections.

• **Simpler bijections ?!** Hopefully there are simpler bijections. On the other hand, there are also no simple non-bijective proofs so far and all of them involve subtractions.

Outline

I. ASMs, DPPs and Bijections 1 & 2

II. Signed sets and sijections

III. Some details of our constructions

I. ASMs, DPPs and Bijections 1 & 2

Alternating Sign Matrices = ASMs

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Square matrix with entries in $\{0, \pm 1\}$ such that in each row and each column

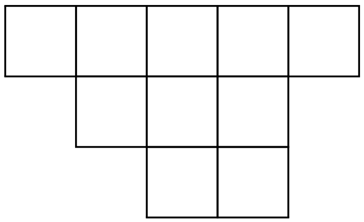
- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

of $n \times n$ ASMs = $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$

Mills, Robbins, Rumsey, Zeilberger, Kuperberg in the 1980s and 1990s.

Descending Plane Partitions = DPPs

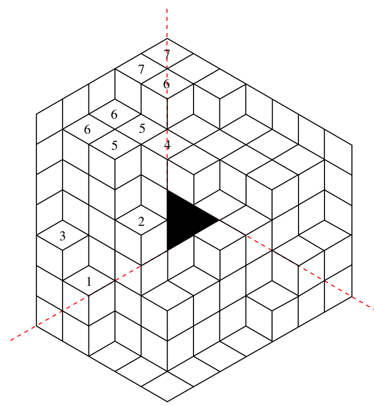
• A strict partition is a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ with distinct parts, i.e., $\lambda_1 > \lambda_2 > \ldots > \lambda_l > 0$. The shifted Young diagram of shape $(5, 3, 2)$ is as follows.



• A column strict shifted plane partition is a filling of a shifted Young diagram with positive integers such that rows decrease weakly and columns decrease strictly.

6	6	5	5	2
	5	4	4	
		3	1	

• A DPP is such a column strict shifted PP where the first part in each row is greater than the length of its row and less than or equal to the length of the previous row. Ugly condition?



• The number of DPPs with parts no greater than n is also $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$ (Andrews).

Bijection 1 (Bijective Proof of the Product Formula)

ASM_n = set of $n \times n$ ASMs

$\mathrm{ASM}_{n,i}$ = set of $n \times n$ ASMs $(a_{p,q})_{1 \le p,q \le n}$ with $a_{1,i} = 1$

B_n = set of $(2n-1)$-subsets of $[3n-2] = \{1, 2, \ldots, 3n-2\}$; $|\mathrm{B}_n| = \binom{3n-2}{2n-1}$

$\mathrm{B}_{n,i}$ = set of elements of B_n whose median is $n+i-1$; $|\mathrm{B}_{n,i}| = \binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}$

DPP_n = set of DPPs with parts no greater than n

We have constructed a bijection between the following sets:

$$\mathrm{DPP}_{n-1} \times \mathrm{B}_{n,1} \times \mathrm{ASM}_{n,i} \longrightarrow \mathrm{DPP}_{n-1} \times \mathrm{ASM}_{n-1} \times \mathrm{B}_{n,i}$$

Then we also have a bijection

$$\mathrm{DPP}_{n-1} \times \mathrm{B}_{n,1} \times \mathrm{ASM}_n \longrightarrow \mathrm{DPP}_{n-1} \times \mathrm{ASM}_{n-1} \times \mathrm{B}_n\,.$$

Iterating this, we obtain a bijection

$$\mathrm{DPP}_0 \times \cdots \times \mathrm{DPP}_{n-1} \times \mathrm{B}_{1,1} \times \cdots \times \mathrm{B}_{n,1} \times \mathrm{ASM}_n \longrightarrow \mathrm{DPP}_0 \times \cdots \times \mathrm{DPP}_{n-1} \times \mathrm{B}_1 \times \cdots \times \mathrm{B}_n\,.$$

Example: $\mathrm{DPP}_2 \times \mathrm{B}_{3,1} \times \mathrm{ASM}_{3,2} \longrightarrow \mathrm{DPP}_2 \times \mathrm{ASM}_2 \times \mathrm{B}_{3,2}$ for $x = 0$

$$\begin{array}{lcl}
\left(\varnothing, 12345, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23457)\\
\left(\varnothing, 12346, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13457)\\
\left(\varnothing, 12347, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12457)\\
\left(\varnothing, 12356, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13456)\\
\left(\varnothing, 12357, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13457)\\
\left(\varnothing, 12367, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13467)\\
\left(2, 12345, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23467)\\
\left(2, 12346, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13467)\\
\left(2, 12347, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12467)\\
\left(2, 12356, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23456)\\
\left(2, 12357, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23457)\\
\left(2, 12367, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23467)
\end{array}$$

$$\begin{array}{lcl}
\left(\varnothing, 12345, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23456)\\
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\left(2, 12345, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23467)\\
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\left(\varnothing, 12347, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12456)\\
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\left(2, 12367, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13467)
\end{array}$$

The python code is available at `https://www.fmf.uni-lj.si/~konvalinka/asmcode.html`.

Bijection 2 (ASMs and DPPs)

$\mathrm{DPP}_{n,i}$ = subset of DPP_n with DPPs that have $i-1$ occurrences of n.

We have constructed a bijection between the following sets:

$$\mathrm{DPP}_{n-1} \times \mathrm{ASM}_{n,i} \longrightarrow \mathrm{ASM}_{n-1} \times \mathrm{DPP}_{n,i}$$

- Once such a bijection is constructed, it follows that

$$|\mathrm{DPP}_{n-1}| \cdot |\mathrm{ASM}_{n,i}| = |\mathrm{ASM}_{n-1}| \cdot |\mathrm{DPP}_{n,i}|.$$

- By induction, we can assume $|\mathrm{DPP}_{n-1}| = |\mathrm{ASM}_{n-1}|$ and so $|\mathrm{ASM}_{n,i}| = |\mathrm{DPP}_{n,i}|$.
- Summing this over all i implies $|\mathrm{DPP}_n| = |\mathrm{ASM}_n|$.

Example $\mathrm{DPP}_3 \times \mathrm{ASM}_{4,2} \longrightarrow \mathrm{ASM}_3 \times \mathrm{DPP}_{4,2}$ for $x = 0$

$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\;2\;1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, 4\;1\;1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, 4\;1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\;2\right)$
$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\;1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, 4\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\;1\;1\right)$
$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\;1\;1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, 4\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\;2\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\right)$
$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, 4\;1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\;1\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\;3\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, 4\;3\right)$
$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}, 4\;2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}, 4\;3\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}, 4\;3\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, 4\;3\right)$
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$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, 4\;2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, 4\;2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}, 4\;1\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}, 4\;1\right)$
$\left(3\;3,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\;2\;1\right)$	$\left(3\;3,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(3\;3,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}, \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(3\;3,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}, 4\;2\;2\right)$
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II. Signed sets and sijections

A short introduction to signed sets

A signed set is a pair of disjoint finite sets: $\underline{S} = (S^+, S^-)$ with $S^+ \cap S^- = \varnothing$.

- The size of a signed set $\underline{S}$ is $|\underline{S}| = |S^+| - |S^-|$.
- The opposite signed set of $\underline{S}$ is $-\underline{S} = (S^-, S^+)$.
- The Cartesian product of signed sets $\underline{S}$ and $\underline{T}$ is

$$\underline{S} \times \underline{T} = (S^+ \times T^+ \cup S^- \times T^-, S^+ \times T^- \cup S^- \times T^+).$$

- The disjoint union of signed sets $\underline{S}$ and $\underline{T}$ is

$$\underline{S} \sqcup \underline{T} = (\underline{S} \times (\{0\}, \varnothing)) \cup (\underline{T} \times (\{1\}, \varnothing)).$$

- The disjoint union of a family of signed sets $\underline{S}_t$ indexed with a signed set $\underline{T}$ is

$$\bigsqcup_{t \in \underline{T}} \underline{S}_t = \bigcup_{t \in \underline{T}} (\underline{S}_t \times \underline{\{t\}}).$$

Addition, subtraction, multiplication but not division

Recall our approach: We translate some of my non-bijective proofs into combinatorics.

Note that

- $|\underline{S} \sqcup \underline{T}| = |\underline{S}| + |\underline{T}|$,
- $|-\underline{S}| = -|\underline{S}|$, and
- $|\underline{S} \times \underline{T}| = |\underline{S}| \cdot |\underline{T}|$.

and so we can "deal" with all arithmetic operations except for division.

The latter explains the "redundant" factors in our bijections.

Crucial example: Signed intervals

For $a, b \in \mathbb{Z}$, we set

$$\underline{[a,b]} = \begin{cases} ([a,b], \varnothing) & \text{if } a \leq b \\ (\varnothing, [b+1, a-1]) & \text{if } a > b+1 \\ (\varnothing, \varnothing) & \text{if } a = b+1 \end{cases},$$

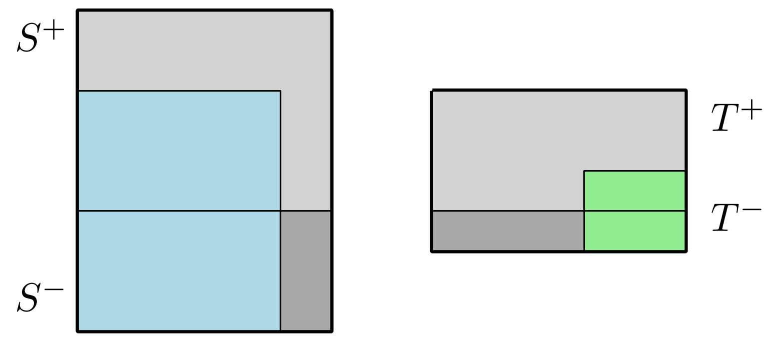
where $[a, b]$ stands for an interval in $\mathbb{Z}$ in the usual sense.

The signed sets in our constructions are typically signed boxes (= Cartesian products of signed intervals) and disjoint unions of signed boxes.

Sijections

The role of bijections for signed sets is played by "signed bijections", which we call sijections. A sijection is a "manifestation" of the fact that two signed sets have the same size.

A sijection is a collection of
- a sign-reversing involution on a subset of $\underline{S}$,
- a sign-reversing involution on a subset of $\underline{T}$,
- a sign-preserving bijection between the remaining elements of $\underline{S}$ and the remaining elements of $\underline{T}$.



Simpler: A sijection φ from $\underline{S}$ to $\underline{T}$, $\varphi\colon \underline{S} \Rightarrow \underline{T}$, is an involution on the set $(S^+ \cup S^-) \sqcup (T^+ \cup T^-)$ with the property $\varphi(S^+ \sqcup T^-) = S^- \sqcup T^+$. This implies:

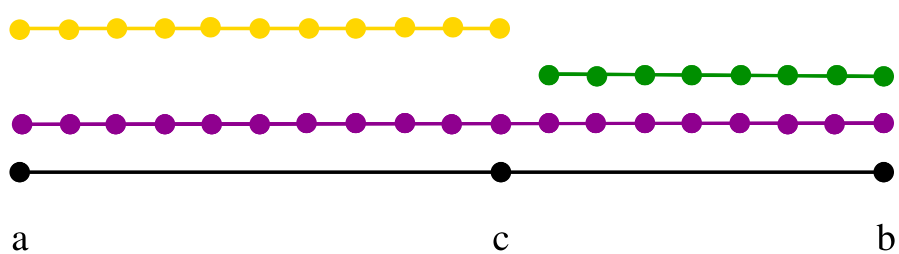
$$|\underline{S}| = |S^+| - |S^-| = |T^+| - |T^-| = |\underline{T}|$$

The fundamental sijection

Given $a, b, c \in \mathbb{Z}$, construct a sijection

$$\alpha = \alpha_{a,b,c} \colon [a,c] \Rightarrow [a,b] \sqcup [b+1,c].$$

Construction: For $a \le b \le c$ and $c < b < a$, there is nothing to prove. For, say, $a \le c < b$, we have that $[b+1,c] = -[c+1,b]$ is "contained" in $[a,b]$, but due to its opposite sign this subset "cancels" and what remains is $[a,c]$.

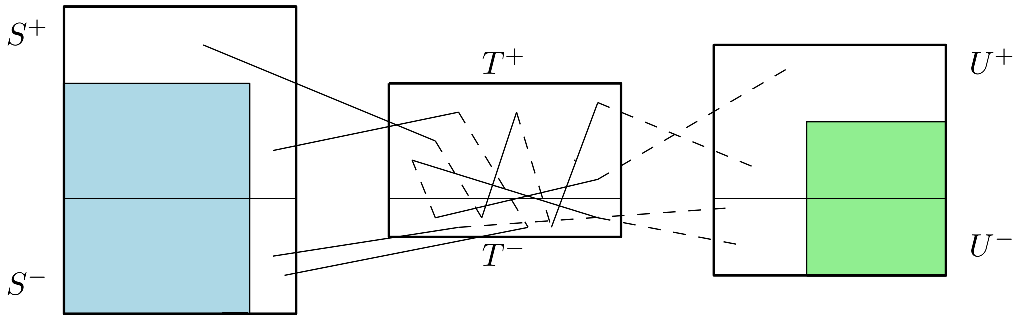


The cases $b < a \le c$, $b \le c < a$, and $c < a \le b$ are analogous.

Use straightforward constructions for the Cartesian product of sijections and the disjoint union to obtain more complicated sijections.

Composition of sijections

Suppose $\underline{S}, \underline{T}, \underline{U}$ are signed sets and $\varphi : \underline{S} \Rightarrow \underline{T}$, $\psi : \underline{T} \Rightarrow \underline{U}$, then we can construct a sijection $\psi \circ \varphi : \underline{S} \to \underline{U}$ by alternating applications of φ (solid lines) and ψ (dashed lines) as sketched next.



The special case $S^- = U^- = \varnothing$ is the **Garsia-Milne involution principle.**

III. Some details of our constructions

ASMs $\longrightarrow$ Monotone Triangles

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 2 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

- A monotone triangle is a triangular array of integers that increases weakly in $\nearrow$-direction and in $\searrow$-direction, and strictly along rows.

- The set of monotone triangles with bottom row $k_1, \ldots, k_n$ is denoted by $\underline{MT}(k_1, \ldots, k_n)$.

- If we drop the condition that rows are strictly increasing, then we obtain the well-known Gelfand-Tsetlin patterns.

Gelfand-Tsetlin patterns with arbitrary bottom row in $\mathbb{Z}^n$

• Gelfand-Tsetlin patterns have weakly increasing rows, in particular this is true for the bottom row $(k_1, \ldots, k_n)$.

• The number of Gelfand-Tsetlin patterns with bottom row $k_1, \ldots, k_n$ is $\prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}$.

• There is a very natural extension of Gelfand-Tsetlin patterns to arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and a very natural notion of a sign.

• The signed set of these extended Gelfand-Tsetlin patterns with bottom row $(k_1, \ldots, k_n)$ is denoted by $\underline{GT}(k_1, \ldots, k_n)$, and we have

$$|\underline{GT}(k_1, \ldots, k_n)| = \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}.$$

• Also monotone triangles can be defined for any bottom row $(k_1, \ldots, k_n) \in \mathbb{Z}^n$. The signed set of these monotone triangles is denoted by $\underline{MT}(k_1, \ldots, k_n)$.

Arrow patterns ...

... are triangular arrays

$$T = \begin{array}{ccccccccc} & & & & t_{1,n} & & & & \\ & & & t_{1,n-1} & & t_{2,n} & & & \\ & & t_{1,n-2} & & t_{2,n-1} & & t_{3,n} & & \\ & \iddots & \vdots & \ddots & \vdots & \iddots & \vdots & \ddots & \\ t_{1,2} & & t_{2,3} & & \dots & & \dots & & t_{n-1,n} \end{array},$$

with $t_{p,q} \in \{\swarrow, \searrow, \swarrow\!\!\!\!\!\searrow\}$. The sign of an arrow pattern is 1 if the number of $\swarrow\!\!\!\!\!\searrow$'s is even and -1 otherwise, and the signed set of arrow patterns of order n is denoted by $\underline{AP}_n$.

The role of an arrow pattern of order n is that it induces a deformation of $(k_1, \ldots, k_n)$:

- Add $k_1, \ldots, k_n$ as bottom row of T (i.e., $t_{i,i} = k_i$).
- For each $\swarrow$ or $\swarrow\!\!\!\!\!\searrow$ which is in the same $\swarrow$-diagonal as k_i add 1 to k_i.
- For each $\searrow$ or $\swarrow\!\!\!\!\!\searrow$ which is in the same $\searrow$-diagonal as k_i subtract 1 from k_i.

$$\begin{array}{ccccccccc} & & & & \searrow & & & & \\ & & & \swarrow\!\!\!\!\!\searrow & & \searrow & & & \\ & & \searrow & & \swarrow & & \swarrow\!\!\!\!\!\searrow & & \\ & \swarrow\!\!\!\!\!\searrow & & \swarrow & & \swarrow\!\!\!\!\!\searrow & & \searrow & \\ k_1-0+2 & & k_2-1+2 & & k_3-1+2 & & k_4-2+0 & & k_5-4+0 \end{array}$$

We let $d(\mathbf{k}, T)$ denote this deformation for $\mathbf{k} = (k_1, \ldots, k_n)$ and $T \in \underline{AP}_n$.

Shifted Gelfand-Tsetlin patterns

For $\mathbf{k} = (k_1, \ldots, k_n)$, a shifted Gelfand-Tsetlin pattern is the disjoint union of deformed Gelfand-Tsetlin patterns over arrow patterns of order n:

$$\underline{SGT}(\mathbf{k}) = \bigsqcup_{T \in \underline{AP}_n} \underline{GT}(d(\mathbf{k}, T)).$$

Given $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $x \in \mathbb{Z}$, we have constructed a sijection

$$\Gamma = \Gamma_{\mathbf{k},x} \colon \underline{MT}(\mathbf{k}) \Rightarrow \underline{SGT}(\mathbf{k}).$$

Example $\mathbf{k} = (1,2,3)$ and $x = 0$

$$\begin{array}{c} 1 \\ 1\ 2 \\ 1\ 2\ 3 \end{array} \leftrightarrow \left(\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 1\ 1 \end{array} , \begin{array}{c} \searrow \\ \searrow\ \searrow \end{array} \right) \qquad \begin{array}{c} 2 \\ 1\ 2 \\ 1\ 2\ 3 \end{array} \leftrightarrow \left(\begin{array}{c} 2 \\ 1\ 2 \\ 1\ 2\ 2 \end{array} , \begin{array}{c} \searrow \\ \searrow\ \swarrow \end{array} \right) \qquad \begin{array}{c} 1 \\ 1\ 3 \\ 1\ 2\ 3 \end{array} \leftrightarrow \left(\begin{array}{c} 1 \\ 1\ 2 \\ 1\ 2\ 2 \end{array} , \begin{array}{c} \searrow \\ \searrow\ \swarrow \end{array} \right)$$

$$\begin{array}{c} 2 \\ 1\ 3 \\ 1\ 2\ 3 \end{array} \leftrightarrow \left(\begin{array}{c} 2 \\ 2\ 3 \\ 2\ 2\ 3 \end{array} , \begin{array}{c} \swarrow \\ \searrow\ \swarrow \end{array} \right) \qquad \begin{array}{c} 3 \\ 1\ 3 \\ 1\ 2\ 3 \end{array} \leftrightarrow \left(\begin{array}{c} 3 \\ 2\ 3 \\ 2\ 2\ 3 \end{array} , \begin{array}{c} \swarrow \\ \searrow\ \swarrow \end{array} \right) \qquad \begin{array}{c} 2 \\ 2\ 3 \\ 1\ 2\ 3 \end{array} \leftrightarrow \left(\begin{array}{c} 2 \\ 2\ 2 \\ 3\ 1\ 2 \end{array} , \begin{array}{c} \swarrow \\ \swarrow\!\!\!\!\!\searrow\ \searrow \end{array} \right)$$

$$\begin{array}{c} 3 \\ 2\ 3 \\ 1\ 2\ 3 \end{array} \leftrightarrow \left(\begin{array}{c} 3 \\ 3\ 3 \\ 3\ 3\ 3 \end{array} , \begin{array}{c} \swarrow \\ \swarrow\ \swarrow \end{array} \right) \qquad \left(\begin{array}{c} 2 \\ 2\ 2 \\ 2\ 2\ 3 \end{array} , \begin{array}{c} \swarrow \\ \searrow\ \swarrow \end{array} \right) \leftrightarrow \left(\begin{array}{c} 2 \\ 2\ 2 \\ 2\ 2\ 2 \end{array} , \begin{array}{c} \swarrow\!\!\!\!\!\searrow \\ \searrow\ \swarrow \end{array} \right)$$

$$\left(\begin{array}{c} 2 \\ 2\ 2 \\ 2\ 3\ 1 \end{array} , \begin{array}{c} \searrow \\ \swarrow\ \swarrow\!\!\!\!\!\searrow \end{array} \right) \leftrightarrow \left(\begin{array}{c} 2 \\ 2\ 2 \\ 2\ 2\ 2 \end{array} , \begin{array}{c} \swarrow \\ \searrow\ \swarrow\!\!\!\!\!\searrow \end{array} \right) \qquad \left(\begin{array}{c} 2 \\ 2\ 2 \\ 1\ 2\ 2 \end{array} , \begin{array}{c} \searrow \\ \searrow\ \swarrow \end{array} \right) \leftrightarrow \left(\begin{array}{c} 2 \\ 2\ 2 \\ 2\ 2\ 2 \end{array} , \begin{array}{c} \searrow \\ \swarrow\!\!\!\!\!\searrow\ \swarrow \end{array} \right)$$

Example $\mathbf{k} = (1,2,3)$ and $x = 1$

$$\begin{array}{c}1\\1\,2\\1\,2\,3\end{array} \leftrightarrow \left(\begin{array}{c}1\\1\,1\\1\,1\,1\end{array}, \begin{array}{c}\searrow\\\searrow\;\searrow\end{array}\right) \qquad \begin{array}{c}2\\1\,2\\1\,2\,3\end{array} \leftrightarrow \left(\begin{array}{c}2\\2\,2\\2\,2\,3\end{array}, \begin{array}{c}\swarrow\\\searrow\;\swarrow\end{array}\right) \qquad \begin{array}{c}1\\1\,3\\1\,2\,3\end{array} \leftrightarrow \left(\begin{array}{c}1\\1\,2\\1\,2\,2\end{array}, \begin{array}{c}\searrow\\\searrow\;\swarrow\end{array}\right)$$

$$\begin{array}{c}2\\1\,3\\1\,2\,3\end{array} \leftrightarrow \left(\begin{array}{c}2\\2\,3\\2\,2\,3\end{array}, \begin{array}{c}\swarrow\\\searrow\;\swarrow\end{array}\right) \qquad \begin{array}{c}3\\1\,3\\1\,2\,3\end{array} \leftrightarrow \left(\begin{array}{c}3\\2\,3\\2\,2\,3\end{array}, \begin{array}{c}\swarrow\\\searrow\;\swarrow\end{array}\right) \qquad \begin{array}{c}2\\2\,3\\1\,2\,3\end{array} \leftrightarrow \left(\begin{array}{c}2\\2\,2\\1\,2\,2\end{array}, \begin{array}{c}\searrow\\\searrow\;\swarrow\end{array}\right)$$

$$\begin{array}{c}3\\2\,3\\1\,2\,3\end{array} \leftrightarrow \left(\begin{array}{c}3\\3\,3\\3\,3\,3\end{array}, \begin{array}{c}\swarrow\\\swarrow\;\swarrow\end{array}\right) \qquad \left(\begin{array}{c}2\\1\,2\\1\,2\,2\end{array}, \begin{array}{c}\searrow\\\searrow\;\swarrow\end{array}\right) \leftrightarrow \left(\begin{array}{c}2\\2\,2\\2\,2\,2\end{array}, \begin{array}{c}\searrow\!\!\!\!\!\!\swarrow\\\searrow\;\swarrow\end{array}\right)$$

$$\left(\begin{array}{c}2\\2\,2\\3\,1\,2\end{array}, \begin{array}{c}\swarrow\\\searrow\!\!\!\!\!\!\swarrow\;\searrow\end{array}\right) \leftrightarrow \left(\begin{array}{c}2\\2\,2\\2\,2\,2\end{array}, \begin{array}{c}\swarrow\\\searrow\;\searrow\!\!\!\!\!\!\swarrow\end{array}\right) \qquad \left(\begin{array}{c}2\\2\,2\\2\,3\,1\end{array}, \begin{array}{c}\searrow\\\swarrow\;\searrow\!\!\!\!\!\!\swarrow\end{array}\right) \leftrightarrow \left(\begin{array}{c}2\\2\,2\\2\,2\,2\end{array}, \begin{array}{c}\searrow\\\searrow\!\!\!\!\!\!\swarrow\;\swarrow\end{array}\right)$$

Rotation of monotone triangles

Given $(k_1, \ldots, k_n)$**, we have constructed a sijection**

$$\underline{MT}(k_1, \ldots, k_n) \Longrightarrow (-1)^{n-1}\, \underline{MT}(k_2, \ldots, k_n, k_1 - n).$$

Using $\Gamma : \underline{MT} \Longrightarrow \underline{SGT}$, it suffices to construct a sijection

$$\underline{SGT}(k_1, \ldots, k_n) \Longrightarrow (-1)^{n-1}\, \underline{SGT}(k_2, \ldots, k_n, k_1 - n).$$

...after several more steps we obtain the Bijections 1 & 2.

Thank you!